Optics 335

HW # 1

Solution

Physics Department

Physics 335	Spring 2022	Assignment#1

1. Find the resultant wave resulted from adding the three waves given below.

$$
\begin{aligned}
y_1 &= 2\cos(kx - \omega t + \frac{\pi}{3}) \\
y_2 &= 2\cos(kx - \omega t + \frac{\pi}{6}) \\
y_3 &= 2\cos(kx - \omega t + \frac{\pi}{9})
\end{aligned}
$$

2. Show that the function $S(x,t) = A\exp(2x - 3t)$ is a solution of the classical wave equation. What is the velocity of the wave and its wavelength. In what direction it is propagating?

3. Let $x = 3 + 2i$, $y = 3 - 4i$. Express the following in both Cartesian form $(a + ib)$ and polar form $Ae^{i\phi}$.
   (a) $x - y$
   (b) x^2
   (c) $\frac{y}{x}$
   (d) e^y

4. For an electric field of the form $\vec{E} = \vec{E_0}\exp i(\vec{k}.\vec{r} - \omega t + \phi)$, show that
   (a) $\nabla.\vec{E} = i\vec{k}.\vec{E}$
   (b) $\nabla \times \vec{E} = i\vec{k} \times \vec{E}$

5. The E-field of an electromagnetis wave is given by

$$\vec{E} = (\hat{i} + \hat{j})E_0 \sin(kz - \omega t + \frac{\pi}{6})$$

.

   (a) Find the associated magnetic field.
   (b) Find the irradiance of the wave.

Q1

$$y_1 = 2\cos\left(kx - \omega t + \frac{\pi}{3}\right) = 2\,Re\, e^{i\left(kx-\omega t+\frac{\pi}{3}\right)}$$

$$y_2 = 2\cos\left(kx - \omega t + \frac{\pi}{6}\right) = 2\,Re\, e^{i\left(kx-\omega t+\frac{\pi}{6}\right)}$$

$$y_3 = 2\cos\left(kx - \omega t + \frac{\pi}{9}\right) = 2\,Re\, e^{i\left(kx-\omega t+\frac{\pi}{9}\right)}$$

$$y_R = y_1 + y_2 + y_3 = 2\,Re\left\{ e^{i(kx-\omega t)} \left[e^{i\frac{\pi}{3}} + e^{i\frac{\pi}{6}} + e^{i\frac{\pi}{9}} \right] \right\}$$

$$= 2Re\left\{ e^{i(kx-\omega t)} \left[\left(\cos\frac{\pi}{3} + \cos\frac{\pi}{6} + \cos\frac{\pi}{9}\right) + i\left(\sin\frac{\pi}{3} + \sin\frac{\pi}{6} + \sin\frac{\pi}{9}\right) \right] \right\}$$

$$= 2Re\left\{ e^{i(kx-\omega t)} \left[4.61 + 3.42i \right] \right\}$$

but $4.61 + 3.42i = \sqrt{(4.61)^2 + (3.42)^2}\; e^{i \tan^{-1}\frac{3.42}{4.61}}$

$$= 5.74\, e^{i\,0.2\pi}$$

hence,

$$y_R = 5.74\cos\left(kx - \omega t + 0.2\pi\right)$$

Q2

$$S(x,t) = Ae^{2x-3t}$$

$$\frac{dS}{dt} = -3Ae^{2x-3t}$$

$$\frac{\partial^2 S}{\partial t^2} = 9Ae^{+2x-3t}$$

$$\frac{\partial S}{\partial x} = 2Ae^{2x-3t}$$

$$\frac{\partial^2 S}{\partial x^2} = 4Ae^{2x-3t}$$

$$\frac{\partial^2 S}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 S}{\partial t^2}$$

$$4Ae^{2x-3t} = \frac{1}{v^2} \cdot 9Ae^{2x-3t}$$

$$\Rightarrow \frac{1}{v^2} = \frac{4}{9} \Rightarrow v = \frac{3}{2} = 1.5$$

$$S(x,t) = f(kx-wt) \Rightarrow k = 2$$

$$k = \frac{2\pi}{\lambda} = 2 \Rightarrow \lambda = \pi$$

$f(kx-wt)$ propagates in the positive x direction

Q3

$x = 3 + 2i \quad , \quad y = 3 - 4i$

a) $x - y = 0 + 6i$

$x - y = \sqrt{0^2 + 6^2} \, e^{i \tan^{-1} \frac{6}{0}}$

$= 6 \, e^{i \tan^{-1} \infty} \qquad = 6 \, e^{i \frac{\pi}{2}}$

b) $x^2 = (3+2i)^2 = 9 + 12i - 4 = 5 + 12i$

$x^2 = \sqrt{5^2 + 12^2} \; e^{i \tan^{-1} \frac{12}{5}}$

$= 13 \, e^{i \, 0.37\pi}$

c) $\frac{y}{x} = \frac{3-4i}{3+2i} = \frac{(3-4i)(3-2i)}{(3+2i)(3-2i)}$

$= \frac{9-8-18i}{3^2+2^2} = \frac{1-18i}{13}$

$\frac{y}{x} = \sqrt{\left(\frac{1}{13}\right)^2 + \left(\frac{18}{13}\right)^2} \; e^{i \tan^{-1} \frac{-18}{1}}$

$= 1.39 \, e^{i \, 0.48\pi}$

d) $e^{y} = e^{3-4i} = e^{3}\left[\cos 4 - i \sin 4\right]$

$= 20.09\left[-0.65 + 0.76i\right]$

$= -13.1 + 15.2i$

$e^{y} = 20 \, e^{-0.37\pi}$

Q4

a) $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t + \phi)}$

$= \hat{i} E_x + \hat{j} E_y + \hat{k} E_z$

$= (\hat{i} E_{0x} + \hat{j} E_{0y} + \hat{k} E_{0z}) e^{i(k_x x + k_y y + k_z z - \omega t + \phi)}$

$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

$\frac{\partial E_x}{\partial x} = \frac{\partial}{\partial x} E_{0x} e^{i(k_x x + k_y y + k_z z - \omega t + \phi)}$

$= i k_x E_x$

Similarly:

$\frac{\partial E_y}{\partial y} = i k_y E_y$, and $\frac{\partial E_z}{\partial z} = i k_z E_z$

Hence,

$\vec{\nabla}\cdot\vec{E} = i(k_x E_x + k_y E_y + k_z E_z)$

$= i(\vec{k}\cdot\vec{E})$

b) $\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$

4-b) cont...

$$= \hat{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{j}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

As in part a), we do the derivative and get

$$\vec{\nabla} \times \vec{E} = i\left[\hat{i}(k_y E_z - k_z E_y) - \hat{j}(k_x E_z - k_z E_x) + \hat{k}(k_x E_y - k_y E_x)\right]$$

$$= i\, \vec{k} \times \vec{E}$$

Q_5

a) $\vec{B} = \frac{1}{\omega} \vec{k} \times \vec{E}$

$= \frac{k}{\omega} \left[\hat{k} \times (\hat{i} + \hat{j}) \right] E_0 \sin\left(kz - \omega t + \frac{\pi}{6}\right)$

$= \frac{E_0}{c} \left[\hat{j} - \hat{i} \right] \sin\left(kz - \omega t + \frac{\pi}{6}\right)$

b) $I = \frac{1}{2} c \epsilon_0 E_0^2$

$= \frac{1}{2} \times 3 \times 10^{8} \times 8.85 \times 10^{-12} E_0^2$

$= 1.3 \times 10^{-3} E_0^2$