Conduction Heat Transfer

Notes for MECH 7210

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Preface

The *Notes on Conduction Heat Transfer* are, as the name suggests, a compilation of lecture notes put together over ~ 10 years of teaching the subject. The notes are not meant to be a comprehensive presentation of the subject of heat conduction, and the student is referred to the texts referenced below for such treatments. A goal of mine, in preparing the notes, has been to address an apparent shortcoming in many of the current texts, in that the texts present the mathematical formulation and analytical solution to a wide variety of conduction problems, yet they spend little if any time on discussing how numerical and graphical results can be obtained from the solutions. As will be seen, this task in itself is not trivial, and to this end mathematical software packages (in particular, the package *Mathematica*) will be used extensively in application of the analytical solutions.

The notes were prepared using the LaTeX typesetting program, which is freely available via internet download. I wish to thank my former students, who have (and continue) to catch the multitude of mistakes and typos in the notes.

These notes are dedicated to the memory of Clifford Cremers, an outstanding teacher of heat transfer and a fine fly fisherman.

References

The 'text' in the course will consist of my lecture notes – which contain few if any literature citations. I will need to fix this if I ever expect to publish the notes as a book. The following reference texts were used to prepare the notes.

1. Carslaw, H. S., and Jaeger, J. C., *Conduction of Heat in Solids*: A compendium of analytical solutions for practically every conceivable problem. Very mathematical and hard to read.

2. Myers, G. E., *Analytical Methods in Conduction Heat Transfer*: most closely follows the lecture notes. A good introduction text.

3. Poulikakos, D., *Conduction Heat Transfer*: A basic graduate–level text, similar to Myers but with more engineering applications.

4. Arpaci, V. S., *Conduction Heat Transfer*

5. Ozisik, M. N., *Heat Conduction*

6. Kakac, S., and Yener, Y., *Heat Conduction*

Contents

Chapter 1

Preliminaries and Review

1.1 The Conduction Equation

The basic objective of this course can be stated as: given an object that is subjected to known temperature and/or heat flux conditions on the surface, predict the distribution of temperature and heat transfer within the object. The fundamental physical principle we will employ to meet this objective is conservation of energy – often referred to as the first law of thermodynamics. Thermodynamics, however, is typically applied to a system at equilibrium, whereas we will be dealing with systems that most definitely are not at equilibrium. For example, you may want to predict how long it takes a rod of hot metal to cool to the ambient temperature, or predict the rate of heat transfer through a slab that is maintained at different temperatures on the opposite faces. In such situations the temperature throughout the medium will, generally, not be uniform – for which the usual principles of equilibrium thermodynamics do not apply. What is needed, therefore, is a first–law statement that applies to the discrete elements within a nonequilibrium system – as opposed to the system as a whole.

In undergraduate heat transfer you were presented with such an analysis – which typically involved applying the first law to a small, 'differential' control volume within the system. Presented here is an alternative (and more mathematically elegant) method for obtaining the differential equation for energy conservation. It starts with an arbitrary system as shown in Fig. 1.1. Assuming that the volume of the system is fixed (so that no work is transferred) and it's mass is constant, energy conservation is simply described by

$$\frac{dE}{dt} = \dot{Q} \tag{1.1}$$

in which $\dot{Q}$ is the rate of heat transfer into the system and E is the energy of the system. If the system is not in equilibrium then E cannot be related to a single temperature of the system[1]. It is

[1] an average temperature could be defined from E, but this would not be of much use in predicting heat transfer

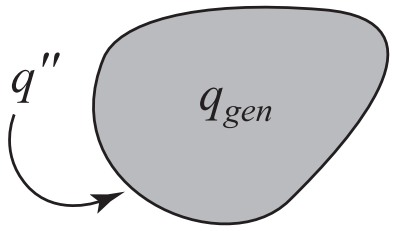


Figure 1.1: an abritrary system

possible, however, to represent E as a sum of energies of small volume elements within the system – with each element assumed to be in thermodynamic equilibrium at any instant. As the volume of the elements go to zero the sum can be expressed as an integral, which gives:

$$\frac{dE}{dt} = \int_V \rho \frac{\partial e}{\partial t}\, dV = \int_V \rho c \frac{\partial T}{\partial t}\, dV \tag{1.2}$$

where e is the specific energy, ρ is the density, c is the specific heat and the integral is over the volume of the system. The heat transfer can also be written in integral form as

$$\dot{Q} = -\int_A \mathbf{q}'' \cdot \mathbf{n}\, dA + \int_V q'''\, dV \tag{1.3}$$

In the first integral $\mathbf{q}''$ is the heat flux vector, $\mathbf{n}$ is the normal outward vector at the surface element dA (which is why the minus sign is present) and the integral is taken over the area of the system. The second integral represents the generation of heat within the system (through chemical or nuclear reactions, radiation absorption/emission, viscous dissapation etc.) which is described by a volumetric heat source function q''' (W/m^3).

The area integral can be transformed into a volume integral by use of the divergence theorem of vector calculus:

$$\int_A \mathbf{q}'' \cdot \mathbf{n}\, dA = \int_V \nabla \cdot \mathbf{q}''\, dV \tag{1.4}$$

The terms in the energy equation are now all in the form of volume integrals. Energy conservation therefore appears as

$$\int_V \left(\rho c \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{q}'' - q''' \right) dV = 0 \tag{1.5}$$

Realize that this equation should hold for integrals over any arbitrary volume within the system. That is, the system could be split into two volumes, and we would expect the integral to hold individually for each of the volumes. The only way that this condition can be met is for the integrand to be identically zero at all points within the system, i.e.,

$$\rho c \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{q}'' - q''' = 0 \tag{1.6}$$

which is a differential equation for energy conservation within the system. It is not of much use in the present form – because it involves two variables (T and $\mathbf{q}''$). An additional, independent means of relating heat flux to temperature is needed to 'close' the problem.

1.1.1 Fourier's Law and the thermal conductivity

Before getting into further details, a review of some of the physics of heat transfer is in order. As you recall from undergraduate heat transfer, there are three basic modes of transferring heat: conduction, radiation, and convection. Conduction is the transfer of heat through a medium by virtue of a temperature gradient in the medium. It is a microscopic–level mechanism, and results from the exchange of translational, rotational, and vibrational energy among the molecules comprising the medium. Radiation, on the other hand, is the transfer of heat via electromagnetic waves (or, equivalently, photons). Unlike conduction, radiation requires no intervening medium to occur as is obvious in the transfer of heat from the sun to the earth. Convection can be viewed as a macroscopic form of energy transfer through a fluid which occurs by the combined processes of conduction in the fluid and the bulk motion (mass transfer) of the fluid.

This course will focus almost exclusively on conduction heat transfer. Radiation and convection will enter the picture only as 'given' conditions on the surfaces to which we are applying our conduction analysis.

The transfer of heat though a medium by conduction can usually be described by Fourier's law, which is stated

$$\mathbf{q}'' = -k\nabla T \tag{1.7}$$

The quantity k is referred to as the thermal conductivity of the medium, and has units of W/m·K. Fourier's law does not have the same 'legal' standing as, say, the first law of thermodynamics. Rather, Eq. (1.7) presents a *phenomenological* linear relationship between $\mathbf{q}''$ and ∇T – which will be highly accurate providing that the characteristic length scale of the temperature gradient is significantly larger than the 'microscopic' length scale of the medium (i.e., the molecular length scale). Practically all engineering applications will fall into this category, with the exception being heat transfer in highly nonequilibrium conditions (for example, the boundary layer in a re–entering space vehicle).

In the most general sense the thermal conductivity is a tensor quantity – in that it relates one vector to another. In a cartesian frame Fourier's law would appear for a tensor k as

$$\begin{pmatrix} q''_x \\ q''_y \\ q''_z \end{pmatrix} = \begin{pmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{xy} & k_{yy} & k_{yz} \\ k_{xz} & k_{yz} & k_{zz} \end{pmatrix} \times \begin{pmatrix} \partial T/\partial x \\ \partial T/\partial y \\ \partial T/\partial z \end{pmatrix} \tag{1.8}$$

Material such as crystals can posses a highly anisotropic structure, and accordingly the thermal conductivity of these materials can be equally anisotropic: heat can be transferred more effectively in one direction than in other directions. The conductivity k is also a (usually weak) function of

temperature. In general, the thermal conductivity of gases increases with temperature, whereas for liquids and solids k decreases with temperature. This temperature dependence can considerably complicate (and usually eliminates) the ability to analyze conduction via analytical means.

Having duly noted the generality of k, we will constrain our attention to cases in which k is a scalar and is independent of temperature.

1.1.2 The form of the conduction equation

Returning to Eq. (1.6), Fourier's law is to eliminate the heat flux, which results in

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot k \nabla T + q''' \tag{1.9}$$

The assumption of constant thermal conductivity simplifies the above to

$$\boxed{\frac{1}{\alpha}\frac{\partial T}{\partial t} = \nabla^2 T + \frac{q'''}{k}} \tag{1.10}$$

where $\alpha = k/\rho c$ (precisely, $k/\rho c_p$) is the thermal diffusivity of the material – which has units of square length by time (m^2/s). As the name implies, the thermal diffusivity can be viewed as a measure of the rate at which heat 'diffuses' through the material[2]. When a thermal perturbation is applied at some point in a medium (say, for example, an instantaneous change in a surface temperature), it generally takes on the order of $t = r^2/\alpha$ for the perturbation to appear at a distance r from the point.

Heat conduction is analogous in many respects to mass diffusion. Similar to heat flux, the diffusion mass flux $\mathbf{j}''_A$ ($kg/m^2 \cdot s$) of a dilute component (or species), denoted species A, through a medium of species B is given by Fick's law of diffusion as

$$\mathbf{j}''_A = -\rho D_{AB} \nabla w_A \tag{1.11}$$

where w_A is the mass fraction of A in B and D_{AB} is the binary diffusion coefficient (m^2/s). Similar to the derivation of the energy equation, the *species conservation* equation for A can be obtained by applying mass conservation laws to the system. The resulting differential equation would be in the same form as Eq. (1.10), with T replaced by w_A, α by D_{AB}, and q'''/k by $\dot{s}'''_A/\rho D_{AB}$, where $\dot{s}'''_A$ is the volumetric creation rate (through chemical reactions) of species A.

The quantity $\nabla^2 T$ is commonly referred to as the Laplacian operator. The particular form of this operator will depend on the coordinate system that best represents the system. It turns out that there are 11 orthogonal coordinate systems in the Laplacian can be cast as a differential

[2] early scientists considered heat to be a substance – consisting of heat 'particles' – which were transported in the same way as mass is transported.

operator. We will deal with the most common geometries of cartesian, cylindrical, and spherical. In these systems, the Laplacian is

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \tag{1.12}$$

$$= \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{1.13}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta \frac{\partial T}{\partial \theta} + \frac{1}{r^2 \sin\theta}\frac{\partial^2 T}{\partial \phi^2} \tag{1.14}$$

1.1.3 Boundary conditions

Much of the course will focus on methods of solving Eq. (1.10). Two principal elements go into the solution method: 1) application of mathematical techniques to obtain a general solution to the differential equation, and 2) application of the given *boundary* and *initial* conditions to obtain a complete solution for the temperature field within the system.

Three basic types of boundary conditions will be encountered. The first (and most basic) type is where the temperature is specified at the surface of the system. In one dimension, this would appear in the form

$$T(x=0) = T_0, \quad T(x=L) = T_L \tag{1.15}$$

where T_0 and T_L are the known boundary temperatures of the system. The second type of boundary condition is specified heat flux at the surface:

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=0} = q_0'', \quad k \left.\frac{\partial T}{\partial x}\right|_{x=L} = q_L'' \tag{1.16}$$

in which q_0''' and q_L''' are the applied heat fluxes at $x = 0$ and L. Note the sign of the derivatives – the flip in sign at $x = L$ reflects the usual convection of heat flux into the surface as positive. Before writing a heat flux boundary condition, use physical reasoning to figure out which way the sign should be: would the temperature be increasing or decreasing into the region for a given flux? A special and common case is the adiabatic (or insulated) surface, for which

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0, \quad \text{adiabatic} \tag{1.17}$$

Finally, the third type of boundary condition is commonly referred to as the convection condition, in which the heat flux to/from the surface is proportional to the difference between the surface temperature and an ambient fluid temperature;

$$\begin{aligned} k \left.\frac{\partial T}{\partial x}\right|_{x=0} &= h(T(x=0) - T_\infty) \\ -k \left.\frac{\partial T}{\partial x}\right|_{x=L} &= h(T(x=L) - T_\infty) \end{aligned} \tag{1.18}$$

The quantity h is the heat transfer (or convection) coefficient ($W/m^2 \cdot K$). Similar to Fourier's law, the convection rate law represents a phenomenological relation between surface heat flux and the difference in surface and ambient temperature. The heat transfer coefficient h is not a property solely of the surface – rather, it depends mostly on the properties and flow conditions of the fluid in contact to the surface.

Determination of h requires a detailed analysis of momentum and heat transfer within the fluid boundary layer adjacent to the surface – which is outside the objectives of this course. We will typically treat h as a given quantity when such boundary conditions are encountered. It is important to understand that the convection boundary condition implies knowledge of neither the temperature nor the gradient at the surface. Rather, the convection condition provides a relationship between the surface temperature and gradient. You should again be able to deduce the proper sign convention of the boundary condition by physical reasoning: if the wall is being cooled ($T(x = 0) > T_\infty$) which way will the temperature gradient be directed?

Heating/cooling via radiation can become significant when the surface temperatures are relatively high. Assuming that the surface is surrounded by an environment at temperature T_∞, heat transfer to the surface via radiation can typically be expressed as

$$q''_{s,rad} = \epsilon\sigma(T_\infty^4 - T_s^4) \tag{1.19}$$

where ϵ is the surface emissivity and σ is the Stefan–Boltzmann constant. The complication with this boundary condition is that temperature appears to the fourth power – which makes the problem nonlinear in T and eliminates most hopes of finding an analytical solution the conduction problem. One way to deal with this is to linearize the radiation rate law via a first–order Taylor series expansion. This process gives

$$T_\infty^4 - T_s^4 \approx 4T_\infty^3(T_\infty - T_s) \tag{1.20}$$

The quantity $4\epsilon\sigma T_\infty^3$ can now be viewed as a linearized radiation heat transfer coefficient, denoted h_{rad}.

1.2 One–Dimensional Steady Conduction

1.2.1 The Thermal Resistance

The most simple conduction situation consists of one dimension, steady heat transfer without sources or sinks of heat. Consider, for example, the plane wall illustrated in Fig. 1.2. In the cartesian system, the conduction equation reduces to the ordinary differential equation:

$$\frac{d^2T}{dx^2} = 0 \tag{1.21}$$

Assume that the boundary conditions have $T = T_1$ and $T = T_2$ for $x = 0$ and L. The solution to this problem is the familiar linear profile:

$$T = T_1 + (T_2 - T_1)\frac{x}{L} \tag{1.22}$$

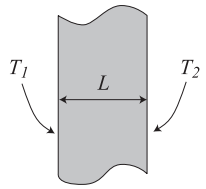


Figure 1.2: plane wall configuration

and the heat transfer through the wall is

$$q = \frac{kA}{L}(T_1 - T_2) \tag{1.23}$$

where A is the wall area.

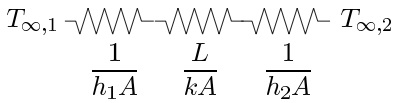
For 1–D steady heat transfer with no heat generation, the heat transfer will be proportional to the temperature difference across the surfaces. This allows for an analogy with current flow in electric circuits – in which the current is proportional to the voltage drop divided by the resistance. Here, heat is current, voltage is temperature, and the resistance is defined from the above equation:

$$q = \frac{T_1 - T_2}{(L/kA)} = \frac{T_1 - T_2}{R_c} \tag{1.24}$$

The electrical analogy allows for a simplified analysis of heat transfer across more complicated 1–D configurations. Say, for example, that convection heat transfer occurs on both faces. As opposed to the surface temperature, the known information would now be the ambient fluid temperatures $T_{\infty,1}$ and $T_{\infty,2}$ on both ends and (hopefully) the heat transfer coefficients h_1 and h_2 which characterize the convective processes. The heat transfer into surface 1 would be

$$q = h_1 A(T_{\infty,1} - T_1) \tag{1.25}$$

and likewise for surface 2. Equation (1.24) would also remain valid, which together with the two convection rate laws would give three equations for the three unknowns q, T_1, and T_2. If one, however, uses the circuit analogy, the system as a whole can be recognized as a series circuit, for which the current q is the total voltage drop across the circuit $(T_{\infty,1} - T_{\infty,2})$ divided by the total resistance;



The heat transfer would simply be given by

$$q = \frac{T_{\infty,1} - T_{\infty,2}}{\sum R} = \frac{T_{\infty,1} - T_{\infty,2}}{1/h_1 A + L/kA + 1/h_2 A} \tag{1.26}$$

The surface temperature T_1 could be obtained by equating the previous two equations. More complicated situations, such as composite walls, can be analyzed in a similar manner.

A problem with the circuit analogy is that it is too easy. Too often, it is used in situations in which it is not valid, and it is important to remember that it applies only to steady, 1–D heat transfer without energy generation. For example, if the wall was not homogeneous in the lateral direction (if, for example, studs are present) then the temperature field would be two–dimensional (i.e., heat flow parallel and perpendicular to the wall surface). You might be tempted to apply a 'parallel' circuit to such a situation – but the accuracy of such an analysis is difficult to estimate. Suffice to say that such a method will not be exact.

One–dimensional cylindrical systems can be examined in a similar manner. The heat conduction equation in cylindrical (r) coordinates is

$$\frac{1}{r}\frac{d}{dr}r\frac{dT}{dr} = 0 \tag{1.27}$$

which, when integrated, gives

$$T = c_1 \ln r + c_2 \tag{1.28}$$

This shows that for r–directed steady heat flow in a cylinder (without generation!) the temperature field is linear in $\ln r$. Take the system to be a pipe with inside and outside temperatures of T_1 and T_2. The temperature field will then be

$$T = T_1 + (T_2 - T_1)\frac{\ln(r/r_1)}{\ln(r_2/r_1)} \tag{1.29}$$

and the heat transfer through the pipe (per unit length) will be

$$q' = \frac{T_1 - T_2}{\ln(r_2/r_1)/2\pi k} \tag{1.30}$$

which identifies the resistance.

1.2.2 Heat generation

Conduction problems are often encountered in which the flow of heat is steady and 1–D, yet heat generation is present. The wall could be absorbing radiation within it's volume, or a wire could be carrying current. Again, the circuit analysis will not be valid under these conditions, and we are forced to formally solve the conduction equation, for the given boundary conditions, to obtain the temperature profile within the object.

Start again with the plane wall. The conduction equation is now

$$\frac{d^2T}{dx^2} = -\frac{q'''}{k} \tag{1.31}$$

Assume that the temperatures are specified on both surfaces. The problem then becomes well–posed, i.e., all information is present to allow a complete prediction of the temperature profile within the wall.

Before proceeding further, the problem will first be re–cast in an *nondimensional* form. This procedure will be used extensively throughout the course. A dimensionless problem offers several advantages; 1) the number of quantities involved in the problem is reduced to the minimum number possible (thus making algebra easier), and 2) the fundamental parameters which govern the flow of heat can be identified. A formal method exists for converting a dimensional problem to a nondimensional one (i.e., the Buckingham π theorem), yet it is probably easiest (for most conduction problems) to convert the problem by simple inspection.

Define the new, dimensionless variables as

$$\overline{T} \equiv \frac{T - T_1}{T_2 - T_1}, \quad \overline{x} \equiv \frac{x}{L} \tag{1.32}$$

Replacing the above into the conduction equation leads to

$$\frac{d^2\overline{T}}{d\overline{x}^2} = -S, \quad S \equiv \frac{q''' L^2}{k(T_2 - T_1)} \tag{1.33}$$

The boundary conditions are now

$$\overline{T}(\overline{x} = 0) = 0, \quad \overline{T}(\overline{x} = 1) = 1 \tag{1.34}$$

Integrating the DE twice results in

$$\overline{T} = -\frac{S\overline{x}^2}{2} + c_1\overline{x} + c_2 \tag{1.35}$$

Applying the boundary conditions gives us two equations for the two integration constants. The final solution is:

$$\overline{T} = \overline{x} + \frac{S\overline{x}}{2}(1 - \overline{x}) \tag{1.36}$$

Upon obtaining a solution to a problem, you should perform a check to see if it is correct. Are the boundary conditions satisfied? Does the solution satisfy the DE?

Consider another example involving heat generation in a slab. Heat is generated uniformly in a wall of thickness $2L$. At both surfaces heat is convected away to the surrounding fluid, and this process is characterized by a heat transfer coefficient h and an ambient temperature T_∞. Note that this situation presents a symmetrical configuration – in that the boundary conditions are the same on both surfaces and the heat generation rate is uniform. The midpoint of the slab is thus a plane of symmetry, and the temperature profile on one side of this plane will be the mirror image of the profile on the other side. No heat can flow across this plane because the temperature gradient will be zero at the plane of symmetry. Consequently, the problem can be recast as a slab of length L with an adiabatic surface at $x = 0$ and convection at $x = L$. it will usually make sense to incorporate any symmetry in a problem into the formulation – because the resulting problem will often be easier to solve.

Identification of dimensionless groups

The problem, on a dimensional basis, is

$$\frac{d^2T}{dx^2} = -\frac{q'''}{k}$$
$$\left.\frac{dT}{dx}\right|_{x=0} = 0$$
$$-k\left.\frac{dT}{dx}\right|_{x=L} = h(T(x=L) - T_\infty)$$

As before, we start by nondimensionalizing the problem. Perhaps it is best to introduce some formalism at this point because the choice of dimensionless variables and parameters is not obvious. In general, the dimensionless length will be the dimensional length divided by the characteristic length of the system, i.e., $\overline{x} = x/L_C$. In this problem the characteristic length is obviously $L_C = L$. Likewise, the dimensionless temperature will be

$$\overline{T} \equiv \frac{T - T_C}{\Delta T_C} \tag{1.37}$$

where T_C and ΔT_C are the characteristic temperature and temperature difference of the system. Usually T_C can be found by inspection – here it will obviously be $T_C = T_\infty$. The characteristic temperature difference is often less obvious to spot. The previous problem had $\Delta T_C = T_2 - T_1$ – but here there is no second temperature such as T_2 to make a difference with T_∞. In this problem, the quantity $q'''L^2/k$ has units of temperature – and it represents the temperature difference across a wall of thickness L and thermal conductivity k that would occur due to a steady heat flux of $q'''L$. This quantity can therefore be used to scale the temperature. The dimensionless temperature is defined as

$$\overline{T} \equiv \frac{(T - T_\infty)k}{q'''L^2} \tag{1.38}$$

and the DE becomes

$$\frac{d^2\overline{T}}{d\overline{x}^2} = -1 \tag{1.39}$$

You might have been tempted to use T_∞ as the characteristic temperature difference. The problem with this approach is, first, it does not represent a temperature difference, and second, it would not reduce the problem down to the fewest number of dimensionless parameters. By using Eq. (1.38) the heat generation rate becomes 'absorbed' into the problem – it no longer explicitly appears in the DE. On the other hand, the use of $\Delta T_C = T_\infty$ would have left the q''' term explicitly in the DE.

The dimensionless boundary conditions are

$$\left.\frac{d\overline{T}}{d\overline{x}}\right|_{\overline{x}=0} = 0, \quad \left.\frac{d\overline{T}}{d\overline{x}}\right|_{\overline{x}=1} = -Bi\,\overline{T}(\overline{x}=1) \tag{1.40}$$

where $Bi = hL/k$ is the Biot number for the system – which represents the ratio of conduction and convection thermal resistances.

The formal solution to Eq. (1.39) is

$$\overline{T} = -\frac{1}{2}\overline{x}^2 + c_1\overline{x} + c_2 \tag{1.41}$$

Using the BC at $\overline{x} = 0$ gives $c_1 = 0$. At $\overline{x} = 1$ the BC gives

$$1 = Bi\left(-\frac{1}{2} + c_2\right) \longrightarrow c_2 = \frac{1}{2} + \frac{1}{Bi} \tag{1.42}$$

and the final solution is

$$\overline{T} = \frac{1}{2}\left(1 - \overline{x}^2\right) + \frac{1}{Bi} \tag{1.43}$$

For certain problems (such as this one) it is often possible to use physical insight to simplify the analysis. Such was done by invoking the symmetry arguments – which led to a problem that is much easier to solve than that for the entire slab. Physics could also be used to simplify the boundary condition at $x = L$. In general, convection–type boundary conditions will involve more complicated algebra than fixed–temperature BC's – yet for this problem the temperature was not initially specified at $x = L$. However, all the heat generated in the slab must be removed from the $x = L$ surface by convection because the slab is in steady state and the $x = 0$ surface is adiabatic. This gives the energy conservation statement of

$$\dot{Q}_{gen} = \int_V q''' \, dV = A\int_0^L q''' \, dx \tag{1.44}$$

$$= LAq''' = hA(T_s - T_\infty) \tag{1.45}$$

from which

$$T_s = T(x = L) = T_\infty + \frac{q'''L}{h} \tag{1.46}$$

By using the definition of dimensionless temperature, this becomes

$$\overline{T}(\overline{x} = 1) = \frac{1}{Bi} \tag{1.47}$$

which agrees perfectly with the solution – as it must. Equation (1.47) could therefore have been used as a boundary condition in the original DE.

Heat generation problems become more complicated when the distribution of heat generation in the system becomes nonuniform. The following example illustrates such a problem.

A sphere, of radius R, and thermal conductivity k contains radioactive material. Heat is being generated within its volume at a rate

$$q''' = q'''_0 e^{-ar/R} \tag{1.48}$$

where q_0''' and a are constants. The above function is chosen as representative of heat generation by nuclear decay (fission). Heat is convected from the surface of the sphere, which is characterized by an ambient temperature and a heat transfer coefficient.

Define the nondimensional variables of the problem as

$$\overline{T} = \frac{(T - T_\infty)k}{R^2 q_0'''}, \quad \overline{r} = \frac{r}{R} \tag{1.49}$$

The problem statement, in spherical coordinates, becomes

$$\begin{aligned} \frac{1}{\overline{r}^2}\frac{d}{d\overline{r}}\overline{r}^2\frac{d\overline{T}}{d\overline{r}} &= -e^{-a\overline{r}} \\ \left.\frac{d\overline{T}}{d\overline{r}}\right|_{\overline{r}=0} &= 0 \\ \left.\frac{d\overline{T}}{d\overline{r}}\right|_{\overline{r}=1} &= -Bi\overline{T}(\overline{r}=1) \end{aligned} \tag{1.50}$$

where $Bi = hR/k$.

The problem is now well–posed. A solution can be obtained by direct integration of the DE followed by substitution of the BCs. Before proceeding, however, it will be worthwhile to simplify the problem where possible. When dealing with radial problems in spherical coordinates, the DE can usually be simplified by substitution of the variable $u = \overline{r}\,\overline{T}$. This gives

$$\begin{aligned} \overline{r}^2\frac{d\overline{T}}{d\overline{r}} &= \overline{r}^2\frac{d(u/\overline{r})}{d\overline{r}} = \overline{r}\frac{du}{d\overline{r}} - u \\ \frac{d}{d\overline{r}}\overline{r}^2\frac{d\overline{T}}{d\overline{r}} &= \overline{r}\frac{d^2u}{d\overline{r}^2} + \frac{du}{d\overline{r}} - \frac{du}{d\overline{r}} = \overline{r}\frac{d^2u}{d\overline{r}^2} \end{aligned}$$

and the DE becomes

$$\frac{d^2u}{d\overline{r}^2} = -\overline{r}e^{-a\overline{r}}$$

The DE is now integrated over $\overline{r}$:

$$\begin{aligned} \frac{du}{d\overline{r}} &= -\int \overline{r}e^{-a\overline{r}}\,d\overline{r} = \frac{\overline{r}}{a}e^{-a\overline{r}} - \frac{1}{a}\int e^{-a\overline{r}}\,d\overline{r} \\ &= \frac{\overline{r}}{a}e^{-a\overline{r}} + \frac{1}{a^2}e^{-a\overline{r}} + c_1 \\ u &= \int\left(\frac{\overline{r}}{a}e^{-a\overline{r}} + \frac{1}{a^2}e^{-a\overline{r}} + c_1\right)d\overline{r} \\ &= -\frac{\overline{r}}{a^2}e^{-a\overline{r}} - \frac{2}{a^3}e^{-a\overline{r}} + c_1\overline{r} + c_2 \end{aligned}$$

The integrations in the above were performed using integration by parts – which is an extremely useful formula for integration of a product of two functions. For the indefinite integrals appearing above, the general formula is

$$\boxed{\int v(x)w'(x)\,dx = vw - \int v'w\,dx} \tag{1.51}$$

in which the prime denotes differentiation. To apply the formula, it was recognized that if $w' = \exp(-a\overline{r})$ then $w = -\exp(-a\overline{r})/a$.

The temperature gradient is zero at the center of the sphere (equivalently, $\overline{T}$ must remain finite at the center). In term of u, the BC at $\overline{r} = 0$ becomes

$$\begin{aligned}\overline{T}'(\overline{r}\to 0) &= \left(\frac{u}{\overline{r}}\right)'_{\overline{r}\to 0} = \left.\frac{u'}{\overline{r}}\right|_{\overline{r}\to 0} - \left.\frac{u}{\overline{r}^2}\right|_{\overline{r}\to 0}\\ &= 0\end{aligned}$$

which implies that

$$u(\overline{r}=0) = 0 \tag{1.52}$$

Likewise, the convection BC at $\overline{r} = 1$ can be posed with u as the dependent variable. Alternatively, the surface temperature can be deduced directly from an energy balance;

$$hA_s(T_s - T_\infty) = \dot{Q}_{tot} = 4\pi\int_0^R q'''r^2\,dr$$

or, in terms of nondimensional variables

$$\begin{aligned}Bi\,\overline{T}(\overline{r}=1) &= \int_0^1 e^{-a\overline{r}}\overline{r}^2\,d\overline{r}\\ &= -\left[\left(\frac{\overline{r}^2}{a} + \frac{2\overline{r}}{a^2} + \frac{2}{a^3}\right)e^{-a\overline{r}}\right]_0^1\\ &= \frac{2}{a^3}\left(1-e^{-a}\right) - \frac{1}{a}\left(1+\frac{2}{a}\right)e^{-a}\end{aligned}$$

which gives

$$u(\overline{r}=1) = \frac{2}{Bia^3}\left(1-e^{-a}\right) - \frac{1}{Bia}\left(1+\frac{2}{a}\right)e^{-a} \tag{1.53}$$

Equations (1.43), (1.52) and (1.53) can now be combined to eliminate the constants c_1 and c_2. The final result is

$$u = \frac{\overline{r}}{a^3}\left[-2 - ae^{-a\overline{r}} + \frac{2}{Bi}\left(1-e^{-a}\right) - \left(\frac{a}{Bi}-1\right)(a+2)\,e^{-a}\right] + \frac{2}{a^3}\left(1-e^{-a\overline{r}}\right) \tag{1.54}$$

The dimensionless temperature is given from $\overline{T} = u/\overline{r}$. Obtaining the center temperature takes some mathematical maneuvers, because the second term, when divided by $\overline{r}$, becomes indeterminate at $\overline{r} = 0$. We then use

$$\lim_{\overline{r}\to 0}\left(\frac{1-e^{-a\overline{r}}}{\overline{r}}\right) = a$$

The center temperature of the sphere is (dimensionlessly)

$$\overline{T}(\overline{r}=0) = \frac{1}{a^3}\left[-2+a+\frac{2}{Bi}\left(1-e^{-a}\right)-\left(\frac{a}{Bi}-1\right)(a+2)\,e^{-a}\right] \tag{1.55}$$

Modern computer technology has almost eliminated most of the subtle mathematical manipulations used in this example. At the end of this chapter the same problem is solved using the symbolic mathematical manipulation package *Mathematica*. This package essentially reduces the solution procedure to a 'black box'. Most linear, ordinary differential equations – which are the type most frequently encountered in 1–D steady heat transfer problems – can be solved using *Mathematica*.

1.3 Extended Surfaces

1.3.1 The fin equation

The purpose of extended surfaces (commonly known as fins) is to enhance convective heat transfer from surfaces. The primary mechanism behind the operation of fins is to increase the effective heat transfer area of a surface. They are commonly used in situations in which cooling is attained via free (or natural) convection – for which the heat transfer coefficients h are relatively small.

Typically fins are much longer than they are thick. Because of this it is common, and fairly accurate, to assume that the temperature varies only in the lengthwise direction. That is, at any point x along the length of the fin the temperature is essentially uniform across the cross section of the fin. What results from this assumption is a one–dimensional heat transfer problem – yet the 1–D DE from the previous section cannot be directly applied to analyze the fin. Rather, an energy conservation equation specific to the fin must be derived.

Consider the arbitrary fin illustrated in Fig. 1.3. The heat flow direction is x, and the cross sectional area of the fin (the area exposed to the heat flow) is taken to be a function of x. Consider the small volume element of the fin of length Δx. An energy balance is performed on this element, in which it is assumed that the element is at a constant and uniform temperature of T. This yields

$$q_{cond,in} - q_{cond,out} - q_{conv} = 0$$

Substitution of the rate laws for convection and conduction gives

$$-kA_c(x)\left.\frac{dT}{dx}\right|_x + kA_c(x+\Delta x)\left.\frac{dT}{dx}\right|_{x+\Delta x} - h\,dA_s(x)(T-T_\infty) = 0$$

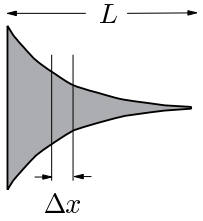


Figure 1.3: fin geometry

where A_c is the cross sectional area of the fin (a function of x) and dA_s is the differential surface area of the fin at position x. The latter can be approximated by the first term in a Taylor series via

$$dA_s \approx \frac{dA_s}{dx}\Delta x + \ldots = P(x)\Delta x$$

in which P is the fin perimeter (again a function of x). The two previous equations are combined and divided by Δx, and the limit of $\Delta x \to 0$ is taken. This gives

$$k\frac{d}{dx}A_c\frac{dT}{dx} - hP(T - T_\infty) = 0 \tag{1.56}$$

which is known as the fin equation.

The typical boundary condition at the base ($x = 0$) is $T = T_B$, i.e., the base temperature is specified. Three forms of boundary condition can be specified at the fin tip, i.e., specified temperature, specified flux, or convection. Before introducing further details, the dependent and independent variables are made dimensionless by the definitions

$$\overline{T} = \frac{T - T_\infty}{T_B - T_\infty}, \quad \overline{x} = \frac{x}{L}$$

for which the fin equation becomes

$$\frac{d}{d\overline{x}}A_c\frac{d\overline{T}}{d\overline{x}} - \frac{hPL^2}{k}\overline{T} = 0 \tag{1.57}$$

This equation is not completely dimensionless – each term has units of area – but further reduction cannot be made until the specific form of A_c has been set. This will depend on the shape of the fin (uniform cross section, triangular, annular, etc.).

The three types of boundary conditions at the tip are:

$$\overline{T}(\overline{x}=1)=\overline{T}_t, \qquad \text{fixed tip } T$$

$$\left.\frac{d\overline{T}}{d\overline{x}}\right|_{\overline{x}=1}=0, \qquad \text{insulated tip}$$

$$-\left.\frac{d\overline{T}}{d\overline{x}}\right|_{\overline{x}=1}=Bi_t,\overline{T}(\overline{x}=1) \qquad \text{tip convection}$$

where $Bi_t = h_tL/k$ is the Biot number characterizing convection at the tip. Of all the boundary conditions the third (tip convection) is the most realistic – yet it is also the most difficult to mathematically analyze.

There is a certain contradiction inherent in the assumptions that led up to the fin equation, Eq. (1.57). It was assumed that temperature varies only with the x direction – yet this cannot be completely true because heat is removed from the sides of the fin by convection. If heat is transferred from a surface, then a temperature gradient must exist normal to the surface to supply the heat. More specifically, if y denotes the direction normal to the surface area, then the energy balance at the surface would give

$$-k\left.\frac{\partial T}{\partial y}\right|_{y=b}=h(T-T_\infty)$$

where b denotes the thickness of the fin at a particular position x. The above clearly indicates that a temperature gradient must exist in the y direction – that is, if the fin is designed to remove heat – and because of this the temperature must be a function of both x and y.

How then, can the y variation in temperature be neglected? To try to get a gauge of the accuracy in the 1–D assumption, we can approximate the derivative in the above equation as $\Delta T/b$, where ΔT represents the average temperature difference across the fin in the y direction. If the surface energy balance is divided by $T_B - T_\infty$ and rearranged, it becomes

$$\Delta\overline{T}=\frac{\Delta T}{T_B-T_\infty}\approx\frac{hb}{k}\,\overline{T}=Bi_b\,\overline{T}$$

where Bi_b is the Biot number based on the fin thickness. The dimensionless temperature $\overline{T}$ ranges between 1 and 0 – and for the 1–D assumption to be correct we would expect that $\Delta\overline{T}\ll\overline{T}$, i.e., the variation in temperature in the y direction is much smaller than the variation in the x direction. In view of the above surface energy balance, this assumption will therefore be valid when $Bi_b \ll 1$. This is consistent with the general interpretation of a small Biot condition; the temperature in an object will essentially be uniform (here in the y direction) because the dominant resistance to heat transfer occurs at the surface from convection. It turns out that Bi_b will typically be a very small number for most fins. To give an example, consider aluminum (a common fin material) for which $k \approx 400$ W/m·K. For free convection in air the heat transfer coefficient is typically no

greater than 10 W/m^2·K. A fin with a thickness of 1 cm (which is a very thick fin) will have $Bi_b = (10)(.01)/400 = 0.0025$ – which is small enough to validate the 1–D approximation of the fin.

1.3.2 Simple fins of uniform cross section

The most simple type of fin has a constant cross sectional area, The fin equation reduces to

$$\overline{T}'' - N^2\overline{T} = 0 \tag{1.58}$$

where the prime denotes differentiation and the dimensionless parameter N is defined

$$N^2 = \frac{hPL^2}{kA_C} \tag{1.59}$$

The reason the DE uses N^2 – rather than N – will soon become obvious. The general solution to the ODE is

$$\overline{T} = Ae^{N\overline{x}} + Be^{-N\overline{x}}$$

where A and B are integration constants. The boundary condition $\overline{T}(\overline{x} = 0) = 1$ gives $B = 1 - A$. Again, the BC at $\overline{x} = 1$ can be posed using either of the three basic forms. The most simple outcome will result from the adiabatic tip condition, which has

$$\left.\frac{d\overline{T}}{d\overline{x}}\right|_{\overline{x}=1} = N\left(Ae^N - Be^{-N}\right) = Ne^{-N} + NA\left(e^N + e^{-N}\right) = 0 \tag{1.60}$$

which leads to

$$\begin{aligned} A &= e^{-N}/\left(e^N + e^{-N}\right) \\ B &= 1 - A = e^N/\left(e^N + e^{-N}\right) \end{aligned}$$

and the final solution is

$$\begin{aligned} \overline{T} &= \frac{e^{N(1-\overline{x})} + e^{-N(1-\overline{x})}}{e^N + e^{-N}} \\ &= \frac{\cosh[N(1-\overline{x})]}{\cosh(N)} \end{aligned} \tag{1.61}$$

Hyperbolic functions

A math digression is in order. The hyperbolic function cosh and sinh are given as

$$\begin{aligned} \cosh(x) &= \frac{1}{2}\left(e^x + e^{-x}\right) \\ \sinh(x) &= \frac{1}{2}\left(e^x - e^{-x}\right) \end{aligned}$$

They are defined as the two linearly independent solutions to the ODE:

$$\frac{d^2y}{dx^2} - y = 0$$

The cosh function is even, in that $\cosh(-x) = \cosh(x)$, whereas $\sinh(x)$ is odd, i.e., $\sinh(-x) = -\sinh(x)$. Also, the functions have the properties

$$\frac{d}{dx}\cosh(x) = \sinh(x), \quad \frac{d}{dx}\sinh(x) = \cosh(x)$$

which is not precisely the same as the triginometric functions cos and sin (although they are similar in that $\cosh(0) = 1$ and $\sinh(0) = 0$). The exact equivalence between the hyperbolic and triginometric functions are

$$\cosh(ix) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cos(x)$$
$$\sinh(ix) = \frac{1}{2}\left(e^{ix} - e^{-ix}\right) = i\sin(x)$$

Likewise,

$$\cos(ix) = \cosh(x), \quad \sin(ix) = -i\sinh(x)$$

In the above, $i = \sqrt{-1}$ is the radical. Complex mathematics will prove to be very useful throughout this course

Returing to the problem of a constant cross section fin, the general solution can appear as

$$\overline{T} = A\cosh(N\overline{x}) + B\sinh(N\overline{x})$$

At $\overline{x} = 1$ the condition is $\overline{T}' = 0$. Consequently, the solution will be in the form

$$\overline{T} = C\cosh[N(1-\overline{x})] \tag{1.62}$$

Recognize that if $\cosh(x)$ is a solution to the DE, then $\cosh(a+x)$ (where a is a constant) is also a solution. This amounts to a shifting of the system origin. The BC at $\overline{x} = 0$ gives the final solution:

$$\overline{T} = \frac{\cosh[N(1-\overline{x})]}{\cosh(N)} \tag{1.63}$$

1.3.3 Measures of fin performance

The temperature distribution in the fin is of limited usefulness to us as engineers. A more relevant and useful quantity is the rate of heat removal by the fin. One method to calculate q would be to compute the total convection from the fin surface, i.e.,

$$q = \int_{A_s} h(T - T_\infty)\, dA_s = hL(T_B - T_\infty)\int_0^1 P\overline{T}\, d\overline{x} \tag{1.64}$$

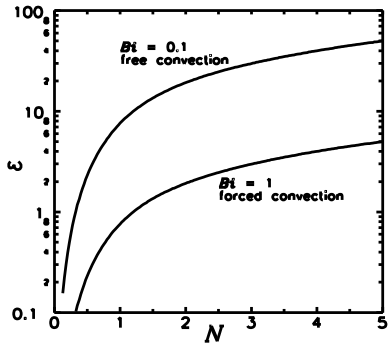


Figure 1.4: fin effectiveness ϵ for A_c = constant

Equivalently, all the heat removed from the fin must be transported into the fin at the base by conduction. This gives

$$q = -kA_{C,B}\left.\frac{dT}{dx}\right|_{x=0} = -\frac{kA_{C,B}(T_B - T_\infty)}{L}\left.\frac{d\overline{T}}{d\overline{x}}\right|_{\overline{x}=0} \tag{1.65}$$

For the specific case of the constant cross section fin, the heat transfer rate becomes

$$q = \frac{kA_C(T_B - T_\infty)}{L}\, N\, \tanh(N) = \sqrt{hPkA_C}\,(T_B - T_\infty)\,\tanh(N) \tag{1.66}$$

Note that $\tanh(N) \to 1$ for $N \gg 1$, and $\tanh(3) \approx 0.995$. Consequently a fin with $N > 3$ is essentially 'infinite' in length. Adding additional length to the fin (and thus increasing N) will not significantly increase the heat transfer from the fin. From a design viewpoint, there is little justification for making a fin with $N > 2$ to 2.5.

Fin performance can measured by two main criteria. The first is the fin effectiveness ϵ, which is defined as the heat transfer from the fin divided by that from the base without the fin. The A_c = constant fin has

$$\begin{aligned}\epsilon &= \frac{\dot{q}_{fin}}{\dot{q}_{w/o\ fin}} = \frac{\sqrt{hPkA_C}\,(T_B - T_\infty)\tanh(N)}{hA_C(T_B - T_\infty)} \\ &= \frac{N}{Bi}\,\tanh(N)\end{aligned} \tag{1.67}$$

where $Bi = hL/k$ is the Biot number of the fin based on the fin length. The effectiveness ϵ gives the engineer an idea of the cooling improvement offered by the fin. One would certainly want $\epsilon \geq 1$ – anything less would mean that the fin is insulating the surface. A rule of thumb is that fins are justified only if $\epsilon \geq 2$. The plot in Fig. 1.4 shows the general behavior of ϵ for two values of Bi, corresponding to typical free and forced convection values.

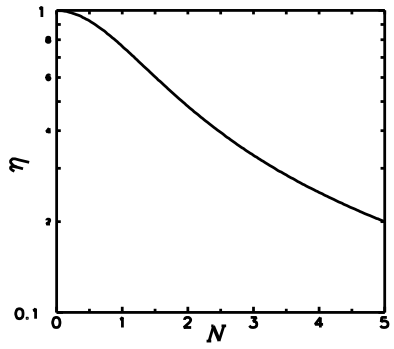


Figure 1.5: fin efficiency η for A_c = constant

Fins are most effective for small h convection conditions – which again correspond to free convective situations. For such conditions heat transfer from a surface can be greatly enhanced by extending the area of the surface. As the plot indicates, when $Bi = 1$ only a fin with relatively large N will be effective – in that ϵ exceeds 2 only for $N > 2$. When $Bi = 0.1$, on the other hand, practically any length of fin will improve the heat transfer from the base. The limiting behavior for large N (say $N \geq 3$) has $\epsilon \approx N/Bi = \sqrt{(kP/hA_C)}$ – which is independent of fin length. The fact that $\epsilon \to 0$ for $N = 0$ is an artifact of our particular solution. We have assumed that the tip is adiabatic – primarily because it simplifies the analysis. If the fin length went to zero ($N \to 0$) the base would become covered by an insulating surface, and $\epsilon \to 0$. In reality, some heat will be convected from the tip. For realistic fins, however, the amount of heat transferred though the tip will be a small fraction of the total fin heat transfer, and because of this it is usually reasonable to approximate the tip as adiabatic.

The quantity N plays a critical role in fin design. A physical interpretation of this parameter can be obtained from the definition:

$$N^2 = \frac{hPL^2}{kA_C} = \frac{hPL(T_B - T_\infty)}{kA_C(T_B - T_\infty)/L} \approx \frac{R_{cond}}{R_{conv}} \tag{1.68}$$

i.e., N^2 is proportional to the ratio of axial conduction resistance to surface convection resistance. Once N exceeds a certain value (around 3) the resistance to heat transfer in the fin is dominated by axial conduction. As mentioned above, there will be little gained by adding more length to the fin when this condition is met.

The second criteria for measuring fin performance is the fin efficiency η. The efficiency is defined as the ratio of the actual to the theoretical maximum heat transfer from the fin. This latter quantity corresponds to the same type of fin, except with a thermal conductivity that goes to infinity. Equivalently, the maximum heat transfer would occur if the fin was entirely at the

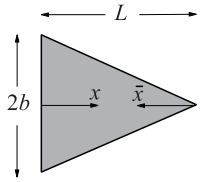


Figure 1.6: triangular fin

temperature of the fin base:

$$q_{max} = hA_{fin}(T_B - T_\infty) = hPL(T_B - T_\infty), \quad \text{uniform } A_C \tag{1.69}$$

The constant A_c fin will have η given by

$$\eta = \frac{\tanh(N)}{N} \tag{1.70}$$

A plot of η vs. N is shown in Fig. 1.5.

The name 'efficiency' is somewhat misleading – it implies that a better fin would have a higher efficiency. This is not necessarily the case. As the fin length decreases the efficiency will increase (and $\eta \to 1$ in the limit of $L \to 0$). However, the heat transfer from the fin decreases as $L \to 0$ – which defeats the purpose of heat removal from a surface. The real advantage of the efficiency is that, for many types of fins, it is a function primarily of the parameter N (and is solely a function of N for the uniform cross section, adiabatic tip fin).

As is encountered in the analysis of most types of heat transfer equipment, there are two basic types of engineering problems when working with fins; rating problems and design problems. The rating problem predicts the heat transfer rate from a given fin configuration and convection conditions – which is straightforward. The design problem, as the name implies, seeks to identify a fin configuration which will remove a specified amount of heat for fixed convection conditions. Unlike the rating problem, there is (in general) no unique solution to a given design problem, although there are so–called optimum configurations which maximize heat transfer for a given mass of fin. An analysis of fin optimization is presented in a following section.

1.3.4 Fins of non uniform cross section

Fins of non–uniform cross section can usually transfer more heat for a given mass than those of a constant cross section. Given in this section are forms of the fin equation for common shapes.

Consider first the triangular shaped fin shown in Fig. 1.6. The fin is W wide (in and out of the paper), and it is assumed that $W \gg L \gg b$. The cross sectional area and perimeter of the fin, for these assumptions, will be

$$A_C = 2bW\left(1 - \frac{x}{L}\right) \quad P = 2(W + b) \approx 2W$$

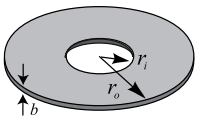


Figure 1.7: annular fin

Redefine the x coordinate origin so that the dimensionless x becomes

$$\overline{x} = 1 - \frac{x}{L} \tag{1.71}$$

The fin differential equation, Eq. (1.57), can now be cast in the form

$$\overline{x}\frac{d^2\overline{T}}{d\overline{x}^2} + \frac{d\overline{T}}{d\overline{x}} - N^2\overline{T} = 0 \tag{1.72}$$

in which N is now defined

$$N^2 = \frac{hL^2}{kb} \tag{1.73}$$

The boundary conditions for the problem are

$$\left.\frac{d\overline{T}}{d\overline{x}}\right|_{\overline{x}=1} = 0, \quad \overline{T}(\overline{x}=0) = 1 \tag{1.74}$$

Equation (1.72) is a form of Bessel's equation – which has a solution involving Bessel functions. Details of the solution will be addressed in the next chapter.

Another common type of nonuniform cross section fin is the annular (or circular) fin, as illustrated in Fig. 1.7. These are typically used in to assist heat transfer to/from pipes. The cross sectional area (in the r direction) is $A_C = 2\pi rb$, and the perimeter is $P = 2 \times 2\pi r$ (remember that $P = dA_S/dr$ if this does not make any sense. The extra 2 comes from including both sides of the fin). By using these definitions in the fin DE, and choosing the nondimensional radial coordinate as $\overline{r} \equiv r/r_o$, the problem becomes

$$\overline{r}\frac{d^2\overline{T}}{d\overline{r}^2} + \frac{d\overline{T}}{d\overline{r}} - \overline{r}N^2\overline{T} = 0 \tag{1.75}$$

where N is defined

$$N^2 = \frac{2r_0h}{kb} \tag{1.76}$$

Assume we have an adiabatic tip at r_o. The boundary conditions are then

$$\overline{T}(\overline{r}=a) = 1, \quad \left.\frac{d\overline{T}}{d\overline{r}}\right|_{\overline{r}=1} = 0 \tag{1.77}$$

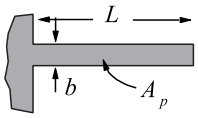


Figure 1.8: rectangular fin

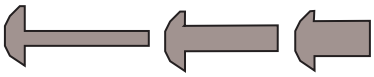


Figure 1.9: Possible 'shapes' of the rectangular fin

where a is the radius ratio r_i/r_o. Just like the triangular fin, the circular fin DE does not have an obvious solution in terms of familiar functions. Again, Eq. (1.75) is a form of Bessel's equation, and the solution will be given in terms of Bessel functions.

1.3.5 Fin optimization

Fins can come in a variety of shapes, e.g., rectangular and circular with constant cross section, and annular and triangular with variable cross section. For a given fin shape, fin material, and convection conditions, there exists an optimized design which transfers the maximum amount of heat for a given mass of the fin. The methodology to finding this optimum design is presented here.

The simplest case to examine is the rectangular fin, as illustrated in Fig. 1.8. The fin is taken to be long in and out of the paper (i.e., $W \gg L$, where W is the width). Since fin mass is proportional to the profile area A_P $(= bL)$ times W, the optimization problem can be stated as finding the thickness b and length L which maximize q/W for a given A_P. Possible shapes of the fin, for a fixed profile area A_P, are illustrated (in a very exaggerated manner) in Fig. 1.9.

