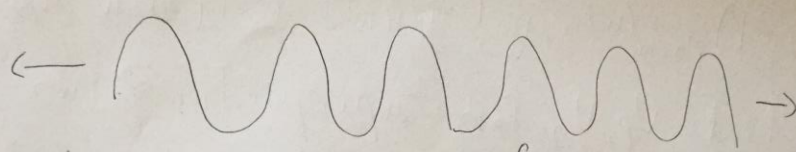
7.3 Frequency Spectrum of Light



Plane wave with one frequency ω has infinite length and infinite duration.

A waveform that does not repeat, like a pulse is given by:

$$\vec{E}(\vec{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}(\vec{r},\omega)\, e^{-i\omega t}\, d\omega$$

Spectrum

gives amplitude and phase of each plane wave component in the waveform.

$$\vec{E}(\vec{r},w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{E}(\vec{r},t)\, e^{iwt}\, dt$$ 1

Fourier Transform

Note: We will not ~~through~~ throw away the imaginary part in $\vec{E}(\vec{r},w)$.

Power Spectrum Definition:

$$\underline{I}(\vec{r},w) = \frac{n\varepsilon_0 c}{2} \vec{E}(\vec{r},w) \cdot \vec{E}^{*}(\vec{r},w)$$

what we see when light goes through a spectrometer; different colors etc.

We will use Parseval's Theorem:

$$\int_{-\infty}^{\infty} \underline{I}(\vec{r},t)\, dt = \int_{-\infty}^{\infty} \underline{I}(\vec{r},w)\, dw$$

In general: $\int_{-\infty}^{\infty} |f(w)|^2 dw = \int_{-\infty}^{\infty} |f(t)|^2 dt$

Example 7.2

let $\vec{E}(\vec{r},t) = \vec{E}_0(\vec{r})\, e^{\frac{-t^2}{2T^2}}\, e^{-i\omega_0 t}$

Sketch of real part:

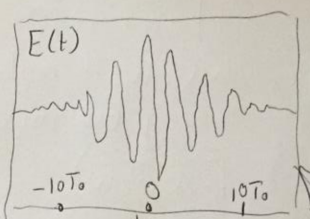


fig 7.3

Find $\vec{E}(\vec{r},\omega)$?

$$\vec{E}(\vec{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{E}(\vec{r},t)\, e^{i\omega t}\, dt$$

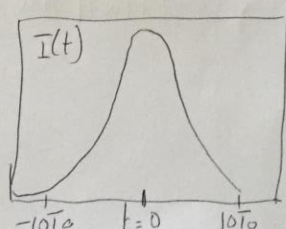
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{E}_0(\vec{r})\, e^{\frac{-t^2}{2T^2}}\, e^{-i\omega_0 t}\, e^{i\omega t}\, dt$$

$$= \frac{\vec{E}_0(\vec{r})}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-t^2}{2T^2} + i(\omega-\omega_0)t}\, dt$$

Using (0.55): $\displaystyle\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\, dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}+c}$

$$\vec{E}(\vec{r},\omega) = T\,\vec{E}_0(\vec{r})\, e^{-\frac{T^2(\omega-\omega_0)^2}{2}}$$

Fig 7.4



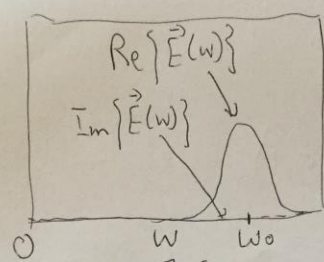


Fig 7.5

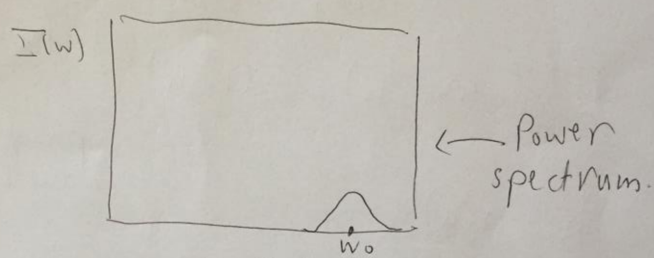


Fig 7.6

Note: If "Pulse" is shorter, then Frequency "Pulse" in frequency space is longer.

7.4 Wave Packet Propagation and group delay

let $\vec{E}(\vec{r}_0,t)$ = electric field at point $\vec{r}_0$ and time t.

$$\vec{E}(\vec{r}_0,w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{E}(\vec{r}_0,t)\,e^{-iwt}\,dt$$

$$\vec{E}(\vec{r}_0+\Delta\vec{r},w) = E(\vec{r}_0,w)\,\underbrace{e^{i\vec{k}(w)\cdot\Delta\vec{r}}}_{\text{phase shift due to } \Delta\vec{r}} \quad ??$$

$$k = n(w)\frac{w}{c}$$

We take the inverse fourier transform of $\vec{E}(\vec{r}_0+\Delta\vec{r},w)$ at the new position $\vec{r}_0+\Delta r$ to get $\vec{E}(\vec{r}_0+\Delta\vec{r},t)$

$$\therefore\ \vec{E}(\vec{r}_0+\Delta\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{E}(\vec{r}_0+\Delta\vec{r},w)\,e^{-iwt}\,dw$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{E}(\vec{r}_0,w)\,e^{i(\vec{k}(w)\cdot\Delta\vec{r}-wt)}\,dw$$