To further simplify the problem the fin assumed to have an adiabatic tip. The heat transfer is

$$q = \sqrt{hPkA_c}(T_b - T_\infty)\tanh N$$

with

$$N^2 = \frac{hPL^2}{kA_c}$$

For a long fin ($W \gg b$), $P \approx 2W$ and $A_C = bW$. Thus

$$q' = \frac{q}{W} = \sqrt{2bhk}\,(T_b - T_\infty)\tanh N \tag{1.78}$$

where now

$$N^2 = \frac{2hL^2}{kb}$$

The length L can be eliminated using $A_P = bL$. The formula for N becomes

$$N^2 = \frac{2hA_P^2}{kb^3}$$

and by inverting this equation,

$$b = \left(\frac{2hA_P^2}{kN^2}\right)^{1/3} \tag{1.79}$$

and replacing this into the formula for the heat transfer gives

$$q' = \left(4h^2k\,A_P\right)^{1/3}\,(T_b - T_\infty)\,N^{-1/3}\,\tanh N \tag{1.80}$$

Observe that we are taking the profile area A_p to be *fixed*. Likewise, the properties h, k, and $T_b - T_\infty$ are assumed to be constants. The above formula therefore indicates that, for these given constraints, there is an optimum value of N which will maximize the fin heat transfer rate.

Obtaining the optimum N is relatively easy at this point. From the previous equation, the heat transfer is functionally related to N via

$$f(N) = N^{-1/3}\,\tanh N$$

and a maximum q' implies that

$$\frac{df}{dN} = 0$$

or, for this particular case,

$$\cosh N\,\sinh N - 3N = 0 \tag{1.81}$$

This is a nonlinear equation for N and can be solved using standard numerical techniques. The solution is

$$N_{opt} = 1.419 = \left(\frac{2hA_P^2}{kb_{opt}^3}\right)^{1/2} \tag{1.82}$$

Once A_P is fixed, the above formula can be used to obtain b_{opt}, and L follows from $L = A_P/b_{opt}$. And by replacing N_{opt} into Eq. (1.80) we get

$$\begin{aligned} q'_{opt} &= \left(\frac{4h^2kA_P}{N_{opt}}\right)^{1/3}(T_b - T_\infty)\tanh N_{opt} \\ &= 1.256\,\left(h^2kA_P\right)^{1/3}(T_b - T_\infty) \end{aligned} \tag{1.83}$$

The same process can be applied to triangular fins (which will require information on Bessel functions that will be presented in the following chapter). Under the assumption of $b^2/L^2 \ll 1$ – which is consistent with the 1–D heat transfer approximation – the following result can be obtained:

$$q'_{opt} = 1.422\,\left(h^2kA_P\right)^{1/3}(T_b - T_\infty) \tag{1.84}$$

Recognize that for a fixed A_P (and consequently, fin mass), an optimized triangular fin can transfer more heat than an optimized rectangular fin.

In designing a fin, one would always want to choose the optimum value of b for a fixed q' and A_P. Fins are often placed in arrays (such as on the head of a motorcycle engine). To maximize the heat transfer coefficient, the spacing between the fins should be somewhat greater than twice the boundary layer thickness. In selecting a material for the fin, it follows from Eqs. (1.83) and (1.84) that

$$A_P \propto \frac{1}{h^2 k}\left(\frac{q'_{opt}}{T_b - T_\infty}\right)^3 \tag{1.85}$$

For a given required heat transfer rate and fixed 'environmental' parameters (h, T_b, T_∞), the optimum profile area will be inversely proportional to the fin thermal conductivity. Consequently, the fin mass (which is $\rho A_P W$) will be proportional to ρ/k, where ρ is the density of the fin material. Aluminum is often a good choice for fins – since it has relatively high k and low ρ. It is also relatively cheap.

Exercises

1. Consider again the derivation of the heat conduction equation, Eq. (1.10). Say that mass transfer occurs at the boundaries and convects an energy flux $\rho \mathbf{u} e$ into/out of the system, where $\mathbf{u}$ is the velocity vector. Using the fact that $\partial e = c\partial T$, and taking the mass of the system to be constant, derive the more general form of the energy equation:

$$\frac{1}{\alpha}\left(\frac{\partial T}{\partial t} + \mathbf{u}\cdot(\nabla T)\right) = \nabla^2 T + \frac{q'''}{k}$$

   Note: this will involve use of the divergence theorem as outlined in Sec. 1. Also, use continuity (system mass = constant) along with the vector identity

$$\nabla\cdot(\rho\mathbf{u}e) = e\nabla\cdot\rho\mathbf{u} + \rho\mathbf{u}\cdot(\nabla e)$$

2. A 1 cm thick copper wire ($k_c = 400$ W/m/K) conducts a large electrical current, which results in a heat dissipation within the wire of 60 W per meter of length. The wire is covered with insulation having $k_{ins} = 1.2$ W/m/K, and the outer surface of the insulation is cooled by convection, with $h = 30$ W/m$^2\cdot K$ and $T_\infty = 25$ °C.

   (a) Formally 'pose' the heat conduction problem for the wire and the insulation regions. What are the boundary conditions at the interface between the wire and the insulation?

   (b) Using a resistance analogy for the insulation, determine, via energy balance principles (not via solution of the DEs) the surface temperature of the wire (i.e., T at $r = 1$ cm) as a function of the insulation outer radius r_{ins}. Identify an appropriate Biot number for

the wire. Based on the magnitude of this Biot number, would you think it is necessary to solve the conduction equation for the wire to obtain the wire centerline temperature?

(c) Plot the wire surface temperature vs. r_{ins}. Physically interpret the minimum in T_s that occurs for a critical value of r_{ins}. Analytically, what is this critical thickness equal to?

3. A rectangular fin of length L, thickness t and width W is mounted on a surface that is maintained at a constant temperature T_b. The fin is exposed to a uniform source of radiant flux q''_R, of which a fraction α is absorbed by the fin. Heat transfer from the fin occurs by convection to an ambient at T_∞ (which is less than T_b), which is characterized by a heat transfer coefficient h. Assuming that 1) the tip of the fin is adiabatic, and 2) thermal emission from the fin is negligible in comparison to convection, determine formulas for the temperature distribution in the fin and the total heat transfer from the base. If thermal emission is not negligible, how is the problem complicated? Devise a way of approximating the effect of thermal emission on the fin by 'linearizing' the thermal emission rate law.

4. Shown that, for an arbitrary fin, Eqs. (1.64) and (1.65) are completely equivalent. Hint: use the DE.

5. A plane wall of thickness L has an insulated boundary at $x = 0$. Radiation is absorbed within the wall, which results in a heat generation characterized by $\kappa q''_0 e^{-\kappa(L-x)}$, where κ is the radiative absorption coefficient of the wall material and q''_0 is the incident radiative flux on the wall surface. The outer surface of the wall is cooled by convection to an ambient temperature T_∞. Formulate and solve the heat conduction equation for the temperature distribution within the wall. What happens to the distribution when $\kappa L \gg 1$ (i.e., the wall becomes highly absorbing)? Show that this case becomes equivalent to a problem in which radiation absorption 'disappears' from the DE (as a heat generation function), yet 'reappears' in the problem in the boundary condition at $x = L$.

Mathematica solution

Given here is the *Mathematica* solution to the spherical coordinate heat transfer problem, presented beginning with Eq. (1.50). A more detailed explanation of the application of *Mathematica* to solution of ODEs will be given in Ch. 2.

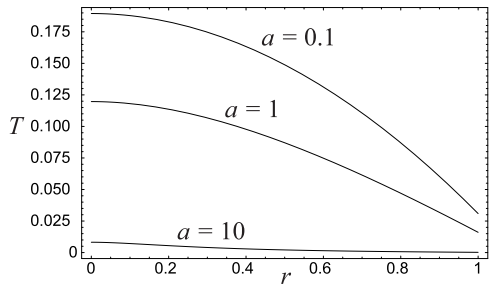
The formulas given below are in nondimensional form. The inner boundary condition – which is typically posed for spherical coordinates as finite T at $r \to 0$ – needs to be given in a more precise mathematical form. This is done by imposing

$$r^2 \left.\frac{dT}{dr}\right|_{r\to 0} = 0$$

which is essentially the same as saying that the total heat transfer rate at the center is zero. The extra r^2 is needed to keep the solution finite at the origin. A functional form of the solution (i.e.,

$T(r, Bi, a)$) which can be used to generate plots for numerical values of the parameters is defined in line [6]. Pay attention to the replacement rules when such definitions are made: the quantities x, bix, and ax are simply dummy variables – problems would arise if the arguments were written as r, bi, and a since these symbols are already used in the symbolic form of the solution.

```
In[1]:=de=D[r^2D[t[r],r],r]/r^2+E^(-a r)==0;
        bc1=Limit[x^2 D[t[x],x],x->0]==0;
        bc2=-t'[1]==bi t[1];
        soln=DSolve[{de,bc1,bc2},t[r],r]
Out[4]={{t[r] ->
    -((2 + 2*a + a^2 - 2*bi - a*bi -
          2*E^a + 2*bi*E^a)/(E^a*a^3*bi)) \
          + (-(1/a^2) - 2/(a^3*r))/E^(-(-a*r)) +
          2/(a^3*r)}}
In[5]:=Simplify[%]
Out[5]={{t[r] ->
    -(1/a^3*((a^2 - a*(-2 + bi) +
          2*(-1 + bi)*(-1 + E^a))/
          (E^a*bi) - 2/r +
          (2 + a*r)/(r/E^(-(-(-a*r))))))}}
In[6]:=tfunc[x_,bix_,ax_]:=t[r]/.soln[[1]]
          /.r->x/.bi->bix/.a->ax
In[7]:=Plot[{tfunc[r,10,.1],tfunc[r,10,1],
          tfunc[r,10,10]},{r,0,1}]
```



Chapter 2

Advanced 1–D Analytical Methods

2.1 Introduction

In a wide variety of heat conduction problems the flow of heat occurs primarily in one direction. Chapter 1 outlined the application of a steady, 1–D conduction analysis to relatively 'simple' configurations, such as the plane wall, the infinite–length cylinder and the sphere (with heat flow in the r direction), and the fin with a uniform cross sectional area. On the other hand, fins of nonuniform cross section present a more difficult analytical problem. The difficulty arises simply from the fact that the ordinary differential equations which describe the heat flow in these situations do not have 'common' analytical solutions.

When I took this course as a graduate student, and when I have taught it in the past, several lectures were devoted to deriving analytical solution to ODEs that are typical of general, 1–D extended surface heat transfer. Such derivations typically begin with a power series representation of the solution, which, when manipulated into the ODE, can be used to define the functional form of the solution. Perhaps you recall such methods in your differential equations courses – one (of many) analytical methods which were painful to comprehend and easy to forget.

It is, in my opinion, no longer necessary to present such derivations in an advanced conduction class. These derivations are inherently mathematical in nature, and do not reveal any of the underlying physics to the problem (such as a grasp of the temperature profile in a fin). In addition, the advanced symbolic mathematics packages (i.e., *Mathematica*) which are available today allow us to completely bypass the painful details to deriving the solution to an ODE – and cut to the chase.

This chapter will examine the solutions to ordinary differential equations that characterize 1–D heat flow in triangular and annular fins (which are forms of Bessel's equation), and introduce the use of *Mathematica* to derive and manipulate the solutions.

2.2 Application of *Mathematica* to the annular fin

2.2.1 Formulation of the problem

From the previous chapter, the governing DE for the annular fin was

$$\overline{r}\frac{d^2\overline{T}}{d\overline{r}^2} + \frac{d\overline{T}}{d\overline{r}} - \overline{r}N^2\overline{T} \tag{2.1}$$

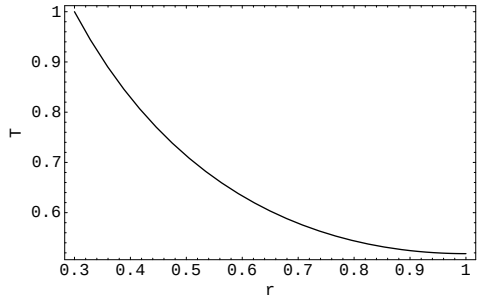
Assuming an adiabatic tip, boundary conditions are

$$\begin{aligned} \overline{T}(\overline{r}=a) &= 1 \\ \left.\frac{d\overline{T}}{d\overline{r}}\right|_{\overline{r}=1} &= 0 \end{aligned} \tag{2.2}$$

where $a = R_1/R_2$. The problem as stated is well–posed and can be given directly to *Mathematica* for solution. The code which obtains the solution is listed below.

```
In[1]:=de=r t''[r]+t'[r]-r n^2 t[r]==0;
bc1=t[a]==1;
bc2=t'[1]==0;
soln=Simplify[DSolve[{de,bc1,bc2},t[r],r][[1,1]]]
Out[1]=t[r] ->
  ((BesselI[-1, Sqrt[n^2]] +
       BesselI[1, Sqrt[n^2]])*
     BesselK[0, Sqrt[n^2]*r] +
    BesselI[0, Sqrt[n^2]*r]*
     (BesselK[-1, Sqrt[n^2]] +
       BesselK[1, Sqrt[n^2]]))/
   ((BesselI[-1, Sqrt[n^2]] +
       BesselI[1, Sqrt[n^2]])*
     BesselK[0, a*Sqrt[n^2]] +
    BesselI[0, a*Sqrt[n^2]]*
     (BesselK[-1, Sqrt[n^2]] +
       BesselK[1, Sqrt[n^2]]))
In[3]:=soln=Simplify[
    soln/.{BesselK[-1,x_]->BesselK[1,x],
       BesselI[-1,x_]->BesselI[1,x]}/.(n^2)^(1/2)->n]
Out[3]=t[r] ->
  (BesselI[1, n]*BesselK[0, n*r] +
     BesselI[0, n*r]*BesselK[1, n])/
   (BesselI[1, n]*BesselK[0, a*n] +
```

```
     BesselI[0, a*n]*BesselK[1, n])
In[29]:=
 Plot[t[r] /. soln/. n -> 1.5/. a ->  .3, {r,  .3, 1},
    Frame -> True, Axes -> False,
    FrameLabel -> {r, T}]
```



The functions `BesselI` and `BesselK` appearing in the solution are *Modified Bessel Functions* of order -1, 0, and 1, and are typically given the symbol I_n and K_n (where n is the order). Some general properties of Bessel functions appear in Sec.(2.3), including formulas for computation of integrals and derivatives – which are needed to calculate properties such as heat transfer rate from the fin. For now, however, we will let *Mathematica* do the symbolic manipulations and numeric calculations.

2.2.2 Explanation of the *Mathematica* code

Several points can be made regarding the *Mathematica* code that was used to obtain the solution.

1. *Mathematica* begins all intrinsic functions and mathematical constants with upper case letters, i.e., `BesselI[n,x]` for $I_n(x)$, `Sin[x]` for $\sin(x)$, `Pi` for π, and `I` for $i = \sqrt{-1}$. It is *strongly advised* that you use lower case letters for all variables and parameters in your solution. By doing so you will avoid any conflict with a *Mathematica*–defined function or constant. For example, the temperature was denoted as `t[x]` and the fin number N was denoted as `n` (the upper case `N` is an intrinsic function in *Mathematica*).

2. The single equal sign '=' refers to an *assignment* in *Mathematica*, whereas the double equals sign '==' denotes the condition of equality. In the first line, the variable `de` is assigned to represent the differential equation, and likewise with the assignment of the boundary condition equations to `bc1` and `bc2`. This assignment is not necessary; the equations could have been written out explicitly in the argument of the `DSolve` function.

3. By writing the dependent variable as `t[r]`, it is implied that T is a function of r.

4. The differential operator `t'[r]` is equivalent in *Mathematica* to `D[t[r],r]`, and `t''[r]` would be `D[t[r],r,2]`. The expression `t'[0]` implies the derivative of T with respect to r evaluated at $r = 0$. This could also appear as `D[t[r],r]/.r->0`.

5. `soln` denotes the simplified solution returned by the *Mathematica* function `DSolve`. The use of `DSolve` should be self–explanatory from the context. The command `Simplify` finds the algebraically reduced form of the solution (if it exists). The solution appears as a *list* in the form of a *replacement rule*.

   (a) A list is a group of one or more quantities, and the list construction and manipulation features of *Mathematica* enable one to perform vector and matrix mathematics. A 3 element vector would appear as `{a,b,c}`, whereas a 2×2 matrix would be `{{a,b},{c,d}}`. The *depth* of the list is the dimensionality (or rank) of the list plus 1; a vector would have a depth of 2 and a matrix would have a depth of 3. A *part*, or element, of a list can be extracted using the double brace format as follows; `{a,b,c}[[2]]` gives `b` and `{{a,b},{c,d}}[[2,1]]` gives `c`. The solution to `DSolve` appears as a depth 3 list with one element – the reason the output appears as a list is because the DE can, in general, have more than one solution. To extract this one element from the list and assign it to `soln`, the part specification `[[1,1]]` is included at the end of the assignment to `soln`. This may or may not be necessary – it is done here to avoid any subsequent problems with the list structure of the solution.

   (b) A replacement rule is in the form `f[a]/. a -> b`. It is somewhat akin to an assignment, in that `a` is replaced by `b`, except that the replacement acts only within the command (or line) with which it is executed. The given line would compute `f[b]` (where `f` is some function) – yet the variable `a` will not be assigned the value `b` in subsequent calculations. This is different than `a=b` followed by `f[b]`; for which `a` has now been assigned to `b` for all subsequent operations. As another example, `D[t[r],r]/.r->0` first computes `t'[r]` and then replaces `r` with zero. The operation `r=0`, followed by `D[t[r],r]`, would try to compute `D[t[0],0]` and would give an error because `r` has been assigned to the constant 0 and is not a valid variable. The solution `soln` to `DSolve` appears as a replacement; `t[r] -> f[r]`, where `f[r]` is shorthand for the actual solution. It *does not* assign `t[r]` to the solution; the operation `t[.5]`, for example, would simply return `t[.5]`. To compute the solution at $r = .5$, one would use `t[r]/.soln/.r->.5`. This command first replaces `t[r]` with the functional form of the solution, and then replaces `r` with 0.5. Any other parameters appearing in the solution (i.e., `a` and `n`) would also have to be given values (via a direct assignment or a replacement) to obtain a numerical answer. The use of this can be seen in the `Plot` argument.

6. *Mathematica* will often not give the most 'simple' form to an equation. For example, it does not automatically recognize that $I_{-1} = I_1$, $K_{-1} = K_1$ (which are properties of the modified

Bessel functions), and $\sqrt{N^2} = N$. In line [6] substitutions are made (via replacement rules) into the solution, and the operation `Simplify` is used to condense the result. The two Bessel function replacement rules are applied first to the result (by appearing together in a list they are applied simultaneously) and the $\sqrt{N^2}$ replacement rule is applied next. The underscore following `x` in `BesselK[-1,x_]->BesselK[1,x]` denotes that x can have any value or form; by doing so the Bessel function for argument N, Na, and Nr are all replaced accordingly.

7. *Mathematica* offers comprehensive online help – which includes a complete hypertext version of the *Mathematica* book. There you can find more information on the strategy used in the code and on other features (such as the plotting function).

2.2.3 Heat transfer

As can be seen in the *Mathematica* plot, the temperature distribution in the annular fin follows similar behavior to that for a straight fin, i.e., exponential–like decay. Indeed, the Modified Bessel functions I and K are analogous to the hyperbolic functions which form the solution for the uniform cross section case, in that they have exponential behavior.

The solution for the temperature distribution, as derived by *Mathematica*, stands as

$$\overline{T} = \frac{I_0(N\overline{r})K_1(N) + K_0(N\overline{r})I_1(N)}{I_0(Na)K_1(N) + K_0(Na)I_1(N)} \tag{2.3}$$

This obviously gives $\overline{T} = 1$ at $\overline{r} = 1$. If we use the fact that $I_0' = I_1$ and $K_0' = -K_1$, we find that the adiabatic BC is satisfied at $\overline{r} = a$.

The heat flux from the fin follows from

$$\begin{aligned} q_{fin} &= -kA_B \left.\frac{dT}{dr}\right|_{r_i} = -\frac{2\pi kbr_i(T_B - T_\infty)}{r_o} \left.\frac{d\overline{T}}{d\overline{r}}\right|_a \\ &= 2\pi akb(T_B - T_\infty)N\frac{K_1(Na)I_1(N) - I_1(Na)K_1(N)}{K_0(Na)I_1(N) + I_0(Na)K_1(N)} \end{aligned} \tag{2.4}$$

Since the heat transfer from the base without the fin is $q_{base} = 2\pi br_i h(T_B - T_\infty)$, it follows that the effectiveness of the fin is

$$\epsilon = \frac{N}{Bi} \cdot \frac{K_1(Na)I_1(N) - I_1(Na)K_1(N)}{K_0(Na)I_1(N) + I_0(Na)K_1(N)} \tag{2.5}$$

where $Bi = hr_o/k$ is the Biot number based on the outer radius. This formula is qualitatively similar to that obtained for the rectangular fin – except the hyperbolic tangent function in the latter is now replaced with a function of modified Bessel functions. The effectiveness will be a function of N, Bi and the radius ratio a.

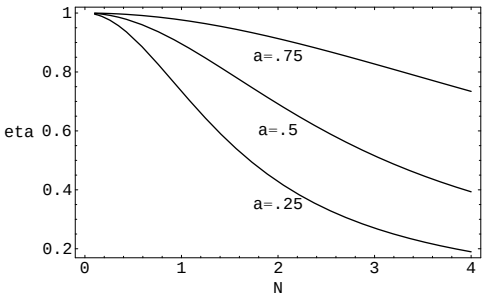


Figure 2.1: annular fin η

The efficiency, on the other hand, is obtained using $q_{max} = 2\pi(r_o^2 - r_i^2)(T_B - T_\infty)$, from which

$$\eta = \frac{2a}{N(1-a^2)} \cdot \frac{K_1(Na)I_1(N) - I_1(Na)K_1(N)}{K_0(Na)I_1(N) + I_0(Na)K_1(N)} \tag{2.6}$$

The efficiency η is a function only of N and the geometrical parameter a – and not a function of Bi. A plot of η vs. N with a as a parameter is given in Fig. 2.1. As $a \to 1$ the fin becomes 'stubbier'. Accordingly, there is less of a temperature drop across the fin and the efficiency becomes closer to unity.

2.3 Ordinary and modified Bessel functions

2.3.1 Definitions and Properties

The ordinary Bessel functions of integer order n, denoted $J_n(x)$ and $Y_n(x)$, are solutions to the ODE

$$x^2u'' + xu + (x^2 - n^2)u = 0 \tag{2.7}$$

i.e.,

$$u(x) = AJ_n(x) + BY_n(x) \tag{2.8}$$

whereas the modified Bessel functions $I_n(x)$ and $K_n(x)$ are solutions to

$$x^2u'' + xu - (x^2 + n^2)u = 0 \tag{2.9}$$

Plots of J_n, Y_n, I_n, and K_n for $n = 0$, 1 and 2 are shown in Fig. 2.2.

The ordinary Bessel functions share some of the same characteristics of the sine and cosine functions, in that they exhibit oscillatory behavior about zero. Indeed, when $x \gg n$ they have the

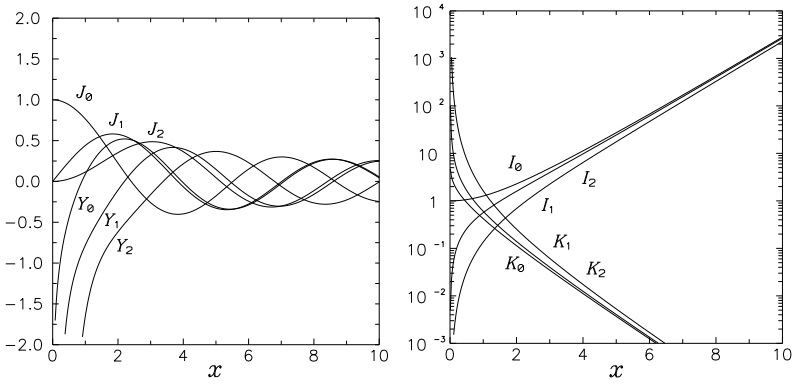


Figure 2.2: ordinary Bessel functions J_n and Y_n, and modified Bessel functions I_n and K_n

asymptotic limit of

$$J_n(x \gg n) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$
$$Y_n(x \gg n) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

which indicates a direct linkage between the Bessel and triginometric functions. Both J_n and Y_n go to zero (at rate $1/\sqrt{x}$) for $x \to \infty$. In the other limit, i.e., $x \to 0$, the properties are

$$J_n(x \ll 1) \sim \frac{1}{n!}\left(\frac{x}{2}\right)^n, \qquad n \geq 0$$
$$Y_0(x \ll 1) \sim \frac{2}{\pi}\ln(x)$$
$$Y_n(x \ll 1) \sim -\frac{(n-1)!}{\pi}\left(\frac{x}{2}\right)^{-n}, \qquad n \geq 1$$

The modified Bessel functions, on the other hand, are cousins to the hyperbolic functions. Recall that sin and cos are related via complex arithmetic to sinh and cosh (from the previous chapter). Likewise, the modified Bessel functions are related to the ordinary Bessel functions by analogous complex relationships.

For large argument ($x \gg n$) the modified Bessel functions behave as

$$I_n(x \gg n) \sim \sqrt{\frac{1}{2\pi x}} \exp(x)$$
$$K_n(x \gg n) \sim \sqrt{\frac{1}{2\pi x}} \exp(-x)$$

which shows the hyperbolic (or exponential) behavior. In the opposite limit, the behavior is

$$\begin{aligned} I_n(x \ll 1) &\sim \frac{1}{n!}\left(\frac{x}{2}\right)^n, && n \geq 0 \\ K_0(x \ll 1) &\sim -\ln(x) \\ K_n(x \ll 1) &\sim \frac{(n-1)!}{2}\left(\frac{x}{2}\right)^{-n}, && n \geq 1 \end{aligned}$$

The derivatives of ordinary Bessel functions can be obtained via the following recurrence relations. Here, C and D represents J, Y or any linear combination of the two:

$$C_{n+1}(x) = \frac{2n}{x}C_n(x) - C_{n-1}(x) \tag{2.10}$$

$$\frac{d}{dx}C_n(x) = \frac{1}{2}\left(C_{n-1}(x) - C_{n+1}(x)\right) \tag{2.11}$$

$$= -C_{n+1}(x) + \frac{n}{x}C_n(x) \tag{2.12}$$

Useful integral relationships are

$$\int xJ_0(x)\,dx = xJ_1(x) \tag{2.13}$$

$$\int xY_0(x)\,dx = xY_1(x) \tag{2.14}$$

$$\int J_1(x)\,dx = -J_0(x) \tag{2.15}$$

$$\int Y_1(x)\,dx = -Y_0(x) \tag{2.16}$$

$$\int x^{2n+1}J_n^2(x)\,dx = \frac{x^{2n+2}}{2(2n+1)}\left[J_n^2(x) + J_{n+1}^2(x)\right] \tag{2.17}$$

$$\int xJ_n^2(x)\,dx = \frac{x^2}{2}\left[J_n^2(x) - J_{n+1}^2(x)J_{n-1}^2(x)\right] \tag{2.18}$$

$$\int x^{n+1}C_n(x)\,dx = x^{n+1}C_{n+1}(x) \tag{2.19}$$

$$\int x^{1-n}C_n(x)\,dx = -x^{1-n}C_{n-1}(x) \tag{2.20}$$

$$\int xC_n(hx)D_n(gx)\,dx = \frac{x}{h^2-g^2}\left[hC_{n+1}(hx)D_n(gx) - gC_n(hx)D_{n+1}(gx)\right] \tag{2.21}$$

$$\int xC_n(hx)D_n(hx)\,dx = -\frac{x^2}{4}\left[C_{n-1}(hx)D_{n+1}(gx) - 2C_n(hx)D_n(hx) + C_{n+1}(hx)D_{n-1}(gx)\right] \tag{2.22}$$

For formulas for the modified Bessel functions, let $C_n = I_n$, $(-1)^n K_n$, or any linear combination of the previous two;

$$C_{n+1}(x) = -\frac{2n}{x} C_n(x) + C_{n-1}(x) \tag{2.23}$$

$$\frac{d}{dx} C_n(x) = \frac{1}{2} \left(C_{n-1}(x) + C_{n+1}(x) \right) \tag{2.24}$$

$$= C_{n+1}(x) + \frac{n}{x} C_n(x) \tag{2.25}$$

The integral formulas are;

$$\int x I_0(x)\, dx = x I_1(x) \tag{2.26}$$

$$\int x K_0(x)\, dx = -x K_1(x) \tag{2.27}$$

$$\int I_1(x)\, dx = I_0(x) \tag{2.28}$$

$$\int K_1(x)\, dx = -K_0(x) \tag{2.29}$$

The above relationships should give you ~90% of the formulas needed to work with ordinary and modified Bessel function problems. *Mathematica* will be able to perform most of the calculations of integrals and derivatives on the Bessel functions, and will also compute numerical values as well. You can get FORTRAN subroutines to calculate Bessel functions from most numerical libraries – contact me if you are interested.

2.3.2 The general Bessel equation

The differential equations describing the temperature profile in the annular and the triangular fins are both forms of Bessel's equation, the solutions of which will involve Bessel functions. In the most general form, Bessel's equation appears as

$$x^2 u'' + \left[(1 - 2A)x - 2Bx^2 \right] u' + \left[C^2 D^2 x^{2C} + B^2 x^2 - B(1 - 2A)x + A^2 - C^2 n^2 \right] u = 0 \tag{2.30}$$

where A, B, C, and n are constants. The solution to the above DE is

$$x^A e^{Bx} \left[C_1 J_n(Dx^C) + C_2 Y_n(Dx^C) \right] \tag{2.31}$$

where C_1 and C_2 are integration constants which are determined from the boundary conditions of the problem. The cookbook approach to using Eq. (2.30) is to start with the particular DE at hand, put it in the form of Eq. (2.30), and deduce the constants A, B, C, D, and n from inspection.

To give an example, start again with the annular fin DE, which is

$$\overline{r}^2\overline{T}'' + \overline{r}\overline{T}' - \overline{r}^2 N^2 \overline{T} = 0 \tag{2.32}$$

Recognize that $\overline{r}$ has been factored through to make the first term consistent with Eq. (2.30). Comparing the second term to Eq. (2.30) shows that both A and B must be zero, and the third term gives $n = 0$, $C = 1$, and $D^2 = -N^2$. This last relation is equivalent to $D = \pm iN$, and the positive root should be taken. In addition, when D is imaginary (i.e., if i is present), then I_n and K_n are substituted in place of J_n and Y_n in Eq. (2.31). The general solution to the circular fin DE is then

$$\overline{T} = C_1 I_0(N\overline{r}) + C_2 K_0(N\overline{r}) \tag{2.33}$$

which is in the same form as obtained by *Mathematica*.

From Chapter 1, the DE for the triangular fin was

$$\overline{x}^2\overline{T}'' + \overline{x}\overline{T}' - \overline{x} N^2 \overline{T} = 0 \tag{2.34}$$

Comparison to the general Bessel equation gives $A = B = 0$ from the second term, $n = 0$ from the third, and $C = 1/2$ (since $C^2 D^2 x^{2C}$ is the only remaining non–zero part of the third term). We then have $-N^2 = D^2/4$, or $D = 2iN$. Replacing J_n and Y_n with I_n and K_n (since D is imaginary), the general solution to the triangular fin problem is

$$\overline{T} = C_1 I_0(2N\overline{x}^{1/2}) + C_2 K_0(2N\overline{x}^{1/2}) \tag{2.35}$$

Boundary conditions are $\overline{T}' = 0$ at $\overline{x} = 0$ (which is the fin tip – recall that the dimensionless x now runs from the tip of the fin to the base) and $\overline{T} = 1$ at $\overline{x} = 1$. Actually, an explicit BC at $\overline{x} = 0$ is not needed for this particular solution. The function K_0 is singular at $\overline{x} = 0$, and C_2 must therefore be zero to keep the temperature finite at the tip. The same reasoning does not apply to the annular fin because the point $\overline{r} = 0$ was not included in the domain, i.e., $\overline{r}$ ranged from a to 1. The BC at $\overline{x} = 1$ provides a simple equation for the remaining constant C_1, and the complete solution is

$$\overline{T} = \frac{I_0(2N\overline{x}^{1/2})}{I_0(2N)} \tag{2.36}$$

It is interesting to compare the above old–fashioned method with the *Mathematica* approach:

```
In[35]:=
  de=x^2 t''[x]+x t'[x]-x n^2 t[x]==0;
bc2=t[1]==1;
soln=Simplify[DSolve[{de,bc2},t[x],x][[1,1]]]
Out[35]=t[x] ->
  (BesselI[0, 2*Sqrt[n^2]]*
```

```
      BesselK[0,
       2*Sqrt[n^2]*Sqrt[x]]*C[2] +
     BesselI[0,
       2*Sqrt[n^2]*Sqrt[x]]*
      (1 - BesselK[0, 2*Sqrt[n^2]]*
         C[2]))/
   BesselI[0, 2*Sqrt[n^2]]
In[43]:=
  soln=Simplify[soln/.C[2]->0/.(n^2)^(1/2)->n]
Out[43]=t[x] ->
  BesselI[0, 2*n*Sqrt[x]]/
   BesselI[0, 2*n]
```

Mathematica would have trouble with an explicit BC at $\overline{x} = 0$ because of the singularity in K_0. Perhaps a BC could be stated using a `Limit` function as was done in an example in the previous chapter. In this case, however, it is easiest to simply leave the tip BC initially unspecified. The returned solution contains an integration constant `C[2]` which appears as a product with the K_0 function. Since this part of the solution must dissapear, the constant `C[2]` is zeroed out by a replacement operation, which is performed in the second line.

Exercises

1. A conical pin fin of circular cross section is shaped (as the name implies) as a circular cone. It has a length L and a radius R at the base. The cross sectional area is zero at the tip (i.e., the tip is a point). Assuming 1–D conduction within the fin, derive the fin differential equation for this geometry, and identify the appropriate fin parameter N. Obtain a solution for this configuration by using the general Bessel equation, and check the solution with that obtained with *Mathematica*. Derive equations for the heat flux though the fin, the fin effectiveness, and the fin efficiency.

2. Perform the fin optimization analysis, presented in the previous chapter, for a triangular fin. Determine the optimum value of N and the optimum heat flux for a fixed profile area. Note that the answer to the latter part is given in the previous chapter.

3. A parabolic–profile fin has a cross sectional area given by

$$A_C = 2bW \left(\frac{x}{L}\right)^2 \tag{2.37}$$

   where W is the fin width, b is the half–thickness at the base, L is the fin length, and x runs from the tip of the fin inwards (i.e., in the same manner used for the triangular fin). It will be assumed here that $W \gg b$.

(a) Formulate the DE and boundary conditions for this particular type of fin, and solve the problem. Note: this is not Bessel's equation. Rather, the solution will be in the form $\overline{T} \propto x^{\alpha}$, where α will be a function of N.

(b) Obtain an equation for the fin heat transfer rate, and determine the optimum heat transfer, per unit width of fin, for a fixed profile area using the procedure in the previous chapter. How does the parabolic fin compare to the straight rectangular and triangular fins with regard to optimum heat transfer per unit mass? Also, what is the form of the temperature profile in the fin for optimum N?

4. A pin fin with a circular cross section has a length L and a radius R, and the cross section $A_C = \pi R^2$ is constant.

(a) Perform an optimization analysis on the fin for constant fin k, h, and base/environment temperatures. What you are looking for is the optimum length/radius ratio of the fin for a fixed fin volume $V = L\,A_C$.

(b) For free convection around a horizontal cylinder, the heat transfer coefficient will vary as $h \approx CR^{-1/4}$, where C is a constant which depends on the fluid properties. For this situation, derive the optimum L/R.

5. Consider now a pin fin with a square cross section, with b denoting the thickness of the fin and L the length. Fins of this type are to be mounted in a rectangular array on a circuit board, subject to the constraint that the gap between adjacent fin surfaces is fixed at a value t. With this constraint, the centers of the fins will be spaced at a distance $b+t$ on the board, and the total number of fins on a square board of width W will be $n_F \approx W^2/(b+t)^2$. Now take both W and the total volume of fin material on the board, $V = b^2 L\, n_F$ to be fixed. By performing an optimization analysis, determine the optimum dimensionless fin thickness b_{opt}/t as a function of the effective board fin number $N_B^2 = 4hS^2/kt$, in which $S = V/W^2$ is an effective length based on the volume of the fins and the board area. Prepare a plot of b_{opt}/t vs. N_B, and explain the limiting behavior for large N_B.

6. A wire, of length L and radius R, is suspended between two posts. The temperature at each post is fixed at T_B, and the wire is exposed to a convection environment characterized by h and T_∞. The wire is carrying current, which results in a heat dissipation rate of q' per m of wire length. Derive the temperature distribution in the wire and the net rate of heat transfer from the wire to the post.

Chapter 3

Transient and One Dimensional Conduction

3.1 Introduction

One–dimensional, steady heat transfer is, more often than not, an idealization of the actual process we are attempting to model. For example, in modeling heat transfer through a plane wall it is typically assumed that the ambient temperatures, on either side of the wall, are at steady, fixed values – when in reality these temperatures will change with the changing environmental conditions (such as night and day). We know that a 1–D, steady analysis would be appropriate for such situations providing the characteristic time for ambient temperature change is significantly larger than the characteristic diffusion time (or thermal relaxation time) of the wall. For such cases, the heat transfer in the wall could be modelled as a quasi–steady process, i.e., a succession of steady processes. This approach formed the basis of the so–called lumped capacity (or small Biot number) approximation for transient heating/cooling of an object. When this criterion is not met – which is of interest here – it becomes necessary to model the interacting dependence of time and position on the temperature field. Analytically, this implies that the governing differential equations for the temperature field will involve partial derivatives, as opposed to ordinary derivatives.

When examining transient heat transfer problems, it is important to recognize the nature (or source) of the transient effect and the spacial dimensionality of the temperature field. The latter can obviously be grouped into zero–dimension problems (which correspond to the lumped capacity approximation) and 1, 2, and 3–D problems. The former can be categorized into *impulse* problems in which the conditions at the boundaries (or within the system) instantaneously change from one state to another, *forced* problems in which the boundary and/or system properties change from an initial state to a final state over a set length of time (as opposed to the instantaneous change in impulse problems), and *periodic* problems in which the boundary conditions and/or system properties cycle over time.

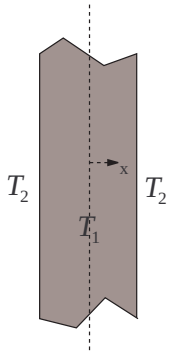


Figure 3.1: plane wall configuration

This chapter will address the modelling of impulse problems in one spacial dimension. By doing so, we will become familiar with the separation–of–variables method (SOV) for solution of certain types of partial differential equations (PDEs). Extension to two and three spacial dimensions, and forced problems, is relatively direct once the basic analytical concepts for transient impulse and 1–D problems are understood. Periodic problems, on the other hand, will require a uniquely different analytical approach. These topics will be addressed in subsequent chapters.

3.2 The transient impulse and 1–D cartesian problem

Perhaps the easiest way to introduce the SOV procedure is to apply it to a relatively simple problem. Consider a plane, symmetrical wall as illustrated in Fig. 3.1. Initially the wall is at a temperature T_1. At time $t = 0$ the boundaries are brought to temperature T_2. The problem is to find the temperature distribution in the wall as a function of x and t.

The heat conduction equation for this situation – which is 1–D and transient, without heat generation – is

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{3.1}$$

The temperature field will, for this problem, remain symmetrical about the midplane of the slab because the boundary conditions are identical at both surfaces. The x coordinate origin can then be placed at the center of the slab and an adiabatic condition can be imposed at $x = 0$. The boundary conditions of the problem are then

$$\left.\frac{\partial T}{\partial x}\right|_0 = 0, \quad T(x = L, t) = T_2 \tag{3.2}$$

and the initial condition is

$$T(x, t = 0) = T_1 \tag{3.3}$$

It is assumed that the surface temperature is T_2 for any finite time into the process – which implies that the solution (once obtained) will be valid only for $t > 0$. Of course, the instantaneous change in surface temperature, which occurs at $t = 0$, is an idealization of physics. Such an impulse would correspond to an infinite surface heat flux at $t = 0$; in any real situation the surface temperature would change over a finite length of time. It turns out, however, that modelling of the real, finite–heat–up–time problem requires a solution to the idealized, impulse problem (which, as was mentioned in the introduction, will be addressed in a future chapter).

The problem is first made dimensionless. Let $\overline{x} = x/L$, $\overline{T} = (T - T_2)/(T - T_1)$, and $\bar{t} = t\alpha/L^2$. The problem becomes

$$\frac{\partial \overline{T}}{\partial \bar{t}} = \frac{\partial^2 \overline{T}}{\partial \overline{x}^2} \tag{3.4}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_0 = 0, \quad \overline{T}(\overline{x} = 1, \bar{t}) = 0 \tag{3.5}$$

$$\overline{T}(\overline{x}, \bar{t} = 0) = 1 \tag{3.6}$$

The solution procedure can be stated as follows: we assume that the solution $\overline{T}$ can be expressed as a product of two functions, each of which depends only on a single variable. In other words,

$$\overline{T}(\overline{x}, \bar{t}) = u(\overline{x}) \cdot v(\bar{t}) \tag{3.7}$$

This approach will be tested by applying it to the problem at hand – if a valid solution is obtained, then the procedure works (at least for this particular problem). By replacing Eq. (3.7) into Eq. (3.4), one obtains

$$\begin{aligned} \frac{\partial (u(\overline{x}) \cdot v(\bar{t}))}{\partial \bar{t}} &= \frac{\partial^2 (u(\overline{x}) \cdot v(\bar{t}))}{\partial \overline{x}^2} \\ u(\overline{x}) \frac{dv(\bar{t})}{d\bar{t}} &= v(\bar{t}) \cdot \frac{d^2 u(\overline{x})}{d\overline{x}^2} \end{aligned} \tag{3.8}$$

Again, u depends only on $\overline{x}$, and v only on $\bar{t}$ – which explains the transformation in the second line. Recognize that the differential operators are now ordinary rather that partial. The equation is rearranged to obtain

$$\frac{v'}{v} = \frac{u''}{u} \tag{3.9}$$

The left hand side of the above is a function only of $\bar{t}$, whereas the right hand side is a function only of $\overline{x}$. The only way that this equality could hold is if both sides of the equation are, in fact, constant. The precise value(s) of this constant will be determined later, but for now the only

distinguishing feature of the constant is whether it is positive, negative, or zero. Therefore, denote the constant as λ^2, and Eq. (3.9) becomes

$$\frac{v'}{v} = \frac{u''}{u} = \pm\lambda^2 \tag{3.10}$$

We write λ^2 (as opposed to simply λ) for a convenience which will soon become evident, and also to fix the fact that λ^2 is a positive number (that is, it is assumed that λ is real valued). Two separate ordinary differential equations are now obtained from Eq. (3.10);

$$v' = \pm\lambda^2 v \tag{3.11}$$

$$u'' = \pm\lambda^2 u \tag{3.12}$$

This last statement illustrates the central objective of the separation of variables method – in that the partial differential equation has been separated into two ordinary differential equations. And the latter can be easily solved.

Perhaps the concern over the sign of the separation constant λ^2 can now become evident. The sign will completely dictate the nature of the solution, and only one choice in sign will lead to a physically meaningful result. Precise mathematical procedures will be introduced in following sections to prescribe the choice of sign on λ^2; yet for this particular problem the sign can be deduced from physical reasoning. The wall will eventually reach a steady state temperature distribution, and the solution must be consistent with this behavior in the limit of $\bar{t} \to \infty$. Because the back boundary is insulated the steady–state profile will simply be $T = T_2$, or (nondimensionally) $\overline{T} = 0$ for $\bar{t} \to \infty$. For non–zero λ, Eq. (3.11) has the solution

$$v = Ce^{\pm\lambda^2\bar{t}} \tag{3.13}$$

Since $\overline{T}$ is $u \cdot v$, only the case of $\lambda^2 < 0$ will give the physically correct result, i.e., an exponentially decaying solution in time. A zero value of λ would give v equal to a constant (or $\overline{T}$ independent of time), and $\lambda^2 > 0$ would give an exponentially–growing solution. Therefore, the cases of $\lambda^2 \geq 0$ can be eliminated.

Having fixed the sign of λ^2, a solution to the ODE for u, Eq. (3.12), can be obtained:

$$u = A\cos(\lambda\overline{x}) + B\sin(\lambda\overline{x}) \tag{3.14}$$

The general solution to the conduction equation is, again, $\overline{T} = v \cdot u$. The constant C, in Eq. (3.13), can be absorbed into A and B in Eq. (3.14) to obtain

$$\overline{T} = u \cdot v = \left[A\cos(\lambda\overline{x}) + B\sin(\lambda\overline{x})\right] e^{-\lambda^2\bar{t}} \tag{3.15}$$

Three constants appear in this solution (A, B, and λ) and three equations exist for their specification (the 2 BCs and the IC) – so it appears that the finishing touches on the solution will be trivial. Unfortunately, this is not the case: the real complications to the method have only begun.

Apply the solution first to the BC at $\overline{x} = 0$:

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_0 = 0 = [0 + \lambda B]\, e^{-\lambda^2 \overline{t}} \tag{3.16}$$

This gives either $\lambda = 0$ or $B = 0$. The first option has already been discounted, so the correct choice is $B = 0$. This result could have been deduced from the symmetry of our problem; the solution must be even in $\overline{x}$ (i.e., $\overline{T}(-\overline{x}, \overline{t}) = \overline{T}(\overline{x}, \overline{t})$), from which the sin part of the solution can be eliminated.

The boundary condition at $x = 1$ has

$$\overline{T}(\overline{x} = 1, \overline{t}) = 0 = A\cos(\lambda) e^{-\lambda^2 \overline{t}} \tag{3.17}$$

Again, the choice $A = 0$ leads to $\overline{T} = 0$ as the solution – which is obviously not correct. Rather, the outcome should be

$$\cos(\lambda) = 0 \tag{3.18}$$

This represents an *transcendental* equation for λ; meaning that an infinite number of roots exist for the equation. For this particular relation the roots can be obtained explicitly as

$$\lambda = \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2}, \ldots \tag{3.19}$$

or, in a more general form

$$\lambda = \lambda_n = \frac{1}{2}(2n-1)\pi,\ \ n = 1, 2, \ldots \tag{3.20}$$

We have no rationale for rejecting any particular value of λ. That is, each value of λ could (in general) have a meaningful and essential contribution to the solution. The integration constant A in Eq. (3.17) would also be expected to depend on the particular value of λ, and can be denoted, for the n^{th} value of λ, as A_n. Because the heat conduction equation is linear, the sum of any number of separate solutions to the equation is also a solution. The most general solution to the problem would include all possible valid solutions. The outcome of this reasoning is that the general solution is

$$\overline{T} = \sum_{n=1}^{\infty} A_n \cos[\frac{1}{2}(2n-1)\pi\overline{x}] \exp\left[-\left(\frac{1}{2}(2n-1)\pi\right)^2 \overline{t}\right] \tag{3.21}$$

Recognize that each term in the series 1) is a solution to the PDE, and 2) satisfies both BCs. The initial condition has yet to be satisfied.

At this point you may be questioning the usefulness of the SOV procedure. One equation remains (the IC) to fix the integration constants of the problem, yet the general solution contains an infinite number of constants (i.e., the *expansion coefficients* A_n for $n = 1, 2, \ldots$). It would

therefore appear that the problem is 'underconstrained', i.e., there is not enough information to give a unique solution. Specifically, the initial condition, when applied to Eq. (3.21), gives

$$\overline{T}(\overline{x}, \overline{t} = 0) = 1 = \sum_{n=0}^{\infty} A_n \cos[\frac{1}{2}(2n-1)\pi\overline{x}] \tag{3.22}$$

The very nature of this equation is somewhat contradictory – the left hand side is obviously constant, yet the right hand side appears to show a clear functionality on $\overline{x}$. The distiguishing feature of the right hand side, however, is the fact that an *infinite* number of terms are included in the series. If this condition is exactly met (in a mathematical sense), it is possible to choose values for the A_n so that the condition in Eq. (3.22) is exactly satisfied for all $\overline{x}$. On the other hand, if only a finite number of terms are included in the series – which would be required for any numerical evaluation of the solution – then the condition can only be met in an approximate sense.

One possible method of fixing the A_n coefficients would be to select a set of 'collocation' points $\overline{x} = \overline{x}_1, \overline{x}_2, \ldots \overline{x}_M$ and use these to obtain equations for the A_n's, i.e.,

$$\sum_{n=1}^{\infty} A_n \cos[\frac{1}{2}(2n-1)\pi\overline{x}_1] = 1$$
$$\sum_{n=1}^{\infty} A_n \cos[\frac{1}{2}(2n-1)\pi\overline{x}_2] = 1$$
$$\sum_{n=1}^{\infty} A_n \cos[\frac{1}{2}(2n-1)\pi\overline{x}_M] = 1$$

If the series is limited (or 'truncated') to M terms, one would obtain a system of M equations for the M coefficients. This method, however, is ill–advised and not very practical.

A much more efficient and elegant approach manages to eliminate every term in the series in Eq. (3.22) except one. And the one remaining term leads to an explicit equation for the corresponding A_n. To demonstrate this method, first multiply Eq. (3.22) through by $\cos[\frac{1}{2}(2m-1)\pi\overline{x}]$, where m is an integer, and integrate over $\overline{x}$ from 0 to 1:

$$\begin{aligned}
&\int_0^1 \cos[\frac{1}{2}(2m-1)\pi\overline{x}]\, d\overline{x} \\
&\quad = \int_0^1 \cos[\frac{1}{2}(2m-1)\pi x] \left(\sum_{n=1}^{\infty} A_n \cos[\frac{1}{2}(2n-1)\pi\overline{x}] \right) d\overline{x} \\
&\quad = \sum_{n=1}^{\infty} A_n \int_0^1 \cos[\frac{1}{2}(2m-1)\pi\overline{x}] \cos[\frac{1}{2}(2n-1)\pi\overline{x}]\, d\overline{x}
\end{aligned} \tag{3.23}$$

The orders of integration and summation can be switched in the above equation because it is assumed that the series is convergent – if it was not, the solution would not be very useful. The

left hand side integral is

$$\int_0^1 \cos[\frac{1}{2}(2m-1)\pi\overline{x}]\,d\overline{x} = \frac{2}{(2m-1)\pi}\sin(\frac{1}{2}(2m-1)\pi)$$
$$= \frac{2}{(2m-1)\pi}(-1)^{m-1}$$

The integral on the right hand side will have two possible outcomes. Denoting λ_n as shorthand for $(2n-1)\pi/2$, one has

$$\int_0^1 \cos[\lambda_m\overline{x}]\cos[\lambda_n\overline{x}]\,d\overline{x}$$
$$= \begin{cases} \dfrac{\lambda_m\cos(\lambda_n)\sin(\lambda_m) - \lambda_n\cos(\lambda_m)\sin(\lambda_n)}{\lambda_m^2-\lambda_n^2}, & \lambda_m \neq \lambda_n \\ \dfrac{1}{2} + \dfrac{\sin(2\lambda_n)}{4\lambda_n}, & \lambda_m = \lambda_n \end{cases}$$

The first case, for $n \neq m$, is zero because $\cos(\lambda_n) = 0$ for all integer n. Likewise, the second case, $n = m$, gives a value of $1/2$ because $\sin(2\lambda_n) = 0$. To summarize, the result is

$$\int_0^1 \cos[\lambda_m\overline{x}]\cos[\lambda_n\overline{x}]\,d\overline{x} = \begin{cases} 0 & , \quad n \neq m \\ \dfrac{1}{2} & , \quad n = m \end{cases} \tag{3.24}$$

Substitution of this result into Eq. (3.23) leads to the removal of all terms in the series except the one for which $n = m$. What remains is simply

$$\frac{2(-1)^{m-1}}{(2m-1)\pi} = A_m \times \frac{1}{2}$$

or

$$A_m = \frac{4(-1)^{m-1}}{(2m-1)\pi} = \frac{2(-1)^{m-1}}{\lambda_m} \tag{3.25}$$

This is the sought explicit equation for A_m. Of course, the index m can be set to any number or symbol we choose, i.e. A_n. The final, complete solution to the temperature field is then

$$\overline{T}(\overline{x},\overline{t}) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos[\lambda_n\overline{x}]}{\lambda_n}\exp\left[-\lambda_n^2\overline{t}\right] \tag{3.26}$$

where $\lambda_n = (2n-1)\pi/2$.

As was mentioned above, the solution is 'formally' exact only when the upper limit of the sum appears as infinity. To obtain a useful numerical result, however, it is only necessary to sum a finite

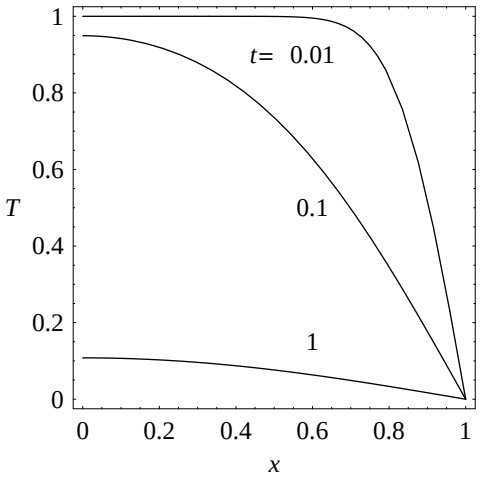


Figure 3.2: Solution to Eq. (3.26)

number of terms (say N) so that the series converges. This number typically corresponds to the point where the relative change in the sum due to the $n = N^{th}$ term is less that a set tolerance. It can be easily seen that the 'truncation limit' N decreases as time $\overline{t}$ increases – which is due to the exponential dependence of the solution on $\lambda_n^2 \overline{t}$.

Series solutions will occur in the significant majority of PDE solutions examined in this course. As engineers, we are obviously interested in obtaining numerical results from the analytical solution, and this will typically require evaluation of the solution in a *Mathematica*, fortran, or other programming language code[1]. A discussion of the aspects of *Mathematica* evaluation of infinite series solutions is presented in Sec. 3.5.

Shown in Fig. 3.2 are calculated results from the solution, in which dimensionless temperature is plotted vs. $\overline{x}$ for various values of dimensionless time. It is important to be able to produce and plot numerical results from an analytical solution – for this allows one to perform a 'reality check' on the solution. The plot shows that the boundary and initial conditions appear to be met. Zero gradient exists at $\overline{x} = 0$, the temperature is zero at $\overline{x} = 1$, and for small values of time (i.e., $\overline{t} = 0.01$) the temperature is nearly uniform in the interior of the wall. It takes a finite amount of time for a temperature 'wave' to propagate through the wall. That is, for dimensionless times somewhat less that 0.1 the temperature at $x = 0$ is essentially unaffected by the sudden alteration of temperature at $x = 1$. Up until this time the wall could be treated as semi–infinite, i.e., of effectively infinite thickness. The advantage of this approximation is that it can lead to somewhat simpler expressions for the temperature field. This (and other) approximation methods will be explored in future chapters. The temperature in the wall has decayed nearly to ambient for $\overline{t} = 1$.

[1]Series solution can also be numerically evaluated with a spreadsheet – but this approach is NOT advised

This, again, is to be expected: an order–of–magnitude analysis would give a characteristic cooling time of $t_c \approx L^2/\alpha$. This corresponds to a dimensionless time of unity, and is consistent with our exact results.

Basic elements of Separation–of–Variables

Here is a summary of what has been covered, with some additional mathematical terminology introduced:

1. Problem Statement: The governing differential equations and boundary conditions are formulated and cast in nondimensional form:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$
$$\left.\frac{\partial T}{\partial x}\right|_0 = 0$$
$$T(x=1,t) = 0$$
$$T(x,t=0) = 1$$

In the above and what follows, it is understood that all quantities – unless specified otherwise – are in a nondimensional form. The differential equation and the two boundary conditions are homogeneous, whereas the initial condition is inhomogeneous. An equation is homogenous if the substitution $T \rightarrow C{\cdot}T$, where C is a constant, leads to the same original equation. The homogeneity and/or inhomogeneity of the DE and BC/ICs are of critical importance in using separation of variables. In the particular problem examined above, the success of the method actually required both the DE and the BCs to be homogeneous. A modified analytical approach would have been required had this not been the case. Inhomogeneities in the DE will typically arise from sources or sinks in the energy equation (i.e., heat generation). An inhomogeneous boundary condition would have resulted had the surfaces been maintained at constant yet unequal temperatures, or if a specified heat flux (as opposed to an adiabatic condition) had been applied at a surface. As you can imagine, such situations will not be uncommon – and because of this it will be necessary to generalize the analytical procedure to situation in which SOV, by itself, cannot work.

A completely homogeneous problem (say an IC of $T(x,t=0) = 0$ along with homogeneous DE and BCs) would have given the trivial solution of $T = 0$ for all x and t. You should be able to prove this to yourself. In any meaningful heat conduction problem, there must be at least one inhomogeneity in the problem statement.

2. Separation of Variables: The dependent variable $T(x,t)$ is split into the product of two functions, with each function being dependent solely on one independent variable. That is, $T(x,t) = u(x){\cdot}v(t)$. The product is swapped back into the partial differential equation, and the equation is rearranged so that one side contains only the u function and its derivatives, and the other side contains only the v function and its derivative. It can then be declared that each side must be constant. The

constant is denoted as $\pm\lambda^2$, and two ordinary differential equations are formed from the original partial differential equation:

$$u'' = \pm\lambda^2 u$$
$$v' = \pm\lambda^2 v$$

3. Choice of the sign of λ^2: This part is critical. In the previous example physics were used to determine that $-\lambda^2$ was the correct choice, with the additional constraint that λ is non–zero. For other problems (such as in two–dimensional, steady conduction and problems in which there is no steady–state solution) the choice of the sign of λ^2 will not be so obvious. For these problems it will be necessary to resort to mathematical, rather than physical, reasoning. More about this will be learned as such problems are encountered. Once the sign of λ^2 has been established, the two characteristic solutions for u and v are obtained;

$$u = A\cos(\lambda x) + B\sin(\lambda x)$$
$$v = e^{-\lambda^2 t}$$

Note that it is not necessary to include a 'C' in front of the solution for v, because this will ultimately be absorbed into the constants A and B when the product $T = u \cdot v$ is formed.

4. Apply the homogeneous BCs: The homogeneous conditions (here at $x = 0$ and 1) are used to eliminate one of the constants A and/or B and establish a condition for λ. In the previous problem the adiabatic wall condition at $x = 0$ was used to set $B = 0$, and the boundary condition at $x = 1$ led to the condition

$$\cos(\lambda) = 0 \quad \longrightarrow \lambda = \lambda_n = \frac{1}{2}(2n-1)\pi,\ n = 1,\ 2, \ldots \tag{3.27}$$

This is known as an eigencondition, and the associated values of λ (referred to as λ_n) are known as eigenvalues. The corresponding functions $\cos(\lambda_n x)$ are known as eigenfunctions. These 'eigen' terms are hybrids of German and English, in which eigen has the meaning 'characteristic'. The general solution for T is then put in the form of an infinite series involving the eigenfunctions:

$$T = \sum_{n=1}^{\infty} A_n \cos(\lambda_n x) e^{-\lambda_n^2 t}$$

in which the A_n's are the expansion coefficients for the series.

5. Apply the inhomogeneous condition: The initial condition, which is the sole inhomogeneous aspect of the previous example, gave the equation

$$T(x, t = 0) = 1 = \sum_{n=0}^{\infty} A_n \cos(\lambda_n x)$$

This result was multiplied through by $\cos(\lambda_m x)$ and integrated over x from 0 to 1. By use of the fact that

$$\int_0^1 \cos(\lambda_n x)\cos(\lambda_m x)\,dx = \begin{cases} 0, & n \neq m \\ \frac{1}{2}, & n = m \end{cases}$$

it was found that

$$A_n = 2\int_0^1 \cos(\lambda_n x)\,dx = -\frac{2(-1)^n}{\lambda_n}$$

Having completely specified the expansion coefficients, a complete solution to the problem has been obtained.

3.3 Orthogonal functions and orthogonality

Perhaps one of the most difficult concepts to grasp in the separation of variables (SOV) method is the 'expansion' of the solution into a series, and the use of the integral relationships presented above to establish the values of the expansion coefficients in the series. It is no lucky coincidence that these integral relationships occurred for the particular problem we addressed; rather, they will be features of any problem that is solved with the SOV method.

The origin and properties of these integral relationships can be better understood by generalization of some of the details presented in the previous section. The x dependence of the dependent variable T was taken to reside in a function $u(x)$, which satisfied the ODE (obtained from the SOV procedure)

$$u'' + \lambda^2 u = 0 \tag{3.28}$$

The boundary conditions in the x direction are homogeneous. In the most general case these homogeneous boundary conditions can be cast as

$$C_1 T + C_2 \cdot \frac{\partial T}{\partial x} = 0, \quad x \text{ on boundaries} \tag{3.29}$$

where the constants C_1 and C_2 may have different values on different boundaries, but are not a function of time. In the previous example we had $C_1 = 0$ and $C_2 = 1$ at $x = 0$ and visa–versa at $x = 1$. Since $T = u \cdot v$, the above BC is equivalent to

$$C_1 uv + C_2 vu' = 0, \quad x \text{ on boundaries} \tag{3.30}$$

or, cancelling out the v,

$$C_1 u + C_2 u' = 0, \quad x \text{ on boundaries} \tag{3.31}$$

The fact that the time dependence part (in v) cancels from the BCs is due entirely to the homogeneity of the BCs. So, for the problem at hand, u has boundary conditions of

$$u'(x = 0) = 0, \qquad u(x = 1) = 0 \tag{3.32}$$

The function that will satisfy the DE and BCs for u is, again, referred to as an eigenfunction, which will be denoted by the general symbol $\phi_n(x)$. It is important to realize that the solution to Eqs. (3.28) and (3.32) is not unique. That is, their exist an infinite number of eigenfunctions ϕ_n, $n = 0, 1, 2, \ldots$, each of which exactly satisfies the DE and the BCs. As was mentioned in the previous section, this may seem like a problem when it comes to 'pinning down' a solution to the PDE. However, the eigenfunctions will exhibit an integration property which will be immensely useful in constructing series solutions to the PDE from the eigenfunctions. Specifically, it will be shown – without resorting to a table of integrals – that

$$\int_0^1 \phi_n(x)\phi_m(x)\,dx = \begin{cases} 0, & n \neq m \\ \text{not necessarily } 0, & n = m \end{cases} \tag{3.33}$$

The original DE (Eq. (3.28)) gives $\phi_n = -\phi_n''/\lambda_n^2$, or

$$\int_0^1 \phi_n(x)\phi_m(x)\,dx = -\frac{1}{\lambda_n^2}\int_0^1 \phi_n''\phi_m\,dx \tag{3.34}$$

Now factor out the λ_n^2 and integrate by parts:

$$\begin{aligned} \lambda_n^2 \int_0^1 \phi_n(x)\phi_m(x)\,dx & \\ &= -\phi_n'\phi_m\Big|_0^1 + \phi_n\phi_m'\Big|_0^1 - \int_0^1 \phi_n\phi_m''\,dx \\ &= -\phi_n'\phi_m\Big|_0^1 + \phi_n\phi_m'\Big|_0^1 + \lambda_m^2\int_0^1 \phi_n\phi_m\,dx \end{aligned}$$

The last part came from replacing ϕ_m'' with $-\lambda_m^2\phi_m$. After rearranging:

$$\int_0^1 \phi_n\phi_m\,dx = \frac{1}{\lambda_n^2 - \lambda_m^2}\left[\phi_n'\phi_m\Big|_0^1 + \phi_n\phi_m'\Big|_0^1\right]$$

The boundary conditions on the problem give $\phi_n' = \phi_m' = 0$ at $x = 0$ and $\phi_n = \phi_m = 0$ at $x = 1$. Consequently, the boundary terms in the above are identically zero. It can then be stated that the above integral is zero for $n \neq m$. However, if $n = m$ then $\lambda_n^2 - \lambda_m^2 = 0$, and the above result gives $0/0$ – which is indeterminate. This only means that integration–by–parts, as used above, cannot determine the integral of $\phi_n^2\,dx$. We can deduce, however, that this integral cannot be zero unless ϕ_n is itself zero for all x: ϕ_n^2 will be a positive quantity, and the definite integral of a quantity that is always greater than zero cannot be zero. General formulas for the integral of ϕ_n^2 are presented in the appendix.

Recognize also that the integral in the above equation would be zero, for $n \neq m$, whenever ϕ_n and ϕ_m satisfy any *homogeneous* boundary condition at the boundaries. That is, if the BCs were

in the general form of Eq. (3.31) (with C_1 and C_2 not depending on n), then the boundary terms in the above equation would have cancelled. You should prove this to yourself.

This integral property exhibited by the ϕ_n functions is known as orthogonality. Orthogonal functions will appear whenever analytical solutions to the conduction equation (and a host of other partial differential equations) are obtained by separation of variables techniques. An orthogonal function will always display *oscillatory* behavior about zero – as is the case with the trigonometric functions encountered here. Because of this the product $\phi_n(x)\,\phi_m(x)$ will also oscillate about zero (providing $n \neq m$) and will do so in such a way that the integral of the product will equal zero.

Expansion in orthogonal functions

The property of orthogonality allows one to expand practically any function into a series of orthogonal functions. For example, say the arbitrary function $f(x)$ is single–valued on the interval (0, 1). An expansion of f over this interval in terms of orthogonal functions ϕ_n would be

$$f(x) = \sum_{n=0}^{\infty} A_n \phi_n(x) \tag{3.35}$$

Explicit formulas for the expansion coefficients (A_n) are obtained by multiplying each side through by $\phi_m(x)$ and integrating from 0 to 1. From Eq. (3.33), each term in the series vanishes upon the integration except the one with $n = m$. The general formula for the A_ns becomes:

$$A_n = \int_0^1 f(x)\phi_n(x)\,dx \cdot \left[\int_0^1 \phi_n^2(x)\,dx\right]^{-1} \tag{3.36}$$

The process described in the previous two equations was used in the example of Sec. 3.2 to expand the initial temperature profile of the slab in terms of the orthogonal functions (or eigenfunctions) $\phi_n(x) = \cos(\lambda_n x)$. In this case the function f was simply unity. Realize, however, that any arbitrary (within reason) initial temperature distribution could have been specified within the wall. For example, the initial distribution could have been linear, quadratic, exponential, discontinuous, etc. If this profile can be represented by a function f, then the expansion coefficients would be given by Eq. (3.36).

An example of this strategy is now presented. Say f is a step function, represented by

$$f(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ 1, & \frac{1}{2} < x \leq 1 \end{cases}$$

and say that the eigenfunctions are defined by

$$\phi_n(x) = \cos(n\pi x), \;\; n = 0,\; 2, \ldots \tag{3.37}$$

Note that, as defined above, ϕ_n satisfies the homogeneous boundary conditions of $\phi_n'(0) = \phi_n'(1) = 0$. Because of this, ϕ_n will be orthogonal on the interval (0,1). The integrals appearing in Eq. (3.36) will be

$$\begin{aligned}
\int_0^1 f(x)\phi_n(x)\,dx &= \int_{1/2}^1 \cos(n\pi x)\,dx \\
&= \begin{cases} \frac{1}{2}, & n = 0 \\ -\frac{1}{n\pi}\sin(n\pi/2), & n > 0 \end{cases} \\
\int_0^1 \phi_n^2(x)\,dx &= \int_0^1 \cos^2(n\pi x)\,dx \\
&= \begin{cases} 1, & n = 0 \\ \frac{1}{2}, & n > 0 \end{cases}
\end{aligned}$$

The formula for the A_n is therefore:

$$A_n = \begin{cases} \frac{1}{2}, & n = 0 \\ -\frac{2}{n\pi}\sin(n\pi/2), & n > 0 \end{cases} \tag{3.38}$$

The A_n will be zero for even n – with the exception of $n = 0$. The index n can then be replaced with $2n - 1$, and

$$A_n = \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{2}{(2n-1)\pi}(-1)^n, & n = 1, 2, \ldots \end{cases} \tag{3.39}$$

The final formula for the function is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \cos[(2n-1)\pi x]}{2n-1} \tag{3.40}$$

Illustrated in Fig. 3.3 is a plot of $f(x)$ calculated using Eq. (3.40) for 1,2,5 and 100 terms retained in the series The series for $N = 100$ terms appears (to the eye) to be a good representation of the actual step function. For this case, the worst accuracy occurs at the points where $f(x)$ undergoes the most rapid change. Indeed, the exact function $f(x)$ has $f' \to \infty$ for $x = 1/2$. At this point the series gives

$$f'(1/2) = -2\sum_{n=1}^{\infty}(-1)^n \sin[(2n-1)\pi/2] = 2\sum_{n=1}^{\infty} 1 \to \infty \tag{3.41}$$

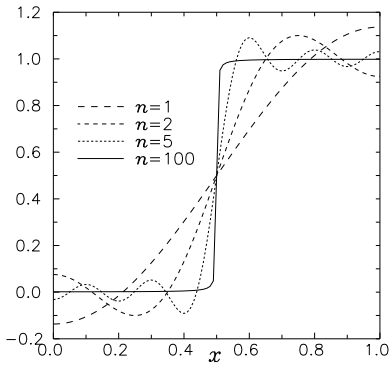


Figure 3.3: series representation of the step function

The series result is consistent with the exact result – in that the singularity in f' at $x = 1/2$ is represented by a non–convergent series. This example points out, however, one of the most powerful aspects of a series expansion in orthogonal function, in that it can express a discontinuous function (such as the $f(x)$ used here) in terms of a series of continuous functions.

The Sturm–Liouville system

The property of orthogonality can now be generalized. Consider an ODE of the form:

$$(p(x)u')' + \left[s(x) + \lambda^2 w(x)\right] u = 0 \tag{3.42}$$

where $p(x)$, $s(x)$, and $w(x)$ are prescribed functions of x. The problem of Sec. 3.2, Eq. (3.28), had $p(x) = w(x) = 1$ and $s(x) = 0$. The boundary conditions on u are also prescibed as homogeneous, i.e.,

$$\begin{aligned} a_1 u(x_1) + b_1 u'(x_1) &= 0 \\ a_2 u(x_2) + b_2 u'(x_2) &= 0 \end{aligned} \tag{3.43}$$

where a and b are constants and x_1 and x_2 are the bounds on the domain of the system. The above system (Eqs. (3.42) and (3.43)) is known as the Sturm–Liouville problem. The solution for u will be in the form of eigenfunctions $\phi_n(x)$, and these eigenfunctions will be orthogonal. The specific form of the orthognality relationship will be

$$\int_{x_1}^{x_2} \phi_n(x)\phi_m(x)w(x)\,dx = 0, \quad n \neq m \tag{3.44}$$

The function $w(x)$, which appeared first in the DE for u, is commonly known as a weighting function. In problems involving cylindrical or spherical coordinates the weighting functions will be other than unity.

3.4 More on transient problems

3.4.1 Convection BCs

A more realistic representation of a transient cooling and/or heating problem is offered by a convection boundary condition. Consider the same problem examined in Sec. 3.2, i.e., a plane wall, of length L, that has an adiabatic boundary at $x = 0$ and is initially at a temperature at $T = T_1$, yet now at $t = 0$ the surface at $x = L$ is instantaneously subjected to convective cooling, characterized by a heat transfer coefficient h and an ambient temperature T_∞. As before, the evolution of temperature within the wall is to be determined.

Nondimensional variables are defined according to $\overline{x} = x/L$, $\overline{T} = (T - T_\infty)/(T - T_1)$, and $\overline{t} = t\alpha/L^2$. The overbar notation, signifying nondimensionality, will now be dropped and it is understood that all quantities (unless otherwise specified) are nondimensional. The corresponding problem is

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{3.45}$$

$$\left.\frac{\partial T}{\partial x}\right|_0 = 0$$

$$\left.\frac{\partial T}{\partial x}\right|_1 = -Bi\, T(x = 1, t) \tag{3.46}$$

$$T(x, t = 0) = 1 \tag{3.47}$$

where $Bi = hL/k$ is the Biot number. The only modification to the problem is the convection BC at $x = 1$ – which is still a homogeneous BC. Both the BCs are homogeneous and the separation of variables can proceed as before.

Following the same procedure based on $T(x, t) = u(x) \cdot v(t)$ the characteristic DE for u is the same as before;

$$u'' + \lambda^2 u = 0$$

which gives us

$$u = A\cos(\lambda x) + B\sin(\lambda x)$$

The adiabatic condition at $x = 0$ gives $B = 0$. At $x = 1$ the BC for u is

$$u'(x = 1) = -Bi\, u(x = 1)$$

or, after substitution of the solution:

$$\lambda\sin(\lambda) = Bi\cos(\lambda) \longrightarrow \lambda\tan(\lambda) - Bi = 0 \tag{3.48}$$

This result represents the eigencondition to the convection BC problem under examination. Similar to the eigencondition that was obtained for the fixed–temperature BC (Eq. (3.18)), Eq. (3.48)

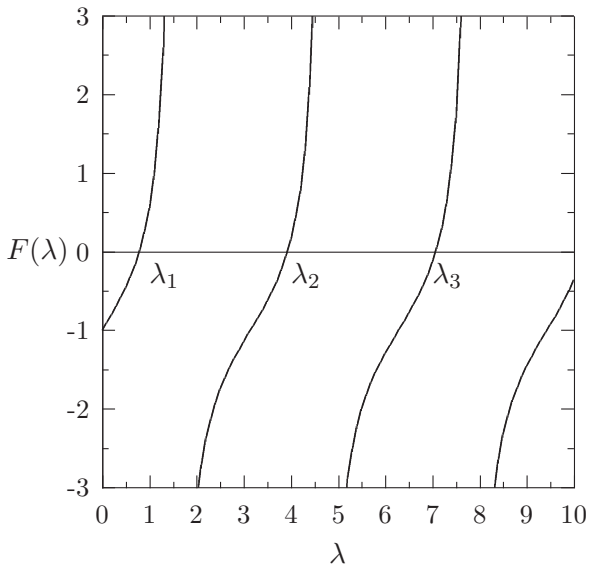


Figure 3.4: plot of $\lambda \tan(\lambda) - 1$

represents a transcendental equation that has an infinite number of roots; $\lambda = \lambda_1,\ \lambda_2, \ldots$. Unlike the previous example, the convection BC eigencondition does not provide a closed–form, *explicit* expression for the roots. Rather, the eigencondition is now an *implicit* relation for λ_n.

To further illustrate, a plot of the function $F(\lambda) = \lambda \tan(\lambda) - 1$ (i.e., Eq. (3.48) with $Bi = 1$) is given in Fig. 3.4. Singular points occur in the function $\tan(\lambda)$ for $\lambda = \pi/2,\ 3\pi/2,\ 5\pi/2 \ldots$. The roots occur where the continuous part of the curve crosses the $F = 0$ line. Determination of the eigenvalues (or roots) from Eq. (3.48) requires use of either a chart (such as illustrated above for $Bi = 1$), a table, or a numerical method for solving the nonlinear equation. The latter is easily coded into *Mathematica*, and this is discussed in Sec. 3.5.

Assume now that the set of eigenvalues λ_n, $n = 1, 2, \ldots$, corresponding to the roots of Eq. (3.48) for a given value of Bi, are known. From this point on the problem follows the same procedure as that used before. The solution will be in the form of the series

$$T = \sum_{n=1}^{\infty} A_n \cos(\lambda_n x) e^{-\lambda_n^2 t} \tag{3.49}$$

At $t = 0$ the initial condition gives

$$1 = \sum_{n=1}^{\infty} A_n \cos(\lambda_n x) \tag{3.50}$$

Multiply through by $\cos(\lambda_m x)$ and integrate from 0 to 1. Use the orthogonality of the eigenfunctions to obtain

$$A_n = \int_0^1 \cos(\lambda_n x)\,dx \cdot \left[\int_0^1 \cos^2(\lambda_n x)\,dx\right]^{-1} \tag{3.51}$$

These integrals will have different values than before because λ_n is different. The first integral is

$$\int_0^1 \cos(\lambda_n x)\,dx = \frac{\sin(\lambda_n)}{\lambda_n} \tag{3.52}$$

This can be simplified somewhat. By taking the square of the eigencondition, the following is obtained:

$$\lambda_n^2 \sin^2(\lambda_n) = Bi^2 \cos^2(\lambda_n) = Bi^2\left[1 - \sin^2(\lambda_n)\right] \tag{3.53}$$

or

$$\sin(\lambda_n) = \pm\frac{Bi}{(\lambda_n^2 + Bi^2)^{1/2}} \tag{3.54}$$

If λ_n is between $2n\pi$ and $(2n+1)\pi$ then $\sin(\lambda_n)$ will be positive; otherwise it will be negative. Looking at the previous plot, it appears that the λ_n's with odd n will be in the positive $\sin(\lambda_n)$ region, and the even λ_n's will give the negative. Actually, you can prove to yourself that this will hold for arbitrary Bi. The first integral becomes

$$\int_0^1 \cos(\lambda_n x)\,dx = -\frac{(-1)^n Bi}{\lambda_n(\lambda_n^2 + Bi^2)^{1/2}} \tag{3.55}$$

Similarly, the second integral is

$$\begin{aligned}\int_0^1 \cos^2(\lambda_n x)\,dx &= \frac{1}{2}\left(1 + \frac{\cos(\lambda_n)\sin(\lambda_n)}{\lambda_n}\right)\\ &= \frac{1}{2}\left(1 + \frac{Bi}{\lambda_n^2 + Bi^2}\right)\end{aligned}$$

Again, the eigencondition was used to eliminate the trigonometric functions. Combining the previous two equations, the formula for the A_n is

$$A_n = -\frac{2(-1)^n Bi\left(\lambda_n^2 + Bi^2\right)^{1/2}}{\lambda_n\left[\lambda_n^2 + Bi(1+Bi)\right]} \tag{3.56}$$

Of course, we could have left the result in terms of the trigonometric functions.

The key feature of convection boundary conditions in separation of variables solutions is that the eigencondition becomes an implicit, rather than explicit, equation for λ_n. This does limit somewhat the accessibility of the solution, in that a numerical root finding method (or tables or charts) is required to obtain the eigenvalues prior to evaluation of the series. Since, however, a code is usually used to evaluate the series, it should be, in principle, little added difficulty to include routine that evaluates the λ_n's from the eigencondition.

Limiting forms of the eigencondition

The convection boundary condition can reduce, for appropriate limiting values of Bi, to isothermal or adiabatic BCs. When $Bi \to \infty$ (large h limit) the (dimensional) surface temperature would go to T_∞ – corresponding to a zero dimensionless temperature. The eigencondition would reduce in this case to

$$\cos(\lambda_n) = \frac{\lambda_n \sin(\lambda_n)}{Bi} \to 0 \quad \text{or} \quad \lambda_n = \frac{1}{2}(2n-1)\pi$$

which is the result of the original problem. Likewise, for $Bi \to 0$ the condition becomes

$$\lambda_n \sin(\lambda_n) \to 0, \quad \text{or} \quad \lambda_n = n\pi, \ n = 0, \ 1, \ldots \tag{3.57}$$

You could prove to yourself analytically (by going though the SOV process) that this particular eigenvalue will give the trivial solution of $\overline{T} = 1$ for all $\bar{t}$ and $\bar{x}$. This, of course, follows from physical insight: if insulated boundaries exist at both the $x = 0$ and L surfaces, then no heat can be removed from the system and the wall will remain in equilibrium at $T = T_1$.

A more realistic situation is the small Bi regime (say $Bi \ll 1$). For this situation an approximation can be obtained to the first eigenvalue. The eigencondition would correspond to

$$\lambda_n \sin(\lambda_n) = Bi \cos(\lambda_n) \ll 1 \tag{3.58}$$

which implies that λ_n has to be near the values of $(n-1)\pi$, with $n = 1, 2, \ldots$. The first eigenvalue λ_1 will therefore be a small number. By expanding $\sin(\lambda_1)$ and $\cos(\lambda_1)$ in powers of λ_1 about $\lambda_1 = 0$, the eigencondition for the first root will become

$$\lambda_1 \left(\lambda_1 - \frac{\lambda_1^3}{3!} + \ldots \right) = Bi \left(1 - \frac{\lambda_1^2}{2!} + \ldots \right) \tag{3.59}$$

Retaining only powers of λ_1^2, one obtains

$$\lambda_1 \approx \left(\frac{2Bi}{2 + Bi} \right)^{1/2} \approx (Bi)^{1/2} \tag{3.60}$$

Eigenvalues for arbitrary n can be obtained by expanding the triginometric functions for the argument λ_n about $(n-1)\pi$ – which would give a quadratic equation for the roots. It will turn out, though, that these higher–order roots are not needed when $Bi \ll 1$.

Given in Fig. 3.5 is a plot of the dimensionless temperature in the wall for three values of Bi (= 0.01, 1, and 100) and three values of a re–scaled nondimensional time $\bar{t} \cdot Bi = th/\rho cL$. Recall that this nondimensional time appears in 'lumped–capacity' (or small–Biot) heating cooling problems. For $Bi \ll 1$ the temperature within the wall will remain nearly uniform and decay at the rate

$$\overline{T} \approx \exp\left(-\frac{th}{\rho cL} \right) \tag{3.61}$$

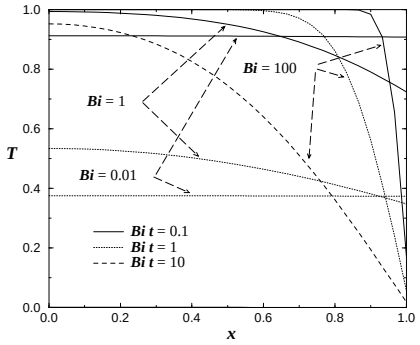


Figure 3.5: T distribution for $Bi = 0.01$, 1 and 100

Inspection of Eq. (3.60) reveals that the $Bi \ll 1$ regime gives $\lambda_1^2 \bar{t} \approx Bi\bar{t}$. Likewise, $\cos(\lambda_1 \bar{x}) \approx 1$ (since $\lambda_1 \ll 1$) and $A_1 \approx 1$. The first term in the exact series solution, Eq. (3.49), therefore reduces for $Bi \ll 1$ to the lumped capacity result. Indeed, the exact result presented in Fig. 3.5 for $Bi = 0.01$ shows that a lumped capacity analysis would be entirely appropriate – in that the temperature distribution remains essentially uniform throughout the cooling process. For $Bi = 1$ the distribution of temperature becomes more nonuniform within the wall, and for $Bi = 100$ the solution approachs the conditions of the first problem examined – i.e., an instantaneous change in the boundary temperature to T_∞.

3.4.2 Heat transfer

Continue with the same symmetrical plane wall configuration. At any instant into the cooling (or heating) process the heat flux from the wall would be given by (in dimensional form)

$$q'' = -k \left. \frac{\partial T}{\partial x} \right|_L$$

or, in dimensionless variables and using Eq. (3.49), the flux is

$$\overline{q}'' \equiv \frac{q'' L}{k(T_1 - T_\infty)} = - \left. \frac{\partial \overline{T}}{\partial \overline{x}} \right|_1 = \sum_{n=1} \lambda_n A_n \sin(\lambda_n) e^{-\lambda_n^2 \bar{t}}$$

Another quantity of interest is the net amount of heat Q that has been removed (or added) to the wall during the time interval $0 - t$. This quantity would be in units of J (or Q'', J/m^2), and could be obtained by integrating q'' over time from 0 to t. Alternatively, this quantity can be obtained

from the First Law of thermodynamics, which states

$$Q = \rho c V (T_{m,1} - T_{m,2}) \tag{3.62}$$

where $T_{m,1}$ and $T_{m,2}$ are the mean temperatures of the wall at the initial and final points of the process. A positive Q is defined as heat removed from the system. By using the definition of the mean temperature, the net heat transfer at time t would be

$$Q(t) = \rho c \int_V [T_1 - T(x,t)]\, dV = \rho c A \int_0^L [T_1 - T(x,t)]\, dx \tag{3.63}$$

where T_1 is the initial temperature of the wall. For the problem at hand, the wall will eventually reach a steady state temperature of T_∞. Consequently $Q_{tot} = \rho c A L (T_1 - T_\infty)$ is the total net heat transfer that would be transferred to/from the wall during the entire process. With this in mind, the above equation can be recast in a dimensionless form;

$$\begin{aligned} \frac{Q(t)}{Q_{tot}} = \overline{Q}(\bar{t}) &= 1 - \int_0^1 \overline{T}\, d\overline{x} \\ &= 1 - \sum_{n=1}^{\infty} \frac{A_n \sin \lambda_n}{\lambda_n} e^{-\lambda_n^2 \bar{t}} \end{aligned} \tag{3.64}$$

This result represents sort of a 'generalized' lumped capacity relation – in that the dimensionless net heat transfer will be proportional to the dimensionless mean temperature, i.e.,

$$\overline{Q} = 1 - \frac{T_m - T_\infty}{T_1 - T_\infty} \tag{3.65}$$

Equation (3.64) (and those like it) can therefore be used to determine characteristic cooling times of the wall.

3.4.3 Non–homogeneous BCs/DEs: Partial solutions

Inhomogeneous boundary conditions

The boundary conditions in the plane wall problem are now modified. As before, the wall is initially at $T = T_1$, yet the surface at $x = 0$ is maintained at $T = T_1$ and the surface at $x = L$ is instantaneously brought to $T = T_2$ at $t = 0$. By defining the non–dimensional temperature as $\overline{T} = (T - T_1)/(T_2 - T_1)$, the problem becomes

$$\frac{\partial \overline{T}}{\partial \bar{t}} = \frac{\partial^2 \overline{T}}{\partial \overline{x}^2} \tag{3.66}$$

$$\begin{aligned} \overline{T}(\overline{x} = 0, \bar{t}) &= 0 \\ \overline{T}(\overline{x} = 1, \bar{t}) &= 1 \\ \overline{T}(\overline{x}, \bar{t} = 0) &= 0 \end{aligned} \tag{3.67}$$

The overbar notation will now be dropped from the variables. For this problem the BC at $x = 1$ is now inhomogeneous, and the IC is homogeneous. Because of this, SOV cannot be directly applied to the problem as stated. This is because the time 'direction' of the problem (which has the homogeneous condition) cannot be put in the Sturm–Liouville form, i.e. time–dependent eigenfunctions would not result from the separated problem.

In a subsequent chapter a more robust method of solving PDEs (know as variation of parameters) will be introduced, which allows one to get around this problem. For now, though, a solution can be obtained by (again) appealing to physics. It is not hard to see that this particular problem will attain a steady–state condition for $t \to \infty$. Denoting the steady–state solution as $s(x)$, this behavior would be

$$T(x, t \to \infty) = s(x) = x \tag{3.68}$$

That is, the steady state temperature profile in the wall is simply the linear profile x. Continuing with this concept, the function defined by $w(x,t) = T(x,t) - s(x)$ would go to zero for $t \to \infty$. Consequently, the solution for the temperature can be split into the two parts:

$$T(x,t) = w(x,t) + s(x) \tag{3.69}$$

in which $w(x,t)$ represents the *transient* portion (that goes to zero for large t) and $s(x)$ is the steady–state portion. The next step is to determine the PDE, BCs, and IC for w. By replacing the above equation into the PDE for T, one obtains

$$\frac{\partial w}{\partial t} + \underbrace{\frac{\partial s}{\partial t}}_{=0} = \frac{\partial^2 w}{\partial x^2} + \underbrace{\frac{d^2 s}{dx^2}}_{=0} \tag{3.70}$$

The second derivative of s is zero in the above because s satisfies the steady–state conduction equation. The homogeneous 1–D and transient conduction PDE is therefore obtained for w. The boundary conditions for w are found in a similar manner, in that $T = w + s$ is substituted into the BCs and IC;

$$T(x=0,t) = w(x=0,t) + \underbrace{s(x=0)}_{=0} = 0 \quad \longrightarrow w(x=0,t) = 0 \tag{3.71}$$

$$T(x=1,t) = w(x=1,t) + \underbrace{s(x=1)}_{=1} = 1 \quad \longrightarrow w(x=0,t) = 0 \tag{3.72}$$

$$T(x,t=0) = w(x,t=0) + s(x) = 0 \quad \longrightarrow w(x=0,t) = -s(x) = -x \tag{3.73}$$

The problem for w is now has homogeneous BCs and an inhomogeneous IC – which is the form that can be tackled by SOV. Note that the IC for w is simply the negative of the steady–state temperature profile – which makes physical sense.

Following the separation of variables procedure, we set $w = u \cdot v$ as before, and find that $v = \exp(-\lambda^2 t)$, as before. The solution for u will be

$$u = A\cos(\lambda x) + B\sin(\lambda x)$$

The result $A = 0$ will satisfy the BC at $x = 0$, and the BC at $x = 1$ gives the eigencondition:

$$\sin(\lambda_n) = 0 \longrightarrow \lambda_n = n\pi,\ n = 1,\ 2, \ldots$$

The specific case of $n = 0$ can be eliminated here, because it does not give the correct time–asymptotic behavior for w. The general solution for w is then

$$w = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x) e^{-\lambda_n^2 t}$$

Application of the initial condition $w = -x$ gives

$$-x = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x)$$

By use of the orthogonality properties of the eigenfunctions, the expansion coefficients are found as

$$\begin{aligned} A_n &= -\int_0^1 x\sin(n\pi x)\,dx \cdot \left[\int_0^1 \sin^2(n\pi x)\,dx\right]^{-1} \\ &= \frac{2\cos(n\pi)}{n\pi} = \frac{2(-1)^n}{n\pi} \end{aligned} \tag{3.74}$$

And the complete solution is $T = w + s$, or

$$T = x + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n}\, e^{-(n\pi)^2 t} \tag{3.75}$$

The method of splitting the solution for T into two parts, i.e., $T = w + s$, is sometimes known as 'partial solutions'. The goal in using this method is to transform a problem that cannot, directly, be solved with SOV into a problem (or problems) that can. The partial solutions technique is one example of a general method known as superposition, in which two or more solutions to a modified problem are *superimposed* (or, equivalently, added) to form a solution to the whole problem (DE, BCs, and IC) under consideration. The feature of the DE and BCs that allows for this method is *linearity* – for which a sum of independent solutions to the DE will also be a solution.

Inhomogeneous DE

Consider now an example in which heat generation occurs in the wall. Say the surface at $x = 0$ is adiabatic and the surface at $x = 1$ is maintained at $T = T_1$. Initially the wall is at a uniform temperature of T_1. At time $t = 0$ uniform heat generation occurs in the wall, of strength q'''_0.

The dimensionless temperature in this case will be $T \to (T - T_1)k/q'''_0 L^2$, and the dimensionless problem is

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + 1 \tag{3.76}$$

$$\left.\frac{\partial T}{\partial x}\right|_0 = 0 \tag{3.77}$$

$$T(x = 1, t) = 0 \tag{3.78}$$

$$T(x, t = 0) = 0 \tag{3.79}$$

Both the BCs and the IC are homogeneous, yet the DE is inhomogeneous. Again, SOV cannot be directly applied to this problem, because the resulting ODE for $u(x)$ will not be in the Sturm-Liouville form. However, the method of partial solutions can be applied because this particular situation will have a steady–state. As before, let

$$T(x, t) = w(x, t) + s(x) \tag{3.80}$$

in which $s(x)$ is the steady state temperature distribution in the wall. The problem presented by s is

$$s'' + 1 = 0, \quad s'(0) = 0, \quad s(1) = 0 \tag{3.81}$$

which has the solution

$$s = \frac{1}{2}\left(1 - x^2\right) \tag{3.82}$$

Using $T = w + s$ in Eqs. (3.76–3.79) leads to

$$\frac{\partial w}{\partial t} + \underbrace{\frac{\partial s}{\partial t}}_{=0} = \frac{\partial^2 w}{\partial x^2} + \underbrace{\frac{d^2 s}{dx^2}}_{=-1} + 1 \tag{3.83}$$

$$\left.\frac{\partial w}{\partial x}\right|_0 + \underbrace{\left.\frac{ds}{dx}\right|_0}_{=0} = 0 \tag{3.84}$$

$$w(x = 1, t) + \underbrace{s(x = 1)}_{=0} = 0 \tag{3.85}$$

$$w(x, t = 0) + \underbrace{s(x)}_{=\frac{1}{2}(1-x^2)} = 0 \tag{3.86}$$

So again, a homogeneous DE and homogeneous BCs are obtained for w, and the initial condition for w is equal to the negative of the steady state solution.

The general solution for w is

$$w = \sum_{n=1}^{\infty} A_n \cos(\lambda_n x) e^{-\lambda_n^2 t} \tag{3.87}$$

in which the eigenvalues are

$$\lambda_n = \frac{(2n-1)\pi}{2}$$

At $t = 0$ the condition is

$$-s(x) = -\frac{1}{2}(1-x^2) = \sum_{n=1}^{\infty} A_n \cos(\lambda_n x)$$

Using the orthogonality of the eigenfunctions gives

$$\begin{aligned} A_n &= -\int_0^1 s(x)\cos(\lambda_n x)\,dx \cdot \left[\int_0^1 \cos^2(\lambda_n x)\,dx\right]^{-1} \\ &= -2\int_0^1 s(x)\cos(\lambda_n x)\,dx \end{aligned}$$

At this point the formula for s could be inserted and the formula integrated (which would be easy with *Mathematica*), yet it is instructive to demonstrate how this integral can be easily evaluated by making use of the DE and BCs for s. Let ϕ_n denote the eigenfunction $\cos(\lambda_n x)$. The previous integral becomes

$$\begin{aligned} \int_0^1 s(x)\phi_n(x)\,dx &= -\frac{1}{\lambda_n^2}\int_0^1 s\phi_n''\,dx \\ &= -\frac{1}{\lambda_n^2}\left[s\phi_n'\Big|_0^1 - s'\phi_n\Big|_0^1 + \int_0^1 s''\phi_n\,dx\right] \\ &= \frac{1}{\lambda_n^2}\int_0^1 \phi_n\,dx \\ &= \frac{1}{\lambda_n^3}\sin\left(\frac{(2n-1)\pi}{2}\right) \\ &= -\frac{8(-1)^n}{(2n-1)^3\pi^3} \end{aligned}$$

Integration by parts was used in evaluating the integrals. The boundary terms are zero by virtue of the BC's on ϕ_n and s. Also, s'' was eliminated using the DE for s, i.e., $s'' = -1$. The formula for the A_n becomes

$$A_n = \frac{16(-1)^n}{(2n-1)^3\pi^3} \tag{3.88}$$

which, together with Eqs. (3.87) and (3.82), gives the complete solution for the temperature distribution in the wall.

3.4.4 Problems with no steady state

The partial solutions method, as shown in the previous examples, relies on the splitting of the sought T solution into a steady state part and a transient part. The transient part will conform to the SOV requirements. However, not all transient problems have a steady state. Consider, for example, a situation in which a wall is initially at T_1, the boundary at $x = 0$ is insulated, and at $t = 0$ a uniform heat flux of q_0'' is applied at $x = L$. If the dimensionless variables are defined as

$$T \to \frac{(T - T_0)k}{q_0'' L}, \quad x \to \frac{x}{L}, \quad t \to \frac{\alpha t}{L^2}$$

the problem statement becomes

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{3.89}$$

$$\left.\frac{\partial T}{\partial x}\right|_0 = 0 \tag{3.90}$$

$$\left.\frac{\partial T}{\partial x}\right|_1 = 1 \tag{3.91}$$

$$T(x, t = 0) = 0 \tag{3.92}$$

Because the wall receives a flux at $x = 1$, yet is insulated at $x = 0$, it will never attain an equilibrium state. Rather, the temperature throughout the wall will continuously increase with time.

The analytical procedure follows the general approach already developed – in that the solution is represented by a superposition of partial solutions. For this case, the superposition is

$$T(x, t) = w(x, t) + s(x) + T_m(t) \tag{3.93}$$

The function $w(x, t)$ has the same meaning as before; it is a transient part which decays to 0 for large t. The quantity $T_m(t)$ is the mean (or average) temperature in the wall – which will be a function solely of time, and $s(x)$ can be interpreted as a *stationary* solution: it is the temperature *profile* in the wall which occurs after the transient portion has decayed. Alternatively, $s(x)$ represents the solution to $T(x, t) - T_m(t)$ for $t \to \infty$. This approach therefore makes the assumption that, for adequately long times past the initial transient, the time and position dependencies on temperature are additive. Such an approach will be valid because the boundary conditions are not functions of time.

The solution for the mean temperature $T_m(t)$ is obtained by integration of the DE, Eq. (3.89), over x from 0 to 1;

$$\int_0^1 \frac{\partial T}{\partial t}\,dx = \int_0^1 \frac{\partial^2 T}{\partial x^2}\,dx$$
$$\frac{d}{dt}\underbrace{\int_0^1 T\,dx}_{=T_m} = \underbrace{\left.\frac{\partial T}{\partial x}\right|_1}_{=1} - \underbrace{\left.\frac{\partial T}{\partial x}\right|_0}_{=0} \quad \rightarrow \quad \frac{dT_m}{dt} = 1$$

Since the mean temperature is zero at $t = 0$, solution of the above DE gives the result $T_m(t) = t$. This could have been anticipated: the rate change in mean temperature will be proportional to the rate of heat addition to the wall. Since the latter is a constant and equal to unity, the mean temperature will be equal to t.

Now replace Eq. (3.93) into the problem statement for T;

$$\frac{\partial w}{\partial t} + \underbrace{\frac{dT_m}{dt}}_{=1} = \frac{\partial^2 w}{\partial x^2} + s''(x)$$
$$\left.\frac{\partial w}{\partial x}\right|_0 + s'(0) = 0$$
$$\left.\frac{\partial w}{\partial x}\right|_1 + s'(1) = 1$$
$$w(x,0) + s(x) + \underbrace{T_m(0)}_{=0} = 0$$

The associated problem for $s(x)$ represents the steady state solution to $T - T_m$, and is

$$s'' = 1, \quad s'(0) = 0, \quad s'(1) = 1$$

which represents, equivalently, a uniform heat sink of unit strength in a wall with a uniform unit flux at one side and adiabatic conditions at the other. The net heat addition to the wall is zero (the sink balances the input flux), and a steady condition is physically realizable. The solution for s is

$$s(x) = \frac{x^2}{2} + C \tag{3.94}$$

where C is an undetermined constant: this arises because both boundary conditions for s are in terms of derivatives of s – the solution to s can therefore be shifted by an arbitrary constant. The constant, however, can be pinned down by invoking the definition of the average temperature; the integral of T over x is defined as T_m and this implies (using $T = T_m + s$ for $t \rightarrow \infty$) that the integral of s over x must be zero. Consequently, $C = -1/6$.

The problem for w is

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2}\\ \left.\frac{\partial w}{\partial x}\right|_0 &= 0\\ \left.\frac{\partial w}{\partial x}\right|_1 &= 0\\ w(x,0) &= -s(x)\end{aligned}$$

which has the general solution

$$w = \sum_{n=1} A_n\,\phi_n(x)\,e^{-\lambda_n^2 t}$$

The eigenfunctions and eigenconditions to the solution are

$$\phi_n(x) = \cos(\lambda_n x), \quad \lambda_n = n\pi$$

Again, the special case of $\lambda = 0$ can be neglected in this problem[2].

The expansion coefficients are determined following the procedures already established;

$$\begin{aligned}A_n &= -2\int_0^1 s(x)\,\phi_n(x)\,dx\\ &= \frac{2}{\lambda_n^2}\int_0^1 s(x)\,\phi_n''(x)\,dx\\ &= \frac{2}{\lambda_n^2}\left[s(x)\phi_n'(x)\Big|_0^1 - s'(x)\phi_n(x)\Big|_0^1 - \int_0^1 s''(x)\phi_n(x)\,dx\right]\\ &= \frac{2}{\lambda_n^2}\left[0 - 1\cdot\phi_n(1) - 0\right]\\ &= -\frac{2(-1)^n}{\lambda_n^2}\end{aligned} \tag{3.95}$$

and the complete solution for the temperature distribution is

$$T = -\frac{1}{6} + t + \frac{x^2}{2} - 2\sum_{n=0}\frac{(-1)^n}{\lambda_n^2}\,\phi_n(x)\,e^{-\lambda_n^2 t} \tag{3.96}$$

The solution is plotted in Fig. 3.6 for times of 0.01, .2, and 1. The evolution to the stationary solution is clearly evident.

[2]Retention of the $\lambda_0 = 0$ term would actually give the zeroth series coefficient $= t$ – which has already been counted for in T_m

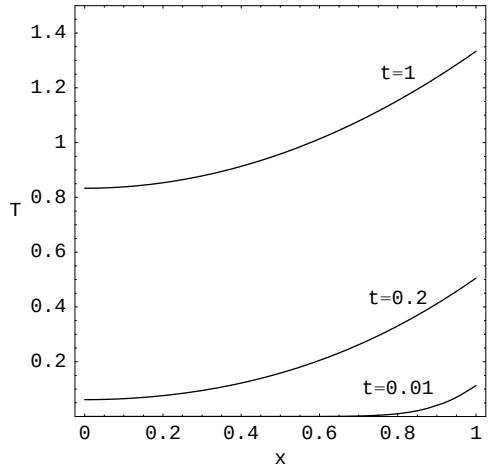


Figure 3.6: temperature profile: constant heat flux BC

3.4.5 Transient problems in radial systems

The solid cylinder

The basic analytical methods introduced above for the 1–D (in space) slab apply directly to problems in cylindrical and spherical coordinates; only the basis functions for the eigenfunctions will change. To illustrate, consider the following problem: A long solid circular cylinder is initially at temperature T_1. At $t = 0$ the surface temperature is instantaneously brought to T_2. Find the temperature distribution within the cylinder as a function of time and radial position.

It is assumed that the cylinder is sufficiently long, relative to the diameter, so that there is no temperature variation in the z (axial) direction. In addition, there is no ϕ dependence in the problem. The problem is therefore 2–D in r and t. The nondimensional variables are defined in the usual manner:

$$T \to \frac{T - T_2}{T_1 - T_2}, \quad r \to \frac{r}{R}, \quad t \to \frac{t\alpha}{R^2} \tag{3.97}$$

where R is the cylinder radius. The dimensionless problem appears as

$$\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \tag{3.98}$$

$$T(r = 0, t) \text{ is finite} \tag{3.99}$$

$$T(r = 1, t) = 0 \tag{3.100}$$

$$T(r, t = 0) = 1 \tag{3.101}$$

The BCs and the DE are homogeneous, and SOV can proceed in the usual manner. Let $T(r,t) = u(r) \cdot v(t)$, and replace this into the DE and separate:

$$\frac{v'}{v} = \frac{1}{ru}(ru')' = \text{constant} = -\lambda^2 \tag{3.102}$$

Choose $-\lambda^2$ as the separation constant because this will give the correct time–decaying solution. The solution for v is the same as before:

$$v = e^{-\lambda^2 t} \tag{3.103}$$

For the u variable, the characteristic DE is

$$(ru')' + r\lambda^2 u = 0 \tag{3.104}$$

or

$$r^2 u'' + ru' + r^2\lambda^2 u = 0 \tag{3.105}$$

From the previous chapter, this is recognized as Bessel's equation and has the solution

$$u = AJ_0(\lambda r) + BY_0(\lambda r) \tag{3.106}$$

The function Y_0 is singular at the origin, so set $B = 0$ to keep the centerline temperature finite. Application of the BC at $r = 1$ gives

$$J_0(\lambda) = 0$$

This condition provides the eigencondition to the problem. A plot of $J_0(x)$ in the previous chapter shows that the function oscillates about 0 much in the same way as the triginometric functions. The eigenvalues λ_n correspond to the first, second, etc. roots to the above equation, i.e.,

$$J_0(\lambda_n) = 0, \quad n = 1,\ 2, 3, \ldots \tag{3.107}$$

An explicit formula for the eigenvalues cannot be obtained for this particular eigenfunction. Rather, one must resort to an appropriate rootfinding method to obtain the λ_n's – much in the same way as for convection–type eigenconditions. A table of the first few roots of the above equation is included in the Appendix.