Example 7.6

$$\vec{E}(0,t) = \vec{E}_0 \, e^{-t^2/2T^2} \, e^{-i w_0 t}$$

Find $\vec{E}(\vec{r} = z\hat{z}, t)$ in vacuum? Propagation in $\hat{z}$ direction.

$$\vec{E}(0,w) = T \vec{E}_0 \, e^{-\frac{T^2 (w-w_0)^2}{2}}$$

$$\vec{E}(\vec{r}_0 + \Delta\vec{r}, w) = \vec{E}(\vec{r}_0, w) \, e^{i \vec{k}(w) \cdot \Delta \vec{r}}$$

$$\vec{k}(w) = k_{vac}(w) \hat{z} = \frac{w}{c} \hat{z}$$

$$\vec{E}(z,w) = \cancel{\vec{E}(0,w)} \, T \vec{E}_0 \, e^{-\frac{T^2 (w-w_0)^2}{2}} \, e^{i \frac{w}{c} z}$$

$$\vec{E}(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{E}_0 \, T e^{-\frac{T^2 (w-w_0)^2}{2}} \, e^{i \frac{w}{c} z} \, e^{-iwt} \, dw$$

$$= \vec{E}_0 \, e^{-\frac{(t - z/c)^2}{2T^2}} \, e^{-i w_0 (t - \frac{z}{c})}$$

This is the same original pulse, only delayed by $\frac{z}{c}$.

Dispersion: Temporal (time).

Different frequency components have different indices of refraction.
Each frequency has its own phase velocity.

The pulse speed is given by group velocity.

$$\vec{E}(\vec{r}_0+\Delta\vec{r},w) = \vec{E}(\vec{r}_0,w)\, e^{i\vec{k}(w)\cdot\Delta\vec{r}}$$

phase delay.

We expand $\vec{k}\cdot\Delta\vec{r}$ about w_0 (center frequency).

$$\vec{k}\cdot\Delta\vec{r} \cong \left[\vec{k}\Big|_{w_0} + \frac{\partial\vec{k}}{\partial w}\Big|_{w_0}(w-w_0) + \frac{1}{2}\frac{\partial^2\vec{k}}{\partial w^2}\Big|_{w_0}(w-w_0)^2 \right]\cdot\Delta\vec{r}$$

Keep 1st and second terms:

$$\vec{E}(\vec{r}_0+\Delta\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{E}(\vec{r}_0,w)\, e^{i\left(\left[\vec{k}(w_0)+\frac{\partial\vec{k}}{\partial w}\Big|_{w_0}(w-w_0)\right]\cdot\Delta\vec{r}-wt\right)} dw$$

$$= e^{i\left[\vec{k}(w_0)-w_0\frac{\partial\vec{k}}{\partial w}\Big|_{w_0}\right]\cdot\Delta\vec{r}}\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\vec{E}(\vec{r}_0,w)\, e^{-iw\left(t-\frac{\partial\vec{k}}{\partial w}\Big|_{w_0}\cdot\Delta\vec{r}\right)} dw$$

$$= e^{i[\vec{k}(w_0)\cdot\Delta\vec{r} - w_0 t']} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \vec{E}(\vec{r}_0, w)\, e^{-iw(t-t')}\, dw$$

Where $t' \equiv \left.\frac{\partial \vec{k}}{\partial w}\right|_{w_0} \cdot \Delta\vec{r}$

$$\therefore \vec{E}(\vec{r}_0 + \Delta\vec{r}, t) = \vec{E}(\vec{r}_0, \underbrace{t-t'}_{})\, e^{i(\vec{k}(w_0)\cdot\Delta\vec{r} - w_0 t')}$$

Let us go back to propagation in vacum in $\hat{z}$:

$$n = 1 \rightarrow k = \frac{w}{c}$$

$$t' \equiv \left.\frac{\partial k}{\partial w}\right|_{w_0} \cdot \underbrace{\Delta\vec{r}}_{z\hat{z}} = \frac{z}{c}$$

$$\vec{E}(\vec{r}_0 + \Delta\vec{r}, t) = \vec{E}(\vec{r}_0, t - \frac{z}{c})$$

In general, the term $e^{i(\vec{k}(w_0)\cdot\Delta\vec{r} - w_0 t')}$ gives a phase shift with a phase velocity of $\frac{w_0}{k(w_0)} = v_p(w_0)$.

$\frac{\partial \vec{k}}{\partial w} \cdot \Delta\vec{r}$ is the groupdelay function.

v_g is obtained from $\frac{\Delta\vec{r}}{t'}$ to get $v_g^{-1}(w_0) = \frac{\left.\frac{\partial \vec{k}}{\partial w}\right|_{w_0} \cdot \Delta\vec{r}}{\Delta\vec{r}} = \left.\frac{\partial k(w)}{\partial w}\right|_{w_0}$

V_g gives the velocity for the center of the pulse.