Continuing with the SOV procedure, the general solution to the problem is given as a series expansion of the eigenfunctions (which are $J_0(\lambda_n r)$) times the time–dependent part of the solution:

$$T(r,t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) e^{-\lambda_n^2 t} \tag{3.108}$$

At $t = 0$ the condition is

$$1 = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \tag{3.109}$$

One could anticipate at this point that $J_0(\lambda_n)$ is orthogonal and proceed to obtain the integral formula for the A_ns – yet the precise nature of the orthogonality relation (in particular, the weighting function) are unknown. Equation (3.104) is in the Sturm–Liouville form (Eq. (3.42)) with $w(r) = r$ as the weighting function. Each side of the equation is therefore multiplied by $rJ_0(\lambda_m r)$ and integrated from 0 to 1. To make the notation compact, denote $J_0(\lambda_n r)$ as $\phi_n(r)$, i.e., the eigenfunction. Again, this function satisfies the DE

$$(r\phi_n')' + r\lambda_n^2\phi_n = 0 \tag{3.110}$$

and the BCs

$$\phi_n'(0) = 0, \quad \phi_n(1) = 0 \tag{3.111}$$

The orthogonality relation is then

$$\begin{aligned}\int_0^1 \phi_n\phi_m r\,dr &= -\frac{1}{\lambda_n^2}\int_0^1 (r\phi_n')'\phi_m\,dr \\ &= -\frac{1}{\lambda_n^2}\left[r\phi_n'\phi_m\Big|_0^1 - r\phi_n\phi_m'\Big|_0^1 + \int_0^1 \phi_n(r\phi_m')'\,dr\right] \\ &= \frac{\lambda_m^2}{\lambda_n^2}\int_0^1 \phi_n\phi_m r\,dr\end{aligned} \tag{3.112}$$

Recognize again how the BC's on ϕ_n were used to eliminate the boundary terms in the above. The integral is

$$\left(1 - \frac{\lambda_m^2}{\lambda_n^2}\right)\int_0^1 J_0(\lambda_n r)J_0(\lambda_m r)r\,dr = 0 \tag{3.113}$$

which provides the desired orthogonality proof.

With this information in hand, the expansion coefficients are obtained from Eq. (3.109) as

$$A_n = \int_0^1 J_0(\lambda_n r)r\,dr \cdot \left[\int_0^1 J_0^2(\lambda_n r)r\,dr\right]^{-1}$$

Using the integral formulas from the previous notes (and the appendix) results in

$$\begin{aligned}\int_0^1 J_0(\lambda_n r)r\,dr &= \frac{1}{\lambda_n^2}\int_0^{\lambda_n} J_0(x)x\,dx \\ &= \frac{1}{\lambda_n^2}[xJ_1(x)]\Big|_0^{\lambda_n} = \frac{J_1(\lambda_n)}{\lambda_n}\end{aligned} \tag{3.114}$$

$$\begin{aligned}\int_0^1 J_0^2(\lambda_n r)r\,dr &= \frac{1}{\lambda_n^2}\int_0^{\lambda_n} J_0^2(x)x\,dx \\ &= \frac{1}{\lambda_n^2}\left[\frac{x^2}{2}\left(J_0^2(x) + J_1^2(x)\right)\right]_0^{\lambda_n} \\ &= \frac{1}{2}J_1^2(\lambda_n)\end{aligned} \tag{3.115}$$

In the second integral the eigencondition was used to eliminate $J_0(\lambda_n)$. Also, $J_1(0) = 0$ by the properties of Bessel functions. The final formula for A_n is

$$A_n = \frac{2}{\lambda_n J_1(\lambda_n)} \tag{3.116}$$

and the complete solution for the temperature is

$$T(r,t) = 2\sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n)} e^{-\lambda_n^2 t} \tag{3.117}$$

Annular cylindrical regions

A more complicated problem is now examined. Consider a cylindrical pipe, as illustrated in Fig. 3.7. Initially the pipe is at a uniform temperature of T_1. At time $t = 0$ a uniform heat flux of q_0'' is applied to the inner surface. The heat is removed from the outer surface by convection, which is characterized by a heat transfer coefficient h and an ambient temperature T_∞.

On a dimensional basis, the problem is

$$\begin{aligned} \frac{\partial T}{\partial t} &= \frac{k}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \\ -k\left.\frac{\partial T}{\partial r}\right|_{R_i} &= q_0'' \\ -k\left.\frac{\partial T}{\partial r}\right|_{R_o} &= h(T(r=R_o,t) - T_\infty) \\ T(r, t=0) &= T_1 \end{aligned}$$

Following the usual procedure, the problem is recast in nondimensional form. The characteristic temperature is obviously T_∞, yet we two choices exist for the characteristic temperature difference, namely $\Delta T_C = q_0'' L/k$ and $T_1 - T_\infty$. The choice is arbitrary – for lack of a better reason choose the first. The variable definitions become

$$T \to \frac{(T - T_\infty)k}{q_0'' R_o}, \quad r \to \frac{r}{R_o}, \quad t \to \frac{t\alpha}{R_o^2} \tag{3.118}$$

Three dimensionless parameters fall out of the problem, which are

$$a = \frac{R_i}{R_o}, \quad Bi = \frac{hR_o}{k}, \quad T_1 \to \frac{(T_1 - T_\infty)k}{q_o'' R_o} \tag{3.119}$$

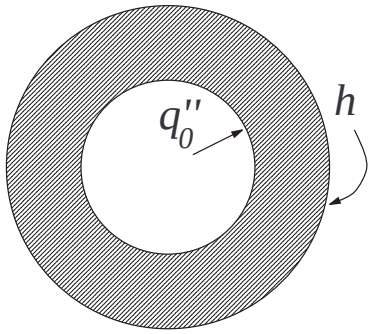


Figure 3.7: annular pipe

The dimensionless problem is now

$$\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial T}{\partial r} \tag{3.120}$$

$$\left.\frac{\partial T}{\partial r}\right|_a = -1 \tag{3.121}$$

$$\left.\frac{\partial T}{\partial r}\right|_1 = -BiT(r=1,t) \tag{3.122}$$

$$T(r, t=0) = T_1 \tag{3.123}$$

The problem has a homogeneous DE, yet the inner BC and the IC are inhomogeneous. Consequently, it cannot be attacked as–is with SOV. Rather, partial solutions are needed to split the problem into sub–problems which, individually, admit solutions amenable to our analtical techniques.

This particular system will eventually attain a steady state, and the solution is therefore formulated in terms of a steady–state part and a transient part, i.e.,

$$T(x,t) = w(r,t) + s(r) \tag{3.124}$$

The steady–state part has a simple, 1–D conduction solution, details of which need not be repeated here. The problem is

$$(rs')' = 0 \tag{3.125}$$

$$s'(a) = -1 \tag{3.126}$$

$$s'(1) = -Bi\, s(1) \tag{3.127}$$

which has the solution

$$s = a\left(\frac{1}{Bi} - \ln(r)\right) \tag{3.128}$$

The superposition $T = w + s$ is now substituted into Eqs. (3.120–3.123);

$$\frac{\partial w}{\partial t} + \frac{\partial s}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial w}{\partial r} + \frac{1}{r}(rs')'$$

$$\left.\frac{\partial w}{\partial r}\right|_a + s'(a) = -1$$

$$\left.\frac{\partial w}{\partial r}\right|_1 + s'(1) = -Bi(w(r=1,t) + s(1))$$

$$w(r, t=0) + s(r) = T_1$$

and the DE and BCs for s are used to cancel terms. The problem statement for w becomes

$$\frac{\partial w}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial w}{\partial r} \tag{3.129}$$

$$\left.\frac{\partial w}{\partial r}\right|_a = 0 \tag{3.130}$$

$$\left.\frac{\partial w}{\partial r}\right|_1 = -Bi\, w(r=1,t) \tag{3.131}$$

$$w(r, t=0) = T_1 - s(r) \tag{3.132}$$

The problem for w now has homogeneous BCs, and can be solved directly with the SOV method.

The time–dependent part to w will be identical to those in previous solutions (exponential decay). The spacial dependence will be of the form,

$$u(r) = AJ_0(\lambda r) + BY_0(\lambda r)$$

The Y_0 Bessel function must now be retained, because the origin ($r = 0$) is not included in the domain. The condition at $r = a$ is

$$u'(a) = 0$$

which gives (after employing formulas for the derivatives of Bessel functions)

$$0 = -\lambda\left[AJ_1(\lambda a) + BY_1(\lambda a)\right] \tag{3.133}$$

A general solution for u, which satisfies the DE and the inner BC, is therefore

$$u = A\left[J_0(\lambda r)Y_1(\lambda a) - J_1(\lambda a)Y_0(\lambda r)\right] \tag{3.134}$$

The condition at $r = 1$ is

$$u'(1) = -Bi\, u(1)$$

or, after using Eq. (3.134) and cancelling constant terms:

$$\lambda_n\left[J_1(\lambda_n)Y_1(\lambda_n a) - J_1(\lambda_n a)Y_1(\lambda_n)\right] = Bi\left[J_0(\lambda_n)Y_1(\lambda_n a) - J_1(\lambda_n a)Y_0(\lambda_n)\right] \tag{3.135}$$

This provides the eigencondition for the problem, in which λ_n represents the n^{th} root to Eq. (3.135). Do not be overly concerned about the apparent complexity of this equation. In general, numerical rootfinding techniques are required to solve the most simple of eigenconditions for radial problems – such as that encountered in Eq. (3.107) from the previous example. Providing that an appropriate rootfinding 'black box' is available in your numerical tools (which is the case in *Mathematica*), the only extra overhead involved in finding the roots to Eq. (3.135) is coding the equation into the package.

To condense some of the notation, let $\phi_n(r)$ denote the eigenfunction of this problem, i.e.,

$$\phi_n(r) = J_0(\lambda r)Y_1(\lambda a) - J_1(\lambda a)Y_0(\lambda r) \tag{3.136}$$

The general solution is then

$$T(r,t) = \sum_{n=1}^{\infty} A_n \phi_n(r) e^{-\lambda_n^2 t} \tag{3.137}$$

and the initial condition is

$$T_1 - s(r) = \sum_{n=1}^{\infty} A_n \phi_n(r) \tag{3.138}$$

The eigenfunctions ϕ_n are orthogonal on the interval $(a,1)$ (not $(0,1)$ as have been all previous problems), with a weighting function r. The expansion coefficients are then

$$A_n = \int_a^1 [T_1 - s(r)]\phi_n(r) r\, dr \cdot \left[\int_a^1 \phi_n^2(r) r\, dr\right]^{-1}$$

Obviously, the difficult part in wrapping up this example is evaluation of the integrals. The eigenfunction ϕ_n represents a linear combination of ordinary Bessel functions of order zero (see Eq. (3.136)), so let

$$\phi_n(r) = J_0(\lambda_n r)Y_1(\lambda_n a) - J_1(\lambda_n a)Y_0(\lambda_n r) \equiv C_0(\lambda_n r) \tag{3.139}$$

where C_0 denotes the linear combination of J_0 and Y_0. The integral formulas for combinations of Bessel functions, given in the previous notes, can now be applied. In particular,

$$\begin{aligned} \int_a^1 C_0^2(\lambda_n r) r\, dr &= \left[\frac{r^2}{2}\left(C_1^2(\lambda_n r) + C_0^2(\lambda_n r)\right)\right]_a^1 \\ &= \frac{1}{2}\left[\left(1 + \frac{Bi^2}{\lambda_n^2}\right)\phi_n^2(1) - a^2\phi_n^2(a)\right] \end{aligned} \tag{3.140}$$

Alternatively, the formulas in the appendix could have been used in the above. As has been done in practically every previous example, the BC at $r = a$ and the eigencondition were used to simplify the result. The other integral appearing in the formula for A_n is

$$\int_a^1 [T_1 - s(r)]\, \phi_n r\, dr$$

This can be split into two parts. The first half is

$$\begin{aligned} T_1 \int_a^1 \phi_n r\, dr &= -\frac{T_1}{\lambda_n^2} \int_a^1 (r\phi_n')'\, dr \\ &= -\frac{T_1}{\lambda_n^2} \left[\phi_n'(1) - a\phi_n'(a)\right] \\ &= \frac{T_1 Bi}{\lambda_n^2} \phi_n(1) \end{aligned} \tag{3.141}$$

and the second half is

$$\begin{aligned} \int_a^1 s(r)\phi_n r\, dr &= -\frac{1}{\lambda_n^2} \int_a^1 s(r\phi_n')'\, dr \\ &= -\frac{1}{\lambda_n^2} \left[rs\phi' \Big|_a^1 - rs'\phi \Big|_a^1 + \int_a^1 (rs')'\phi_n\, dr \right] \\ &= -\frac{1}{\lambda_n^2} \left[s(1)\phi_n'(1) - as(a)\phi_n'(a) - s'(1)\phi_n(1) + as'(a)\phi_n(a) \right] \\ &= \frac{a\phi_n(a)}{\lambda_n^2} \end{aligned} \tag{3.142}$$

Pay attention to the fact that the explicit forms of s (Eq. (3.128)) and ϕ_n (Eq. (3.136)) are never used in evaluation of the integral: all that is needed are the DEs (which provides the rules for integration by parts) and BCs (which are used to evaluate the boundary terms) that s and ϕ_n satisfy. This property has been exploited in practically every example of the chapter. The final formula for the A_n is

$$A_n = \frac{2[T_1 Bi\phi_n(1) - a\phi_n(a)]}{[(\lambda_n^2 + Bi^2)\phi_n^2(1) - a^2\lambda_n^2\phi_n^2(a)]} \tag{3.143}$$

and the dimensionless temperature in the cylinder is

$$T = a\left(\frac{1}{Bi} - \ln r\right) + \sum_{n=1}^{\infty} A_n \phi_n(r) e^{-\lambda_n^2 t} \tag{3.144}$$

Calculation results are presented in Fig. 3.8 for a system with $a = 1/2$, $Bi = 5$, and $T_1 = 1$. Perhaps the most noteworthy aspect of the results is the initial increase in temperature at the inner wall ($r = a = 0.5$), followed by a decrease in temperature to a steady–state value that is less than the starting value of unity. Such behavior is easily explained from a physics perspective: the initial increase in temperature at the inner surface results from the instantaneous application of the heat flux at $t = 0$. On the other hand, the steady–state inner surface temperature ($= a(1/Bi - \ln a) = 0.446$) is less than the initial temperature of unity for the conditions used here.

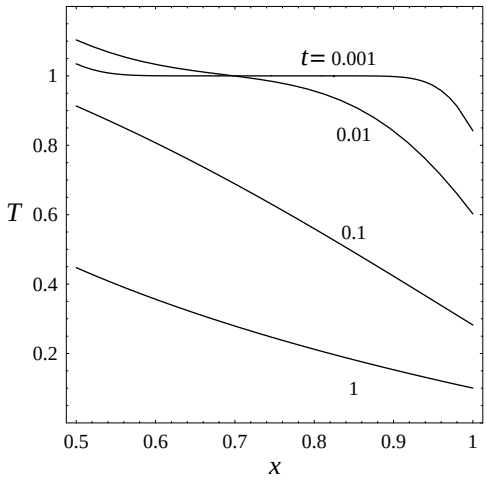


Figure 3.8: solution of Eq. (3.144), $Bi = 5$, $a = 0.5$, $T_1 = 1$

Consequently, the inner surface temperature will attain a maximum value at a finite time period into the process, and will then relax to a minimum as $t \to \infty$.

It is not easy, however, to anticipate this behavior solely from inspection of the analytical solution, Eq. (3.144). As is the case will all solutions in this chapter that have a time–independent steady–state limit, the time dependence in the solution appears in the series terms as decaying exponentials. Since the magnitude of each term in the series must therefore monotonically decease in time, one might expect that the series, as a whole, would display the same monotonic behavior – which would imply that the maxima and minima in the series would occur at $t = 0$ and/or $t \to \infty$. This behavior does not occur because of the delicate balancing of terms in the series; the terms have different signs (some are +, others -) and they decay at different rates. With this property, it is entirely possible for the series to have a local maximum/mininum in time.

3.5 Computational Strategies in *Mathematica*

3.5.1 Evaluation of simple series

A series solution for a problem in which the eigenvalues are explicitly known, such as Eq. (3.26), is relatively easy to evaluate in *Mathematica*. For such solutions evaluation of the terms in the series is 'cheap' (i.e., not involving a great deal of computational time), and the series can be summed using the *Mathematica* function `Sum` for a fixed number of terms. For example, a *Mathematica*-defined function to evaluate Eq. (3.26) could appear as

```
tsoln[x_,t_]:=-2 Sum[((-1)^n Cos[lambdan x]
     E^(-lambdan^2 t)/lambdan
    /.lambdan->(2n-1)Pi/2), {n,1,20}]
```

This function will sum 20 terms in the series – which is assumed beforehand to provide adequate precision. The total number of required terms, of course, could be smaller or greater depending on the value of t. Typically only the first term is required for $t > 0.4$ and evaluation of 20 terms would certainly be overkill. *Mathematica*, however, will (usually) not complain about underflow errors – which could occur in compiled programming languages (e.g., fortran) when the exponential function is evaluated for a large negative argument. Make note also of the coding convention in the *Mathematica* code: the replacement operator `lambdan->(2n-1)Pi/2` was used to avoid explicitly writing out each λ_n where it occured in the formula.

3.5.2 Eigencondition evaluation

A more difficult numerical problem is to evaluate non–explicit eigenconditions, such as those occuring for convection–type BCs in cartesian geometries and any SOV solution in cylindrical geometries. *Mathematica* provides an intrisic rootfinding algorithm, `FindRoot`, which uses Newton's method to find a root of a nonlinear equation. This function requires the equation to be solved (say $f(x) = 0$), the independent variable (x), and the starting point of the variable for the search (x_1). This last piece of information (the starting point) is the most challenging quantity to pin down. Recognize that the eigencondition intrinsically has an infinite number of roots, and to select any one root via `FindRoot` requires that the search begin at a point relatively 'close' to the desired root.

A way to overcome this problem is to find a pair of points, $x = x_p$ and x_m, so that $f(x_p)$ and $f(x_m)$ bisect the desired solution $f(x = x_R) = 0$. This bisection implies that $f(x_p) \cdot f(x_m) \leq 0$ – because one function must be in the region where $f \geq 0$ and the other in the region $f \leq 0$. Two such points are relatively easy to find; one would begin with two initial points $x_m = x_1$ and $x_p = x_m + \Delta x$, where Δx is a chosen step size for x, and make the test $f(x_m) \cdot f(x_p) \leq 0$. If this is not the case, then set $x_m = x_p$, $x_p = x_p + \Delta x$, and reperform the test. Essentially, the algorithm steps (or marches) x along until a bisection in the function is obtained.

Once the bisection is identified, the root can be approximated by linear interpolation;

$$x_R \approx x_p - \frac{f(x_p)\Delta x}{f(x_p) - f(x_m)}$$

This approximation can then be given to the *Mathematica* `FindRoot` function to finish the job.

A *Mathematica* function which performs this strategy is given below;

```
eigenroot[lamstart_]:=Module[{dlam = 0.2,
  lam0,eigen0,eigen1,lamr,lamroot},
  lam0=lamstart+1*^-6;
  eigen0=eigencond[lam0];
  lam0=lam0+dlam;
  eigen1=eigencond[lam0];
  While[eigen0 eigen1>0,
     lam0=lam0+dlam;
     eigen0=eigen1;
     eigen1=eigencond[lam0];
  ];
  lamr=lam0-eigen1 dlam/(eigen1-eigen0);
  lamroot=
     lam/.FindRoot[eigencond[lam]==0,{lam,lamr}];
  lamroot]
```

The function returns the first root to the user–defined equation `eigencond[lam]` that occurs after the point `lam=lamstart`. It uses a hard–wired step size of $\Delta\lambda = 0.2$, which is adequately small for all problems encountered in this chapter. The command `Module` provides a way of coding a 'subroutine' in *Mathematica*. The variable names enclosed in brackets in the first line of `Module` are the 'temporary variables' of the subroutine; assignments to these variables are made only within the subroutine and are not 'global'. The rest of the subroutine consists of individual statements, with each statement ended by a semicolon except the last one. The quantity returned by `Module` is the last statement (i.e., the value of `lamroot`. Refer to *Mathematica* help for more information on coding with `Module`.

The algorithm used by the function follows that described above; `eigen0` and `eigen1` correspond to $f(x_m)$ and $f(x_p)$, and the first value of x_m is set to `lamstart` $+ 10^{-6}$. The `While` block continues the marching process until the bisection point is found. `lamr` corresponds to the interpolated root from the bisection, and `lamroot` is the root returned by `FindRoot` using the interpolated approximation as the starting point.

The addition of the small number 10^{-6} to `lamstart`, to derive the first evaluation point, is included so that `lamstart` can correspond to a known root; when, for example, `lamstart` is set at λ_1 (where λ_1 is known), the function will return the next root λ_2. The first root λ_1 will be returned by setting `lamstart=0`, providing that the root is larger than 10^{-6}.

The code requires the user to define beforehand the function `eigencond[lam]`, which must return a numerical value of the eigencondition relation for a given numerical value of $\lambda =$ `lam`. The eigencondition corresponds to `eigencond[lam]==0`. This function should be coded to avoid singularities for $\lambda > 0$. For example, the plane wall convective BC eigencondition should appear

```
eigencond[lam_]:=lam Sin[lam] -bi Cos[lambda]
```

as opposed to the equivalent form $\lambda \tan(\lambda) - Bi$, which has singularities at $\lambda = (2n-1)\pi/2$. Recognize that the constant `bi` in the eigencondition must be assigned a numerical value prior to execution of the functions, e.g., by executing the line `bi = 5` for a Bi value of 5. This would apply to any numerical parameters that appear in the eigencondition.

Series evaluation with eigencondition roots

The most simple – and the computationally most expensive (and most stupid) – method to incorporate the eigencondition roots into series evaluation would be to modify the series summation code via

```
tsoln[x_,t_]:=-2Sum[((-1)^n Cos[lambdan x]
    E^(-lambdan^2 t)/lambdan
    /.lambdan->lambda[n], {n,1,20}]
lambda[n_]:=eigenroot[lambda[n-1]]/;n>1
lambda[n_]:=eigenroot[0]/;n<=1
```

It is easy to see that the function `lambda[n]` returns the n^{th} root of the eigencondition by recursively calling the eigenroot function for $\lambda_{n-1},\ \lambda_{n-2}, \ldots \lambda_1$ – and the first root is obtained from `eigenroot[0]`. The *Mathematica* convention '/;', following a function definition, represents a conditional test; the first definition of `lambda` is used if $n > 1$, otherwise the second definition is used. Of course, such a method will be tremendously time–consuming; evaluation of each root requires evaluation of all previous roots. Furthermore, this process would be performed every time the `tsoln` function is called – even though the all the roots may have been found in a previous function evaluation.

An easy way around this problem is to calculate the roots beforehand and store them in an array. The code could now appear as;

```
lambda[1]=eigenroot[0];
Do[lambda[n]=eigenroot[lambda[n-1]],{n,2,20}];
tsoln[x_,t_]:=-2Sum[((-1)^n Cos[lambdan x]
    E^(-lambdan^2 t)/lambdan
    /.lambdan->lambda[n], {n,1,20}]
```

The quantity `lambda[n]` now denotes an array element which contains the numerical value of the n^{th} root: note that the operator '=' was used to assign values to `lambda[1]`, `lambda[2]`, etc., as opposed to the function definition ':=' used in the previous code.

3.5.3 Series terms that are expensive to computute: advanced summation methods

Evaluation of series using the methods described above took the cheap approach; which was to calculate an adequately large number of eigencondition roots and series terms so that the sum will

always converge within the chosen number of terms. An obvious problem with this method is that it provides no real test of the series convergence; mostly likely one is either including unnecessary terms in the series or not including enough terms. This can be a problem if the terms are expensive to calculate (i.e., if they involve special functions such a Bessel functions) and can lead to long run times and/or inaccurate answers.

To overcome this problem, it is necessary to explicitly perform a convergence test on the series – so that only enough terms are included to obtain a desired precision. This is performed in the following code;

```
seriessum[x_, t_] := Module[{ssum, n, err,
         sumold, lambda=0.},
         If[t > .4,
            (* if t > .4 only the first term is used*)
            lambda = eigenroot[lambda];
            ssum= seriesterm[lambda, x, t]
         ,
            (* the following lines are for t <= .4 *)
            n = 1; ssum = 0; err = 1;
            While[err > 1*^-5 && n < 100,
               sumold = ssum;
               lambda = eigenroot[lambda];
               ssum = ssum
                     + seriesterm[lambda, x, t];
               n++;
               lambda = eigenroot[lambda];
               ssum = ssum
                      + seriesterm[lambda, x, t];
               n++;
               err = Abs[ssum - sumold];
            ];
         ];
      ssum]
seriesterm[lambda_,x_,t_]:=an[lambda]
    phi[lambda,x] E^(-lambda^2 t)
```

The code adds terms, two at a time, until the relative change in the sum is less than 10^{-5}. The two–at–a–time approach is used to avoid the possible case of $\phi_n(x) \approx 0$, for a particular n and x, prematurely terminating the series. When $t > 0.4$ the code includes only the first term in the series. The user–defined function `seriesterm[lambda,x,t]` calculates the term in the series that corresponds to the eigenvalue λ as a function of the independent variables (here x and t). An example of this function is given on the last line. Prior to each evaluation of `seriesterm`, the code

calculates the current eigenvalue via `lambda=eigenroot[lambda]`. This could be modified if the λ_n values were evaluated beforehand and stored in an array. Note also that the summation code is limited to a maximum of 100 terms (see the `While` statement, where && denotes 'and') – this is to prevent a runaway loop.

One would have to define functions for the eigenfunction `phi[lambda,x]`, the expansion coefficients `an[lambda]` and the eigencondition `eigencond[lambda]` prior to use of `seriesterm` and `seriessum`.

For the annular pipe problem discussed in the text, the eigenfunctions, eigencondition, and solution are defined by

```
j0[r_]:=BesselJ[0,r] y0[r_]:=BesselY[0,r] j1[r_]:=BesselJ[1,r]
y1[r_]:=BesselY[1,r]

phi[lam_,r_]:=j0[lam r] y1[lam a]
     -j1[lam a] y0[lam r]

eigencond[lam_]:=lam (j1[lam] y1[lam a]-j1[lam a] y1[lam])
     -bi (j0[lam]y1[lam a]-j1[lam a] y0[lam])

an[lam_]:=(2(t1 bi phi[lam,1]-a phi[lam,a]))/
((lam^2+bi^2)phi[lam,1]^2-(a lam phi[lam,a])^2)

tempsoln[r_,t_]:=a(1/bi-Log[r])+seriessum[r,t]
```

The code was used to generate the curves in Fig. 3.8. It's relatively slow – due to the computational effort required to evaluate the eigenvalues each and every time the solution is called. Given below is a code which stores the λ_n and A_n values in an array. I've redefined `seriessum` to do this. Note the timing comparisons – the new code is almost 50 times faster than the old.

```
bi = 5; a = .5; t1 = 1;

In[218]:=Timing[tempsoln[.5, .01]]

Out[218]={7.96 Second, 1.10367}

lam = 0; Clear[andat, lamdat]; Do[lam = eigenroot[lam]; lamdat[n]
= lam;
  andat[n] = an[lam], {n, 1, 102}]
```

```
seriessum[r_, t_] := Module[{sum, oldsum, n, lam},
  If[t > 0.4, n = 1;
    lam = lamdat[1];
    sum = andat[1]phi[lam , r]E^(-lam^2 t)
  ,
    oldsum = 0;
    lam = lamdat[1];
    sum = andat[1]phi[lam , r]E^(-lam^2 t);
    err = 1; n = 1;
    While[err > 1*^-5 && n <= 100,
      oldsum = sum;
      n++; lam = lamdat[n];
      sum = sum + andat[n]phi[lam , r]E^(-lam^2 t);
      n++; lam = lamdat[n];
      sum = sum + andat[n]phi[lam , r]E^(-lam^2 t);
      err = Abs[sum - oldsum];
    ];
  ];
  sum]

In[257]:=Timing[tempsoln[.5, .01]]

Out[257]={0.17 Second, 1.10367}
```

3.6 Summary

This chapter has introduced the basics of applying the separation of variables method to solution of 1–D and transient heat conduction problems. Key concepts are 1) the series nature of the solution; 2) eigen–conditions, functions, and values; 3) the property of orthogonality and how it can be used, and 4) methods of re–arranging problems so that can be fit into the SOV framework (i.e., partial solutions).

This material, no doubt, will take time and practice to master. I know of only one way of doing this: practice via the solution of problems. This is the type of material which, on the surface, may seem relatively straightforward, yet it is also relatively easy to complicate things by changing, for example, the type of boundary conditions or the form of the initial condition. Also, many

of these problems can take considerable time and paper to work out – as was seen with the last example. Again, there are few shortcuts. Codes such as *Mathematica* can help with much of the manipulation, yet the main analytical aspects to the solution must still be done by human brains[3].

In the next chapter the analysis will be extended to multidimensional heat conduction problems. For the most part the procedure will be similar to what was introduced here – except that the addition of extra dimensions will result in more complicated series solutions to the problem. Nevertheless, most of the concepts (such as eigenfunctions, orthogonality, etc.) will be the same.

Eigenfunctions and Integral Relationships

Provided here are some general forms and integrals of the eigenfunctions for transient and 1–D problems. In what follows ϕ_n refers to the n^{th} eigenfunction and ϕ_n' denotes the derivative of ϕ_n with respect to the independent variable.

Cartesian Systems

The eigenfunction DE will be

$$\phi_n'' + \lambda_n^2 \phi_n = 0 \tag{3.145}$$

with the general solution

$$\phi_n(x) = A\cos(\lambda_n x) + B\sin(\lambda_n x) \tag{3.146}$$

The normalization is

$$\int_a^b \phi_n^2\, dx = \frac{1}{2}\left[x\left(\phi_n^2 + \frac{1}{\lambda_n^2}(\phi_n')^2\right) - \frac{1}{\lambda_n^2}\phi_n\phi_n'\right]_a^b \tag{3.147}$$

Cylindrical Systems

Eigenfunction DE:

$$\left(r\phi_n'\right)' + r\lambda_n^2\phi_n = 0 \tag{3.148}$$

Solution:

$$\phi_n(r) = A\,J_0(\lambda_n r) + B\,Y_0(\lambda_n r) \tag{3.149}$$

Normalization:

$$\int_a^b \phi_n^2\, r\, dr = \left[\frac{r^2}{2}\left(\frac{1}{\lambda_n^2}(\phi_n')^2 + \phi_n^2\right)\right]_a^b \tag{3.150}$$

[3]It should be noted that *Mathematica* CANNOT solve the PDEs examined in this chapter – at least not yet

Spherical Systems

Eigenfunction DE:

$$\left(r^2\phi_n'\right)' + r^2\lambda_n^2\phi_n = 0 \tag{3.151}$$

Solution (by making the substitution $\phi_n = u_n/r$, the problem will reduce to the cartesian case):

$$\phi_n(x) = \frac{1}{r}\left(A\cos(\lambda_n r) + B\sin(\lambda_n r)\right) \tag{3.152}$$

Normalization:

$$\int_a^b \phi_n^2\, r^2\, dr = \frac{1}{2}\left[r^3\left(\frac{1}{\lambda_n^2}\left(\phi_n'\right)^2 + \phi_n^2\right) + \frac{r^2}{\lambda_n^2}\phi_n'\phi_n\right]_a^b \tag{3.153}$$

Eigencondition roots

The following numerical roots can be found in most heat transfer textbooks, they are included here for your reference and to provide checks in numerical codes you may develop.

Roots of Bessel Functions

n	$J_0(\lambda_n) = 0$	$J_1(\lambda_n) = 0$
1	2.4048	3.8317
2	5.5201	7.0156
3	8.6537	10.1735
4	11.7915	13.3237
5	14.9309	16.4706
6	18.0711	19.6159
7	21.2116	22.7601
8	24.3525	25.9037
9	27.4935	29.0468
10	30.6346	32.1897

Exercises

1. A plane wall of thickness L is initially at temperature of T_1. At time $t = 0$ a uniform heat flux of strength q'' is applied to the surface at $x = L$. The surface at $x = 0$ is maintained at temperature T_1. Formulate the problem in appropriate nondimensional variables. Using separation of variables and the partial solutions procedure, determine the temperature distribution in the wall as a function of position and time.

2. Another plane wall, thickness L, is initially at T_∞. At time $t = 0$ heat generation is applied within the wall. The heat generation is a function of x and is given by

$$q'''(x) = q'''_0 e^{-ax/L}$$

where q'''_0 and a are constants. The surface at $x = 0$ is adiabatic, and convection occurs at the $x = L$ surface, characterized by a heat transfer coefficient h and an ambient temperature T_∞. Non–dimensionalize the problem, and, using the SOV and partial solutions methods, find the solution for the nondimensional temperature field in the wall. Finally, prepare a plot of the nondimensional temperature at $x = 0$ and $x = L$ as a function of dimensionless time. Use values of $a = 2$ and $Bi = 10$ in calculating the numbers for this plot.

3. A pipe has an inner radius of R_i and an outer radius of R_0. Initially the pipe is at a temperature of T_1. At $t = 0$ the inside temperature of the pipe is brought to the steady value of T_2. Convection occurs on the outer surface, characterized by h and T_∞. Formulate the problem in appropriate non–dimensional variables, and use SOV to determine the solution to the problem.

4. A solid sphere, of radius R, is initially at a temperature of T_1. At time $t = 0$ the surface of the sphere is instantaneously brought to the temperature T_2. Formulate the nondimensional problem, and use SOV to determine the solution. Prepare a plot of the dimensionless temperature distribution in the sphere vs. dimensionless radius for several values of dimensionless time. NOTE: the eigenfunctions for this problem can be easy or difficult to calculate. If you go the difficult (and cookbook) route, you will wind up with Bessel functions of order 1/2. The easy route is to use the substitution that was discussed in Ch. 1 in the example involving heat generation in the sphere. That is, use something like $u(r) = f(r)/r$ when you solve the ODE for the separated function $u(r)$. If you do this, you should get an ODE that has a simple solution.

5. An egg has been sitting in a pan of boiling water for 12 minutes, at which point it can be considered hard–boiled. The egg is removed from the water and placed under the cold–water tap. Estimate the minimum amount of time the egg must remain under the flow of cold water so that, when the egg is removed from the flow and allowed to sit in air, the surface temperature of the egg will not exceed 50°C. Make whatever assumptions are necessary, and obtain correlations for the convection coefficients and thermophysical property data from any undergraduate heat transfer text.

Chapter 4

Two Dimensional Steady–State Conduction

4.1 Introduction

This chapter carries on from the previous one. The focus, as before, is on obtaining analytical solutions to the heat conduction equation, with an emphasis on steady, two–dimensional heat transfer. The same separation–of–variables (SOV) method developed in Ch. 3 for the 2–D in space and time problem can be applied directly to the 2–D in space problem, and the solutions will contain many of the same features. As before, the presentation will start with relatively simple problems, and will gradually be built up and generalized to arbitrarily complex (within reason) 2–D situations.

4.2 2–D Cartesian configurations

4.2.1 Specified temperature boundary conditions

Consider the 2–D square region illustrated in Fig. 4.1. This could represent a cross section in a long square rod, for example. The sides and bottom surfaces of the rod are maintained at temperature T_1, and the top surface is held at T_2. It will be shown below that these particular boundary conditions are physically impossible, but for now the problem is posed simply in a mathematical sense. The objective is to determine the temperature distribution in the rod.

We begin by casting the problem in non–dimensional form. The characteristic temperature difference is obviously $T_2 - T_1$, and let T_1 be the characteristic temperature. This gives

$$\overline{T} = \frac{T - T_1}{T_2 - T_1}, \quad \overline{x} = \frac{x}{L}, \quad \overline{y} = \frac{y}{L} \tag{4.1}$$

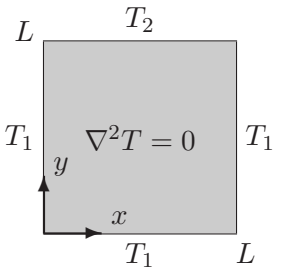


Figure 4.1: square region, specified surface T

The problem statement becomes

$$\frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} = 0 \tag{4.2}$$

$$\overline{T}(\overline{x} = 0, \overline{y}) = 0 \tag{4.3}$$

$$\overline{T}(\overline{x} = 1, \overline{y}) = 0 \tag{4.4}$$

$$\overline{T}(\overline{x}, \overline{y} = 0) = 0 \tag{4.5}$$

$$\overline{T}(\overline{x}, \overline{y} = 1) = 1 \tag{4.6}$$

Note that the choice of T_1 as the characteristic temperature resulted in homogeneous BCs on all surface except the top surface. This is desired – the more homogeneous the problem, the easier it is to solve.

As was done before, the temperature $\overline{T}(\overline{x}, \overline{y})$ is split into the product of two functions, each of which depends only on one variable:

$$\overline{T}(\overline{x}, \overline{y}) = u(\overline{x}) \cdot v(\overline{y}) \tag{4.7}$$

Replacing the above into Eq. (4.2) gives

$$vu'' + uv'' = 0 \tag{4.8}$$

or, after separating the variables,

$$\frac{u''}{u} = -\frac{v''}{v} = \pm\lambda^2 \tag{4.9}$$

The quantity λ^2 is the separation constant, which will ultimately be put in the form of an eigenvalue. Again, the rationale for introducing the constant λ^2 is that the separated equation has a function of x on one side, and a function of y on the other side. Both sides must therefore be constant. Equation (4.9) leads to the two ODEs for the u and v variables:

$$u'' \mp \lambda^2 u = 0$$

$$v'' \pm \lambda^2 v = 0$$

By choosing $+\lambda^2$ as the separation constant, and restricting the situation to non–zero λ, the following solutions are obtained for u and v;

$$u = A\cosh(\lambda\overline{x}) + B\sinh(\lambda\overline{x}) \tag{4.10}$$
$$v = C\cos(\lambda\overline{y}) + D\sin(\lambda\overline{y}) \tag{4.11}$$

and for $-\lambda^2$ just the opposite is obained:

$$u = A\cos(\lambda\overline{x}) + B\sin(\lambda\overline{x}) \tag{4.12}$$
$$v = C\cosh(\lambda\overline{y}) + D\sinh(\lambda\overline{y}) \tag{4.13}$$

Finally, for $\lambda^2 = 0$ the solutions are

$$u = A + B\overline{x} \tag{4.14}$$
$$v = C + D\overline{y} \tag{4.15}$$

Only one set of solutions will lead to a physically correct solution, yet it is not obvious which set should be chosen. When dealing with transient problems we could use physical reasoning to decide the sign of the separation constant. This is not the case with the steady–state problem; there is no intuitive information which would appear to rule out the functional dependencies obtained from $+\lambda^2$ or $-\lambda^2$ – although the case of $\lambda = 0$ appears suspicious simply because it is too simple.

Mathematical, rather than physical, reasoning must therefore be utilized to decide on the proper sign. Specifically, the sign of λ^2 is chosen so that a Sturm–Liouville form of an ODE is obtained in the direction with the homogeneous boundary conditions. Recall that the Sturm–Liouville ODE has solutions in terms of eigenfunctions (or orthogonal functions) such as sin, cos, and the ordinary Bessel functions. The hyperbolic functions and modified Bessel functions, on the other hand, will not be orthogonal functions. For the problem at hand the x direction has the homogeneous boundary conditions. Therefore, we want the x–direction function ODE (which is the one for u) to be in the Sturm–Liouville form – which is the one that has Eq. (4.12) as a solution.

The solutions for $\lambda^2 = 0$ can always be considered a special case for a positive or negative separation constant. Again, such cases were usually inconsistent with transient behavior in time–dependent problems. In steady–state problems, however, more attention will have to be given to the $\lambda^2 = 0$ case. Specifically, if it is determined that the eigenvalue λ_n can have a value of zero, then the unique solution for $\lambda^2 = 0$ will have to be included into the analysis. This situation will be discussed further when it is encountered. To briefly summarize, we want to obtain the eigenfunctions (and eigencondition) in the direction with homogeneous boundary conditions – which is the x direction for the problem at hand.

The BCs in the x direction can now be used to eliminate one of the constants A or B and determine the eigencondition. At $\overline{x} = 0$ the condition is

$$T(0, \overline{y}) = u(0) \cdot v(\overline{y}) = 0$$

which gives

$$u(0) = 0 = A \longrightarrow A = 0$$

At $\overline{x} = 1$ the BC gives

$$T(1, \overline{y}) = u(1) \cdot v(\overline{y}) = 0$$

which leads to

$$u(1) = 0 = B \sin(\lambda)$$

This provides the eigencondition for the problem:

$$\lambda = n\pi, \quad n = 0,\ 1,\ 2, \ldots \tag{4.16}$$

The correspondition eigenfunctions are:

$$\phi_n(\overline{x}) = \sin(\lambda_n \overline{x}) = \sin(n\pi\overline{x}) \tag{4.17}$$

The unique case of a zero eigenvalue, i.e., $\lambda_0 = 0$, does occur for this eigencondition. We therefore need to consider this case separately. When $\lambda = 0$ the solution for u is, again,

$$u = A + B\overline{x}$$

The zero-temperature condition at $\overline{x} = 0$ eliminates A, and the zero temperature condition at $\overline{x} = 1$ eliminates B. The corresponding solution for λ_0 is zero, and we need not consider the λ_0 case further. Another way of arriving at this conclusion is to note that the eigenfunction for λ_0 is itself zero. This procedure may have seemed trivial – but it is important to go through it. Situations will occur in which the $\lambda = 0$ condition plays a role in the solution.

The general solution to the problem will be in the form of a series expansion of the v function times the corresponding eigenfunctions. In other words,

$$\boxed{\overline{T} = \sum_{n=1}^{\infty} \left(A_n \cosh(\lambda_n \overline{y}) + B_n \sinh(\lambda_n \overline{y})\right) \phi_n(\overline{x})} \tag{4.18}$$

This equation is the most general form of the solution for 2–D steady conduction in cartesian coordinates when the homogeneous direction is x. Had the homogeneous direction been y, the roles of $\overline{x}$ and $\overline{y}$ would simply be switched.

The final step is to determine formulas for the expansion coefficients A_n and B_n, which is done by using the BCs in the $\overline{y}$ direction and the orthogonality properties of the eigenfunctions. At $\overline{y} = 0$ the condition is

$$0 = \sum_{n=1}^{\infty} \left(A_n + 0\right) \phi_n(\overline{x})$$

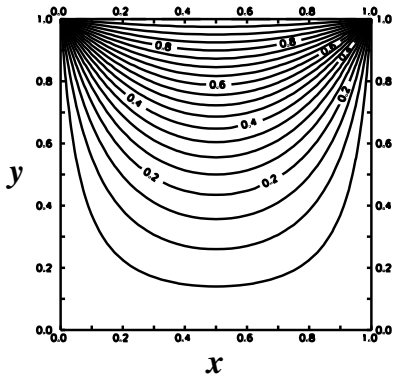


Figure 4.2: isotherms for Eq. (4.21)

which is satisfied by $A_n = 0$ for all n. The inhomogeneous boundary condition at $\overline{y} = 1$ has

$$1 = \sum_{n=1}^{\infty} B_n \phi_n(\overline{x}) \sinh(\lambda_n) \tag{4.19}$$

The eigenfunctions are orthogonal over the interval (0,1) with a weighting function of unity. Each side is therefore multiplied by $\phi_m(x)$ and integrated over $\overline{x}$ from 0 to 1. All terms in the series disappear except the one for $n = m$. The result is

$$\begin{aligned} B_n &= \int_0^1 \sin(n\pi\overline{x})\, d\overline{x} \cdot \left[\sinh(n\pi) \int_0^1 \sin^2(n\pi\overline{x})\, d\overline{x}\right]^{-1} \\ &= \frac{2[1-(-1)^n]}{n\pi \sinh(n\pi)} \end{aligned}$$

This gives zero B_n for even n. Therefore, the index n can be replaced with $2n-1$, and

$$B_n = \frac{4}{(2n-1)\pi \sinh[(2n-1)\pi]}, \quad n = 1,\ 2, \ldots \tag{4.20}$$

The final solution for the temperature field is

$$T = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi\overline{x}] \sinh[(2n-1)\pi\overline{y}]}{(2n-1)\sinh[(2n-1)\pi]} \tag{4.21}$$

Shown in Fig. 4.2 is a contour plot of the dimensionless temperature in the square region. The lines in the plot correspond to isotherms. The temperature field shows the expected symmetry about $\overline{x} = 1/2$. It could have been recognized, at the onset, that the problem had a plane of

symmetry, and this information could have been exploited in the solution. The equivalent domain would have been a rectangular rod, of height equal to twice the width, with an adiabatic condition at $\overline{x} = 0$.

Observe also the large temperature gradient at the top left and right corners. This results from the discontinuous jump in surface temperature from the side to the top – which is a physical impossibility. The consequence of this behavior becomes evident when we calculate the net heat transfer at the top surface. The heat flux at the top is given by

$$\begin{aligned} q''_y(\overline{x}) &= \frac{k(T_1 - T_2)}{L} \left.\frac{\partial \overline{T}}{\partial \overline{y}}\right|_1 \\ &= \frac{4k(T_1 - T_2)}{L} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)\pi\overline{x}]\cosh[(2n-1)\pi]}{\sinh[(2n-1)\pi]} \end{aligned}$$

and the total heat transfer (per unit length of rod) through the top surface will be

$$\begin{aligned} q' &= \int_0^L q''\, dx = k(T_1 - T_2) \int_0^1 \left.\frac{\partial \overline{T}}{\partial \overline{y}}\right|_1 d\overline{x} \\ &= \frac{8k(T_1 - T_2)}{\pi} \sum_{n=1}^{\infty} \frac{\cosh[(2n-1)\pi]}{(2n-1)\sinh[(2n-1)\pi]} \\ &= \frac{8k(T_1 - T_2)}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\tanh[(2n-1)\pi]} \\ &= \infty \end{aligned} \tag{4.22}$$

This series does not converge – i.e, it has a value of infinity. This is because $\tanh[(2n-1)\pi] \to 1$ for $n \gg 1$, which leaves for large n the simple series of $\sum 1/(2n-1)$ – which will not converge to a constant. Therefore, the net heat transfer rate through the upper face would be infinite.

The impossibility in the problem arises from the chosen boundary conditions. It would take an infinite amount of heat transfer to maintain the top and side surfaces at the precisely uniform values of T_2 and T_1, respectively. In reality, the heat transfer through the rod would be finite, and the temperature at the upper corners would vary continuously from T_1 to T_2.

4.2.2 Convection boundary conditions

Continue with the same square region, yet impose now convective boundary condtions on the sides and bottom. The convection on all surfaces is characterized by a heat transfer coefficient h and an ambient temperature T_∞. The upper surface temperature remains at T_2.

Define the dimensionless temperature as $\overline{T} = (T - T_\infty)/(T_2 - T_\infty)$ – which is formulated to provide homogeneous boundary conditions on the bottom and sides. The symmetry along the $\overline{x} = 1/2$ plane is also exploited, and the origin is placed in the middle of the bottom surface. The

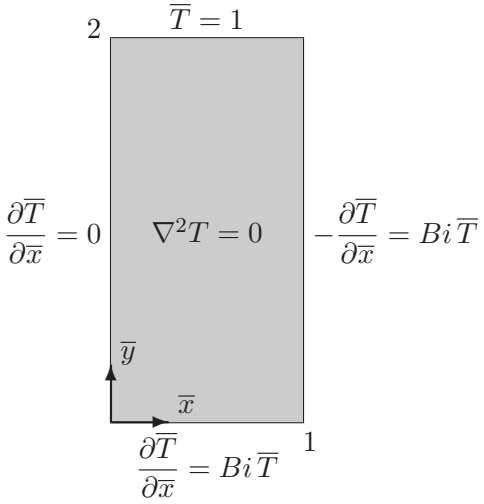


Figure 4.3: region for convective BC problem

domain becomes a rectangle of height equal to twice the width, and the dimensionless coordinates $\overline{x} = x/(L/2)$ and $\overline{y} = y/(L/2)$ run from 0 to 1 and 0 to 2, respectively[1]. The problem configuration appears in Fig. 4.3.

The boundary conditions for the problem appear as

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_0 = 0 \tag{4.23}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_1 = -Bi\,\overline{T}(0,\overline{y}) \tag{4.24}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{y}}\right|_0 = Bi\,\overline{T}(\overline{x},0) \tag{4.25}$$

$$\overline{T}(\overline{x},2) = 1 \tag{4.26}$$

Again, the boundary conditions in the $\overline{x}$ direction are homogeneous, so we choose the sign of the separation constant so that eigenfunctions are obtained in the $\overline{x}$ direction. Using $\overline{T} = u(\overline{x}) \cdot v(\overline{y})$ gives

$$u = A\cos(\lambda \overline{x}) + B\sin(\lambda \overline{x}) \tag{4.27}$$

$$v = C\cosh(\lambda \overline{y}) + D\sinh(\lambda \overline{y}) \tag{4.28}$$

[1] it is good practice to use the same characteristic length to scale the position variables

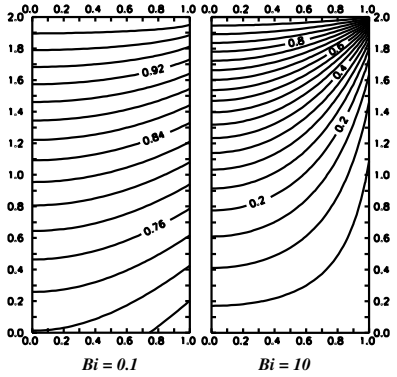


Figure 4.4: convective BC isotherms: $Bi = 0.1$ (left), $= 10$ (right)

At $\overline{x} = 0$, Eq. (4.23) results in $B = 0$. At $\overline{x} = 1$, Eq. (4.24) gives the eigencondition:

$$\lambda_n \sin(\lambda_n) = Bi \cos(\lambda_n) \tag{4.29}$$

Note that $\lambda = 0$ is not a solution to the above eigencondition – at least not for non–zero Bi – and can be eliminated from the solution. The corresponding eigenfunction for the problem is

$$\phi_n(\overline{x}) = \cos(\lambda_n \overline{x}) \tag{4.30}$$

The general form of our solution will be the same as before;

$$\overline{T} = \sum_{n=1}^{\infty} \left(A_n \cosh(\lambda_n \overline{y}) + B_n \sinh(\lambda_n \overline{y}) \right) \phi_n(\overline{x})$$

The convection boundary condition at $\overline{y} = 0$ is homogeneous. When a homogeneous BC occurs in the non–homogeneous direction of the problem (here, $\overline{y}$), the boundary condition can be applied to each term in the series without 'formally' applying the orthogonality procedure. This is equivalent to application of the BC directly to the $v(\overline{y})$ function in Eq. (4.28), and was (trivially) done in the previous example. The result for this case is

$$\lambda_n B_n = Bi\, A_n \;\rightarrow\; B_n = A_n \frac{Bi}{\lambda_n} \tag{4.31}$$

With this, the general solution reduces to

$$\overline{T} = \sum_{n=1}^{\infty} A_n \phi_n(\overline{x}) \left(\cosh(\lambda_n \overline{y}) + \frac{Bi}{\lambda_n} \sinh(\lambda_n \overline{y}) \right) \tag{4.32}$$

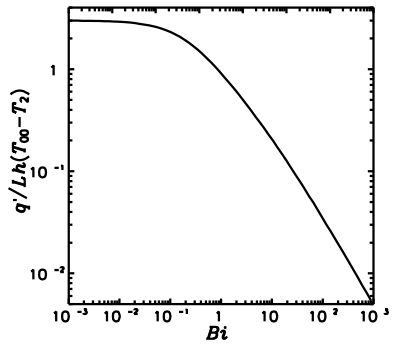


Figure 4.5: q' vs. Bi

Finally, the inhomogeneous BC at $\overline{y} = 2$ is

$$1 = \sum_{n=1}^{\infty} A_n \phi_n(\overline{x}) \left(\cosh(2\lambda_n) + \frac{Bi}{\lambda_n} \sinh(2\lambda_n) \right) \tag{4.33}$$

The orthogonality of ϕ_n can now be used to obtain the expansion coefficients:

$$A_n = \int_0^1 \phi_n(\overline{x})\, d\overline{x}$$
$$\times \left[\left(\cosh(2\lambda_n) + \frac{Bi}{\lambda_n} \sinh(2\lambda_n) \right) \int_0^1 \phi_n^2(\overline{x})\, d\overline{x} \right]^{-1}$$

The integrals have already been worked out in Ch. 3; ϕ_n and λ_n are the same eigenfunction and eigenvalues as that occurring for the transient, plane–wall convective cooling problem. The result is

$$A_n = -\frac{2(-1)^n Bi \left(\lambda_n^2 + Bi^2\right)^{1/2}}{\left[\lambda_n^2 + Bi(1+Bi)\right] \left[\lambda_n \cosh(2\lambda_n) + Bi \sinh(2\lambda_n)\right]} \tag{4.34}$$

Replacing the above in Eq. (4.32) gives the complete solution for the temperature field.

Results of Eq. (4.32) are plotted in Fig. 4.4 using Bi values of 0.1 and 10. As the Biot number gets large the temperature field approaches the previous solution of uniform temperature on the side and bottom surfaces. In the opposite limit ($Bi \to 0$) the temperature of the rod would become uniform at the value of unity.

The total heat transfer from the top surface is obtained from the same procedure given in the

previous problem:

$$q' = 2k(T_\infty - T_2) \int_0^1 \left.\frac{\partial \overline{T}}{\partial \overline{y}}\right|_2 d\overline{x} \tag{4.35}$$

$$= 2k(T_\infty - T_2) \sum_{n=1}^{\infty} A_n \sin(\lambda_n) \left(\sinh(2\lambda_n) + \frac{Bi}{\lambda_n} \cosh(2\lambda_n) \right) \tag{4.36}$$

The terms in the series still approach $1/n$ as $n \to \infty$ – yet they will not all have the same sign and the series will converge for $Bi < \infty$. Presented in Fig. 4.5 is the dimensionless heat transfer rate $\overline{q}'$, defined by

$$\overline{q}' = \frac{q'}{2k(T_\infty - T_2)Bi} = \frac{q'}{hL(T_\infty - T_2)},$$

vs. the Biot number Bi. This particular scaling for q' is chosen so that in the limit of $Bi \to 0$, the dimensionless $\overline{q}' \to 3$. This limit has a relatively simple physical explanation which you should be able to figure out. The quantity $\overline{q}'$ goes to zero for $Bi \to \infty$ – because $\overline{q}' \sim q'/Bi$ – yet the actual heat transfer would also go to infinity in this limit.

4.3 Superposition

It is not uncommon to have conduction problems that have inhomogeneous boundary conditions in both directions or that have an inhomogeneous DE (through heat generation). Separation of variables, however, can only work if a separated ODE and the associated BCs conform to the Sturm–Liouville system, which constrains the method to conduction problems with a homogeneous DE and homogeneous BCs in all but one direction. The way around this dilemma is to apply the technique of *superposition* – which is a generalized form of the partial solutions technique that was learned in Ch. 3. The superposition method exploits the fact that the heat conduction equation is linear. Because of this, any sum of solutions to the DE is also a solution. If solutions to the DE can be found which, when summed together, satisfy all the BCs to the original problem, then the original problem has been solved.

Devising a superimposed solution involves some creativity and imagination. In general, the procedure is to sequentially replace the inhomogeneous conditions with homogeneous conditions of the same type, and then solve the resulting problems with appropriate techniques. If N inhomogeneous conditions appear in the original problem, then the complete solution will need at most N partial solutions that are pasted together. Often this number can be reduced by inspection or appropriate combinations.

4.3.1 Superposition example #1

The application of the method is best seen through application. Consider a square region of width = height = L. The surface at $y = 0$ is maintained at T_1, and convection occurs at the $y = L$

surface, characterized by h and T_∞. The surface at $x = 0$ is adiabatic, and a uniform heat flux of q_0'' is applied at $x = L$. Finally, a uniform heat generation, of strength q_0''' occurs within the region.

The dimensional problem is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{q_0'''}{k} = 0$$
$$\left.\frac{\partial T}{\partial x}\right|_0 = 0$$
$$\left.\frac{\partial T}{\partial x}\right|_L = \frac{q_0''}{k}$$
$$T(x, 0) = T_1$$
$$\left.\frac{\partial T}{\partial y}\right|_L = -h(T(x, L) - T_\infty)$$

Define the dimensionless temperature by

$$\overline{T} = \frac{T - T_\infty}{T_1 - T_\infty} \tag{4.37}$$

and the dimensionless problem becomes

$$\frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} + \overline{q}_0''' = 0 \tag{4.38}$$
$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_0 = 0 \tag{4.39}$$
$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_1 = \overline{q}'' \tag{4.40}$$
$$\overline{T}(\overline{x}, 0) = 1 \tag{4.41}$$
$$\left.\frac{\partial \overline{T}}{\partial \overline{y}}\right|_1 = -Bi\,\overline{T}(\overline{x}, 1) \tag{4.42}$$

in which the dimensionless parameters are

$$\overline{q}'' = \frac{q_0'' L}{k(T_1 - T_\infty)}, \quad \overline{q}''' = \frac{q_0''' L^2}{k(T_1 - T_\infty)}, \quad Bi = \frac{hL}{k}$$

The dimensionless domain is shown in Fig. 4.6

The problem has an inhomogeneous DE and inhomogeneous BCs at $\overline{x} = 1$ and $\overline{y} = 0$. SOV cannot be applied to the total problem at hand, but the problem can be split into several sub–problems which, individually, can be solved using SOV or more simple methods.

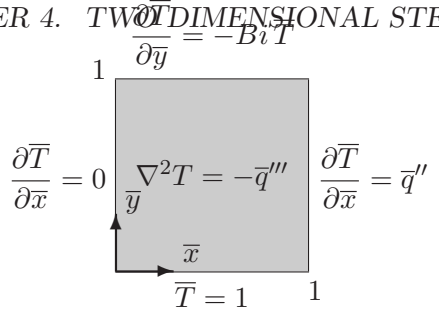


Figure 4.6: a complicated problem

For this particular case, it will now be demonstrated the problem can be split into the two superimposed problems illustrated in Fig. 4.7. Problem A consists of a square region with adiabatic conditions at $\overline{x} = 0$ and 1, uniform temperature of unity at $\overline{y} = 0$, convection at $\overline{y} = 1$ and uniform heat generation of $\overline{q}'''$. Problem B has an adiabatic condition at $\overline{x} = 0$, uniform flux of $\overline{q}''$ at $\overline{x} = 1$, zero temperature at $\overline{y} = 0$, convection at $\overline{y} = 1$, and no heat generation.

Denote as $\overline{T}_A$ and $\overline{T}_B$ the solutions to these individual problems, which satisfy

$$\frac{\partial^2 \overline{T}_A}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}_A}{\partial \overline{y}^2} + \overline{q}''' = 0 \tag{4.43}$$

$$\left.\frac{\partial \overline{T}_A}{\partial \overline{x}}\right|_0 = 0 \tag{4.44}$$

$$\left.\frac{\partial \overline{T}_A}{\partial \overline{x}}\right|_1 = 0 \tag{4.45}$$

$$\overline{T}_A(0, \overline{y}) = 1 \tag{4.46}$$

$$-\left.\frac{\partial \overline{T}_A}{\partial \overline{y}}\right|_1 = Bi\,\overline{T}_A(0, \overline{y}) \tag{4.47}$$

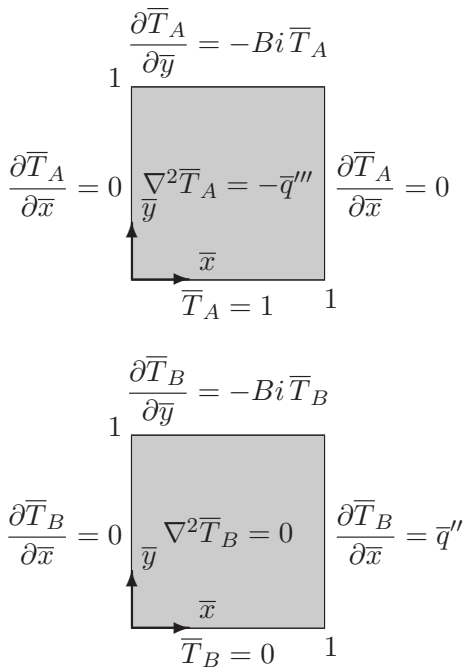


Figure 4.7: the two superimposed problems

and

$$\frac{\partial^2 \overline{T}_B}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}_B}{\partial \overline{y}^2} = 0 \tag{4.48}$$

$$\left.\frac{\partial \overline{T}_B}{\partial \overline{x}}\right|_0 = 0 \tag{4.49}$$

$$\left.\frac{\partial \overline{T}_B}{\partial \overline{x}}\right|_1 = \overline{q}'' \tag{4.50}$$

$$\overline{T}_B(0, \overline{y}) = 0 \tag{4.51}$$

$$-\left.\frac{\partial \overline{T}_B}{\partial \overline{y}}\right|_1 = Bi\,\overline{T}_B(0, \overline{y}) \tag{4.52}$$

$$\tag{4.53}$$

You should be able to convince yourself, by substitution of the above equations into the original problem in Eqs. (4.38–4.42), that the superposition $\overline{T} = \overline{T}_A + \overline{T}_B$ solves the original problem. Specifically, $\overline{T}_A$ takes care of the inhomogeneous bondary condition at $\overline{y} = 0$ and the heat generation term, whereas $\overline{T}_B$ takes care of the inhomogeneous boundary condition at $\overline{x} = 1$.

It is important to recognize that in formulating the problem for A, the heat flux condition at $\overline{x} = 1$ was replaced with the corresponding homogeneous condition – which is an adiabatic condition.

Likewise, in formulating the problem for B the inhomogenous, unit–temperature condition at $\overline{y} = 0$ was replaced with the corresponding homogeneous condition of $\overline{T}_B = 0$. This procedure is critical: always replace an inhomogeneous BC in a superimposed solution with a homogenous on of the same type.

It turns out that the problem for A has a relatively simple solution because the A configuration presents a one dimensional problem. That is, the adiabatic conditions at $\overline{x} = 0$ and 1 eliminates any heat flow in the $\overline{x}$ direction, for which the temperature will depend only $\overline{y}$. The corresponding 1–D solution for $\overline{T}_A$ is

$$\overline{T}_A = 1 - \frac{\overline{q}'''\overline{y}^2}{2} + \frac{\overline{q}'''(2 + Bi) - 2Bi}{2 + 2Bi}\overline{y} \tag{4.54}$$

The B problem must be solved by separation of variables. The $\overline{y}$ direction has the homogeneous boundary conditions, and the separation constant is chosen to give eigenfunctions in the $\overline{y}$ direction. Since $\overline{T}_B(\overline{x}, 0) = 0$, the eigenfunction will be of the form

$$\phi_n(\overline{y}) = \sin(\lambda_n \overline{y}) \tag{4.55}$$

The convection condition at $\overline{y} = 1$ provides the eigencondition

$$\lambda_n \cos(\lambda_n) + Bi \sin(\lambda_n) = 0, \;\; n = 1, \;\; 2, \ldots \tag{4.56}$$

The special case of $\lambda = 0$ is a solution to the above eigencondition, and this case must therefore be examined further. The zeroth eigenfunction would be (from solution of the characteristic ODE in the homogeneous direction)

$$\phi_0(\overline{y}) = A + B\overline{y}$$

The constant $A = 0$ from the BC at $\overline{y} = 0$. At $\overline{y} = 1$ the convection condition has

$$B = -Bi\, B$$

which can be satisfied for arbitrary Bi only if $B = 0$. Consequently, the zeroth eigenfunction is zero, and can be dismissed from the solution. Recognize that λ_1 will now denote the first non–zero root to Eq. (4.56).

In the $\overline{x}$ direction the zero–gradient condition at $\overline{x} = 0$ will eliminate the sinh term in the general solution of Eq. (4.18). The solution for $\overline{T}_B$ will then appear as

$$\overline{T}_B = \sum_{n=1}^{\infty} A_n \cosh(\lambda_n \overline{x}) \sin(\lambda_n \overline{y})$$

At $\overline{x} = 1$ the specified heat flux BC gives

$$\overline{q}'' = \sum_{n=1}^{\infty} \lambda_n A_n \sinh(\lambda_n) \sin(\lambda_n \overline{y}) \tag{4.57}$$

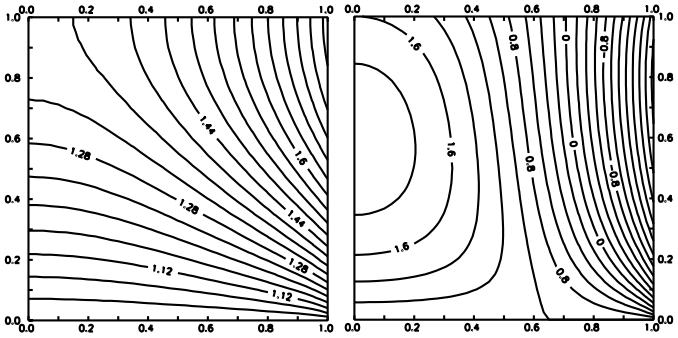


Figure 4.8: superimposed solution: $\overline{q}''' = Bi = 0,\ \overline{q}'' = 1$ (left), $\overline{q}''' = 10,\ \overline{q}'' = -10,\ Bi = 1$ (right)

from which the expansion coefficients are obtained as

$$\begin{aligned} A_n &= \frac{\overline{q}'' \displaystyle\int_0^1 \sin(\lambda_n \overline{y})\, d\overline{y}}{\lambda_n \sinh(\lambda_n) \displaystyle\int_0^1 \sin^2(\lambda_n \overline{y})\, d\overline{y}} \\ &= \frac{2\overline{q}''[1 - \cos(\lambda_n)]}{\lambda_n \sinh(\lambda_n)[\lambda_n - \cos(\lambda_n)\sin(\lambda_n)]} \end{aligned} \tag{4.58}$$

The complete solution to the problem is now given by $\overline{T} = \overline{T}_A + \overline{T}_B$, or

$$\overline{T} = 1 - \frac{\overline{q}'''\overline{y}^2}{2} + \frac{\overline{q}'''(2 + Bi) - Bi}{2 + 2Bi}\overline{y} + 2\overline{q}'' \sum_{n=1}^{\infty} \frac{[1 - \cos(\lambda_n)]\cosh(\lambda_n \overline{x})\sin(\lambda_n \overline{y})}{\lambda_n \sinh(\lambda_n)[\lambda_n - \cos(\lambda_n)\sin(\lambda_n)]} \tag{4.59}$$

in which the eigenvalues λ_n are obtained from the roots of Eq. (4.56).

The solution to this problem is plotted in Fig. 4.8 for $Bi = \overline{q}''' = 0$, $\overline{q}'' = 1$ (left plot) and $Bi = 1$, $\overline{q}'' = -10$, $\overline{q}''' = 10$ (right). Setting $Bi = 0$ results in an adiabatic top surface, and this is observed in the perpendicular intersection of the isotherms with the upper surface. Recognize that the first set of parameters results in $\overline{T}_A = 1$ throughout the domain; this result therefore provides a check on the correctness of the SOV solution for $\overline{T}_B$. It is difficult to gauge from a contour plot whether or not a constant, non–zero flux condition is satisfied at a boundary (as occurs along $\overline{x} = 1$); to examine the results in a different perspective a plot of $\overline{T}$ vs. $\overline{x}$, using $q'' = Bi = 0$ and $q'' = 1$, is shown in Fig. 4.9. The sloping lines represent the temperature profile at a constant values of $\overline{y}$; increasing height of the lines correspond to increasing $\overline{y}$. This view shows that the flux condition is met at $\overline{x} = 1$, in that the lines uniformly intersect the surface with a slope of unity. The one point where this condition would fail would be at $\overline{y} = 0$ and $\overline{x} = 1$ – which is a result of the contradiction implied in maintaining the bottom surface at $\overline{T} = 1$ while providing a uniform

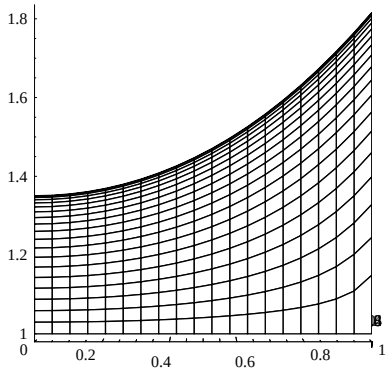


Figure 4.9: $\overline{T}$ vs. $\overline{x}$ for $\overline{q}''' = Bi = 0,\ \overline{q}'' = 1$

flux from the side. This is solely a mathematical artifact and, for reasons discussed above, would not occur physically; it would be impossible to maintain the uniform surface temperature and/or the constant flux in the vicinity of this point.

4.3.2 Superposition example #2

The domain for this example consists of a square region of width = height = $2L$ in which all four surfaces are maintained at $T = T_1$. Uniform heat generation occurs within the region of strength q'''_0. This problem is completely symmetric about the $x = L$ and $y = L$ planes. Therefore, we need only consider one quadrant of the region – say the first quadrant.

By defining the dimensionless variables as as

$$\overline{T} = \frac{(T - T_1)k}{q'''_0 L^2}, \quad \overline{x} = \frac{x}{L}, \quad \overline{y} = \frac{y}{L}$$

the dimensionless problem becomes

$$\frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} + 1 = 0 \tag{4.60}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_0 = 0 \tag{4.61}$$

$$\overline{T}(1, \overline{y}) = 0 \tag{4.62}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{y}}\right|_0 = 0 \tag{4.63}$$

$$\overline{T}(\overline{x}, 1) = 0 \tag{4.64}$$

All BCs in the problem are homogeneous, yet the DE is inhomogeneous. Because of the latter, we cannot apply SOV directly to the problem.

The approach to this problem is to use a superposition method that is similar to the partial solutions technique that was developed for transient problems. The procedure is to let $\overline{T} = w(\overline{x}, \overline{y}) + s(\overline{x})$, where s is chosen to 'absorb' the heat generation term in the DE. That is, s satisfies the problem

$$s'' + 1 = 0 \tag{4.65}$$

$$s'(0) = 0, \quad s(1) = 0 \tag{4.66}$$

The solution for s is simply

$$s = \frac{1}{2}\left(1 - \overline{x}^2\right) \tag{4.67}$$

By replacing $\overline{T}$ with $w(\overline{x}, \overline{y}) + s(\overline{x})$ in Eqs. (4.60–4.64), the following system is obtained for w.

$$\frac{\partial^2 w}{\partial \overline{x}^2} + \frac{\partial^2 w}{\partial \overline{y}^2} = 0 \tag{4.68}$$

$$\left.\frac{\partial w}{\partial \overline{x}}\right|_0 = 0 \tag{4.69}$$

$$w(1, \overline{y}) = 0 \tag{4.70}$$

$$\left.\frac{\partial w}{\partial \overline{y}}\right|_0 = 0 \tag{4.71}$$

$$w(\overline{x}, 1) = -s(\overline{x}) \tag{4.72}$$

As is evident from the above, the problem for w has a homogeneous DE and an inhomogeneous BC at $\overline{y} = 1$ – and this problem can be solved directly with SOV.

The homogeneous direction of the problem is $\overline{x}$. In view of the BC at $\overline{x} = 0$, the eigenfunction will be

$$\phi_n(\overline{x}) = \cos(\lambda_n \overline{x}) \tag{4.73}$$

and the BC at $\overline{x} = 1$ delivers the eigencondition:

$$\cos(\lambda_n) = 0 \longrightarrow \lambda_n = \frac{1}{2}(2n-1)\pi, \quad n = 1,\ 2, \ldots \tag{4.74}$$

The specific case of $\lambda_n = 0$ does not contribute to the solution. The BC at $\overline{y} = 0$ will eliminate the sinh part to the $\overline{y}$ functional dependence, and the general solution for w becomes

$$w = \sum_{n=0} A_n \phi_n(\overline{x}) \cosh(\lambda_n \overline{y}) \tag{4.75}$$

At $\overline{y} = 1$ the condition is

$$-s(\overline{x}) = \sum_{n=1} A_n \phi_n(\overline{x}) \cosh(\lambda_n) \tag{4.76}$$

from which we obtain

$$\begin{aligned} A_n &= -\frac{\displaystyle\int_0^1 s\,\phi_n\,d\overline{x}}{\cosh(\lambda_n)\displaystyle\int_0^1 \phi_n^2\,d\overline{x}} \\ &= -\frac{2}{\cosh(\lambda_n)}\int_0^1 s\,\phi_n\,d\overline{x} \end{aligned}$$

The remaining integral is evaluated using integration by parts:

$$\begin{aligned} \int_0^1 s\,\phi_n\,d\overline{x} &= -\frac{1}{\lambda_n^2}\int_0^1 s\phi_n''\,d\overline{x} \\ &= -\frac{1}{\lambda_n^2}\left[s\phi_n'\Big|_0^1 - s'\phi_n\Big|_0^1 + \int_0^1 s''\phi_n\,d\overline{x}\right] \\ &= \frac{1}{\lambda_n^2}\int_0^1 \phi_n\,d\overline{x} = \frac{\sin(\lambda_n)}{\lambda_n^3} \\ &= -\frac{(-1)^n}{\lambda_n^3} \end{aligned}$$

The complete solution to the problem is

$$\overline{T} = \frac{1}{2}\left(1-\overline{x}^2\right) + 2\sum_{n=1}^{\infty}\frac{(-1)^n\cos(\lambda_n\overline{x})\cosh(\lambda_n\overline{y})}{\lambda_n^3\cosh(\lambda_n)} \tag{4.77}$$

The solution to the problem is plotted in Fig. 4.10. The solution shows the expected symmetry – observe that the $\overline{x} = \overline{y}$ line is an adiabat. It would have been entirely possible and valid to pose the partial solution s in the y direction, as opposed to the $\overline{x}$ direction. This would have given exactly the same solution, expect that $\overline{x}$ and $\overline{y}$ would have been interchanged.

What you should have gathered from the previous two examples is that the goal of superposition methods is to split a problem into a number of sub–problems, each of which can be solved by SOV or simpler analytical methods. The SOV method requires that the problem have a homogeneous DE and only one inhomogeneous BC. In the second example we transformed the inhomogeneous DE/homogeneous BCs problem into a homogeneous DE problem with one inhomogeneous BC. This could be solved with SOV. Likewise, in the first example we developed a relatively simple partial solution which took care of the source term in the DE and the inhomogeneous BC at $\overline{y} = 0$. The remaining problem (for $\overline{T}_B$) had a homogeneous DE and only one inhomogeneous BC, which again could be solved with SOV.

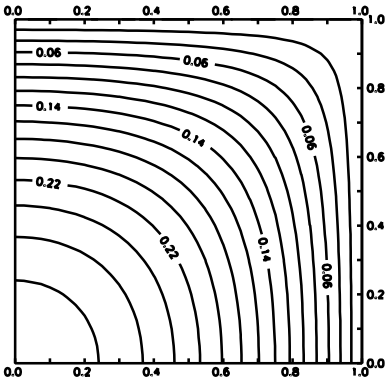


Figure 4.10: result of Eq. (4.77)

Application of SOV requires some imagination, but the procedure is fairly straightforward. When a problem is split into sub (or partial) problems, it is important to insure that the partial solutions, when superimposed (or added), satisfy both the DE and the BCs of the original problem.

4.3.3 Superposition example #3

The region for this example consists of a rectangle with width W and height H. A uniform heat flux of q'' is applied to the left face, convection occurs on the right face, and the bottom and top surfaces are maintained at T_1 and T_2, respectively.

As is usally the case, one of the boundary conditions can be made homogeneous by defining the right dimensionless temperature. The following definitions will be used;

$$\overline{T} = \frac{(T - T_\infty)k}{q''W}, \quad \overline{x} = \frac{x}{W}, \quad \overline{y} = \frac{y}{W}$$

The rationale for this choice is 1) the convection BC at the right face will become homogenous, and 2) the heat flux gets absorbed into the definition of $\overline{T}$, so a unit flux BC will occur at the left

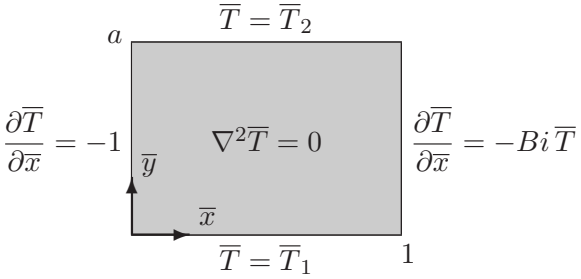


Figure 4.11: problem for Ex. 4.3.3

face. The dimensionless problem becomes

$$\frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} = 0 \tag{4.78}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_0 = -1 \tag{4.79}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_1 = -Bi\,\overline{T}(1,\overline{y}) \tag{4.80}$$

$$\overline{T}(\overline{x},0) = \overline{T}_1 \tag{4.81}$$

$$\overline{T}(\overline{x},a) = \overline{T}_2 \tag{4.82}$$

in which the dimensionless parameter $a = H/W$ is the aspect ratio of the region, $Bi = hW/k$ is the Biot number, $\overline{T}_1 = (T_1 - T_\infty)k/q''W$ and likewise for $\overline{T}_2$. It would have been possible to make the BCs on either the top or bottom homogeneous by appropriate definition of $\overline{T}$ – yet this would have resulted in an inhomogeneous convection condition on the right side. Inhomogeneous convection BCs should be avoided when possible.

A set of partial solutions must now be devised in which 1) each have completely homogeneous BCs in at least one direction (so that SOV can be applied to obtain the solution), and 2) satistfy BCs that, when added together, represent the complete set of BCs for the original problem.

As before, let $\overline{T} = \overline{T}_A + \overline{T}_B$, where $\overline{T}_A$ and $\overline{T}_B$ satisfy the following boundary value problems:

$$\frac{\partial^2 \overline{T}_A}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}_A}{\partial \overline{y}^2} = 0 \qquad \frac{\partial^2 \overline{T}_B}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}_B}{\partial \overline{y}^2} = 0$$

$$\left.\frac{\partial \overline{T}_A}{\partial \overline{x}}\right|_0 = 0 \qquad \left.\frac{\partial \overline{T}_B}{\partial \overline{x}}\right|_0 = -1$$

$$\left.\frac{\partial \overline{T}_A}{\partial \overline{x}}\right|_1 = -Bi\,\overline{T}_A(1,\overline{y}) \qquad \left.\frac{\partial \overline{T}_B}{\partial \overline{x}}\right|_1 = -Bi\,\overline{T}_B(1,\overline{y})$$

$$\overline{T}_A(\overline{x},0) = \overline{T}_1 \qquad \overline{T}_B(\overline{x},0) = 0$$

$$\overline{T}_A(\overline{x},a) = \overline{T}_2 \qquad \overline{T}_B(\overline{x},a) = 0$$

You should prove to yourself that addition of the boundary conditions for A and B recovers the original problem. Each problem is completely homogeneous in one direction ($\overline{x}$ and $\overline{y}$, respectively), and SOV can be applied to each problem in turn.

Minimal detail will be given to the solution procedure for the partial solutions – because the procedure should be familiar by now. Start with the problem for A. The homogeneous direction is $\overline{x}$, and the eigenfunctions and eigencondition must have zero $\overline{x}$–gradient at $\overline{x} = 0$ and convection at $\overline{x} = 1$. This leads to

$$\phi_n(\overline{x}) = \cos(\lambda_n \overline{x}) \tag{4.83}$$

$$\lambda_n \sin(\lambda_n) = Bi\, \cos(\lambda_n), \quad n = 1,\ 2, \ldots \tag{4.84}$$

Again, the general solution will be in the form

$$\overline{T}_A = \sum_{n=1}^{\infty} \left(A_n \cosh(\lambda_n \overline{y}) + B_n \sinh(\lambda_n \overline{y})\right) \phi_n(\overline{x})$$

At $\overline{y} = 0$ the condition is

$$\overline{T}_1 = \sum_{n=1}^{\infty} A_n \phi_n(\overline{x})$$

Multiplying through by ϕ_m, integrating over $\overline{x}$, and borrowing some previous results for the integrals will result in

$$A_n = -\frac{2(-1)^n \overline{T}_1 Bi \left(\lambda_n^2 + Bi^2\right)^{1/2}}{\lambda_n \left[\lambda_n^2 + Bi(1 + Bi)\right]}$$

At $\overline{y} = a$ the second inhomogeneous BC gives

$$\overline{T}_2 = \sum_{n=1}^{\infty} \left(A_n \cosh(\lambda_n a) + B_n \sinh(\lambda_n a)\right) \phi_n(\overline{x})$$

Again, the orthogonality of ϕ_n is used to solve for B_n. The result is

$$\begin{aligned} B_n &= \overline{T}_2 \frac{\displaystyle\int_0^1 \phi_n \, d\overline{x}}{\sinh(\lambda_n a) \displaystyle\int_0^1 \phi_n^2 \, d\overline{x}} - \frac{A_n}{\tanh(\lambda_n a)} \\ &= C_n \left(\overline{T}_2 - \overline{T}_1 \cosh(\lambda_n a)\right) \end{aligned}$$

where the coefficient C_n is defined

$$C_n = -\frac{2(-1)^n Bi \left(\lambda_n^2 + Bi^2\right)^{1/2}}{\lambda_n \sinh(\lambda_n a) \left[\lambda_n^2 + Bi(1 + Bi)\right]} \tag{4.85}$$

By using some identities for the hyperbolic functions, the complete solution for $\overline{T}_A$ can be made to appear as

$$\overline{T}_A = \sum_{n=1}^{\infty} C_n \Big[\overline{T}_1 \sinh[\lambda(a - \overline{y})] + \overline{T}_2 \sinh(\lambda_n \overline{y})\Big] \tag{4.86}$$

This form could have been anticipated from the start – note how the boundary condition in the $\overline{y}$ direction are satisfied by the $\sinh(0) = 0$ property.

The homogeneous direction for the B problem is $\overline{y}$, and the eigenfunctions will have zero temperature at $\overline{y} = 0$ and a. The corresponding functions are

$$\psi_n(\overline{y}) = \sin(\beta_n \overline{y}) \tag{4.87}$$

$$\beta_n = \frac{n\pi}{a} \tag{4.88}$$

where ψ_n and β_n denote the eigenfunction and eigenvalue, respectively. The change in symbols is to avoid confusion with the functions for $\overline{T}_A$. Once again, the solution will be in the form

$$\overline{T}_B = \sum_{n=1}^{\infty} \left(A_n \cosh(\beta_n \overline{x}) + B_n \sinh(\beta_n \overline{x}) \right) \psi_n(\overline{y})$$

By applying first the homogeneous BC at $\overline{x} = 1$, the above solution can reduce to

$$\overline{T}_B = \sum_{n=1}^{\infty} A_n \left(\cosh[\beta_n(1-\overline{x})] + \frac{Bi}{\beta_n} \sinh[\beta_n(1-\overline{x})] \right) \tag{4.89}$$

and at $\overline{x} = 0$ the inhomogeneous, unit–gradient BC gives

$$-1 = -\sum_{n=1}^{\infty} A_n \left(\beta_n \sinh(\beta_n) + Bi \cosh(\beta_n) \right) \sin(\beta_n \overline{y})$$

Orthogonality of the sin functions gives

$$\begin{aligned} A_n &= \frac{2}{a} \cdot \frac{(1-\cos(\beta_n))}{\beta_n \left(\beta_n \sinh(\beta_n) + Bi \cosh(\beta_n) \right)} \\ &= \frac{2a(1-(-1)^n)}{\pi n \left(\pi n \sinh(n\pi/a) + Bi\, a \cosh(n\pi/a) \right)} \end{aligned} \tag{4.90}$$

Note that the A_n are zero for even n. We usually find this sort of cancellation whenever we have failed to fully exploit the symmetry in a problem – in this case the adiabatic plane that occurs at $\overline{y} = a/2$.

The solution for $\overline{T}_B$ is given by Eq. (4.89), with A_n given by Eq. (4.90). And the complete solution to the problem is given by $\overline{T} = \overline{T}_A + \overline{T}_B$.

Results for the solution are shown in Fig. 4.12. The parameters used in the plot correspond to $\overline{T}_1 = 1.5$, $\overline{T}_2 = 0.5$, $Bi = 10$, and $a = 1.5$. Shown are the individual partial solutions ($\overline{T}_A$ and $\overline{T}_B$) along with the superimposed solution. This plot gives a good graphical view of the workings of superposition. The normal gradient at $\overline{x} = 0$ for A is identically zero, and it is -1 for B. Adding the two solutions therefore gives the desired normal gradient at the wall, i.e., the A solution does not contribute anything to the gradient at $\overline{x} = 0$. Likewise, the B solution has zero temperature at $\overline{y} = 0$ and a, yet the A solution has temperatures of 1.5 and 0.5 here. Finally, each solution obeys the homogeneous convection BC at $\overline{x} = 1$ – so the sum of the solutions obeys the same BC.

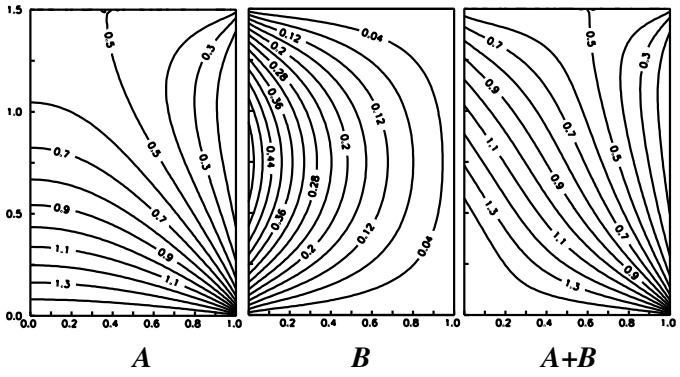


Figure 4.12: superposition solution #3

4.4 Two dimensional problems in cylindrical coordinates

4.4.1 2–D heat transfer in a circular fin

A good place to begin a discussion on steady, 2–D conduction in cylindrical coordinate systems is to examine heat transfer in a circular pin fin. The 1–D analysis for fin heat transfer, developed in Ch. 1, was based on the assumption of $Bi_R = hR/k \ll 1$ (where R is the fin radius). When this condition is not met the temperature distribution in the fin will be a function of both axial position z and radial position r, and the prediction of the temperature field (and heat transfer rate) will require solution of the 2–D steady conduction equation in cylindrical coordinates.

Assume that the fin, of length L, is mounted to a base that is maintained at a uniform temperature of T_B, and take the tip of the fin to be adiabatic. The sides of the fin are cooled by convection.

The nondimensional temperature is obviously $\overline{T} = (T - T_\infty)/(T_B - T_\infty)$. Let L be the characteristic length, so that $\overline{z} = z/L$ and $\overline{r} = r/L$. Again, it is good practice to non–dimensionalize all spacial variables by the same characteristic length. The boundary value problem is

$$\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\overline{r}\frac{\partial \overline{T}}{\partial \overline{r}} + \frac{\partial^2 \overline{T}}{\partial \overline{z}^2} = 0 \tag{4.91}$$

$$\overline{T}(\overline{r}, 0) = 1 \tag{4.92}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{z}}\right|_1 = 0 \tag{4.93}$$

$$\overline{T}(0, \overline{z}) \text{ is finite} \tag{4.94}$$

$$-\left.\frac{\partial \overline{T}}{\partial \overline{r}}\right|_a = Bi\,\overline{T}(a, \overline{z}) \tag{4.95}$$

in which $Bi = hL/k$ and $a = R/L$.

The problem is completely homogeneous in the $\overline{r}$ direction, so we can proceed with the SOV method. By separating $\overline{T} = u(\overline{r}) \cdot v(\overline{z})$ the following characteristic ODEs are obtained;

$$(\overline{r}u')' \pm \lambda^2 \overline{r}u = 0$$
$$v'' \mp \lambda^2 v = 0$$

The choice of $+\lambda^2$ in the first ODE will give a solution involving the ordinary Bessel functions of order zero, and $-\lambda^2$ will return the modified Bessel functions of order zero. The homogeneous direction to the problem is $\overline{r}$, and eigenfunction are therefore required in this direction. The ordinary Bessel functions (which oscillate about zero) can serve as eigenfunctions, whereas the modified Bessel functions cannot. Alternatively, one could seek to obtain the Sturm–Liouville problem in the $\overline{r}$ direction – which would require the choice of $+\lambda^2$.

The solution for u is therefore

$$u = AJ_0(\lambda\overline{r}) + BY_0(\lambda\overline{r})$$

The BC at $\overline{r} = 0$ eliminates B because the Y function is singular at the origin. The eigenfunctions to the problem become

$$\phi_n(\overline{r}) = J_0(\lambda_n\overline{r}) \tag{4.96}$$

and the convection BC gives the eigencondition:

$$\phi_n'(a) = -Bi\,\phi_n(a)$$

By use of the formulas for the derivative of J_0, the eigencondition becomes:

$$\lambda_n J_1(\lambda_n a) = Bi\,J_0(\lambda_n a), \quad n = 1,\ 2, \ldots \tag{4.97}$$

The solution of the characteristic DE in the $\overline{z}$ direction will be in terms of the hyperbolic functions. The general, 2–D form of the solution will then be

$$\overline{T} = \sum_{n=1}^{\infty} \left(A_n \cosh(\lambda_n\overline{z}) + B_n \sinh(\lambda_n\overline{z})\right) \phi_n(\overline{r}) \tag{4.98}$$

The derivative with respect to $\overline{z}$ must vanish at $\overline{z} = 1$, so the general solution can be put in the form

$$T = \sum_{n=1}^{\infty} A_n \cosh[\lambda_n(1-\overline{z})]\,\phi_n(\overline{r}) \tag{4.99}$$

Recognize again that $\cosh(\lambda_n\overline{z})$ and $\cosh[\lambda_n(1-\overline{z})]$ are both valid solutions to the DE. The second form provides a simplified means for satisfying the homogeneous BC at $\overline{z} = 1$. At $\overline{z} = 0$ the remaining inhomogeneous BC gives

$$1 = \sum_{n=1}^{\infty} A_n \cosh(\lambda_n)\,\phi_n(\overline{r}) \tag{4.100}$$

The orthogonality of ϕ_n is now used to obtain the A_n's. Recall that the orthogonality relationship for Bessel functions has a weighting function of $w(\overline{r}) = \overline{r}$. Each side of Eq. (4.100) is multiplied by $\overline{r}\phi_m(\overline{r})$ and integrated over the $\overline{r}$ domain – which extends from 0 to a. The integration kills every term in the series except the one for which $n = m$, and the A_n expansion coefficients are

$$A_n = \frac{\displaystyle\int_0^a J_0(\lambda_n \overline{r})\overline{r}\, d\overline{r}}{\cosh(\lambda_n)\displaystyle\int_0^a J_0^2(\lambda_n \overline{r})\overline{r}\, d\overline{r}} = \frac{2J_1(\lambda_n a)}{a\lambda_n \cosh(\lambda_n)\left(J_0^2(\lambda_n a) + J_1^2(\lambda_n a)\right)}$$

By substitution of Eq. (4.97) in the above, the formula reduces to

$$A_n = \frac{2Bi}{a\cosh(\lambda_n)J_0(\lambda_n a)\,(\lambda_n^2 + Bi^2)} \tag{4.101}$$

and the complete solution is

$$\overline{T} = \frac{2Bi}{a}\sum_{n=1}^{\infty}\frac{\cosh[\lambda_n(1-\overline{z})]J_0(\lambda_n\overline{r})}{\cosh(\lambda_n)J_0(\lambda_n a)\,(\lambda_n^2 + Bi^2)} \tag{4.102}$$

Predicted isotherms from Eq. (4.102) are given in Fig. 4.13. The results correspond to an aspect ratio a of 0.75 (a stubby fin) and a Biot number of 20. This would not represent a very effective fin – but it does show that the solution is valid. In particular, notice how the unit temperature BC at $\overline{z} = 0$ and the adiabatic BC at $\overline{z} = 1$ are satisfied.

The heat transfer from the fin would be obtained from the net conduction at the base:

$$q = \int_{A_B} q_z''(r,0)\, dA = -2\pi k \int_0^R \left.\frac{\partial T}{\partial z}\right|_0 r\, dr$$

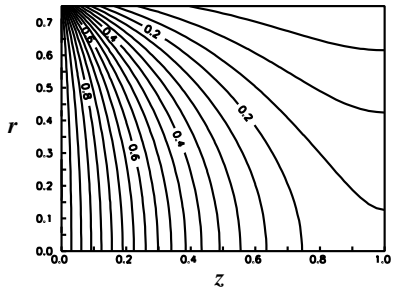


Figure 4.13: isotherms in a stubby fin

or, using the dimensionless quantities:

$$\begin{aligned} q &= -2\pi Lk(T_B - T_\infty) \int_0^a \left.\frac{\partial \overline{T}}{\partial \overline{z}}\right|_0 \overline{r}\, d\overline{r} \\ &= 2\pi Lk(T_B - T_\infty) \sum_{n=1}^{\infty} \lambda_n A_n \sinh(\lambda_n) \int_0^a J_0(\lambda_n \overline{r})\, \overline{r}\, d\overline{r} \\ &= 2\pi\, L\, a\, k(T_B - T_\infty) \sum_{n=1}^{\infty} A_n \sinh(\lambda_n) J_1(\lambda_n a) \\ &= 2\pi\, R\, k\, Bi(T_B - T_\infty) \sum_{n=1}^{\infty} \frac{A_n}{\lambda_n} \sinh(\lambda_n) J_0(\lambda_n a) \\ &= 2\pi\, R\, L\, h(T_B - T_\infty) \sum_{n=1}^{\infty} \frac{A_n}{\lambda_n} \sinh(\lambda_n) J_0(\lambda_n a) \end{aligned}$$

The maximum heat transfer from the fin is $2\pi RLh(T_B - T_\infty)$. Using this in the above along with Eq. (4.101), the fin efficiency becomes:

$$\begin{aligned} \eta &= \sum_{n=1}^{\infty} \frac{A_n}{\lambda_n} \sinh(\lambda_n) J_0(\lambda_n a) \\ &= \frac{2Bi}{a} \sum_{n=1}^{\infty} \frac{\tanh(\lambda_n)}{\lambda_n\, (\lambda_n^2 + Bi^2)} \\ &= 2Bi\, a \sum_{n=1}^{\infty} \frac{\tanh(\lambda_n)}{\lambda_n\, [(\lambda_n a)^2 + (Bi\, a)^2]} \end{aligned} \tag{4.103}$$

This is starting to look somewhat like the 1–D formula for the uniform cross section fin, which is

$$\eta_{1-D} = \frac{\tanh(N)}{N}, \quad N = \sqrt{\frac{2hL^2}{kR}} \tag{4.104}$$

It is now shown that the exact solution, in Eq. (4.103), reduces to the 1–D approximation of Eq. (4.104) for $Bi\, a = hr/k \ll 1$. By multiplying the eigencondition (Eq. (4.97)) by a, one obtains

$$(\lambda_n a) J_1(\lambda_n a) = Bi_R\, J_0(\lambda_n a) \tag{4.105}$$

where $Bi_R = Bi\, a$. This provides an equation for the roots $\lambda_n a$ in terms of Bi_R. Now, for $Bi_R \ll 1$, we would expect that only the first term in the series in Eq. (4.103) would be important – because the solution has to reduce to the simple analytical expression in Eq. (4.104). From the Bessel function relations in Ch. 2, it can be shown, for small x, that $J_0(x) \approx 1$ and $J_1(x) \approx x/2$.

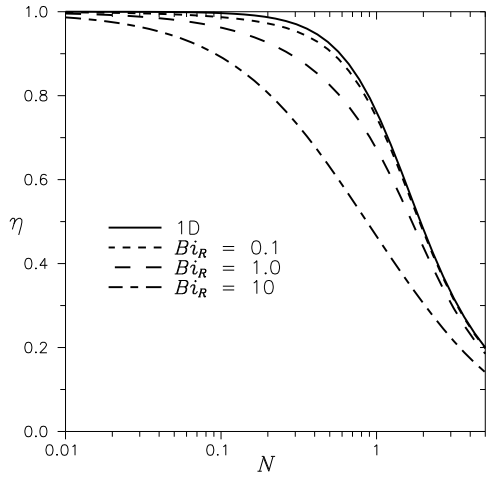


Figure 4.14: efficiency η for a circular pin fin

Therefore, for $Bi_R \to 0$, the above equation would predict that $\lambda_1 a$ would become a small number. Specifically,

$$\lambda_1 a \approx \sqrt{2\,Bi_R}, \quad Bi_R \ll 1$$

or

$$\lambda_1 \approx \sqrt{\frac{2Bi_R}{a^2}} = \sqrt{\frac{2hL^2}{kR}} = N$$

So – in the limit of $Bi_R \ll 1$ the first eigenvalue λ_1 goes to the fin parameter N. By using this in Eq. (4.103) and retaining only the first term in the series, the 1–D approximate solution for η is recovered.

Presented in Fig. 4.14 is a plot of the fin efficiency η, calculated from the 2–D model, vs. the fin parameter N with Bi_R as a parameter. The aspect ratio a is obtained from N and Bi_R from $a = \sqrt{2Bi_R}/N$. As expected, once Bi_R becomes less than unity the 2–D, exact results correspond closely to the 1–D approximate result. Perhaps surprisingly, the poorest agreement between the 1–D and the 2–D models occurs for $N \sim 0.1 - 1$, i.e., for a stubby (or short) fin. This behavior is an artifact of the mathematical model of the fin – and you should be able to explain why such behavior occurs.

4.4.2 The long, annular cylinder: problems in r and ϕ

Two dimensional conduction problems in spherical coordinates typically occur for the independent variable pairs (r, z) and (r, ϕ). An example of the first type was encountered in the previous section. The latter case will occur whenever the surface of the cylinder is subjected to an uneven heating/cooling situation.

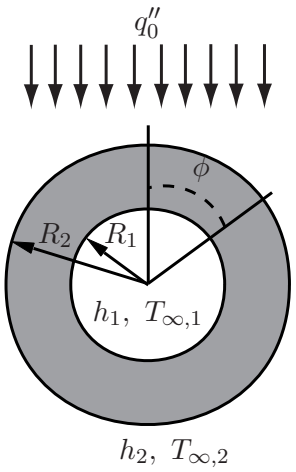


Figure 4.15: pipe configuration

To pose an example, consider an annular pipe, having inner and outer radii of R_1 and R_2, that is carrying a fluid at temperature $T_{\infty,1}$. Convection occurs between the inner surface and the fluid with a heat transfer coefficient h_1, and likewise the outer surface is exposed to a convection environment characterized by h_2 and $T_{\infty,2}$. In addition , the outside of the pipe is exposed to a collimated source of thermal radiation, which has a flux of q_0''. The problem is schematically illustrated in Fig. 4.15.

Taking into account the angle between surface normal and the incident radiation, the normal component of the absorbed heat flux at the exterior pipe surface is

$$q_r'' = \begin{cases} \alpha q_0'' \cos(\phi), & -\dfrac{\pi}{2} \leq \phi \leq \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} < \phi < \dfrac{3\pi}{2} \end{cases}$$

where α is the surface absorptivity of the pipe. The objective of the problem is to predict the temperature distribution in the pipe and the heat flux at the inner wall.

Assuming that the temperature varies only in the r and ϕ directions, the dimensional problem

is

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}=0$$
$$k\left.\frac{\partial T}{\partial r}\right|_{R_1}=h_1(T(R_1,\phi)-T_{\infty,1})$$
$$-k\left.\frac{\partial T}{\partial r}\right|_{R_2}=-q_r''(\phi)+h_2(T(R_2,\phi)-T_{\infty,2})$$
$$T(r,\phi)=T(r,\phi+2\pi)$$

The BC on ϕ simply states a 'continuation' principle – in that the temperature at ϕ has to be the same as the temperature at $\phi+2\pi$ (i.e., once around the pipe).

Let the nondimensional radial position and temperature be defined

$$\overline{r}=\frac{r}{R_2},\quad \overline{T}=\frac{(T-T_{\infty,1})k}{\alpha q_0'' R_2}$$

The dimensionless problem becomes

$$\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\overline{r}\frac{\partial \overline{T}}{\partial \overline{r}}+\frac{1}{\overline{r}^2}\frac{\partial^2 T}{\partial \phi^2}=0 \tag{4.106}$$
$$\left.\frac{\partial \overline{T}}{\partial \overline{r}}\right|_{a}=\frac{Bi_1}{a}\overline{T}(a,\phi) \tag{4.107}$$
$$\left.\frac{\partial \overline{T}}{\partial \overline{r}}\right|_{1}=f(\phi)-Bi_2\left(\overline{T}(1,\phi)-\overline{T}_{\infty,2}\right) \tag{4.108}$$
$$\overline{T}(\overline{r},\phi)=\overline{T}(\overline{r},\phi+2\pi) \tag{4.109}$$

where $a=R_1/R_2$, $Bi_1=h_1R_1/k$, $Bi_2=h_2R_2/k$, $\overline{T}_{\infty,2}=(T_{\infty,2}-T_{\infty,1})k/\alpha q_0'' R_2$, and

$$f(\phi)=\begin{cases}\cos(\phi), & -\dfrac{\pi}{2}\le\phi\le\dfrac{\pi}{2}\\ 0, & \dfrac{\pi}{2}<\phi<\dfrac{3\pi}{2}\end{cases} \tag{4.110}$$

The dimensionless problem has a homogeneous DE and homogeneous BCs in the ϕ direction, and an inhomogeneous convection BC occurs at $\overline{r}=1$. The $1/a$ in Eq. (4.107) comes from the fact that we are defining Bi_1 with respect to R_1 instead of R_2.

Separation of variables can be applied directly to this problem. Let $\overline{T}=u(\overline{r})v(\phi)$, which leads to

$$r(ru')'\pm\lambda^2 u=0$$
$$v''-(\pm\lambda^2)v=0$$

The homogeneous direction of the problem is ϕ and eigenfunctions are required in the ϕ direction. We then choose $-\lambda^2$ in the above which will give a solution for v in terms of the trigonometric functions. Specifically,

$$v = A\cos(\lambda\phi) + B\sin(\lambda\phi)$$

Because of the symmetry of the problem, the temperature field must have $\overline{T}(\overline{r},\phi) = \overline{T}(\overline{r},-\phi)$ – i.e, the temperature is even in ϕ. The constant B can therefore be set to zero. From Eq. (4.109) the eigencondition of the problem is, simply,

$$\lambda_n = n, \quad n = 0,\ 1,\ 2, \ldots$$

This will give the desired periodic behavior in ϕ.

The ODE for u becomes

$$\overline{r}^2 u'' + \overline{r}u' - n^2 u = 0$$

Two cases need to be examined – depending on whether or not n is zero. If $n \neq 0$ the DE takes the form of an *equidimensional* equation, in which each term has the same net dimension in $\overline{r}$. The solution to such equations is typically obtained by setting $u = C r^{\beta}$, where β is a constant. Replacing this into the DE, the solution to the above becomes

$$u = A_n \overline{r}^n + B_n \overline{r}^{-n}, \quad n = 1,\ 2, \ldots$$

For the case of $n = 0$ the DE for u will appear as

$$(\overline{r}u')' = 0$$

Integrating twice over $\overline{r}$ gives

$$u = A_0 + B_0 \ln(\overline{r}), \quad n = 0$$

which is recognized simply as the 1–D radial temperature profile in cylindrical coordinates.

At this point the general solution to the problem appears as

$$\overline{T} = A_0 + B_0 \ln(\overline{r}) + \sum_{n=1}^{\infty} \left(A_n \overline{r}^n + B_n \overline{r}^{-n}\right) \cos(n\phi) \tag{4.111}$$

The BC at $\overline{r} = a$ is now used to eliminate the B coefficients. Since this BC is homogeneous it can be applied to each individual term in the series. For $n > 0$ the result is

$$n\left(A_n a^{n-1} - B_n a^{-n-1}\right) = \frac{Bi_1}{a}\left(A_n a^n + B_n a^{-n}\right)$$

which gives

$$B_n = A_n a^{2n} \frac{n - Bi_1}{n + Bi_1}$$

and for $n = 0$:

$$\frac{B_0}{a} = \frac{Bi_1}{a}\left(A_0 + B_0 \ln(a)\right)$$

or

$$B_0 = A_0 \frac{Bi_1}{1 - Bi_1 \ln(a)}$$

To make the notation more compact, define the function $g_n(\overline{r})$ as

$$g_n(\overline{r}) = \overline{r}^n + a^{2n}\frac{n - Bi_1}{n + Bi_1}\,\overline{r}^{-n} \tag{4.112}$$

With this convention, the general solution can now be written as

$$\overline{T} = A_0\left(1 + \frac{Bi_1 \ln(\overline{r})}{1 - Bi_1 \ln(a)}\right) + \sum_{n=1}^{\infty} A_n g_n(\overline{r}) \cos(n\phi) \tag{4.113}$$

Application of the remaining inhomogeneous BC at $\overline{r} = 1$ results in

$$A_0 \frac{Bi_1}{1 - Bi_1 \ln(a)} + \sum_{n=1}^{\infty} A_n g_n'(1) \cos(n\phi)$$
$$= f(\phi) - Bi_2 A_0 - Bi_2 \sum_{n=1}^{\infty} A_n g_n(1) \cos(n\phi) + Bi_2 \overline{T}_{\infty,2} \tag{4.114}$$

where

$$g_n'(1) = n\left(1 - a^{2n}\frac{n - Bi_1}{n + Bi_1}\right) \tag{4.115}$$

Multiply Eq. (4.114) by $\cos(m\phi)$ and integrate over ϕ from 0 to π. The orthogonality relation is

$$\int_0^{\pi} \cos(m\phi)\cos(n\phi)\,d\phi = \begin{cases} 0, & n \neq m \\ \dfrac{\pi}{2}, & n = m, \quad m > 0 \\ \pi, & n = m = 0 \end{cases}$$

Three distinct cases are obtained from Eq. (4.114), depending on whether $m = 0$, 1, or > 1. For $m = 0$ we find that

$$\pi A_0 \frac{Bi_1}{1 - Bi_1 \ln(a)} = \int_0^{\pi/2} \cos(\phi)\,d\phi - \pi Bi_2 A_0 + \pi Bi_2 \overline{T}_{\infty,2}$$

which simplifies to

$$A_0 = \frac{(1 + \pi Bi_2 \overline{T}_{\infty,2})(1 - Bi_1 \ln(a))}{\pi\left[Bi_1 + Bi_2(1 - Bi_1 \ln(a))\right]} \tag{4.116}$$

The $m = 1$ case gives

$$\frac{\pi}{2} A_1 g_1'(1) = \int_0^{\pi/2} \cos^2(\phi)\, d\phi - \frac{\pi}{2} Bi_2 A_1 g_1(1)$$

The integral has the value $\pi/4$, so the formula for A_1 is

$$A_1 = \frac{1}{2\left(g_1'(1) + Bi_2 g_1(1)\right)} \tag{4.117}$$

Finally, for $m > 1$ we have

$$\frac{\pi}{2} A_n g_n'(1) = \int_0^{\pi/2} \cos(\phi)\cos(n\phi)\, d\phi - \frac{\pi}{2} Bi_2 A_n g_n(1)$$

The value of the integral is now

$$\int_0^{\pi/2} \cos(\phi)\cos(n\phi)\, d\phi = \frac{\cos(n\pi/2)}{1 - n^2}$$

and the corresponding formula is

$$A_n = \frac{2\cos(n\pi/2)}{\pi\left(1 - n^2\right)\left(g_n'(1) + Bi_2 g_n(1)\right)}, \qquad n > 1 \tag{4.118}$$

Observe that the A_n in the above equation will be zero for odd n. Taking this into consideration, the complete solution for the temperature is

$$\begin{aligned} \overline{T}(\overline{r}, \phi) = {} & \frac{1 + \pi Bi_2 \overline{T}_{\infty,2}}{\pi\left[Bi_1 + Bi_2(1 - Bi_1 \ln(a))\right]} \left(1 + Bi_1 \ln\left(\frac{\overline{r}}{a}\right)\right) \\ & + \frac{1}{2\left(g_1'(1) + Bi_2 g_1(1)\right)} g_1(\overline{r}) \cos(\phi) \\ & + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n g_{2n}(\overline{r}) \cos(2n\phi)}{\left(1 - 4n^2\right)\left(g_{2n}'(1) + Bi_2 g_{2n}(1)\right)} \end{aligned} \tag{4.119}$$

Some typical results are in Fig. 4.16, in which $a = 0.5$ and $Bi_1 = 5$, $Bi_2 = 0$ (left plot) and $Bi_1 = 1000$, $Bi_2 = 5$ (right plot). The external ambient temperature is $\overline{T}_{\infty,2} = 0$ for both plots. The left plot (for which external convection is absent) shows how the exterior surface becomes adiabatic for $\phi > \pi/2$. The second plot illustrates a case that would be typical of a liquid flow in the pipe (high Bi) and free convection to air on the outside.

The total rate of heat transfer into the fluid would be (per unit length)

$$\begin{aligned} q' &= 2R_2 k \int_0^{\pi} \left.\frac{\partial T}{\partial r}\right|_{R_2} d\phi \\ &= 2R_2 \alpha q_0'' \int_0^{\pi} \left.\frac{\partial \overline{T}}{\partial \overline{r}}\right|_1 d\phi \end{aligned}$$

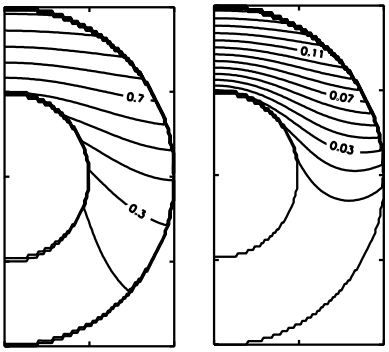


Figure 4.16: isotherms in the pipe: $Bi_1 = 5$, $Bi_2 = 0$ (left); $Bi_1 = 1000$, $Bi_2 = 5$ (right)

By replacing the solution for T into the above and integrating, it turns out that the only term in the series that contributes anything to the total heat transfer is the one associated with $\ln(r)$. The final result is

$$q' = \frac{2\alpha q_0'' R_2 \left(1 + \pi Bi_2 \overline{T}_{\infty,2}\right) Bi_1}{\left[Bi_1 + Bi_2 \left(1 - Bi_1 \ln(a)\right)\right]} \tag{4.120}$$

What may be remarkable to you is the fact that the net heat transfer does not depend, in any way, on the SOV–derived expansion coefficients for the temperature distribution. Indeed, the above formula could have been derived by integration of the governing DE (Eq. (4.106)) over ϕ and solution of the subsequent 1–D boundary value problem. This is left as an exercise.

By setting $Bi_2 = 0$ (i.e., no exterior convection), the result shows (as it must) that $q' = 2R_2\alpha q_0''$, i.e., the net heat transfer to the fluid is equal to the incident flux times the absorptivity times the projected area of the pipe. This makes sense since all of the absorbed radiant energy has to go to the fluid – none can escape back to the environment. Conversely, by setting $Bi_1 = 0$ (insulated inner surface) the result $q' = 0$ is obtained – because none of the absorbed heat can reach the fluid.

4.4.3 Math digression: 2–D in r and ϕ solutions

The general solution for the steady–state temperature field in r and ϕ cylindrical coordinates can be deduced by application of complex variables, as opposed to the SOV procedure used above. To illustrate, consider the cartesian conduction equation in two dimensions:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

It is readily shown that a solution to the above is obtained from

$$T = (x + iy)^n \tag{4.121}$$

where n is a constant. In application of the complex variables it would be understood that we are ultimately interested in the real part of the solution – but for now it is useful to retain the complex formulation. Differentiation of the solution twice with respect to x results in

$$\frac{\partial^2 T}{\partial x^2} = n(n-1)(x+iy)^{n-2}$$

Likewise, differentiation of Eq. (4.121) twice with respect to y gives

$$\frac{\partial^2 T}{\partial y^2} = i \cdot in(n-1)(x+iy)^{n-2} = -n(n-1)(x+iy)^{n-2}$$

The conduction equation is therefore satisfied. Now, from complex variables, the variable $x+iy$ can be represented on the complex plane by

$$x + iy = re^{i\phi} \tag{4.122}$$

where

$$r = \left(x^2+y^2\right)^{1/2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{4.123}$$

The variables r and ϕ are the cylindrical coordinate equivalents to x and y. Realize that a solution to Laplace's equation (i.e., the steady conduction equation) must be *invariant* with respect to the coordinate system. That is, if the solution is valid in cartesian coordinates, it must also be valid in cylindrical coordinates. Consequently, the solution in cylindrical coordinates becomes

$$(x+iy)^n = r^n e^{in\phi} = r^n[\cos(n\phi) + i\sin(n\phi)] \tag{4.124}$$

Using the continuation principle ($T(\phi) = T(\phi+2\pi)$) fixes the constant n as the integers $0, \pm 1, \pm 2, \ldots$. Finally, an additional solution is obtained from

$$\ln\left[(x^2+y^2)^{1/2}\right] = \frac{1}{2}\ln(x^2+y^2) = \ln(r) \tag{4.125}$$

By differentiation of this solution twice with respect to x and with respect to y, it will be seen that the cartesian 2–D conduction equation is satisfied. The general solution to the problem in radial coordinates would then be

$$T = A_0 + B_0\ln(r) + \sum_{n=1}\Big[\cos(n\phi)\left(A_n r^n + A_{-n}r^{-n}\right) + i\sin(n\phi)\left(B_n r^n - B_{-n}r^{-n}\right)\Big]$$

In general, the expansion coefficients B_n would be imaginary in the above equation – because the temperature field must be real–valued. One could simply eliminate i from the above and fix B_n as real.

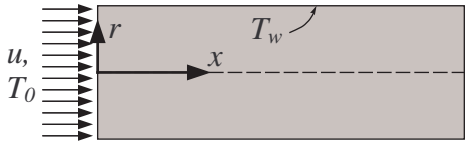


Figure 4.17: Convective–diffusion pipe flow situation

4.5 Convection–Diffusion Problems

The methods developed to examine steady–state heat transfer in a 2–D region can also be applied to convection–diffusion heat transfer in situations in which the velocity of the convective flow is fixed and known. Such a problem is illustrated in Fig. 4.17. Say flow enters a circular pipe with a uniform and constant velocity of u and an inlet temperature of T_0. The walls of the pipe are maintained at a uniform temperature of T_w. We want to predict the temperature distribution of the fluid in the pipe and the rate of heat transfer to/from the wall.

This is not a fluid mechanics problem: we know the velocity distribution throughout the pipe. This particular example, in which the velocity in the pipe is assumed uniform and constant, would be referred to as a *plug–flow*. A more realistic model, for laminar flow conditions, would be to use a parabolic velocity profile, yet the plug flow model will offer mathematical simplicity and would be more representative of certain turbulent flow conditions (in which the average velocity profile is mostly uniform over the cross sectional area of the pipe).

The steady form of the energy equation (in dimensional coordinates) will be

$$\frac{u}{\alpha}\frac{\partial T}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial x^2} \tag{4.126}$$

with $\alpha = k/\rho c_p$ being the thermal diffusivity of the fluid. The term on the left, as you probably recognize, accounts for the axial convection of enthalpy in the flow. There is no corresponding r–directed term because there is no r component of velocity.

Take the radius of the pipe, R, to be the characteristic length. The problem can be made dimensionless by defining the variables as

$$T \to \frac{T - T_w}{T_0 - T_w}, \quad r \to \frac{r}{R}, \quad x \to \frac{x}{R}$$

so that the DE now appears as

$$Pe\,\frac{\partial T}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial x^2} \tag{4.127}$$

in which $Pe = uR/\alpha$ is the *Peclet* number of the flow. The Peclet number is analogous to the Reynolds number; it is a ratio of the characteristic rates of axial convective and radial diffusive

heat transfer in the pipe. Often the Peclet number is defined using diameter D instead of radius R; we'll use the radius definition to keep things more simple.

In pipe flow problems, the axial diffusion term (the second term on the right hand side) is often neglected – because the gradients in the axial direction (for developed flow conditions) will be relatively small compared to the radial gradients. If we removed this term, we would simply get a DE in the same form as the 1–D and transient problems in the previous chapter (i.e., a parabolic DE, with x taking the place of the time variable). However, for this developing flow problem we cannot, in general, neglect the axial diffusion term.

Boundary conditions to the dimensionless problem are

$$T(0, x) \text{ is finite} \tag{4.128}$$

$$T(1, x) = 0 \tag{4.129}$$

$$T(r, 0) = 1 \tag{4.130}$$

$$T(r, x \to \infty) = 0 \tag{4.131}$$

Note that the last boundary condition simply states that the fluid temperature will go to the wall temperature after sufficient distance in the pipe.

This problem can be solved with SOV methods, the solution of which will be outlined here. The problem has a homogeneous direction (r), and we anticipate that the solution will be in the form

$$T(r, x) = \sum_{n=1}^{\infty} A_n \phi_n(r)\, v_n(x) \tag{4.132}$$

Characteristic ODEs for $\phi_n(r)$ and $v_n(x)$ are obtained by substituting $\phi_n(r) \cdot v_n(x)$ into the PDE, Eq. (4.127);

$$\frac{1}{r}(r\phi_n')' = -\lambda_n^2 \phi_n \tag{4.133}$$

$$Pe\, v_n' - v_n'' = -\lambda_n^2 v_n \tag{4.134}$$

The eigenfunction ODE is in the same form as previous cylindrical problems, and we obtain

$$\phi_n(r) = J_0(\lambda_n r) \tag{4.135}$$

$$J_0(\lambda_n) = 0 \tag{4.136}$$

Two independent solutions exist to Eq. (4.134), only one of which satisfies the zero condition at $x \to \infty$. This solution is

$$v_n(x) = \exp\left(\frac{x}{2}\left[Pe - \left(4\lambda_n^2 + Pe^2\right)^{1/2}\right]\right) \tag{4.137}$$

The expansion coefficients A_n in the solution are obtained from the inhomogeneous condition at the inlet:

$$1 = \sum_{n=1}^{\infty} A_n \phi_n(r)\, v_n(0)$$

or, using $v_n(0) = 1$ and the orthogonality of the eigenfunctions,

$$A_n = \frac{\int_0^1 \phi_n \, r \, dr}{\int_0^1 \phi_n^2 \, r \, dr} \tag{4.138}$$

This completes the solution for the temperature profile. In convective/diffusive problems such as this, it is often of interest to *define* the heat transfer coefficient h from the solution to the temperature profile. The heat transfer coefficient, for this pipe flow problem, is defined so that the heat flux to the pipe wall is given by

$$q''(x) = h(x)(T_m(x) - T_w) \tag{4.139}$$

where T_m is the mean temperature of the fluid at position x (the quantities in the above equation are now *dimensional*). The mean temperature is defined as

$$T_m = \frac{2}{R^2} \int_0^R T(r, x) \, r \, dr \tag{4.140}$$

and the heat flux is

$$q'' = -k \left. \frac{\partial T}{\partial r} \right|_{r=R} \tag{4.141}$$

If we now return to dimensionless coordinates and use the above three equations, we will obtain

$$\frac{hR}{k} \equiv \frac{1}{2} Nu_D = -\frac{\left. \dfrac{\partial T}{\partial r} \right|_{r=1}}{2 \displaystyle\int_0^1 T \, r \, dr} \tag{4.142}$$

The dimensionless quantity $Nu_D = hD/k$ is the *Nusselt* number based on pipe diameter. It has the same grouping of quantities as the Biot number, yet it has a fundamentally different interpretation. The Nusselt number, as the above equation shows, is basically a dimensionless temperature gradient at the surface, scaled by a nondimensional mean temperature of the flow. Essentially, it is a nondimensional way of expressing the heat transfer coefficient h. The Biot number, on the other hand, relates conduction resistance within a *solid* to convection resistance *from* the solid.

A plot of Nu_D vs. dimensionless axial position x is given in Fig. 4.18, in which $Pe = 10$. At the entrance to the pipe $Nu_D \to \infty$, due to the instantaneous change in temperature of the wall. As the flow progresses into the pipe both the heat flux and the mean temperature decrease to zero. The ratio of the two, however, approaches a constant. In the large x limit it is easy to show that

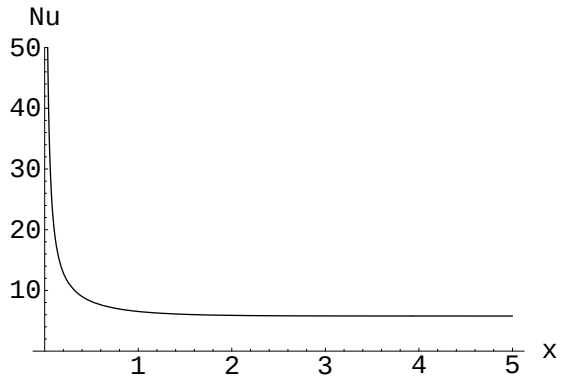


Figure 4.18: Nu_D vs. length into pipe x for plug flow conditions

only one term is retained in the series solution for temperature. For this case the Nusselt number will become independent of both x and Pe and equal to

$$Nu_D\Big|_{x\to\infty} = \lambda_1^2 = 5.783 \tag{4.143}$$

This would be considered the *fully developed flow* limit. You might recall that, for fully developed laminar flow in a pipe (with a parabolic velocity profile), the Nusselt number for isothermal wall conditions is the constant value of 3.66. We get a different value here because of the assumed plug flow velocity distribution.

4.6 Summary

A whole lot of details remain to be covered; such as superposition techniques in radial problems and spherical coordinate solutions. The latter topic will be addressed in a later chapter, and the former would extend directly from the examples in cartesian coordinates.

As a reference, a list of the general solution forms for problems in cartesian and cylindrical 2–D steady conditions is given below.

In cartesian coordinates, with homogeneous boundary conditions in the x direction, the solution will take the form

$$\overline{T} = \sum_{n=1}^{\infty} \left[A_n \cosh(\lambda_n \overline{y}) + B_n \sinh(\lambda_n \overline{y})\right] \phi_n(\overline{x}) \tag{4.144}$$

where ϕ_n is the eigenfunction for the x direction and λ_n is the eigenvalue. The eigenfunction will involve combinations of the trigonometric functions, i.e.,

$$\phi_n = [\cos(\lambda_n \overline{x}),\ \sin(\lambda_n \overline{x})] \tag{4.145}$$

in which the square brackets denote a linear combination of the two functions.

In cylindrical $\overline{r} - \overline{z}$ coordinates with $\overline{r}$ as the homogeneous direction, the solution will appear in the same form as before:

$$\overline{T} = \sum_{n=1}^{\infty} \left[A_n \cosh(\lambda_n \overline{z}) + B_n \sinh(\lambda_n \overline{z}) \right] \phi_n(\overline{r}) \tag{4.146}$$

except the eigenfunctions $\phi_n(\overline{r})$ will now involve the ordinary Bessel functions of order 0:

$$\phi_n(\overline{r}) = \left[J_0(\lambda_n \overline{r}), Y_0(\lambda_n \overline{r}) \right] \tag{4.147}$$

If the homogeneous direction is $\overline{z}$, the general solution will appear

$$\overline{T} = \sum_{n=1}^{\infty} \left[A_n I_0(\lambda_n \overline{r}) + B_n K_0(\lambda_n \overline{r}) \right] \phi_n(\overline{z}) \tag{4.148}$$

where the eigenfunctions $\phi_n(\overline{z})$ will involve the trigonometric functions as in Eq. (4.145).

And as always, the constants A_n and B_n are obtained from the BCs in the non–homogeneous direction and (when needed) the orthogonality relations for the eigenfunctions.

Exercises

1. A square, 2–D rod is exposed to identical convection conditions on the left and right faces. The bottom surface is insulated, and the top surface receives a nonuniform heat flux given by

   $$q''(x) = q''_0 \exp\left[-a^2 \left(x - L/2 \right)^2 \right]$$

   in which q''_0 and a are constants. Formulate the problem for the temperature distribution in appropriate dimensionless form, and derive a solution using the SOV method. Note: the boundary conditions can be simplified by exploiting the symmetry of the problem.

2. A solid circular rod, of length L and radius R, has the ends at $z = 0$ and L maintained at T_1 and T_2. Electrical current flows through the rod which results in a uniform heat generation rate of q''' within the rod. The surface at $r = R$ is cooled by convection to T_∞.

   (a) Using the superposition and SOV methods, determine the solution for the temperature distribution in the rod. Be sure to cast the problem in dimensionless variables.

   (b) Re–derive the solution for the case of $L \to \infty$. Recognize that in this limit there will no longer be an explicit BC stated at L, rather, the solution must asymptote to the correct behavior for $z \to \infty$. Derive a formula for the total rate of heat transfer to/from the $z = 0$ surface of the wire, and plot the result (in appropriate dimensionless form) as a function of dimensionless generation rate $\overline{q}'''$ using $Bi_R = hR/k = 1$. Discuss the physical significance of your results.

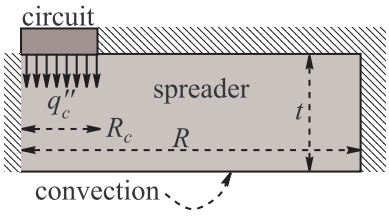


Figure 4.19: circular spreader, side view

3. A *thermal spreader* is a device used to transfer heat from a (typically small) integrated circuit to a cooling environment. Often they are designed to provide a larger heat transfer area to the convection environment than that occupied by the circuit, thereby 'spreading' out the heat much like a fin. A circular disk spreader is illustrated in Fig. 4.19, which has a radius of R and a thickness of t. The circular IC, of radius R_c, is centered on the top of the spreader, and a uniform flux of q_c'' enters the spreader from the IC. The bottom surface of the spreader is cooled by convection, and all other surfaces are adiabatic.

   (a) Formulate the problem in appropriate dimensionless variables, and derive the analytical solution for the temperature distribution in the spreader. You will want to use R as the characteristic length and $q_c''R/k$ as the characteristic temperature difference. Please note that this problem will admit a zero eigenvalue ($\lambda_0 = 0$) with a non–zero zeroth eigenfunction ($\phi_0 \neq 0$). It is critically important that you include the contribution of these terms.

   (b) In the limit of $Bi = hR/k \to 0$, and assuming $\bar{t} = t/R < 1$, your solution should give the result of $\overline{T} \to (R_c/R)^2/Bi$. Explain, using physical arguments, why this is the case.

   (c) Make a plot of $\overline{T}(\overline{x} = \bar{t}, \overline{r} = 0) \cdot Bi$ (i.e., the dimensionless spreader temperature directly under the IC, multiplied by Bi) vs. dimensionless thickness $\bar{t}$ for $Bi = 0.5,\ 1$, and 5 and with $R_c/R = 0.25$. Using this plot, identify an *optimum* thickness of the spreader, i.e., that which minimizes the IC temperature for the fixed heat dissipation rate. Explain, using physical principles, why such an optimum occurs.

4. Consider the convective–diffusion problem examined in the last section, but now the boundary condition at the wall is a constant heat flux condition, i.e.,

$$k \left.\frac{\partial T}{\partial r}\right|_{r=R} = q_w'' = \text{constant}$$

   Develop a solution for the dimensionless temperature profile in the tube as a function of r and x. Note that the zeroth eigenfunction will play a role in this situation. Also obtain

series solutions for the mean temperature $T_m(x)$ and the Nusselt number $Nu_D(x)$, and plot these quantities as a function of x using $Pe = 10$ Note that the mean temperature will not $\rightarrow 0$ for $x \rightarrow \infty$, since heat is continuously being added to the fluid. Finally, obtain the fully–developed value for the Nusselt number.

Chapter 5

General Multidimensional Conduction

5.1 Introduction

The problems examined in the previous chapters were restricted to two dimensions (space + time or space + space) and could be solved with the SOV/superposition methods. The first item of business in this chapter will be to extend SOV/superposition to problems involving three or more dimensions. In doing so we will retain, for time–dependent problems, the transient impulse model of the initial condition; this restriction will be lifted in the following chapter. We will also examine the variation of parameters method for solution of multidimensional conduction-type PDEs – which is somewhat more generalized than the SOV/superposition approach. Finally, we will introduce the concept of a semi–infinite domain and examine the steady flow of heat in such situations. A fundamentally different analytical method, known as the Fourier cosine transform, will be developed to describe the temperature field in the semi–infinite domain.

5.2 Transient and 2–D conduction

5.2.1 Reduction to 1–D

Here we combine the material from chapters three and four to address the problem in which we have a two–dimensional spacial domain that is undergoing a transient conduction process. There are a number of different ways that these problems can be tackled – depending on the type of problem.

The most fundamental problem is a 2–D domain that has homogeneous boundary conditions at all boundaries and an inhomogeneous IC. This would simply be the generalization to 2–D of the problem initially presented at the beginning of Ch. 3. Consider, for example, a square rod of width=height=$2W$, and a length much longer than the width. Initially the rod is at a uniform temperature of T_1. At $t = 0$ the surfaces are exposed to a convection environment. The objective

is to determine the transient and spacial distribution of the temperature in the rod.

The nondimensional variables are defined in the usual manner;

$$\overline{T} = \frac{T - T_\infty}{T_1 - T_\infty}, \quad \overline{x} = \frac{x}{W}, \quad \overline{y} = \frac{y}{W}, \quad \overline{t} = \frac{t\alpha}{W^2} \tag{5.1}$$

Recognizing that the problem is symmetrical in each quadrant of the rod, the origin can be taken at the center of the rod. The problem is then

$$\frac{\partial \overline{T}}{\partial \overline{t}} = \frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} \tag{5.2}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_{\overline{x}=0} = 0 \tag{5.3}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{y}}\right|_{\overline{y}=0} = 0 \tag{5.4}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_{\overline{x}=1} = -Bi\,\overline{T}(1, \overline{y}, \overline{t}) \tag{5.5}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{y}}\right|_{\overline{y}=1} = -Bi\,\overline{T}(\overline{x}, 1, \overline{t}) \tag{5.6}$$

$$\overline{T}(\overline{x}, \overline{y}, 0) = 1 \tag{5.7}$$

In general, the solution method would begin with the basic SOV procedure, in which $\overline{T}(\overline{x}, \overline{y}, \overline{t}) = u(\overline{x}) \cdot v(\overline{y}) \cdot w(\overline{t})$. This approach will be employed later, but there is a simpler route for this particular problem. This is because the BCs (and the DE) are all homogeneous. For this particular case, the 2–D spacial and transient problem will reduce to the product of two 1–D and transient problems.

To see this, let the solution be given by the product $\overline{T} = \overline{T}_x(\overline{x}, \overline{t}) \cdot \overline{T}_y(\overline{y}, \overline{t})$, where $\overline{T}_x$ does not depend on $\overline{y}$ and $\overline{T}_y$ does not depend on $\overline{x}$. Replacing this into the DE in Eq. (5.2) results in

$$\overline{T}_y \frac{\partial \overline{T}_x}{\partial \overline{t}} + \overline{T}_x \frac{\partial \overline{T}_y}{\partial \overline{t}} = \overline{T}_y \frac{\partial^2 \overline{T}_x}{\partial \overline{x}^2} + \overline{T}_x \frac{\partial^2 \overline{T}_y}{\partial \overline{y}^2} \tag{5.8}$$

Now let $\overline{T}_x$ and $\overline{T}_y$ each satisfy the 1–D and transient conduction equation:

$$\frac{\partial \overline{T}_x}{\partial \overline{t}} = \frac{\partial^2 \overline{T}_x}{\partial \overline{x}^2} \tag{5.9}$$

$$\frac{\partial \overline{T}_y}{\partial \overline{t}} = \frac{\partial^2 \overline{T}_y}{\partial \overline{y}^2} \tag{5.10}$$

By doing so, the product $\overline{T} = \overline{T}_x \overline{T}_y$ will satisfy the conduction equation for our entire problem. The BCs for $\overline{T}_x$ and $\overline{T}_y$ are deduced in a similar manner. At $\overline{x} = 0$ the condition is

$$\frac{\partial}{\partial \overline{x}} \left[\overline{T}_x \overline{T}_y\right]_{\overline{x}=0} = \overline{T}_y \left.\frac{\partial \overline{T}_x}{\partial \overline{x}}\right|_{\overline{x}=0} = 0$$

which is satisfied if $\overline{T}_x$ obeys

$$\left.\frac{\partial \overline{T}_x}{\partial \overline{x}}\right|_{\overline{x}=0} = 0 \tag{5.11}$$

Likewise, at $\overline{x} = 1$

$$\overline{T}_y(\overline{y},\overline{t})\left.\frac{\partial \overline{T}_x}{\partial \overline{x}}\right|_{\overline{x}=1} = -Bi\,\overline{T}_x(1,\overline{t})\cdot\overline{T}_y(\overline{y},\overline{t}) \tag{5.12}$$

Cancelling out $\overline{T}_y$ gives

$$\left.\frac{\partial \overline{T}_x}{\partial \overline{x}}\right|_{\overline{x}=1} = -Bi\,\overline{T}_x(1,\overline{t}) \tag{5.13}$$

This shows that $\overline{T}_x$ satisfies the same homogeneous BCs in the $\overline{x}$ direction as did $\overline{T}$. The same result for $\overline{T}_y$ would be obtained with the $\overline{y}$ direction homogeneous BCs. Finally, the IC is

$$1 = \overline{T}(\overline{x},\overline{y},0) = \overline{T}_x(\overline{x},0)\cdot\overline{T}_y(\overline{y},0)$$

This can be satisfied by imposing ICs on $\overline{T}_x$ and $\overline{T}_y$ of

$$\overline{T}_x(\overline{x},0) = 1 \tag{5.14}$$

$$\overline{T}_y(\overline{y},0) = 1 \tag{5.15}$$

For $\overline{T}_x$ we therefore get the 1–D and transient problem given by Eqs. (5.9–5.14). This problem was encountered in Ch. 3, and has the solution

$$\overline{T}_x = \sum_{n=1}^{\infty} A_n\phi_n(\overline{x})e^{-\lambda_n^2\overline{t}}$$

in which the eigenfunctions and eigencondition are

$$\phi_n(\overline{x}) = \cos(\lambda_n\overline{x}) \tag{5.16}$$

$$\lambda_n\sin(\lambda_n) = Bi\,\cos(\lambda_n) \tag{5.17}$$

and the expansion coefficients are

$$A_n = \frac{\displaystyle\int_0^1 \phi_n\,d\overline{x}}{\displaystyle\int_0^1 \phi_n^2\,d\overline{x}}$$

This specific problem requires absolutely no new mathematical formulation; the existing 1–D and transient solutions can be multiplied together to form the 2–D and transient solution. This technique is known as reduction to 1–D, and can be applied to problems that have a unit initial temperature and completely homogeneous BCs and DE.

Reduction to 1–D can also be applied to non–cartesian problems and 3–D configurations, providing the restrictions on the BCs and DE are met. For example, a finite–length cylindrical rod, of length L and radius R, that is initially at a temperature of $\overline{T} = 1$ and exposed at $\bar{t} = 0$ to a convection environment, could be split into the two problems of 1) an infinite length rod of radius R with the same BCs and ICs (a 1–D and transient problem in cylindrical coordinates), and a slab of length L with the same BCs and IC (another 1–D and transient problem – except in cartesian coordinates).

5.2.2 Separation of Variables

It is usually not possible to reduce a problem to a product of 1–D and transient solutions if the BCs and/or DE contain inhomogeneities. The general SOV method, with superposition techniques, will have to be employed for such cases.

The SOV procedure can be applied directly to transient + multidimensional problems that have homogeneous BCs and a homogeneous DE – which are the same class of problems that are amenable to 1–D reduction methods. To illustrate, consider again the 2–D square region used in the previous section. Instead of taking $\overline{T}(\overline{x}, \overline{y}, \bar{t}) = \overline{T}_x(\overline{x}, \bar{t}) \cdot \overline{T}_y(\overline{y}, \bar{t})$, now use $\overline{T}(\overline{x}, \overline{y}, \bar{t}) = u(\overline{x}) \cdot v(\overline{y}) \cdot w(\bar{t})$ per the standard SOV procedure.

Replacing this into Eq. (5.2) and separating the variables results in

$$\frac{w'}{w} = \frac{u''}{u} + \frac{v''}{v} \tag{5.18}$$

Each term in the above equation depends only on one variable. Therefore, to satisfy the equation each term must be constant. We could equate each side, at this point, to a constant (i.e., $\pm\lambda^2$) and proceed, yet it will work out better in the end (as you will see) if two constants are introduced into the equation. Specifically, let $u''/u = -\lambda^2$ and $v''/v = -\beta^2$, in which λ and β are constants, and replace this back into the separated equation;

$$\frac{w'}{w} = -\left(\lambda^2 + \beta^2\right) = \text{constant} \tag{5.19}$$

This maintains SOV principle in that w'/w is contant. You can also see that the choice of sign on λ^2 and β^2 will lead to the desired time–decaying solution, i.e.,

$$w = e^{-\left(\lambda^2+\beta^2\right)\bar{t}} \tag{5.20}$$

The characteristic solutions for $u(\overline{x})$ and $v(\overline{y})$ are now examined. Since the problem in the $\overline{x}$ direction is completely homogeneous, we can pose the problem for u as

$$u'' + \lambda^2 u = 0 \tag{5.21}$$

$$u'(0) = 0 \tag{5.22}$$

$$u'(1) = -Bi\, u(1) \tag{5.23}$$

The solution to this system provides the eigenfunction and the eigencondition:

$$u = \phi_n(\overline{x}) = \cos(\lambda_n \overline{x}) \tag{5.24}$$

$$\lambda_n \sin(\lambda_n) = Bi\, \cos(\lambda_n), \qquad n = 1,\ 2, \ldots \tag{5.25}$$

Likewise, an identical problem for v will be obtained in the $\overline{y}$ direction;

$$v'' + \beta^2 v = 0 \tag{5.26}$$

$$v'(0) = 0 \tag{5.27}$$

$$v(1) = 0 \tag{5.28}$$

This has the same solution as for u;

$$v = \psi_m(\overline{y}) = \cos(\beta_m \overline{y}) \tag{5.29}$$

$$\beta_m \sin(\beta_m) = Bi\, \cos(\beta_m), \qquad m = 1,\ 2, \ldots \tag{5.30}$$

Even though the eigenfunctions for $\overline{x}$ and $\overline{y}$ are identical for this problem, I use the different symbol $\psi_m(\overline{y})$ to denote the $\overline{y}$ eigenfunction; this is to emphasize that the functions need not be the same – as would be the case if the BCs had had different forms in the $\overline{x}$ and $\overline{y}$ directions.

The general solution to the problem is the sum of all possible solutions. To include all possible combinations of the $\overline{x}$ and $\overline{y}$ eigenfunctions, it is necessary to let n and m run *independently* of each other. That is, the general solution will now be in the form of the double infinite series

$$\boxed{\overline{T} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \phi_n(\overline{x}) \psi_m(\overline{y}) e^{-\left(\lambda_n^2 + \beta_m^2\right)\overline{t}}} \tag{5.31}$$

in which the expansion coefficient A_{nm} depends on both n and m. The above equation is the most general form of the transient and 2 cartesian dimensions SOV solution. The double series reflects the fact that the problem contains two independent spacial directions.

The expansion coefficients A_{nm} are obtained from the initial condition by employing the orthogonality properties of both $\phi_n(\overline{x})$ and $\psi_m(\overline{y})$. At $\overline{t} = 0$ the solution becomes

$$1 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \phi_n(\overline{x}) \psi_m(\overline{y})$$

Multiply each side through by $\phi_{n'}$ and integrate over $\overline{x}$ from 0 to 1. Each term in the series over n dissapears except the one in which $n = n'$. Likewise, multiply what's left by $\psi_{m'}$ and integrate over $\overline{y}$. Each term in the series over m dissapears except the one with $m = m'$. The end result is

$$A_{nm} = \frac{\displaystyle\int_0^1 \phi_n(\overline{x})\, d\overline{x} \int_0^1 \psi_m(\overline{y})\, d\overline{y}}{\displaystyle\int_0^1 \phi_n^2(\overline{x})\, d\overline{x} \int_0^1 \psi_m^2(\overline{y})\, d\overline{y}} \tag{5.32}$$

which can be evaluated using the formulas for ϕ_n and ψ_m.

For this particular problem, Eq. (5.32) can be written as the product of two quantities $A_{nm} = B_n C_m$, each given by

$$B_n = \frac{\int_0^1 \phi_n \, d\overline{x}}{\int_0^1 \phi_n^2 \, d\overline{x}}$$

$$C_m = \frac{\int_0^1 \psi_m \, d\overline{y}}{\int_0^1 \psi_m^2 \, d\overline{y}}$$

By replacing this into Eq. (5.31), the double series can be split into the product of two single series:

$$\begin{aligned}\overline{T} &= \sum_{n=1}^{\infty} B_n \phi_n(\overline{x}) e^{-\lambda_n^2 \overline{t}} \times \sum_{m=1}^{\infty} C_m \psi_m(\overline{y}) e^{-\beta_n^2 \overline{t}} \\ &= \overline{T}_x(\overline{x}, \overline{t}) \cdot \overline{T}_y(\overline{y}, \overline{t})\end{aligned}$$

We therefore recover our original solution that was obtained using reduction to 1–D.

The ability to split A_{nm} into $B_n \cdot C_m$ – which provides the basis for reduction to 1–D – relied on a 'separable' initial condition. That is, the form of the initial condition could be represented as a function of $\overline{x}$ times a function of $\overline{y}$, i.e., $\overline{T}(\overline{x}, \overline{y}, 0) = f_x(\overline{x}) \cdot f_y(\overline{y})$. The problem examined above had the trivial case of $f_x = f_y = 1$. Reduction to 1–D, however, could be utilized for arbitrary f_x and f_y – so long as the initial condition can be represented in this form (and providing that the DE and the BCs are homogeneous). The corresponding 1–D and transient problems, used to form the 2–D solution, would obey the following initial conditions,

$$\overline{T}_x(\overline{x}, 0) = f_x(\overline{x})$$
$$\overline{T}_y(\overline{y}, 0) = f_y(\overline{y})$$

5.2.3 Inhomogeneous problems

A slight modification of the BCs in the previous problem results in a situation that cannot be handled with the 1–D reduction method. Specifically, assume that the $\overline{x} = 1$ face is instantaneously brought to T_∞ yet the face at $\overline{y} = 1$ is mainained at $\overline{T} = 1$. Using the same definitions for the nondimensional variables, the problem becomes

$$\frac{\partial \overline{T}}{\partial \overline{t}} = \frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} \tag{5.33}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_{\overline{x}=0} = 0 \tag{5.34}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{y}}\right|_{\overline{y}=0} = 0 \tag{5.35}$$

$$\overline{T}(1, \overline{y}, \overline{t}) = 0 \tag{5.36}$$

$$\overline{T}(\overline{x}, 1, \overline{t}) = 1 \tag{5.37}$$

$$\overline{T}(\overline{x}, \overline{y}, 0) = 1 \tag{5.38}$$

An inhomogeneous BC now occurs at $\overline{y} = 1$, and SOV cannot be applied directly to the problem. Rather, superposition techniques are needed get the problem into a form (or forms) that can be individually handled with SOV.

The procedure used for this problem follows that developed in Ch. 3. Specifically, the solution will be cast as the sum of a decaying part $w(\overline{x}, \overline{y}, \overline{t})$ and a steady–state part $s(\overline{x}, \overline{y})$. Let s satisfy the same DE and BCs as $\overline{T}$ in Eqs. (5.33–5.37) – including the inhomogeneous condition at $\overline{y} = 1$. That is,

$$\frac{\partial^2 s}{\partial \overline{x}^2} + \frac{\partial^2 s}{\partial \overline{y}^2} = 0 \tag{5.39}$$

$$\left.\frac{\partial s}{\partial \overline{x}}\right|_{\overline{x}=0} = 0 \tag{5.40}$$

$$\left.\frac{\partial s}{\partial \overline{y}}\right|_{\overline{y}=0} = 0 \tag{5.41}$$

$$s(1, \overline{y}) = 0 \tag{5.42}$$

$$s(\overline{x}, 1) = 1 \tag{5.43}$$

The solution to s will be (from the previous chapter)

$$s = -2 \sum_{n=1}^{\infty} \frac{(-1)^n \phi_n(\overline{x}) \cosh(\lambda_n \overline{y})}{\lambda_n \cosh(\lambda_n)} \tag{5.44}$$

where the eigenfunction ϕ_n and eigenvalue λ_n are given by

$$\phi_n(\overline{x}) = \cos(\lambda_n \overline{x}) \tag{5.45}$$

$$\lambda_n = \frac{\pi}{2}(2n - 1), \qquad n = 1,\ 2, \ldots \tag{5.46}$$

By using $w = \overline{T} - s$, the following problem for w will be obtained:

$$\frac{\partial w}{\partial \overline{t}} = \frac{\partial^2 w}{\partial \overline{x}^2} + \frac{\partial^2 w}{\partial \overline{y}^2} \tag{5.47}$$

$$\left.\frac{\partial w}{\partial \overline{x}}\right|_{\overline{x}=0} = 0 \tag{5.48}$$

$$\left.\frac{\partial w}{\partial \overline{y}}\right|_{\overline{y}=0} = 0 \tag{5.49}$$

$$w(1, \overline{y}, \overline{t}) = 0 \tag{5.50}$$

$$w(\overline{x}, 1, \overline{t}) = 0 \tag{5.51}$$

$$w(\overline{x}, \overline{y}, 0) = 1 - s(\overline{x}, \overline{y}) \tag{5.52}$$

This problem has completely homogeneous BCs. Following the procedure given in the previous example, the solution will be

$$w = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \phi_n(\overline{x}) \psi_m(\overline{y}) e^{-\left(\lambda_n^2 + \beta_m^2\right)\overline{t}} \tag{5.53}$$

in which the $\overline{y}$–directed eigenfunction and eigenvalue, denoted ψ_m and β_m, have the same form as ϕ_n and λ_n given in Eq. (5.45) and (5.46). The expansion coefficients are

$$A_{nm} = \frac{\displaystyle\int_0^1 \left(1 - s(\overline{x}, \overline{y})\right) \phi_n(\overline{x}) \, \psi_m(\overline{y}) \, d\overline{x} \; d\overline{y}}{\displaystyle\int_0^1 \phi_n^2(\overline{x}) \, d\overline{x} \int_0^1 \psi_m^2(\overline{y}) \, d\overline{y}} \tag{5.54}$$

Observe that the initial condition cannot (at least not obviously) be split into a product of two functions, each of which depend on only one spacial variable. This prevents us from using a reduction of order method.

For our particular eigenfunctions the denominator in Eq. (5.54) will have the value 1/4. The integral of the first term in the numerator (the one with '1') will be $(-1)^n(-1)^m/\lambda_n\beta_n$. The second (with s) would appear to be more difficult to evaluate. However, a remarkably simple answer for this part can be obtained by using the tried–and–true method of integration by parts. First, use $\phi_n = -\phi_n''/\lambda_n^2$ and integrate over $\overline{x}$:

$$\begin{aligned}
\int_0^1 \int_0^1 s\phi_n\psi_m \, d\overline{x} \, d\overline{y} &= -\frac{1}{\lambda_n^2} \int_0^1 \int_0^1 s\phi_n'' \, d\overline{x} \, \psi_m \, d\overline{y} \\
&= -\frac{1}{\lambda_n^2} \int_0^1 \left[s\phi_n' \Big|_0^1 - \frac{\partial s}{\partial \overline{x}} \phi_n \Big|_0^1 + \int_0^1 \frac{\partial^2 s}{\partial \overline{x}^2} \phi_n \, d\overline{x} \right] \psi_m \, d\overline{y}
\end{aligned}$$

The boundary terms are identically zero because s and $\phi_n(\overline{x})$ both satisfy homogeneous BCs. Now use Eq. (5.39) in the integrand to turn the partial of s with respect to $\overline{x}$ into a partial with respect to $\overline{y}$, and integrate by parts over $\overline{y}$:

$$\begin{aligned}\int_0^1\int_0^1 s\phi_n\psi_m\,d\overline{x}\,d\overline{y} &= -\frac{1}{\lambda_n^2}\int_0^1\int_0^1 \frac{\partial^2 s}{\partial\overline{x}^2}\phi_n\,d\overline{x}\,\psi_m\,d\overline{y}\\ &= \frac{1}{\lambda_n^2}\int_0^1\int_0^1 \frac{\partial^2 s}{\partial\overline{y}^2}\psi_m\,d\overline{y}\,\phi_n\,d\overline{x}\\ &= \frac{1}{\lambda_n^2}\int_0^1\left[\frac{\partial s}{\partial\overline{y}}\psi_m\Big|_0^1 - s\psi_m'\Big|_0^1 + \int_0^1 s\psi_m''\,d\overline{y}\right]\phi_n\,d\overline{x}\end{aligned}$$

The boundary conditions on s and $\psi_m(\overline{y})$ are now used to evaluate the boundary terms. The only nonzero term will occur for the inhomogeneous condition at $\overline{y}=1$, for which $s(\overline{x},1)=1$ by Eq. (5.43) and $\psi(1)=0$ by definition. Finally, use $\psi_m''=-\beta_m^2\psi_m$ in the integrand – and observe that the resulting integral is the same as the one we started with. Multiplying through by λ_n^2, collecting the terms, and using the formulas for ϕ_n and ψ_m, the final result is

$$\begin{aligned}\int_0^1\int_0^1 s\phi_n\psi_m\,d\overline{x}\,d\overline{y} &= -\frac{\psi_m'(1)}{\lambda_n^2+\beta_m^2}\int_0^1 s(\overline{x},1)\,\phi_n\,d\overline{x}\\ &= -\frac{\psi_m'(1)}{\lambda_n^2+\beta_m^2}\int_0^1 \phi_n\,d\overline{x}\\ &= \frac{(-1)^{n+m}\beta_m}{\lambda_n\,(\lambda_n^2+\beta_m^2)}\end{aligned}$$

Putting it together, the formula for the A_{nm} expansion coefficients are

$$\begin{aligned}A_{nm} &= \frac{4(-1)^{n+m}}{\lambda_n}\left(\frac{1}{\beta_m}-\frac{\beta_m}{\lambda_n^2+\beta_m^2}\right)\\ &= \frac{4(-1)^{n+m}\lambda_n}{\beta_m\,(\lambda_n^2+\beta_m^2)}\end{aligned}\tag{5.55}$$

You should recognize that it was not necessary to use, in any way, the explicit formula for s (Eq. (5.44)) in the above manipulations. The application of *Mathematica* to a problem of this type – which avoids most of the subtle mathematical manipulations – is given at the end of the chapter.

The complete solution to the problem is given by $\overline{T}=w+s$, or

$$\overline{T} = 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{n+m}\lambda_n\phi_n(\overline{x})\psi_m(\overline{y})}{\beta_m\,(\lambda_n^2+\beta_m^2)}e^{-(\lambda_n^2+\beta_m^2)\overline{t}} - 2\sum_{n=1}^{\infty}\frac{(-1)^n\phi_n(\overline{x})\cosh(\lambda_n\overline{y})}{\lambda_n\cosh(\lambda_n)}\tag{5.56}$$

Shown in Fig. 5.1 are surface plots for the dimensionless temperature in the region for dimensionless times of $\overline{t}=0.02$, 0.2, and steady–state. The results are consistent with the boundary conditions and expected behavior of the model problem – note in particular the steep gradient about $\overline{x}=\overline{y}=1$, which results from the discontinuous change in surface temperature at this point.

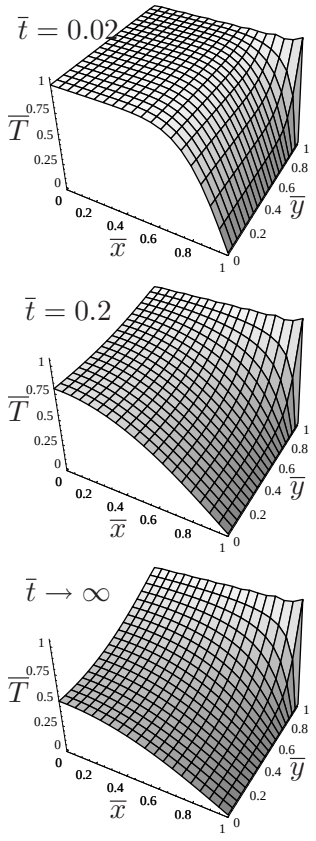


Figure 5.1: transient response in the 2–D slab

5.2.4 Cylindrical geometry example

A thick solid wire of length $2L$ and radius R is suspended between two bases, which are both maintained at the temperature T_1. Heat is removed from the surface of the wire by convection. This configuration exists long enough for a steady–state temperature distribution to develop in the wire. At $t = 0$ current is passed through the wire, which results in a uniform heat generation within the wire of strength q'''. As the wire heats up the temperatures at the ends are maintained at T_1, and the convection conditions remain unaltered. Determine the transient, 2–D temperature distribution in the wire.

This problem will be symmetrical in axial position z about $z = L$ – so we can consider only half the wire length and impose an adiabatic condition at $z = L$. The dimensional problem statement

is

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{q'''}{k}$$
$$T(0,z,t) \text{ is finite}, \quad -k\left.\frac{\partial T}{\partial r}\right|_R = h(T(R,z,t) - T_\infty)$$
$$T(r,0,t) = T_1, \quad \left.\frac{\partial T}{\partial z}\right|_L = 0$$
$$T(r,z,0) = T_s(r,z)$$

in which T_s is the initial steady–state temperature distribution, which is obtained from solution of the same system without the time derivative and the heat generation.

Define the nondimensional variables as

$$\overline{T} = \frac{T - T_\infty}{T_1 - T_\infty}, \quad \overline{r} = \frac{R}{L}, \quad \overline{z} = \frac{z}{L}, \quad \overline{t} = \frac{t\alpha}{L^2}$$

The dimensionless parameters in the problem are

$$Bi = \frac{hL}{k}, \quad \overline{q}''' = \frac{q'''L^2}{(T_1 - T_\infty)k}, \quad a = \frac{R}{L}$$

and the dimensionless problem becomes

$$\frac{\partial \overline{T}}{\partial \overline{t}} = \frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\overline{r}\frac{\partial \overline{T}}{\partial \overline{r}} + \frac{\partial^2 \overline{T}}{\partial \overline{z}^2} + \overline{q}''' \tag{5.57}$$
$$\overline{T}(0,\overline{z},\overline{t}) \text{ is finite} \tag{5.58}$$
$$-\left.\frac{\partial \overline{T}}{\partial \overline{r}}\right|_a = Bi\,\overline{T}(a,\overline{z},\overline{t}) \tag{5.59}$$
$$\overline{T}(\overline{r},0,\overline{t}) = 1 \tag{5.60}$$
$$\left.\frac{\partial \overline{T}}{\partial \overline{z}}\right|_1 = 0 \tag{5.61}$$
$$\overline{T}(\overline{r},\overline{z},0) = \overline{T}_s(\overline{r},\overline{z}) \tag{5.62}$$

The initial temperature distribution also satisfies the following system:

$$\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\overline{r}\frac{\partial \overline{T}_s}{\partial \overline{r}} + \frac{\partial^2 \overline{T}_s}{\partial \overline{z}^2} = 0 \tag{5.63}$$
$$\overline{T}_s(0,\overline{z}) \text{ is finite} \tag{5.64}$$
$$-\left.\frac{\partial \overline{T}_s}{\partial \overline{r}}\right|_a = Bi\,\overline{T}_s(a,\overline{z}) \tag{5.65}$$

$$\overline{T}_s(\overline{r},0) = 1 \tag{5.66}$$

$$\left.\frac{\partial \overline{T}_s}{\partial \overline{z}}\right|_1 = 0 \tag{5.67}$$

The problem for $\overline{T}_s$ can be solved directly using SOV for 2–D, steady configurations – because it has a homogeneous DE and homogenous BCs in $\overline{r}$. The problem for $\overline{T}$, on the other hand, has an inhomogeneous DE and an inhomogeneous BC in $\overline{z}$.

As before, we seek a superimposed solution in the form $\overline{T} = w(\overline{r},\overline{z},\overline{t}) + s(\overline{r},\overline{z})$, where s is the steady–state solution (with heat generation) and w is a time decaying part. Let s satisfy the following system:

$$\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\overline{r}\frac{\partial s}{\partial \overline{r}} + \frac{\partial^2 s}{\partial \overline{z}^2} + \overline{q}''' = 0 \tag{5.68}$$

$$s(0,\overline{z}) \text{ is finite} \tag{5.69}$$

$$-\left.\frac{\partial s}{\partial \overline{r}}\right|_a = Bi\, s(a,\overline{z}) \tag{5.70}$$

$$s(\overline{r},0) = 1 \tag{5.71}$$

$$\left.\frac{\partial s}{\partial \overline{z}}\right|_1 = 0 \tag{5.72}$$

By direct substitution of $w = \overline{T} - s$, the following system is obtained for w:

$$\frac{\partial w}{\partial \overline{t}} = \frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\overline{r}\frac{\partial w}{\partial \overline{r}} + \frac{\partial^2 w}{\partial \overline{z}^2} \tag{5.73}$$

$$w(0,\overline{z},\overline{t}) \text{ is finite} \tag{5.74}$$

$$-\left.\frac{\partial w}{\partial \overline{r}}\right|_a = Bi\, w(a,\overline{z},\overline{t}) \tag{5.75}$$

$$w(\overline{r},0,\overline{t}) = 0 \tag{5.76}$$

$$\left.\frac{\partial w}{\partial \overline{z}}\right|_1 = 0 \tag{5.77}$$

$$w(\overline{r},\overline{z},0) = \overline{T}_s(\overline{r},\overline{z}) - s(\overline{r},\overline{z}) \tag{5.78}$$

The problem for w now has a homogeneous DE and completely homogeneous BCs, and can be solved with SOV.

Obtaining a solution for s, however, will take one more superposition step – because of the inhomogeneous DE in Eq. (5.68). Let $s = u(\overline{r}) + v(\overline{r},\overline{z})$ (I'm running out of symbols), and let u

satisfy the following 1–D problem:

$$\frac{1}{\overline{r}}(ru')' + \overline{q}''' = 0 \tag{5.79}$$

$$u(0) \text{ is finite} \tag{5.80}$$

$$u'(a) = -Bi\, u(a) \tag{5.81}$$

If $s = v - u$ is substituted into Eqs. (5.68–5.72), the problem for v becomes

$$\frac{1}{\overline{r}}\frac{\partial}{\partial \overline{r}}\overline{r}\frac{\partial v}{\partial \overline{r}} + \frac{\partial^2 v}{\partial \overline{z}^2} = 0 \tag{5.82}$$

$$v(0, \overline{z}) \text{ is finite} \tag{5.83}$$

$$-\left.\frac{\partial v}{\partial \overline{r}}\right|_a = Bi\, v(a, \overline{z}) \tag{5.84}$$

$$v(\overline{r}, 0) = 1 - u(\overline{r}) \tag{5.85}$$

$$\left.\frac{\partial v}{\partial \overline{z}}\right|_1 = 0 \tag{5.86}$$

The problem for v now has a homogeneous DE and homogeneous BCs in $\overline{r}$. Note that the substitution $s = v + u$ effectively 'moved' the inhomogeneous term from the DE into the BC at $\overline{z} = 0$.

Our complete solution would therefore be $\overline{T} = u(\overline{r}) + v(\overline{r}, \overline{z}) + w(\overline{r}, \overline{z}, \overline{t})$, i.e., three superpositions. Only an outline of the solution procedure will be given at this point. The initial steady–state temperature distribution T_s will be in the form

$$\overline{T}_s = \sum_{n=1} A_n \cosh[\lambda_n(1 - \overline{z})]\phi_n(\overline{r}) \tag{5.87}$$

with an eigenfunction and eigencondition of

$$\phi_n(\overline{r}) = J_0(\lambda_n \overline{r}) \tag{5.88}$$

$$\lambda_n J_1(\lambda_n a) = Bi\, J_0(\lambda_n a) \tag{5.89}$$

Note that an 'origin shift' in the $\overline{z}$ direction has been used in Eq. (5.87). The $1 - \overline{z}$ argument in the cosh function appears simply because the $\overline{z}$ derivative of the solution must vanish at $\overline{z} = 1$. The expansion coefficients A_n are obtained from the inhomogeneous BC at $\overline{z} = 0$ via

$$A_n = \frac{\displaystyle\int_0^a \phi_n(\overline{r})\overline{r}\, d\overline{r}}{\cosh(\lambda_n)\displaystyle\int_0^a \phi_n^2(\overline{r})\overline{r}\, d\overline{r}} \tag{5.90}$$

Recognize that the eigenfunctions ϕ_n are orthogonal on the interval $(0, a)$ – not $(0, 1)$.

The solution for $u(\overline{r})$ is

$$u(\overline{r}) = \frac{\overline{q}'''}{4\,Bi}\left[2a + Bi\left(a^2 - \overline{r}^2\right)\right] \tag{5.91}$$

The solution for v will be in the same form as $\overline{T}_s$ – note that the only difference between the systems for $\overline{T}_s$ (Eqs. (5.63–5.67)) and v (Eqs. (5.82–5.86)) is the BC at $\overline{z} = 0$. Therefore, v will be given by

$$v = \sum_{n=1}^{\infty} B_n \cosh[\lambda_n(1-\overline{z})]\phi_n(\overline{r})$$

in which the eigenfunctions and eigenvalues are the same as in Eqs. (5.88) and (5.89), and the B_n coefficients are obtained from

$$\begin{aligned} B_n &= \frac{\displaystyle\int_0^a [1-u(\overline{r})]\phi_n(\overline{r})\overline{r}\,d\overline{r}}{\cosh(\lambda_n)\displaystyle\int_0^a \phi_n^2(\overline{r})\overline{r}\,d\overline{r}} \\ &= A_n - \frac{\displaystyle\int_0^a u(\overline{r})\phi_n(\overline{r})\overline{r}\,d\overline{r}}{\cosh(\lambda_n)\displaystyle\int_0^a \phi_n^2(\overline{r})\overline{r}\,d\overline{r}} \end{aligned}$$

The integral involving u can be readily integrated using integration by parts:

$$\begin{aligned} \int_0^a u\,\phi_n\overline{r}\,d\overline{r} &= -\frac{1}{\lambda_n^2}\int_0^a u(\overline{r}\phi_n')'\,d\overline{r} \\ &= -\frac{1}{\lambda_n^2}\left[u\,\overline{r}\phi_n'\Big|_0^a - u'\overline{r}\phi_n\Big|_0^a + \int_0^a (\overline{r}u')'\phi_n\,d\overline{r}\right] \\ &= \frac{1}{\lambda_n^2}\int_0^a \overline{r}\overline{q}'''\phi_n\,d\overline{r} \end{aligned}$$

Comparing the above with Eq. (5.90) gives

$$B_n = A_n\left(1 - \frac{\overline{q}'''}{\lambda_n^2}\right) \tag{5.92}$$

Finally, the solution for w will be in the general form

$$w = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\phi_n(\overline{r})\psi_m(\overline{z})e^{-\left(\lambda_n^2+\beta_m^2\right)\overline{t}} \tag{5.93}$$

The eigenfunctions and eigencondition in $\overline{r}$ are the same as before. The $\overline{z}$ direction requires a trigonometric eigenfunction that gives zero temperature at $\overline{z} = 0$ and zero gradient at $\overline{z} = 1$. The

function that satisfies this is

$$\psi_m(\overline{z}) = \sin(\beta_m \overline{z})$$
$$\beta_m = \frac{2m-1}{2}\pi$$

Alternatively, we could state the eigenfunctions in a manner consistent with the $1-\overline{z}$ argument of the hyperbolic functions in the previous solutions:

$$\psi_m(\overline{z}) = \cos[\beta_m(1-\overline{z})] \tag{5.94}$$
$$\beta_m = \frac{2m-1}{2}\pi \tag{5.95}$$

The expansion coefficients will be obtained from the initial condition in Eq. (5.78) via

$$C_{nm} = \frac{\displaystyle\int_0^1 \int_0^a [T_s(\overline{r},\overline{z}) - u(\overline{r}) - v(\overline{r},\overline{z})]\phi_n(\overline{r})\overline{r}\,\psi_m(\overline{z})\,d\overline{r}\,d\overline{z}}{\displaystyle\int_0^a \phi_n^2(\overline{r})\overline{r}\,dr \int_0^1 \psi_m^2(\overline{z})\,d\overline{z}}$$

Again, integration by parts, combined with substitution of the DE/BC relationships that are satisfied by $\overline{T}_s$, u, v and the eigenfunctions, would be used to evaluate the integrals. You certainly would not want to directly substitute into the integrands the equations for $\overline{T}_s$, u, and v. That would create a mess.

5.3 3–D steady conduction

5.3.1 Cartesian geometries

As you would guess by now, SOV can be applied directly to steady problems in 3 dimensions providing that the DE and the BCs in all but one direction are homogeneous. In cartesian coordinates, and assuming that the homogeneous directions are x and y, the general form of the solution will be be

$$\boxed{\overline{T} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} [A_{mn}\cosh(\gamma_{mn}\overline{z}) + B_{mn}\sinh(\gamma_{mn}\overline{z})]\,\phi_n(\overline{x})\psi_m(\overline{y})} \tag{5.96}$$

in which $\phi_n(\overline{x})$ and $\psi_m(\overline{y})$ are the eigenfunctions in the $\overline{x}$ and $\overline{y}$ directions, which have corresponding eigenvalues of λ_n and β_m. The eigenvalue γ_{mn} is defined

$$\gamma_{mn} = \left(\lambda_n^2 + \beta_m^2\right)^{1/2} \tag{5.97}$$

As before, formulas for the A_{mn} and B_{mn} coefficients are obtained by using the BCs in the inhomogeneous direction along with orthogonality of the eigenfunctions.

Non–homogeneous problems – which occur more often than not – can be handled via superposition techniques. The procedure here is exactly the same as what was covered in previous sections and chapters. Nothing fundamentally different is created by the 3–D nature of the problems.

5.3.2 Cylindrical geometries

Unlike the cartesian case, the general 3–D solution in cylindrical coordinates takes on different forms depending on the inhomogeneous direction. A conduction domain that consists of a circular cylinder (of a given radius and length) will always have homogeneous BCs in the ϕ direction – because the surfaces of the domain do not correspond to $\phi =$ constant. Recall that the ϕ BCs for circular cylinders were formulated as 'continuation conditions', in which the solution was forced to obey $\overline{T}(\overline{r},\overline{z},\phi) = \overline{T}(\overline{r},\overline{z},\phi+2\pi)$.

The SOV procedure is now outlined for circular cylinders with homogeneous BCs in the $\overline{r}$ and ϕ directions. The governing DE is

$$\frac{1}{\overline{r}}\frac{\partial}{\partial\overline{r}}\overline{r}\frac{\partial\overline{T}}{\partial\overline{r}} + \frac{1}{\overline{r}^2}\frac{\partial^2\overline{T}}{\partial\phi^2} + \frac{\partial^2\overline{T}}{\partial\overline{z}^2} = 0$$

The separated solution is taken as $\overline{T}(\overline{r},\phi,\overline{z}) = u(\overline{r})\cdot v(\phi)\cdot w(\overline{z})$ and replaced into the DE. Rearranging the terms leads to

$$\underbrace{\frac{1}{\overline{r}\,u}(\overline{r}\,u')' + \frac{v''}{\overline{r}^2\,v}}_{=-\gamma^2} + \underbrace{\frac{w''}{w}}_{=\gamma^2} = 0 \tag{5.98}$$

where γ is a separation constant, the sign of which is chosen to give Sturm–Liouville systems in the $\overline{r}$ and ϕ directions (this will become evident later). Rearranging the first two terms in the above – which are equal to $-\gamma^2$ – leads to

$$\underbrace{\frac{\overline{r}}{u}(\overline{r}\,u')' + \overline{r}^2\,\gamma^2}_{=n^2} + \underbrace{\frac{v''}{v}}_{=-n^2} = 0$$

The separation constant of n^2, where n is an integer, is chosen to make the characteristic solution for v periodic in 2π. Specifically, v will appear in the most general form as

$$v(\phi) \equiv \psi_n(\phi) = [\cos(n\phi), \sin(n\phi)] \tag{5.99}$$

where the square brackets denote a linear combination. The particular nature of the eigenfunction ψ_n would depend on the form of the temperature profile; if, for example, the temperature field is even in ϕ (i.e., $\overline{T}(-\phi) = \overline{T}(\phi)$), then the eigenfunction would consist only of the cosine function.

The characteristic DE for u becomes

$$(\overline{r}\,u')' + \left(\overline{r}\,\gamma^2 - \frac{n^2}{\overline{r}}\right)u = 0$$

This is in the Sturm–Liouville form, and has the general solution

$$u(\overline{r}) \equiv \phi_{nm}(\overline{r}) = [J_n(\gamma_{nm}\overline{r}), Y_n(\gamma_{nm}\overline{r})] \tag{5.100}$$

where J_n and Y_n are the ordinary Bessel functions of order n. The integer indices m and n have now been appended to γ to denote that γ_{nm} is an eigenvalue, and would correspond to the m^{th} root of the $\overline{r}$–direction eigencondition using the n^{th}–order eigenfunction. To illustrate via an example, suppose that the domain included the origin $\overline{r} = 0$, and the boundary condition at $\overline{r} = 1$ was $\overline{T} = 0$. The Y_n part of the eigenfunction would be eliminated to avoid singular behavior at $\overline{r} = 0$, and the eigencondition would become

$$J_n(\gamma_{nm}) = 0$$

The characteristic solution for w, from Eq. (5.98), will involve the hyperbolic functions, and the general form of the solution becomes

$$\overline{T} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left(A_{nm} \cosh(\gamma_{nm}\overline{z}) + B_{nm} \sinh(\gamma_{nm}\overline{z})\right) \phi_{nm}(\overline{r})\, \psi_n(\phi) \tag{5.101}$$

Derivation of the general form of the solution when the homogeneous BCs are in the ϕ and $\overline{z}$ directions is left as an exercise.

A cylindrical domain that consisted of an angular section (or wedge) of a cylinder could have inhomogeneous BCs in the ϕ direction – because for this case ϕ = constant describes a surface. Such problems admit analytical solutions only for specified $\overline{T}$ and/or adiabatic conditions on the ϕ = constant surface.

5.3.3 Spherical coordinates

A conduction region that takes the form of a sphere (solid or hollow) will, almost exclusively, have homogeneous boundary conditions in the θ and ϕ directions. Similar to the cylindrical case, the BCs for θ and ϕ take the form of continuation conditions – in which the solution must be even in θ and periodic in ϕ about 2π.

The general form of the solution to Laplace's equation in spherical coordinates appears as

$$T = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(A_{mn}\overline{r}^{-(n+1)} + B_{mn}\overline{r}^{n}\right) P_n^m(\cos\theta) e^{im\phi} \tag{5.102}$$

The function P_n^m appearing above is referred to as the *associated Legendre function* of order n and degree m. This function is a solution to Legendre's equation (which arises from the SOV procedure);

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dP_n^m(\cos\theta)}{d\theta} + \left(n(n+1) - \frac{m^2}{\sin^2\theta}\right) P_n^m(\cos\theta) = 0$$

The Legendre function is non–zero only for $|m| \leq n$ – which explains why the sum over m in Eq. (5.102) runs from $-n$ to n. Observe also that the complex function $e^{im\phi}$ has been used in the general solution. This could be split into $\cos(m\phi) + i\sin(m\phi)$ – from which the real part of the solution could be derived – but the complex form results in a much more compact formulation of the solution. Use of the complex formulation also implies that the expansion coefficients A_{mn} and B_{mn} are themselves complex – yet in the end it is understood that only the real part of the solution is meaningful.

The functions $P_n^m(\cos\theta)$ and $e^{im\phi}$ are the the eigenfunctions for the problem. Each of these functions have orthogonality properties when integrated over their domain. The existence of complex functions in the solution requires a generalization of the orthogonality relation. Specifically, for $e^{im\phi}$ the relation becomes

$$\int_0^{2\pi} e^{im\phi}\left(e^{im'\phi}\right)^* d\phi = \int_0^{2\pi} e^{i(m-m')\phi}\, d\phi = \begin{cases} 0, & m \neq m' \\ 2\pi, & m = m'' \end{cases} \tag{5.103}$$

The superscript $*$ in the above denotes *complex conjugate* – in which the sign of the imaginary parts of the function are reversed. For complex eigenfunctions one would therefore integrate the product of one eigenfunction, of degree m, with the conjugate of another, of degree m', to obtain the orthogonality relation.

The domain in the θ direction is from 0 to π, and the orthogonality relation is

$$\int_0^{\pi} P_n^m(\cos\theta) P_{n'}^m(\cos\theta)\sin\theta\, d\theta = \begin{cases} 0, & n \neq n' \\ \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!}, & n = n' \end{cases} \tag{5.104}$$

Observe that the product $P_n^m(\cos\theta)e^{im\phi}$ can be viewed as a single eigenfunction that has orthogonality properties when integrated over the surface of a sphere. That is, the eigenfunction defined by

$$\psi_{mn}(\theta,\phi) = P_n^m(\cos\theta)e^{im\phi} \tag{5.105}$$

has the property

$$\int_0^{2\pi}\int_0^{\pi} \psi_{mn}\psi^*_{m'n'}\sin\theta\, d\theta\, d\phi = \begin{cases} 0, & n \neq n' \text{ or } m \neq m' \\ \dfrac{4\pi}{2n+1}\dfrac{(n+m)!}{(n-m)!}, & n = n' \text{ and } m = m' \end{cases} \tag{5.106}$$

The eigenfunction $\psi_{mn}(\theta,\phi)$ is often referred to as a *spherical harmonic*. By use of the orthogonality of these functions, the expansion coefficients A_{mn} and B_{mn} appearing in Eq. (5.102) can be obtained from the boundary conditions.

As a very general example, suppose that a solid sphere has a surface temperature distribution given by $\overline{T}(\overline{r}=1,\theta,\phi)=f(\theta,\phi)$, where f is a known function. The A_{mn} coefficients in Eq. (5.102) would be zero by virtue of the singular behavior of $\overline{r}^{-(n+1)}$ at the origin (this would not be the case if the sphere was hollow). The B_{mn} coefficients would be obtained from

$$B_{mn}=\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}\int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)\,\psi^*_{mn}(\theta,\phi)\,\sin(\theta)\,d\theta\,d\phi$$

Formulas would be needed for the integrals of Legendre functions to complete the problem. These can be obtained in several standard mathematical texts or generated (for specified m and n) by *Mathematica*.

5.4 Variation of Parameters

5.4.1 Transient problems

As has been observed in this and the previous two chapters, the solution of arbitrarily inhomogeneous problems usually involves the superposition of several partial solutions, each of which can be solved using SOV or simpler techniques. The result is often that the final solution is a fairly complicated quilt of the individual solutions. One consequence of this is that the functional dependence of a particular independent variable can appear in several different forms within a solution. For example, a 2–D steady–state problem with inhomogeneous BCs in all directions can be split into two separate superimposed problems, with one having eigenfunctions in the x direction and the other having eigenfunctions in the y direction. Because of this, the dependence of the final solution for $\overline{T}$ on $\overline{x}$ would be represented by both the trigonometric functions (i.e., the eigenfunctions) and the hyperbolic functions. This can complicate things if certain manipulations are required on the solution, such as integration over space and/or time.

There is a method that can generate a solution to an arbitrarily inhomogeneous problem in terms of a single series expansion – and which therefore obviates any need to construct superimposed solutions. This method is known as variation of parameters (VOP). In many respects it is similar to SOV – in that the solution to the problem is obtained as a series expansion of eigenfunctions. Variation of parameters, however, is not constrained by the homogeneity of the DE or the BCs.

To introduce the method we will solve a simple 1–D cartesian and transient problem. In particular, a plane wall, initially at temperature $\overline{T}=0$, has the surface at $\overline{x}=1$ instantaneously brought to $\overline{T}=1$ while the surface at $\overline{x}=0$ is maintained at $\overline{T}=0$. The formal problem statement is

$$\frac{\partial\overline{T}}{\partial\overline{t}}=\frac{\partial^2\overline{T}}{\partial\overline{x}^2} \tag{5.107}$$

$$\overline{T}(0,\overline{t})=0 \tag{5.108}$$

$$\overline{T}(1,\overline{t})=1 \tag{5.109}$$

$$\overline{T}(\overline{x},0)=0 \tag{5.110}$$

This problem, as you may recall, was solved in Ch. 3 via the superposition $\overline{T}=w(\overline{x},\bar{t})+s(\overline{x})$, where s was the steady–state solution and w was a time–decaying partial solution. In VOP, on the other hand, the solution is not split into separate parts. Rather, the solution, in a general form, is posed as a single expansion. The terms in the expansion will be the eigenfunctions for the corresponding homogeneous problem times a undetermined function of the remaining variable.

For this particular problem, the general solution is therefore stated as

$$\overline{T}=\sum_{n=1}^{\infty}A_n(\bar{t})\phi_n(\overline{x}) \tag{5.111}$$

where ϕ_n are the eigenfunctions of the corresponding homogeneous problem and $A_n(\bar{t})$ is an undetermined function of $\bar{t}$. The eigenfunctions are identified by replacing all inhomogeneous terms in the DE and/or the $\overline{x}$ BCs by homogeneous ones of the same type. One inhomogeneous BC occurs in the problem at $\overline{x}=1$, and the corresponding homogeneous BC would have zero temperature at both $\overline{x}=0$ and $\overline{x}=1$. Therefore, the eigenfunctions would be

$$\phi_n(\overline{x})=\sin(\lambda_n\overline{x}) \tag{5.112}$$

$$\lambda_n=n\pi \tag{5.113}$$

The key difference between SOV and VOP is that the characteristic function in the inhomogeneous direction is not identified at the start of the procedure. In SOV, for example, it is recognized at the onset that the time dependence of the solution appears as $\exp(-\lambda_n^2\bar{t})$ (if, that is, the problem had homogeneous BCs and DE). In VOP the functional form of the time dependence is initially unknown – and is contained in the function(s) $A_n(\bar{t})$ for $n=1,\ 2,\ldots$. The obvious objective of the method is to specify the function $A_n(\bar{t})$. This will be done by identifying an ODE and an IC (or BCs, for multidimensional steady–state problems) that describe the $A_n(\bar{t})$ functions, and solving the ODE to obtain the $A_n(\bar{t})$.

The first step in the process is to use the orthogonality properties of the ϕ_n eigenfunctions to 'formally' solve for the $A_n(\bar{t})$ functions. That is $A_n(\bar{t})$ in Eq. (5.111) is given by by

$$A_n(\bar{t})=\frac{\displaystyle\int_0^1\overline{T}(\overline{x},\bar{t})\,\phi_n(\overline{x})\,d\overline{x}}{\displaystyle\int_0^1\phi_n^2(\overline{x})\,d\overline{x}}=2\int_0^1\overline{T}(\overline{x},\bar{t})\,\phi_n(\overline{x})\,d\overline{x} \tag{5.114}$$

This relationship is perfectly valid – it's just not very useful at this point because the integrand is an unknown quantity.

The next step is to derive an ODE for $A_n(\bar{t})$ by differentiating the above formula with respect to $\bar{t}$;

$$A_n'(\bar{t})=2\int_0^1\frac{\partial\overline{T}}{\partial\bar{t}}\phi_n(\overline{x})\,d\overline{x}$$

The derivative operator $\partial/\partial\bar{t}$ can be brought inside the integral because the limits on the integral do not depend on $\bar{t}$. The *original* PDE to the problem, Eq. (5.107), is used next to eliminate the partial with respect to $\bar{t}$;

$$A_n'(\bar{t}) = 2\int_0^1 \frac{\partial^2 \overline{T}}{\partial \overline{x}^2}\phi_n(\overline{x})\, d\overline{x} \tag{5.115}$$

Had heat generation been present, the source term would also have appeared in the integrand. Integration by parts is now employed:

$$\begin{aligned} A_n'(\bar{t}) &= 2\int_0^1 \frac{\partial^2 \overline{T}}{\partial \overline{x}^2}\phi_n(\overline{x})\, d\overline{x} \\ &= 2\left[\frac{\partial \overline{T}}{\partial \overline{x}}\phi_n\Big|_0^1 - \overline{T}\phi_n'\Big|_0^1 + \int_0^1 \overline{T}\phi_n''\, d\overline{x}\right] \\ &= 2\left[\frac{\partial \overline{T}}{\partial \overline{x}}\phi_n\Big|_0^1 - \overline{T}\phi_n'\Big|_0^1 - \lambda_n^2\int_0^1 \overline{T}\phi_n\, d\overline{x}\right] \\ &= 2\left[\frac{\partial \overline{T}}{\partial \overline{x}}\phi_n\Big|_0^1 - \overline{T}\phi_n'\Big|_0^1\right] - \lambda_n^2 A_n(\bar{t}) \end{aligned}$$

The characteristic DE for ϕ_n was used in the second–to–last line to eliminate ϕ_n'', and the definition of A_n, in Eq. (5.114), was employed in the last step.

We now evaluate the boundary terms that resulted from integration by parts. To do this, the *original* BCs on $\overline{T}$, and the boundary properties for ϕ_n, are substituted into the above equation. The eigenfunction ϕ_n is zero at 0 and 1, and the temperature is also zero at $\overline{x} = 0$. However, from Eq. (5.109), the temperature is unity at $\overline{x} = 1$. Using this information, the only surviving boundary term is $T(1,\bar{t})\phi_n'(1) = 1\cdot\phi_n'(1)$. The ODE for A_n is now

$$\begin{aligned} A_n'(\bar{t}) &= 2\phi_n'(1) - \lambda_n^2 A_n(\bar{t}) \\ &= 2\lambda_n\cos(\lambda_n) - \lambda_n^2 A_n(\bar{t}) \\ &= 2(-1)^n\lambda_n - \lambda_n^2 A_n(\bar{t}) \end{aligned}$$

We now have a complete ODE for A_n. The general solution is

$$A_n(\bar{t}) = C_1 e^{-\lambda_n^2\bar{t}} - \frac{2(-1)^n}{\lambda_n} \tag{5.116}$$

To evaluate the integration constant an initial condition is needed for A_n. This is obtained by application of the original initial condition, Eq. (5.110), directly to the definition of A_n in Eq. (5.114):

$$A_n(0) = 2\int_0^1 \overline{T}(\overline{x},0)\phi_n\, d\overline{x} = 0 \tag{5.117}$$

Using this in Eq. (5.116), the complete solution for $A_n(\bar{t})$ is

$$\begin{aligned} A_n(\bar{t}) &= \frac{2(-1)^n}{\lambda_n}\left(e^{-\lambda_n^2\bar{t}}-1\right) \\ &= \frac{2(-1)^n}{\pi n}\left(e^{-(n\pi)^2\bar{t}}-1\right) \end{aligned} \tag{5.118}$$

and the complete solution for the temperature, from Eq. (5.111), is

$$\overline{T} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(e^{-(n\pi)^2\bar{t}}-1\right)\sin(n\pi\overline{x}) \tag{5.119}$$

As a reference, the SOV/superposition solution derived for this problem (from Ch. 3) is

$$\overline{T} = \overline{x} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi\overline{x})e^{-(n\pi)^2\bar{t}} \tag{5.120}$$

At first glance, it does not appear that the solutions are the same – as they must be because they both satisfy the same DE, BCs, and IC. The equivalence, however, is established by expanding the function $\overline{x}$ into a series of eigenfunctions via

$$\overline{x} = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi\overline{x}) \tag{5.121}$$

A key feature of the VOP method is that the solution is forced to take on the form of a single expansion of the eigenfunctions. As mentioned above, this has certain mathematical advantages. The solution in Eq. (5.119) has the $\overline{x}$ dependence embedded entirely within the eigenfunctions $\phi_n = \sin(n\pi\overline{x})$. Numerically, however, the VOP solutions often are more time–consuming to evaluate than the equivalent SOV/superposition solutions. This is because the eigenfunctions, by themselves, do not naturally obey the inhomogeneous BCs of the problem. For example, the above solution must give $\overline{T} = 1$ at $\overline{x} = 1$. However, replacing $\overline{x} = 1$ into the solution would appear to give zero – because the eigenfunctions are defined to be identically zero at $\overline{x} = 1$. On a more precise level, we would find that in the *limit* of $\overline{x} \to 1$, the solution would yield $\overline{T} \to 1$. Obtaining this limit, however, would require evaluation of an increasingly large number of terms in the series.

5.4.2 Steady problems

To illustrate further the application of VOP, consider now a 2–D steady–state problem, in which we have a square region with adiabatic BCs at $\overline{x} = 0$ and $\overline{y} = 0$, zero temperature at $\overline{x} = 1$, and unit temperature at $\overline{y} = 1$.

$$\frac{\partial^2 \overline{T}}{\partial \overline{x}^2} + \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} = 0 \tag{5.122}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_{\overline{x}=0} = 0 \tag{5.123}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{y}}\right|_{\overline{y}=0} = 0 \tag{5.124}$$

$$\overline{T}(1, \overline{y}) = 0 \tag{5.125}$$

$$\overline{T}(\overline{x}, 1) = 1 \tag{5.126}$$

The SOV solution to this problem is

$$\overline{T} = -2 \sum_{n=1}^{\infty} \frac{(-1)^n \phi_n(\overline{x}) \cosh(\lambda_n \overline{y})}{\lambda_n \cosh(\lambda_n)} \tag{5.127}$$

where the eigenfunction and eigenvalues are

$$\phi_n(\overline{x}) = \cos(\lambda_n \overline{x}) \tag{5.128}$$

$$\lambda_n = \frac{(2n-1)\pi}{2} \tag{5.129}$$

The above solution can be obtained from direct application of the SOV method – because the direction ($\overline{x}$) has completely homogeneous BCs and the DE is homogeneous. SOV would dictate that we define the eigenfunctions in the $\overline{x}$ direction. VOP, on the other hand, makes no restrictions on our choice of eigenfunctions – aside from that they represent the corresponding homogeneous problem. For example, eigenfunctions for this particular problem can be chosen to run in the $\overline{y}$ direction – which would violate the SOV procedure. This would not be the smart choice, but we still will be able to get a valid solution out of the problem.

Choosing $\overline{y}$ as the direction of our eigenfunctions, we would then pose the solution to the problem as

$$\overline{T} = \sum_{n=1}^{\infty} A_n(\overline{x}) \phi_n(\overline{y}) \tag{5.130}$$

The eigenfunctions satisfy the corresponding homogeneous BCs to the problem, which would have zero gradient at $\overline{y} = 0$ and zero value at $\overline{y} = 1$. The form is the same as that for $\overline{x}$:

$$\phi_n(\overline{y}) = \cos(\lambda_n \overline{y}) \tag{5.131}$$

$$\lambda_n = \frac{(2n-1)\pi}{2} \tag{5.132}$$

Using the orthogonality of the eigenfunctions, the $A_n(\overline{x})$ functions are obtained from Eq. (5.130) via

$$A_n(\overline{x}) = \frac{\displaystyle\int_0^1 \overline{T}(\overline{x},\overline{y})\,\phi_n(\overline{y})\,d\overline{y}}{\displaystyle\int_0^1 \phi_n^2(\overline{y})\,d\overline{y}} = 2\int_0^1 \overline{T}(\overline{x},\overline{y})\,\phi_n(\overline{y})\,d\overline{y} \tag{5.133}$$

We now construct an ODE for $A_n(\overline{x})$. Differentiating the above twice with respect to $\overline{x}$ and using the original DE and BCs results in

$$\begin{aligned}
A_n''(\overline{x}) &= 2\int_0^1 \frac{\partial^2 \overline{T}}{\partial \overline{x}^2}\phi_n(\overline{y})\,d\overline{y} \\
&= -2\int_0^1 \frac{\partial^2 \overline{T}}{\partial \overline{y}^2}\phi_n(\overline{y})\,d\overline{y} \\
&= -2\left[\frac{\partial \overline{T}}{\partial \overline{y}}\phi_n\Big|_0^1 - \overline{T}\phi_n'\Big|_0^1 + \int_0^1 \overline{T}\phi_n''\,d\overline{y}\right] \\
&= -2\left[\frac{\partial \overline{T}}{\partial \overline{y}}\phi_n\Big|_0^1 - \overline{T}\phi_n'\Big|_0^1\right] + \lambda_n^2 A_n(\overline{x}) \\
&= 2\phi_n'(1) + \lambda_n^2 A_n(\overline{x}) \\
&= -2\lambda_n \sin(\lambda_n) + \lambda_n^2 A_n(\overline{x}) \\
&= 2\lambda_n(-1)^n + \lambda_n^2 A_n(\overline{x})
\end{aligned}$$

This is the desired ODE for the $A_n(\overline{x})$ functions. The general solution is

$$A_n(\overline{x}) = C_1\cosh(\lambda_n\overline{x}) + C_2\sinh(\lambda_n\overline{x}) - \frac{2(-1)^n}{\lambda_n} \tag{5.134}$$

Boundary conditions for the A_n are obtained by applying the original BCs in the problem, Eqs. (5.123) and (5.125), into Eq. (5.133). Since the normal gradient of $\overline{T}$ is zero at $\overline{x} = 0$ and $\overline{T}$ is zero at $\overline{x} = 1$, the BCs for A are $A_n'(0) = A_n(1) = 0$. The BC at $\overline{x} = 0$ gives $C_2 = 0$, and at $\overline{x} = 1$ the solution gives

$$C_1 = \frac{2(-1)^n}{\lambda_n\cosh(\lambda_n)}$$

The complete solution for A_n is

$$A_n(\overline{x}) = \frac{2(-1)^n}{\lambda_n}\left(\frac{\cosh(\lambda_n\overline{x})}{\cosh(\lambda_n)} - 1\right) \tag{5.135}$$

The VOP solution to the temperature profile is therefore

$$\overline{T} = 2\sum_{n-1}^{\infty} \frac{(-1)^n}{\lambda_n} \left(\frac{\cosh(\lambda_n \overline{x})}{\cosh(\lambda_n)} - 1 \right) \phi_n(\overline{y}) \tag{5.136}$$

Compare this solution to the one obtained directly with SOV – Eq. (5.127). There are certain similarities. The VOP solution appears as the sum of two separate functions. The first (involving the hyperbolic and trig functions) is essentially the same as the SOV solution, except that the $\overline{x}$ and $\overline{y}$ dependencies are switched and the sign is changed. The second part to the VOP solution is a function solely of $\overline{y}$. Similar to the previous example, this part of the solution represents the eigenfunction expansion of a relatively simple function. Specifically, for the given eigenfunction and eigenvalue, it's relatively easy to show that

$$1 = -2\sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda_n} \phi_n(\overline{y}) \tag{5.137}$$

Consequently, the VOP solution is equivalent to 1 minus the SOV solution of the same problem, except with the $\overline{x}$ and $\overline{y}$ BCs switched. That is:

$$\overline{T}_{VOP}(\overline{x}, \overline{y}) = 1 - \overline{T}_{SOV}(\overline{y}, \overline{x}) \tag{5.138}$$

You should prove to yourself that the superposition of the two partial solutions given in the right hand side will satisfy the given problem.

Again, a problem with the VOP solution is that it often leads to a slowly–converging series – especially compared to that obtained from SOV/superposition. To illustrate, the table in Fig. 5.2 lists the results from numerical evaluation of the VOP solution, as written in Eq. (5.136), compared to the results from the SOV solution in Eq. (5.127). Included are the number of terms in the series needed to obtain 3–digit accuracy in the computed results. Five terms are evaluated at a time in the series, so the n_{max} results are evenly divisible by 5. All results are for $\overline{x} = 0.5$.

The numbers in the table clearly show different convergence characteristics between the two solutions. Observe also that the VOP solution does not return the correct result at $\overline{y} = 1$. Again – this is a numerical artifact rather than a mathematical flaw. The correct result would be obtained in the limit of $\overline{y} \to 1$ – it's just that an infinite number of terms would be required to get it.

To summarize, VOP is (in my opinion) a more *methodical* approach to finding a solution to a problem than is SOV/superposition. In the former a cookbook procedure can basically be followed, whereas in the latter a superposition scheme must be invented that satisfies the problem. VOP can also lead to a more compact form of the solution – because everything is embedded into a single eigenfunction expansion. However, VOP solutions often have poor convergence rates, and also do not obviously satisfy all of the BCs to the problem.

For the most part, you will probably be better off sticking to the SOV procedure – especially if you ultimately want to calculate numerical results from your solution. We'll only use VOP when it provides a mathematical form to the solution that offers clear advantages over the SOV/superposition solution. We'll discuss such applications when we get to them

	VOP		SOV	
y	$\overline{T}$	n_{max}	$\overline{T}$	n_{max}
0.0	0.3640	6370	0.3641	10
0.1	0.3692	3230	0.3691	10
0.2	0.3818	135	0.3843	10
0.3	0.4102	3580	0.4103	10
0.4	0.4484	7875	0.4484	10
0.5	0.5001	4510	0.5000	10
0.6	0.5639	170	0.5671	10
0.7	0.6514	7020	0.6515	15
0.8	0.7539	20605	0.7538	15
0.9	0.8721	20355	0.8720	20
1.0	0.0000	0	1.0001	4510

Figure 5.2: VOP and SOV convergence results

5.5 Application of *Mathematica* to multidimensional problems

Mathematica can greatly simplify the derivation and computation of multidimensional problems. To illustrate its use, an example will be worked out entirely within the *Mathematica* framework.

Consider a square rectangular region which is initially at T_∞. The surfaces at $x = 0$ and $y = 0$ are adiabatic, that at $x = 1$ is cooled by convection, and at $t = 0$ the surface at $y = 1$ is exposed to a uniform heat flux of q_0''. Determine the transient temperature response in the region.

The dimensionless problem is

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \tag{5.139}$$

$$\left.\frac{\partial T}{\partial x}\right|_0 = \left.\frac{\partial T}{\partial y}\right|_0 = 0 \tag{5.140}$$

$$\left.\frac{\partial T}{\partial x}\right|_1 = -Bi\, T(1, y, t) \tag{5.141}$$

$$\left.\frac{\partial T}{\partial y}\right|_1 = 1 \tag{5.142}$$

$$T(x, y, 0) = 0 \tag{5.143}$$

This problem has an inhomogeneous BC and must be broken up into superimposed parts per the usual procedure. Let $T(x, y, t) = w(x, y, t) + s(x, y)$, in which s satisfies the steady problem and w satisfies the corresponding homogeneous BC problem with an initial condition of $-s$. The form of

s will be

$$s(x,y) = \sum_{n=1}^{\infty} A_n\, \phi_n(x)\, \cosh(\lambda_n y) \tag{5.144}$$

$$\phi_n(x) = \cos(\lambda_n x) \tag{5.145}$$

$$\lambda_n\, \sin(\lambda_n) = Bi\, \cos(\lambda_n) \tag{5.146}$$

$$A_n = \frac{\displaystyle\int_0^1 \phi_n(x)\, dx}{\lambda_n\, \sinh(\lambda_n) \displaystyle\int_0^1 \phi_n^2(x)\, dx} \tag{5.147}$$

and w will have

$$w(x,y) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} B_{nm}\, \phi_n(x)\, \psi_m(y)\, e^{-(\lambda_n^2+\beta_m^2)t} \tag{5.148}$$

$$\phi_n(x) = \cos(\lambda_n x) \tag{5.149}$$

$$\lambda_n\, \sin(\lambda_n) = Bi\, \cos(\lambda_n) \tag{5.150}$$

$$\psi_m(y) = \cos(\beta_m y) \tag{5.151}$$

$$\beta_m = m\,\pi \tag{5.152}$$

$$B_{nm} = -\frac{\displaystyle\int_0^1 \int_0^1 s(x,y)\phi_n(x)\, \psi_m(y)\, dx\, dy}{\displaystyle\int_0^1 \phi_n^2(x)\, dx \int_0^1 \psi_m^2(y)\, dy} \tag{5.153}$$

The *Mathematica* code is given below

```
eigencond[l_]:=l Sin[l]-bi Cos[l]

eigenroot[lamstart_]:=Module[
   {dlam=0.2,lam0,eigen0,eigen1,lamr,
      lamroot},
   lam0=lamstart+1*^-6;
   eigen0=eigencond[lam0];
   lam0=lam0+dlam;
   eigen1=eigencond[lam0];
   While[eigen0 eigen1>0,
```

```
      lam0=lam0+dlam;
      eigen0=eigen1;
      eigen1=eigencond[lam0];
   ];
   lamr=lam0-eigen1 dlam/(eigen1-eigen0);
   lamroot=
      lam/.FindRoot[eigencond[lam]==0,{lam,lamr}];
   lamroot]

ntot=100;bi=2;
lambda[1]=eigenroot[0];
Do[lambda[n]=eigenroot[lambda[n-1]],{n,2,ntot}]

phi[n_,x_]:=Cos[lambda[n] x]
psi[n_,y_]:=Cos[n Pi y]

an[n_]:=Evaluate[Integrate[phi[n, x],{x,0,1}]
 /(lambda[n] Sinh[lambda[n]]Integrate[phi[n,x]^2,{x,0,1}])]

(* this shows the definition of an[n]*)
In[51]:=an[n]
Out[51]=(Csch[lambda[n]]*Sin[lambda[n]])/
  (lambda[n]^2*(1/2 + Sin[2*lambda[n]]/(4*lambda[n])))

sterm[n_,x_,y_]:=an[n] phi[n,x] Cosh[lambda[n] y]

bnm[n_,m_]:=
 Evaluate[Simplify[
  Integrate[-sterm[n,x,y]phi[n,x]psi[m,y],{x,0,1},{y,0,1}]
   /(Integrate[phi[n,x]^2,{x,0,1}]
     *Integrate[psi[m,y]^2,{y,0,1}])]]

(* this is the special case for m=0*)
bnm[n_,0]:=
 Evaluate[Simplify[
  Integrate[-sterm[n,x,y]phi[n,x]psi[0,y],{x,0,1},{y,0,1}]
   /(Integrate[phi[n,x]^2,{x,0,1}]
     *Integrate[psi[0,y]^2,{y,0,1}])]]

(* here are the definitions of Bnm *)
```

```
In[88]:=bnm[n, m]
Out[88]=-((16*m*Pi*Csch[lambda[n]]*Sin[lambda[n]]*
    (m*Pi*Cosh[lambda[n]]*Sin[m*Pi] + Cos[m*Pi]*
      lambda[n]*Sinh[lambda[n]]))/
      (lambda[n]*(-I*m*Pi + lambda[n])*
      (I*m*Pi + lambda[n])*(2*m*Pi + Sin[2*m*Pi])*
      (2*lambda[n] + Sin[2*lambda[n]])))

In[89]:=bnm[n, 0]
Out[89]=-(Sin[lambda[n]]/(lambda[n]^3*
    (1/2 + Sin[2*lambda[n]]/(4*lambda[n]))))


wterm[n_,m_,x_,y_,t_]:=
  Re[bnm[n,m]] phi[n,x] psi[m,y] E^(-(lambda[n]^2+(Pi m)^2) t)

s[x_,y_]:=Module[{sum,n,err,oldsum},
  sum=0;n=1;err=1;
  While[err>.00001&&n<ntot-2,
    oldsum=sum;
    sum=sum+sterm[n,x,y]+sterm[n+1,x,y];
    err=Abs[sum-oldsum];
    n=n+2;
  ];sum]

w[x_,y_,t_]:=Module[{sum,n,err,oldsum},
  sum=0;n=1;err=1;
  While[err>.00001&&n<ntot-2,
    oldsum=sum;
    sum=sum+wmsum[n,x,y,t]+wmsum[n+1,x,y,t];
    err=Abs[sum-oldsum];
    n=n+2;
  ];sum]

wmsum[n_,x_,y_,t_]:=Module[{sum,m,err,oldsum},
  sum=0;m=0;err=1;
  While[err>.00001&&m<100,
    oldsum=sum;
    sum=sum+wterm[n,m,x,y,t]+wterm[n,m+1,x,y,t];
```

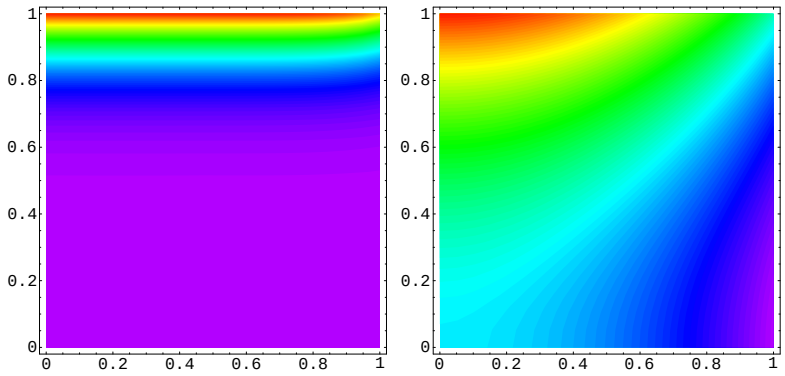


Figure 5.3: Temperature contours for $Bi = 2$; $t = 0.02$ (left) and $t = 1$ (right)

```
    err=Abs[sum-oldsum];
    m=m+2;
  ];sum]

temp[x_,y_,t_]:=s[x,y]+w[x,y,t]

cf[x_] := Hue[.8(1 - x)]

ContourPlot[temp[x, y, .02], {x, 0, 1}, {y, 0, 1},
  ContourLines -> False,PlotPoints -> 20,
  ColorFunction -> cf, Contours -> 100]
```

The code begins by calculation of λ_n, for $n = 1, 2, \ldots 100$ and $Bi = 2$, using the algorithm discussed in Ch. 3. It then calculates formulas for the A_n and B_{nm} expansion coefficients directly from their integral definitions in Eqs. (5.147) and (5.153). Note that, in the definition of B_{mn}, the integral of $s(x, y)\phi_n$ over x would give an orthogonal relation because s is expanded in the ϕ eigenfunctions. The numerator of Eq. (5.153) would therefore only include the n^{th} term in the series for s, and this is used in the *Mathematica* formula. I use the *Mathematica* command `Evaluate` to define the A_n and B_{nm} coefficients via their integral definitions; by doing so the symbolic formula for the integrals is evaluated first and then used in the function definition. The function returned by this strategy is seen in the lines immediately following the function definition (i.e., when I have *Mathematica* evaluate `an[n]`). Without the `Evaluate` present, *Mathematica* would perform the integration each and every time the function is called – and this would be very time consuming.

The formula for B_{mn} has two distinct forms depending on whether $m = 0$ or $m > 0$; these two cases need to be explicitly defined. Even though the result for B_{nm} is real valued, the symbolic

result contains the radical i (seen as `I` in the code); to avoid the inclusion of the zero imaginary part (i.e., 1.+0.I) in the solution the real value of B_{nm} must be taken where it is used (via `Re[bnm[n,m]]`).

The evaluation of the double series for w proceeds directly from the algorithm used to evaluate a single series. To do this, I write the formula for w as

$$w(x,y,t) = \sum_{n=1}^{\infty} w_n(x,y,t)$$

$$w_n(x,y,t) = \sum_{m=0}^{\infty} B_{nm}\,\phi_n(x)\,\psi_m(y)\,e^{-(\lambda_n^2+\beta_m^2)t}$$

The algorithm used to sum a single series, discussed in Ch. 3, is then employed in a nested procedure to calculate w and w_n.

The results are shown in Fig. 5.3 and correspond to $Bi = 2$; the contour plot on the left has $t = 0.02$ and the one on the right has $t = 1$ – which essentially is the steady–state condition.

5.6 Semi–Infinite Regions

All of the problems encountered up to this point have been of the class of two–point boundary value problems. Specifically, the region to which the analysis is applied is bounded by two distinct points, i.e., the inner and outer surfaces of the slab or the cylinder.

Semi–Infinite (SI) regions correspond to situations in which the medium is unbounded in one or more directions. A common example is the ground – for all practical purposes (at least those involving length scales on the order of human scales) we can assume that the earth extends downwards to infinity. In a more general sense we can consider a boundary to be at 'infinity' when the length to the boundary is significantly larger than all other characteristic length scales in the problem. For example, a circular pin fin (a.k.a. an extended surface) of length L and radius R can be taken to be infinite in length whenever $L \gg R$ and $L \gg \sqrt{kR/2h}$. Likewise, a plane wall of thickness L can be considered semi–infinite following an instantaneous change in surface temperature providing that $L \gg \sqrt{\alpha t}$. Analytical methods for treating transient SI problems will be examined in the next chapter; here we are concerned only with the steady state condition.

Certain steady SI problems can be handled with the same techniques used in proceeding examples. Specifically, the SOV/superposition methods can be applied directly to problems in which only one direction extends to infinity – which implies that the other directions are bounded. As an example, consider a square cross section fin of unit width with a length L sufficiently long to meet the SI criterion. Per the usual fin model, the base ($z = 0$) is maintained at unit temperature and the sides ($x, y = 1$) are cooled by convection to a zero ambient temperature. The boundary condition at $z \to \infty$ would be $T \to 0$, i.e., at sufficiently far distances from the base the fin would cool to the ambient temperature.

The general solution to this problem will be in the same form as Eq. (5.96), except that the hyperbolic functions are replaced by the decaying exponential in z, i.e.,

$$\overline{T} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn}\, \phi_n(\overline{x}) \psi_m(\overline{y})\, e^{-\gamma_{mn}\, z} \tag{5.154}$$

where $\gamma_{mn} = (\lambda_n^2 + \beta_m^2)^{1/2}$ as before. Recall that the hyperbolic functions are defined as combinations of exponentials with positive and negative argument; to satisfy the boundary condition at $z \to \infty$ we want to include only the decaying part of the solution. Of course, we could have retained the hyperbolic–function formulation and used the fact that

$$e^{-\gamma_{mn} z} = \cosh(\gamma_{mn} z) - \sinh(\gamma_{mn} z)$$

This would provide a perfectly valid mathematical solution in the form of Eq. (5.96) yet numerically it would be a computational disaster – in that a very small number (the left hand side) would be produced from the difference of two very large numbers.

Inhomogeneous problems in regions with one SI direction can be handled per the superpostion methods. For example, say that the fin in the above example had, in addition to the stated BCs, a uniform internal heat generation rate of q'''. The solution approach would be to split the problem into $T(x, y, z) = s(x, y) + w(x, y, z)$, where the problem for s contains the generation term. The resulting problem for w would be amenable to SOV solution techniques.

5.6.1 SI problems in two directions: Fourier transform techniques

Consider now the problem illustrated in Fig. 5.4. An electronic circuit (a 'chip'), of square cross section with width $2W$, is mounted onto the surface of a large heat–conducting plate. The plate can be assumed to extend to infinity in the $\pm x$, $\pm y$, and z directions. The chip dissapates a uniform heat flux of strength q''_C into the plate, and the rest of the plate surface (that not under the chip) is adiabatic. At distances far from the chip the temperature of the plate is T_∞.

The problem here is to calculate the temperature distribution in the plate. Define the dimensionless variables in the usual way;

$$T \to \frac{T - T_\infty}{kW/q''_C}, \quad x \to \frac{x}{W}, \quad y \to \frac{y}{W}$$

The temperature distribution in the plate will be symmetrical in each quadrant of the plate, and

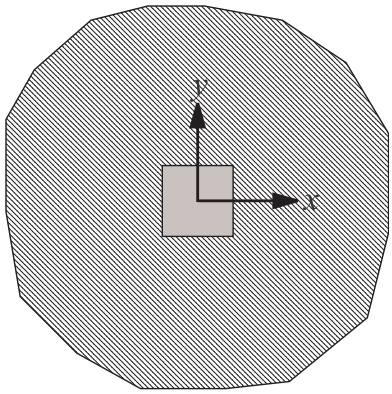


Figure 5.4: chip on a semi–infinite plate; the z direction extends downwards

the problem statement becomes

$$\begin{aligned}
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} &= 0 \\
\left.\frac{\partial T}{\partial x}\right|_{x=0} &= 0 \\
\left.\frac{\partial T}{\partial y}\right|_{y=0} &= 0 \\
\left.\frac{\partial T}{\partial z}\right|_{z=0} &= \begin{cases} -1 & x \leq 1 \text{ and } y \leq 1 \\ 0 & x > 1 \text{ or } y > 1 \end{cases} \\
T(x \to \infty, y, z) &\to 0 \\
T(x, y \to \infty, z) &\to 0 \\
T(x, y, z \to \infty) &\to 0
\end{aligned} \tag{5.155}$$

Even though the boundary conditions in the x and y directions are homogeneous for this problem, the SOV method cannot be applied to the problem. That this is the case should be easy to see; the eigenfunctions would have to be in the form $\cos(\lambda_n x)$, yet the eigencondition would have $\cos(\lambda_n \cdot \infty) = 0$ – which does not make any mathematical sense. A fundamentally new mathematical procedure is needed to tackle this problem.

The new procedure will involve what is known as a *Fourier Cosine transformation* of the dependent variable (temperature). The workings of this method can be seen by starting with our familiar Fourier series expansion method and 'stretching' the domain to infinity – for which the series will reduce to an integral.

Say we have some arbitrary function $f(x)$ that is defined between $0 \leq x \leq L$. The Fourier cosine expansion of f between 0 and L would be given by

$$f(x) = \sum_{n=0}^{\infty} A_n \cos(n\pi\, x/L)$$

where

$$A_n = \frac{\displaystyle\int_0^L f(x)\, \cos(n\pi\, x/L)\, dx}{\displaystyle\int_0^L \cos^2(n\pi\, x/L)\, dx} = \frac{2}{L}\int_0^L f(x)\, \cos(n\pi\, x/L)\, dx$$

The series expansion is, therefore,

$$f(x) = \frac{2}{L}\sum_{n=0}^{\infty}\left(\int_0^L f(x')\, \cos(n\pi\, x'/L)\, dx'\right)\cos(n\pi\, x/L)$$

where x' denotes a dummy variable of integration. Now define $\Delta\lambda \equiv \pi/L$ and substitute into the above

$$f(x) = \frac{2}{\pi}\sum_{n=0}^{\infty}\left(\int_0^L f(x')\, \cos(n\Delta\lambda\, x')\, dx'\right)\cos(n\Delta\lambda\, x)\, \Delta\lambda$$

If we now take the limit of $\Delta\lambda \to 0$ (for which $L \to \infty$), we see that the summation becomes equivalent to the definition of an integral over the variable λ. That is, $n\Delta\lambda \to \lambda$ and $\Delta\lambda \to d\,\lambda$, and the formula for $L \to \infty$ becomes

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\left(\int_0^{\infty} f(x')\, \cos(\lambda\, x')\, dx'\right)\cos(\lambda\, x)\, d\lambda \tag{5.156}$$

Equation (5.156) defines both a Fourier cosine transform and an inverse transform of a function f, those being

$$\hat{f}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\, \cos(\lambda\, x)\, dx \tag{5.157}$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \hat{f}(\lambda)\, \cos(\lambda\, x)\, d\lambda \tag{5.158}$$

The transform variable λ can be viewed as the 'continuous' analog to the discrete eigenvalues $\lambda_n = n\pi/L$. The only restriction on f is that $|f(x)| \to 0$ for $x \to \infty$.

Application of the Fourier cosine transform to the problem at hand follows this basic strategy: We first apply the tranform to the homogeneous directions (x and y) in the governing DE and BCs of the problem to obtain a DE for the transformed variable $\hat{T}$ – which will be a function of z and

the transform variables. The boundary conditions at x and $y \to \infty$ will be automatically satisfied by the transform procedure. We solve the resulting DE for $\hat{T}$ and (this is the hard part) perform the inverse transform on $\hat{T}$ to obtain the desired solution T.

The transformed variable $\hat{T}$ will be defined by the 2–D analog to Eq. (5.157), i.e.,

$$\hat{T}(\lambda, \beta, z) = \frac{2}{\pi} \int_0^\infty \int_0^\infty T(x, y, z)\, \cos(\lambda\, x)\, dx\, \cos(\beta\, y)\, dy \tag{5.159}$$

Transformation of the governing DE for T is straightforward. By using integration by parts, you should see that

$$\frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{\partial^2 T}{\partial x^2}\, \cos(\lambda\, x)\, dx\, \cos(\beta\, y)\, dy$$
$$= -\frac{2}{\pi} \int_0^\infty \int_0^\infty \lambda^2\, T\, \cos(\lambda\, x)\, dx\, \cos(\beta\, y)\, dy = -\lambda^2\, \hat{T}(\lambda, \beta, z)$$

and likewise for the partial with respect to y. We assume, in application of integration by parts, that the gradient of T vanishes at $x \to \infty$ – which is logical since T goes to the constant 0 in this limit. To transform the z–differentiated term in the DE we switch the order of differentiation and integration, leading to

$$\frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{\partial^2 T}{\partial z^2}\, \cos(\lambda\, x)\, dx\, \cos(\beta\, y)\, dy = \frac{2}{\pi} \frac{\partial^2}{\partial z^2} \int_0^\infty \int_0^\infty T\, \cos(\lambda\, x)\, dx\, \cos(\beta\, y)\, dy = \frac{d^2 \hat{T}}{dz^2}$$

We take the z derivative of the transformed temperature to be ordinary rather than partial, because the transformed DE will have only z as the independent variable. Specifically, the transformed equation is

$$\frac{d^2 \hat{T}}{dz^2} - \left(\lambda^2 + \beta^2\right)\, \hat{T} = 0 \tag{5.160}$$

which has the solution

$$\hat{T} = A\, e^{-\gamma\, z} + B\, e^{\gamma z}$$

where

$$\gamma = \left(\lambda^2 + \beta^2\right)^{1/2}$$

The temperature must vanish at $z \to \infty$ – and consequently $\hat{T}$ must also vanish in this limit. This condition is met by setting $B = 0$. We now transform the inhomogeneous boundary condition at $z = 0$, Eq. (5.155), which gives

$$\left.\frac{d\hat{T}}{dz}\right|_{z=0} = -A\, \left(\lambda^2 + \beta^2\right)^{1/2}$$
$$= -\frac{2}{\pi} \int_0^1 \int_0^1 \cos(\lambda\, x)\, dx\, \cos(\beta\, y)\, dy = -\frac{2 \sin(\lambda) \sin(\beta)}{\pi \lambda \beta}$$

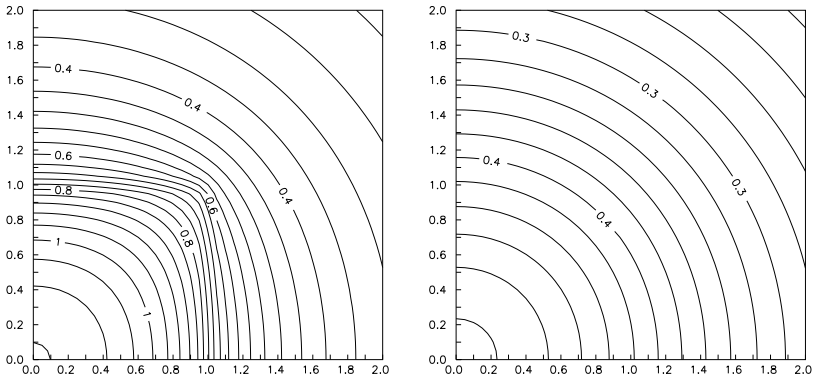


Figure 5.5: temperature contours under the chip: $z = 0$ (left) and $z = 1$ (right)

or

$$A = \frac{2\sin(\lambda)\sin(\beta)}{\pi\lambda\beta\left(\lambda^2+\beta^2\right)^{1/2}}$$

Recognize that the limits on the integration over x and y in the right–hand–side of the above equation extend only to 1, as opposed to ∞, because the heat flux into the plate is non–zero only directly under the chip.

The complete solution to the transformed temperature is therefore

$$\hat{T}(\lambda,\beta,z) = \frac{2\sin(\lambda)\sin(\beta)}{\pi\lambda\beta\left(\lambda^2+\beta^2\right)^{1/2}}\,\exp\left[-\left(\lambda^2+\beta^2\right)^{1/2}\,z\right] \tag{5.161}$$

and the temperature is given by the inverse transform,

$$T(x,y,z) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \hat{T}(\lambda,\beta,z)\,\cos(\lambda\,x)\cos(\beta\,y)\,d\lambda\,d\beta \tag{5.162}$$

Mathematica issues

Equations (5.161) and (5.162) provide the complete analytical solution for the temperature field in the plate. Unlike the series solutions that we have derived in previous problems, the solution here now appears as an integral equation. The difficult part in utilizing the solution, as mentioned above, is evaluating the inverse transform. It turns out that integral in Eq. (5.162), for the transformed variable given in Eq. (5.161), does not have an analytical solution. This is, more often than not, the case when using the Fourier transform method for solution of heat conduction problems.

Making matters worse, the integral posed by Eq. (5.162) is notoriously difficult to compute numerically using quadrature methods – especially for large x and y. The troublesome nature of the integral actually arises from the oscillatory $\cos(\lambda x)\cos(\beta y)$ function in the integrand. Increasing x and y results in a shorter wavelength of oscillation, which in turn requires an increasing number of quadrature points to obtain an accurate result. Recognize that it is the cancellation of positive/negative cycles in the oscillations, during integration over λ and β, that make $T \to 0$ as x and y become $\gg 1$.

Some attention and user–intervention is therefore required to evaluate Eq. (5.162) by the `NIntegrate` command in *Mathematica*. The following code is very slow – yet it manages to obtain around 3–digit accuracy in the temperature solution with a minimum of complaints from *Mathematica*.

```
hatt[lam_,bet_,z_]:=2 Sin[lam]Sin[bet]/
       (Pi lam bet (lam^2+bet^2)^(1/2))*
       E^(-(lam^2+bet^2)^(1/2) z)
t[x_,y_,z_]:=2/Pi NIntegrate[hatt[lam,bet,z] Cos[lam x]
       Cos[bet y],{lam,1*^-6,30},{bet,1*^-6,30},
       PrecisionGoal->4,AccuracyGoal->4,MaxRecursion->20]
```

Observe that the limits of integration have been changed to 10^{-6} and 30 in the numerical scheme. The lower limit avoids the singularity in $\hat{T}$ at λ or $\beta = 0$ (which is integratable yet difficult to work with) and the upper limit is chosen simply to be sufficiently large so that $\hat{T} \ll 1$ at the limit value. Both these values have to be set by inspection – one experiments with increasingly smaller lower and larger upper limits until the results no longer change significantly.

The *Mathematica* code is good if you need only a few temperature values from the solution. For example, the temperatures directly under the center of the chip and at the corner of the chip are $T(0,0,0) = 1.122$ and $T(1,1,0) = 0.561$ – which would be desired information from an electronic cooling point of view. Production runs of numerical results – such as that needed for a contour plot – would probably be best obtained from a compiled fortran code employing a numerical integration package.

Contour plots in the $x-y$ plane of the temperature distribution are shown in Fig. 5.5, in which the left plot has $z = 0$ (the surface) and the right has $z = 1$. The surface temperature plot clearly shows the presence of the square chip, yet these shape effects have largely disappeared at $z = 1$ for which the contours are essentially circular about the symmetry axis of the plate.

Exercises

1. A rectangular pin fin, having length L, thickness b and a very wide width, is initially at the ambient temperature T_∞. At $t = 0$ the base temperature is instantaneously brought to T_b.

Obtain the solution for the transient and 2–D dimensionless temperature distribution in the fin. Show that the solution reduces to the 1–D and transient form when $hb/k \ll 1$.

2. A long solid circular cylinder of radius R is initially at a temperature of T_∞. At $t = 0$ one side of the cylindrical surface is exposed to a source of thermal radiation which results in a heat flux into the cylinder. The distribution of flux is given by

$$q''_s = \begin{cases} q''_0 \cos\phi, & -\frac{\pi}{2} \le \phi \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le \phi \le \frac{3\pi}{2} \end{cases}$$

where q''_0 is a constant. The cylindrical surfaces also transfer heat by convection to T_∞.

   (a) Formulate the problem in dimensionless variables and parameters.
   (b) Devise a partial solutions approach to the problem, involving a transient solution and a steady–state solution.
   (c) Derive the transient and steady parts to the solution. Note that a similar form of the steady problem was examined in Ch. 4.

3. A long square rod, of width 30 cm, is to be heated in a high–temperature convection furnace. The rod is at an initial temperature of 30°C and has a thermal conductivity and thermal diffusivity of 60 W/m·K and 18×10^{-6} m^2/s, respectively. The rod will remain in the furnace until the center temperature reaches 300°C. To minimize thermal stresses in the rod, the surface temperature of the rod cannot exceed 600°C during the heating. The objective of this problem is to develop a pair of design curves for the furnace. One plot would give h_{max} vs. oven temperature T_∞, where h_{max} is the maximum allowable heat transfer coefficient which maintains the surface temperature constraint. For example, if $T_\infty \le 600$°C then h_{max} would $\to \infty$ – since the surface temperature could not exceed 600°C for this condition. For any T_∞ greater than 600°C the value of h_{max} will be finite, and will decrease as T_∞ increases. The second plot would show the required heating time t as a function of T_∞. Considering that h values for forced convection in air are around 100–1000 W/m^2·K, comment on the likely operating conditions for the furnace.

4. A solid cylinder has a length L and a radius R. The surface of the cylinder at $r = R$ has a prescribed temperature distribution given by $f(z, \phi)$, whereas the surfaces at $z = 0$ and L are adiabatic. Derive the series expansion for the steady–state temperature distribution in the cylinder, and give formulas for the expansion coefficients.

5. Consider the problem examined in Sec. (5.6.1), except now take the plate to have a finite depth of L in the z direction. At $z = L$ (the bottom) the plate is cooled by convection to T_∞. All other boundary conditions remain the same. Formulate the problem in non–dimensional variables and use the Fourier cosine transform method to derive the analytical solution to the problem.

Chapter 6

General Time–Dependent Conduction

6.1 Introduction

The transient problems that we have encountered so far have all dealt with an instantaneous perturbation of a system – such as a sudden change in wall temperature – followed by a relaxing of the system to a new equilibrium condition. We have not dealt with problems in which we apply a time–dependent 'forcing' to the system. For example, perhaps we change the wall temperature from T_0 to T_1 over a time interval Δt. Or perhaps the system is exposed to a time–periodic heat flux. What we want to do in this chapter is cover some of the analytical techniques that are used to predict the time-dependent temperature fields and heat fluxes for systems that have more general, time–dependent boundary conditions and sources.

6.2 Initial value problems with time–dependent BCs and/or sources

Here we want to extend the transient SOV analysis that was developed in Ch. 3 and 5 to include problems in which the BCs and/or heat source are a non–periodic function of time. This class of initial–value problems include all situations in which the BCs and/or source function are 'forced' (i.e., changed by external means) from one fixed state to another fixed state over a *finite* (i.e., non–zero) time interval.

The transient impulse problems examined so far had BCs that were, in a sense, a function of time – in that at $t = 0$ the conditions at one or more of the surfaces *instantaneously* changed from one state to another. The same type of problem was encountered for heat generation – at $t = 0$ the source function q''' would be turned on (or off). Mathematically, we would say that the time–dependence of the BCs or q''' in such problems would be represented by a step function $H(t)$, for which

$$H(t) = \begin{cases} 0, & t \leq 0 \\ 1 & t > 0 \end{cases} \tag{6.1}$$

For example, consider a problem in which we have a unit–width slab with zero temperature maintained at $x = 0$, zero initial temperature, and at $t = 0$ the temperature at $x = 1$ is instantaneously brought to unity. We can write the generalized problem statement using the step function as

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{6.2}$$

$$T(0,t) = 0 \tag{6.3}$$

$$T(1,t) = H(t) \tag{6.4}$$

$$T(x, t \leq 0) = 0 \tag{6.5}$$

So, 'formally', the BC is now represented by a time–dependent function – although we have simply introduced some new mathematical terminology into a problem that has already been solved with SOV and superposition methods.

The objective at hand is to develop a solution method for a forcing that is an arbitrary function of time. With regard to the example problem given above, the condition at $x = 1$ would now appear in the general form of

$$T(1,t) = F(t) \tag{6.6}$$

in which F is a prescribed function of t. The only restriction we place on F is that it is zero for $t \leq 0$.

Problems with time–dependent BCs in the form of Eq. (6.6) (or sources of equivalent form) can be solved (in principle) by the method of variation of parameters, and an example of the application of this approach to such problems will be presented in the following section. A different approach – which has greater recognition among the analytical community – is to use a method that is analogous to the superposition technique developed to handle arbitrarily inhomogeneous transient impulse problems.

6.2.1 Time–dependent superposition: Duhamel's theorem

Duhamel's theorem is an analytical method in which the solution for a forced system is obtained from an integral transformation of the solution to the *fundamental problem* that corresponds to the system. The systems to which this method can be applied must have an initial state of zero temperature, and be completely homogeneous except for a single time–dependent forcing function appearing in a BC or the source. This forcing function can be only a function of time (not of position). The corresponding fundamental solution, which is given the symbol $U(x,t)$, represents the solution to the same system except with the forcing function replaced by an equivalent unit step function at $t = 0$.

The fundamental solutions are essentially the types of problems we have been solving up to now – but again with the restriction that the initial temperature is zero and the problem contains a single inhomogenous term. The example problem presented in the previous section (Eq. (6.2)–(6.5)) would represent the corresponding fundamental problem to the situation in which the BC at

$x = 1$ is Eq. (6.6). The forcing in this example appears as a time–dependent surface temperature; accordingly, the equivalent step function at the boundary appears as an instantaneous change in surface temperature from 0 to 1 at $t = 0$. Similarly, a forcing which appeared as a time–dependent heat flux at the boundary would have an instantaneous change in flux from 0 to 1 in the fundamental problem. And a time–dependent source term q''' would be represented in the fundamental problem by a unit step change in generation rate at $t = 0$.

Providing that the problem at hand meets the above restrictions, Duhamel's theorem states that the solution $T(x,t)$ for the forced problem will be given by

$$\boxed{T(x,t) = \int_0^t U(x, t-\tau)\frac{dF}{d\tau}\,d\tau} \tag{6.7}$$

The variable τ in the above is simply a dummy variable of integration. The above integral – in which the integrand has one function evaluated at $t - \tau$ and another evaluated at τ – is known as a convolution integral.

The proof of Duhamel's theorem is straightforward. To determine if T satisfies the conduction equation, first differentiate Eq. (6.7) with respect to t. Leibnitz' rule for differentiation of an integral is needed to do this because the upper limit (t) is a function of t. For our purposes, this rule is

$$\frac{\partial}{\partial t}\int_0^{f(t)} u(t,\tau)\,d\tau = u[t, f(t)]\frac{df}{dt} + \int_0^{f(t)} \frac{\partial}{\partial t}u(t,\tau)\,d\tau$$

Application of this to Eq. (6.7) gives

$$\frac{\partial T}{\partial t} = U(x, t-t)\frac{dF}{dt} + \int_0^t \frac{\partial}{\partial t}\left[U(x, t-\tau)\right]\frac{dF}{d\tau}\,d\tau$$

Our definition of the fundamental solution has $U(x,0) = 0$ for all x and this eliminates the first term. Likewise, the second derivative of T with respect to x is

$$\frac{\partial^2 T}{\partial x^2} = \int_0^t \frac{\partial^2}{\partial x^2}\left[U(x, t-\tau)\right]\frac{dF}{d\tau}\,d\tau$$

Subtracting the two previous equations and combining the integrands gives

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = \int_0^t \left(\frac{\partial}{\partial t}\left[U(x, t-\tau)\right] - \frac{\partial^2}{\partial x^2}\left[U(x, t-\tau)\right]\right)\frac{dF}{d\tau}\,d\tau$$

The quantity $t - \tau$ is ≥ 0 and can be denoted as a new time variable t'. If no heat generation is present, then $U(x, t')$ must satisfy the homogeneous conduction equation for all times. The integral on the right hand side must therefore be zero – regardless of F – which shows that T satisfies the homogeneous conduction equation.

If heat generation is present, then the conduction equations for the forced and fundamental problems would have

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = F(t), \qquad \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 1$$

for all $t' \geq 0$. Replacing these into the previous integral would give

$$F(t) = \int_0^t \frac{dF}{d\tau}\, d\tau = F(t) - \underbrace{F(0)}_{=0} = F(t)$$

which uses the fact that $F(0)$ is constrained to be zero – because the system must commence from a zero temperature initial state. Therefore, the solution given by Eq. (6.7) correctly satisfies the inhomogeneous conduction equation when heat generation is present.

If, on the other hand, the forcing appeared as a time–dependent BC at $x = 1$, Eq. (6.7) would have

$$\begin{aligned} T(1,t) &= \int_0^t \underbrace{U(1,t-\tau)}_{=1} \frac{dF}{d\tau}\, d\tau \\ &= \int_0^t \frac{dF}{d\tau}\, d\tau = F(t) - \underbrace{F(0)}_{=0} \\ &= F(t) \end{aligned}$$

which now uses the fact that $U(1,t') = 1$ for all $t' \geq 0$. Equation (6.7) is therefore consistent with the time–dependent BC. The remaining BC in the problem must be homogeneous and application of this BC to the right–hand–side of Eq. (6.7) must give zero – which will automatically provide the same homogeneous BC on T.

Duhamel's theorem is essentially a superposition solution – in which the desired solution for $T(x,t)$ is constructed from the addition of partial solutions. To see this, approximate the integral in Eq. (6.7) as a sum over M segments, of width $\Delta\tau$, via

$$T(x,t) \approx \sum_{i=1}^{M} U(x, t-\tau_i)\Delta F(\tau_i)$$

where $\tau_i = i \cdot \Delta\tau$ and $\Delta F(\tau_i)$ represents the change in F that occurs at time τ_i over the time step $\Delta\tau$. Recall now that the fundamental solution $U(x,t)$ represents the response of the system at time t to a *unit* step change at $t = 0$. It then follows that $\Delta F(\tau_i)U(x, t-\tau_i)$ would represent the response of the system at time $t - \tau_i$ to a step of $\Delta F(\tau_i)$ at time $t = \tau_i$. The summation formula given above can therefore be interpreted as a superposition of a multitude of infinitesimal step–change events, each occuring further and further 'back' from the present time. In the limit of $\Delta\tau \to 0$ (and $M \to \infty$), we would recover the integral formula in Eq. (6.7).

6.2.2 Discontinuous and piecewise continuous forcing functions

Some attention must be paid to the implementation of Eq. (6.7) when the forcing function F changes discontinuously within the range of integration. The simplest case of this occurrence is when the forcing function takes the form of a step function $H(t)$, i.e., the forcing represents a unit step change in the BC or source at $t = 0$. In general, however, the forcing function could have an arbitrary number of discontinuous jumps at $t = t_1, t_2, \ldots$. The value of $dF/d\tau$ at the jumps would goto infinity – which might seem to imply that the solution predicted by Eq. (6.7) would become singular for discontinuous forcing functions.

To show that this is not the case, consider the unit step $F(t) = H(t)$. The derivative of H at $t = 0$ is modelled by a delta function $\delta(t)$, which is defined by

$$\frac{dH}{dt} = \delta(t) \quad \begin{cases} = 0, & t \neq 0 \\ \to \infty, & t = 0 \end{cases}$$

The delta function, when appearing in an integrand, has the following 'combing' property,

$$\int_0^t f(\tau)\delta(t_1 - \tau)\, d\tau = f(t_1)$$

in which f is an arbitrary continuous function and the jump point t_1 is in between 0 and t. This integration property can be proved by combination of the previous two equations and applying integration by parts;

$$\begin{aligned} \int_0^t f(\tau)\delta(t_1 - \tau)\, d\tau &= -\int_0^t f(\tau)\frac{dH(t_1 - \tau)}{d\tau}\, d\tau \\ &= -f(t)\underbrace{H(t_1 - t)}_{=0} + f(0)\underbrace{H(t_1 - 0)}_{=1} + \underbrace{\int_0^t \frac{df(\tau)}{d\tau} H(t_1 - \tau)\, d\tau}_{= f(t_1) - f(0)} \\ &= f(t_1) \end{aligned}$$

where we've used the fact that $H(t - \tau)$ is 0 for $t \leq \tau$ and 1 for $t > \tau$.

With this information in hand, we can now generalize Eq. (6.7) to situations in which the forcing function changes discontinuously. First we need to formulate a sufficiently general equation for the forcing function – which is provided by the 'piecewise continuous' form of

$$F(t) = \begin{cases} 0 & , \quad t = 0 \\ F_1(t) & , \quad 0 < t \leq t_1 \\ F_2(t) & , \quad t_1 < t \leq t_2 \\ \vdots \end{cases} \tag{6.8}$$

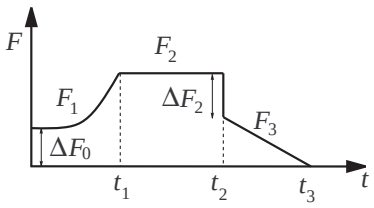


Figure 6.1: an arbitrary piecewise continuous forcing function

in which F_1, $F_2, \ldots$ are taken to be continuous (or more precisely, first–order differentiable) in the specified range. This form can also be written using the step functions as

$$F(t) = (H(t) - H(t-t_1))\ F_1(t) + (H(t-t_1) - H(t-t_2))\ F_2(t) + \ldots$$

and the derivative of F is obtained using the chain rule and Eq. (6.8);

$$\begin{aligned}\frac{dF}{dt} &= \Delta F_0\,\delta(t) + (H(t) - H(t-t_1))\ \frac{dF_1}{dt} \\ &\quad + \Delta F_1\,\delta(t-t_1) + (H(t-t_1) - H(t-t_2))\ \frac{dF_2}{dt} + \ldots\end{aligned}$$

where $\Delta F_0 = F_1(t \to 0)$, $\Delta F_1 = F_2(t \to \bar{t}_1) - F_1(t_1)$, etc. An arbitrary forcing function, consisting of three continuous segments and two jumps at $t = 0$ and t_2, is illustrated in Fig. 6.1. For this particular example the jumps in the function at t_1 and t_3 are zero – even though the function changes its characteristic form at these points.

When applied to Eq. (6.7) and using the integration properties of the delta function, the temperature solution for the generalized forcing function becomes

$$\begin{aligned}T(x,t) &= \Delta F_0\,U(x,t) \\ &\quad + \sum_{i=1}^{M}\left(\Delta F_i\,U(x,t-t_i) + \int_{t_{i-1}}^{t_i} U(x,t-\tau)\,\frac{dF_i}{d\tau}\,d\tau\right) \\ &\quad + \int_{t_M}^{t} U(x,t-\tau)\,\frac{dF_{M+1}}{d\tau}\,d\tau \qquad (6.9)\end{aligned}$$

in which the limit M in the sum is defined by the largest t_M such that $t > t_M$. That is, the solution includes all forcings that have occured *prior* to time t – we would not expect the solution at t to be affected by an event in the future!.

The workings of Eq. (6.9) can be seen by application to the situation in which the forcing function is $F(t) = H(t)$, i.e., the unit step at $t = 0$. The only surviving term in Eq. (6.9) would be

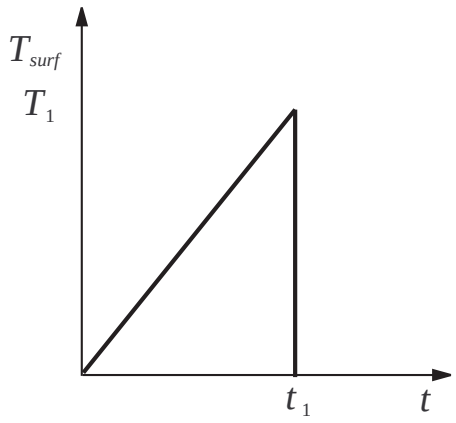


Figure 6.2: ramped surface temperature

the jump at $t = 0$, for which $\Delta F_0 = 1$. This would give $T(x,t) = U(x,t)$ – which it must since this problem represents the fundamental solution by definition.

A more challenging – yet still simple – example of a forcing function is given in Fig. 6.2, in which the boundary temperature linearly changes from 0 to T_1 over a period of t_1, and then instantaneous drops to 0 at $t = t_1$ and remains at zero for all time. Mathematically, the forcing function is represented by

$$F(t) = \begin{cases} T_1 \cdot \dfrac{t}{t_1}, & 0 < t \le t_1 \\ 0, & t > t_1 \end{cases} \tag{6.10}$$

The initial jump $\Delta F_0 = 0$, and Eq. (6.9) would give

$$T(x,t) = \begin{cases} \dfrac{T_1}{t_1} \displaystyle\int_0^t U(x, t-\tau)\, d\tau, & t \le t_1 \\ -T_1\, U(x, t-t_1) + \dfrac{T_1}{t_1} \displaystyle\int_0^{t_1} U(x, t-\tau)\, d\tau, & t > t_1 \end{cases} \tag{6.11}$$

Recognize that the forcing function changes from T_1 to 0 at $t = t_1$; consequently $\Delta F_1 = -T_1$.

To complete the problem we first need the fundamental solution U. The fundamental problem is given by Eqs. (6.2)–(6.5) – which consists again of zero temperature at $x = 0$ and unit temperature at $x = 1$. From superpositon/SOV, U is

$$U(x,t) = x + \sum_{n=1}^{\infty} A_n \phi_n(x) e^{-\lambda_n^2 t} \tag{6.12}$$

in which

$$\phi_n(x) = \sin(\lambda_n x)$$
$$\lambda_n = n\pi$$
$$A_n = \frac{2(-1)^n}{\lambda_n}$$

Consequently, $U(x, t-\tau)$ is

$$U(x, t-\tau) = x + \sum_{n=1}^{\infty} A_n \phi_n(x) e^{-\lambda_n^2 t} e^{\lambda_n^2 \tau} \tag{6.13}$$

For $t \le t_1$, Eq. (6.11) will give

$$\begin{aligned} T &= x\frac{T_1 t}{t_1} + \frac{T_1}{t_1}\sum_{n=1}^{\infty} \frac{A_n\phi_n(x)}{\lambda_n^2} e^{-\lambda_n^2 t}\left(e^{\lambda_n^2 t} - 1\right) \\ &= x\frac{T_1 t}{t_1} + \frac{T_1}{t_1}\sum_{n=1}^{\infty} \frac{A_n\phi_n(x)}{\lambda_n^2}\left(1 - e^{-\lambda_n^2 t}\right), \quad t < t_1 \end{aligned} \tag{6.14}$$

and for $t > t_1$ the solution is

$$\begin{aligned} T &= xT_1 + \frac{T_1}{t_1}\sum_{n=1}^{\infty} \frac{A_n\phi_n(x)}{\lambda_n^2} e^{-\lambda_n^2 t}\left(e^{\lambda_n^2 t_1} - 1\right) \\ &\quad - xT_1 - T_1\sum_{n=1} A_n\phi_n(x) e^{-\lambda_n^2 t} e^{\lambda_n^2 t_1} \\ &= T_1\sum_{n=1}^{\infty} A_n\phi_n(x)\left[\frac{1}{t_1\lambda_n^2}\left(e^{\lambda_n^2 t_1} - 1\right) - e^{\lambda_n^2 t_1}\right] e^{-\lambda_n^2 t} \end{aligned} \tag{6.15}$$

Observe that the solution for $t \gg t_1$ decays to 0 – which it must since the surface temperature at $x = 1$ has been brought back to 0.

Surface plots of the solution are shown in Fig. 6.3 for $t_1 = 0.05$ (left) and $t_1 = 0.5$ (right). In both cases $T_1 = 1$. Notice that when $t_1 = 0.05$ the region next to the $x = 0$ surface is essentially unaffected by the changing conditions at $x = 1$; for this case the characteristic time of the disturbance $\sim t_1 = 0.05$ is less than the characteristic diffusion time of the wall (which corresponds, dimensionlessly, to $t_d \sim 0.1$). Consequently, the disturbance is diffused (or damped out) before it can affect the back wall conditions. Indeed, the wall could be modelled as *semi–infinite* for this condition – and this topic will be addressed in a forthcoming section. On the other hand, the case for $t_1 = 0.5$ shows that during the ramp–up period the temperature distribution in the wall is nearly linear at any moment. The reason for this behavior follow the same logic; the time of the disturbance is now much larger than the diffusion time of the wall and, consequently, the wall attains essentially a 'quasi'–steady–state temperature distribution during the heat up process.

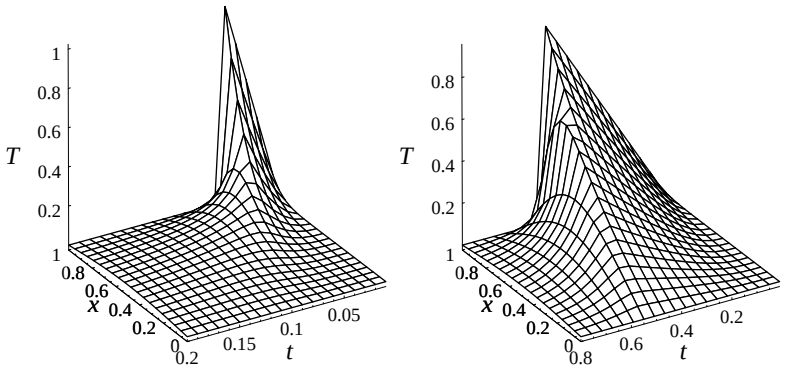


Figure 6.3: effect of the linear ramp in surface temperature

Alternative form to Duhamel's theorem

One way of getting around the need to 'specially' treat the discontinous parts of F in Eq. (6.7) is to integrate Eq. (6.7) by parts – as was done when we defined the integral property of the delta function. Doing this, we find

$$\begin{aligned} T(x,t) &= \int_0^t U(x,t-\tau)\frac{dF}{d\tau}\,d\tau \\ &= F(t)U(x,0) - F(0)U(x,t) - \int_0^t \frac{\partial U(x,t-\tau)}{\partial \tau}F(\tau)\,d\tau \end{aligned}$$

Since both U and F are zero at $t=0$ the first two terms disappear. We can also use

$$\frac{\partial U(x,t-\tau)}{\partial \tau} = -\frac{\partial U(x,t-\tau)}{\partial t}$$

which gives the alternative form of Duhamel's theorem:

$$\boxed{T(x,t) = \int_0^t \frac{\partial U(x,t-\tau)}{\partial t}F(\tau)\,d\tau} \tag{6.16}$$

The above equation is completely equivalent to the original form in Eq. (6.7). The convenient aspect of the above, however, is that we no longer need to be concerned about jumps in F. Rather, for the arbitrary piecewise continuous forcing function, Eq. (6.16) appears

$$T(x,t) = \sum_{i=1}^{M}\int_{t_{i-1}}^{t_i} \frac{\partial U(x,t-\tau)}{\partial t}F_i(\tau)\,d\tau + \int_{t_M}^{t} \frac{\partial U(x,t-\tau)}{\partial t}F_{M+1}(\tau)\,d\tau \tag{6.17}$$

subject to the same restriction that $t > t_M$.

The solutions obtained from Eq. (6.17), however, will often look completely different than the corresponding solution from Eq. (6.9). To illustrate, consider the same example as before in which the fundamental solution is given by Eq. (6.12). The dervative of U with respect to time is

$$\frac{\partial U(x, t-\tau)}{\partial t} = -\sum_{n=1}^{\infty} \lambda_n^2 A_n \phi_n(x) e^{-\lambda_n^2 t} e^{\lambda_n^2 \tau}$$

For $t < t_1$, Eqs. (6.17) and (6.10) yield

$$\begin{aligned} T &= -\frac{T_1}{t_1} \sum_{n=1}^{\infty} \lambda_n^2 A_n \phi_n(x) e^{-\lambda_n^2 t} \int_0^t e^{\lambda_n^2 \tau} \cdot \tau \, d\tau \\ &= -\frac{T_1}{t_1} \sum_{n=1}^{\infty} A_n \phi_n(x) e^{-\lambda_n^2 t} \left[t e^{\lambda_n^2 t} - \frac{1}{\lambda_n^2} \left(e^{\lambda_n^2 t} - 1 \right) \right] \\ &= \frac{T_1}{t_1} \sum_{n=1}^{\infty} A_n \phi_n(x) \left[\frac{1}{\lambda_n^2} \left(1 - e^{-\lambda_n^2 t} \right) - t \right], \quad t < t_1 \end{aligned} \tag{6.18}$$

and for $t \geq t_1$ we get

$$\begin{aligned} T &= -\frac{T_1}{t_1} \sum_{n=1}^{\infty} \lambda_n^2 A_n \phi_n(x) e^{-\lambda_n^2 t} \int_0^{t_1} e^{\lambda_n^2 \tau} \cdot \tau \, d\tau \\ &= T_1 \sum_{n=1}^{\infty} A_n \phi_n(x) \left[\frac{1}{t_1 \lambda_n^2} \left(e^{\lambda_n^2 t_1} - 1 \right) - e^{\lambda_n^2 t_1} \right] e^{-\lambda_n^2 t}, \quad t \geq t_1 \end{aligned} \tag{6.19}$$

Equation (6.19) is identical to what was obtained from our previous formula, which appears in Eq. (6.15). However, Eqs. (6.18) and (6.14) appear different. The equivalence between the two solutions can be established if the function $T_1 x$ – which appears in Eq. (6.14) – is expanded into a series of the eigenfunctions.

Cylindrical system example

Another example is now examined. Say we have a long, solid cylindrical rod, of radius R, that initially is at a temperature of T_1. At time zero a uniform yet time–dependent heat source is applied to the rod, given by

$$q'''(t) = q_0''' \left(1 - e^{-t/t_c} \right)$$

where t_c is a constant. Convection is maintained at the surface of the rod. Determine the transient temperature distribution in the rod.

The first thing we need to do here is determine the fundamental solution $U(r,t)$ for the problem in which 1) the system is initially at zero temperature, and 2) a unit–step heat source is applied at $t = 0$. To make the initial temperature zero, we would define our nondimensional temperature with T_1 as the reference temperature. Therefore, let

$$\overline{T} = \frac{(T - T_1)k}{q'''_0 R^2}, \quad \overline{r} = \frac{r}{R}, \quad \overline{t} = \frac{t\alpha}{R^2}$$

Our dimensionless problem is then

$$\frac{\partial \overline{T}}{\partial \overline{t}} = \frac{1}{\overline{r}} \frac{\partial}{\partial \overline{r}} \overline{r} \frac{\partial \overline{T}}{\partial \overline{r}} + \left(1 - e^{-\overline{t}/\overline{t}_1}\right) \tag{6.20}$$

$$\overline{T}(0, \overline{t}) \text{ is finite} \tag{6.21}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{r}}\right|_1 = -Bi\,[\overline{T}(1, \overline{t}) - \overline{T}_\infty] \tag{6.22}$$

$$\overline{T}(\overline{r}, 0) = 0 \tag{6.23}$$

where $\overline{T}_\infty = (T_\infty - T_1)k/q'''_0 R^2$. The fundamental problem is when we replace the time–dependent source with 1, giving

$$\frac{\partial U}{\partial \overline{t}} = \frac{1}{\overline{r}} \frac{\partial}{\partial \overline{r}} \overline{r} \frac{\partial U}{\partial \overline{r}} + 1 \tag{6.24}$$

$$U(0, \overline{t}) \text{ is finite} \tag{6.25}$$

$$\left.\frac{\partial U}{\partial \overline{r}}\right|_1 = -Bi\,[U(1, \overline{t}) - \overline{T}_\infty] \tag{6.26}$$

$$U(\overline{r}, 0) = 0 \tag{6.27}$$

A complication arises from the inhomogeneous BC at $\overline{r} = 1$. We can easily fix this by letting

$$S(\overline{r}, \overline{t}) = U(\overline{r}, \overline{t}) - \overline{T}_\infty$$

which gives us the homogenous BC problem:

$$\frac{\partial S}{\partial \overline{t}} = \frac{1}{\overline{r}} \frac{\partial}{\partial \overline{r}} \overline{r} \frac{\partial S}{\partial \overline{r}} + 1 \tag{6.28}$$

$$S(0, \overline{t}) \text{ is finite} \tag{6.29}$$

$$\left.\frac{\partial S}{\partial \overline{r}}\right|_1 = -Bi\, S(1, \overline{t}) \tag{6.30}$$

$$S(\overline{r}, 0) = -\overline{T}_\infty \tag{6.31}$$

$$\tag{6.32}$$

Using superposition methods, the general solution to S is

$$S = s(\overline{r}) + \sum_{n=1}^{\infty} A_n \phi_n(\overline{r}) e^{-\lambda_n^2 \overline{t}}$$

where $s(\overline{r})$ is the steady–state solution given by

$$s(\overline{r}) = \frac{1}{2}\left(\frac{1}{Bi} + \frac{1}{2}(1-\overline{r}^2)\right)$$

and with eigenfunctions and eigenconditions given by

$$\phi_n(\overline{r}) = J_0(\lambda_n \overline{r})$$
$$\lambda_n J_1(\lambda_n) = Bi\, J_0(\lambda_n)$$

The expansion coefficients are obtained from

$$A_n = -\frac{\displaystyle\int_0^1 \left(\overline{T}_\infty + s(\overline{r})\right) \phi_n \overline{r}\, d\overline{r}}{\displaystyle\int_0^1 \phi_n^2 \overline{r}\, d\overline{r}}$$

which can be evaluated using techniques developed in the previous notes or with *Mathematica*. The fundamental solution would therefore be $U = S + \overline{T}_\infty$, or

$$U(\overline{r}, \overline{t}) = s(\overline{r}) + \overline{T}_\infty + \sum_{n=1}^{\infty} A_n \phi_n(\overline{r}) e^{-\lambda_n^2 \overline{t}}$$

We now let the time–dependent forcing function be

$$F(\tau) = \left(1 - e^{-\tau/\overline{t}_c}\right) \tag{6.33}$$

Since F is continuous, it is probably easiest to obtain $\overline{T}$ using Eq. (6.7). We would then have

$$\begin{aligned}
\overline{T} &= \int_0^t U(\overline{r}, \overline{t} - \tau) \frac{dF}{d\tau}\, d\tau \\
&= \frac{1}{t_c} \int_0^t \left(s(\overline{r}) + \overline{T}_\infty + \sum_{n=1}^{\infty} A_n \phi_n(\overline{r}) e^{-\lambda_n^2 \overline{t}} e^{\lambda_n^2 \tau} \right) e^{-\tau/\overline{t}_c}\, d\tau \\
&= \left(s(\overline{r}) + \overline{T}_\infty\right)\left(1 - e^{-\overline{t}/t_c}\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{A_n \phi_n(\overline{r})}{\overline{t}_c \lambda_n^2 - 1}\left(e^{-\overline{t}/\overline{t}_c} - e^{-\lambda_n^2 \overline{t}}\right)
\end{aligned} \tag{6.34}$$

It's easy to see from the above that in the limit of $\overline{t}_c \to 0$ – for which the source function becomes instantaneous – the solution reduces to that obtained for the fundamental problem.

6.2.3 Solution by variation of parameters

The analytically–robust variation of parameters (VOP) method, introduced in Ch. 5, can be directly applied to time–dependent forcing problem. Unlike Duhamel's theorem, the VOP method is not constrained to problems where the sole inhomogeneity occurs in the forcing function and the system is initially at zero temperature. Furthermore, VOP could be used to examine problems in which the forcing function was dependent on position as well as time – such as in a space and time–dependent heat source function.

As before, the starting point of this method is to formulate the solution as an eigenfunction expansion in which the expansion coefficients are initially–undetermined functions of time. The eigenfunction and eigencondition are identified as those for the corresponding homogeneous problem in which the BCs (including the time dependent one) are replaced by homogeneous ones *of the same type*. Consider, for example, the plane wall problem posed in Sec. 2, in which the initial and $x = 0$ temperatures are 0 and at $x = 1$ the temperature is given by

$$T(1,t) = F(t) \tag{6.35}$$

Since the temperature is specified in Eq. (6.35), the corresponding homogeneous condition would have zero temperature. The eigenfunction and eigencondition are therefore

$$\phi_n(x) = \sin(\lambda_n x), \quad \lambda_n = n\pi \tag{6.36}$$

Continuing with the same example, the eigenfunction expansion of the temperature field would be

$$T(x,t) = \sum_{n=1}^{\infty} A_n(t)\,\phi_n(x) \tag{6.37}$$

and the expansion functions $A_n(t)$ are

$$A_n(t) = \frac{\displaystyle\int_0^1 T(x,t)\,\phi_n(x)\,dx}{\displaystyle\int_0^1 \phi_n^2(x)\,dx} = 2\int_0^1 T(x,t)\,\phi_n(x)\,dx \tag{6.38}$$

We now obtain the ODE for $A_n(t)$ by differentiation and application of the governing DE and BCs;

$$\begin{aligned}
\frac{dA_n}{dt} &= 2\int_0^1 \frac{\partial T}{\partial t}\,\phi_n\,dx \\
&= 2\int_0^1 \frac{\partial^2 T}{\partial x^2}\,\phi_n\,dx \\
&= 2\left[\frac{\partial T}{\partial x}\,\phi_n\Big|_0^1 - T\,\phi_n'\Big|_0^1 + \int_0^1 T\,\phi_n''\,dx\right] \\
&= -2\,T(1,t)\,\phi_n'(1) - 2\lambda_n^2\int_0^1 T\,\phi_n\,dx \\
&= -2(-1)^n\,\lambda_n\,F(t) - \lambda_n^2\,A_n
\end{aligned} \tag{6.39}$$

By use of an integrating factor, the DE for A_n can appear as

$$e^{-\lambda_n^2 t}\frac{d}{dt}\left(e^{\lambda_n^2 t}\,A_n\right) = -2(-1)^n\,\lambda_n\,F(t)$$

This can be directly integrated from 0 to t to give

$$e^{-\lambda_n^2 t}A_n(t) - A_n(0) = -2(-1)^n\,\lambda_n\int_0^t F(\tau)\,e^{\lambda_n^2\tau}\,d\tau$$

We would now use the definition in Eq. (6.38) to generate an initial condition on $A_n(0)$; the initial condition on T does not have to be zero but assume here that it is. This would give $A_n(0) = 0$, and the complete solution for the expansion functions becomes

$$A_n(t) = -2(-1)^n\,\lambda_n\int_0^t F(\tau)\,e^{-\lambda_n^2(t-\tau)}\,d\tau \tag{6.40}$$

Observe that the VOP approach gives a convolution integral solution for A_n that is analogous to Eq. (6.16); if F is a unit step at $t = 0$ the solution would give

$$\begin{aligned}
A_n(t) &= -2(-1)^n\,\lambda_n\int_0^t e^{-\lambda_n^2(t-\tau)}\,d\tau \\
&= -2\frac{(-1)^n}{\lambda_n}\left(1 - e^{-\lambda_n^2 t}\right) \\
&\equiv U_n(t)
\end{aligned}$$

in which U_n denotes the fundamental solution for A_n, i.e., that corresponding to the unit step forcing. By direct substitution, you should see that

$$A_n(t) = \int_0^t F(\tau)\,\frac{dU_n(t-\tau)}{dt}\,d\tau$$

which is identical in form to Eq. (6.16). If we now specified F as the linear ramp function used in the example of Sec. 3, we would ultimately get a solution for T identical to that obtained from Eq. (6.16). Alternatively, we could formally integrate the above equation using integration by parts, and arrive at the analogous VOP formula to Eq. (6.7).

In the end, the VOP method, when applied to the class of problems admissible by Duhamel's theorem, will give solutions that are identical to those obtained from Duhamel's theorem. That this is the case should not be surprising – there is only going to be one solution to a particular problem (at least physically). VOP, on the other hand, does not have the restrictions on homogeneity and initial conditions that does Duhamel's theorem.

6.3 Time–harmonic boundary conditions and sources

It is not uncommon to find situations in which the heating conditions on a surface oscillate in time. For example, a side of a building could be heated during the day by the sun, and cooled at night, or a thin electically–conducting wire could be subjected to the an oscillating heat dissapation rate corresponding to the 60–cycle alternating current. For such problems we will never attain a steady–state solution – since the conditions at the surface or the heat generation function are constantly changing. However, we would expect that, given enough time from a 'starting' point, the system would eventually 'forget' everything about the initial state. In this situation, the temperature at discrete points in the system would oscillate at the same frequency as the temperature oscillation at the surface – yet the phase and magnitude of the oscillations would not necessarily be the same for all points in the system.

To pose a simple example, say we have a plane wall of thickness L, for which the surface at $x = 0$ is maintained at T_1 and the surface at $x = L$ oscillates over a range ΔT about T_1 during a time period of t_1 seconds. Specifically, say that the surface temperature is given by

$$T(L, t) = T_1 + \Delta T \cos(2\pi t/t_1)$$

The cosine representation of the temperature variation is known as a time–harmonic boundary condition. This form is representative of many oscillatory BCs – such as daytime/nighttime temperature variations or AC electrical heating processes. The dimensional problem statement could then be posed as

$$\begin{gathered} \frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \\ T(0, t) = T_1 \\ T(L, t) = T_1 + \Delta T \cos(2\pi t/t_1) \end{gathered}$$

There is no initial condition – and because of this the problem could not be considered 'well–posed', i.e., we don't have enough information to obtain a complete solution. However, the item of interest

is, again, the limiting form of the solution for $t \to \infty$. The solution will be a *periodic* function of time and consequently will have the same behavior for $t \to \infty$ and $t \to -\infty$.

First nondimensionalize the problem using

$$\overline{T} = \frac{T - T_1}{\Delta T}, \quad \overline{x} = \frac{x}{L}, \quad \bar{t} = \frac{t\alpha}{L^2}, \quad \omega = \frac{2\pi L^2}{t_1 \alpha} \tag{6.41}$$

The dimensionless quantity ω can be viewed as a dimensionless frequency of oscillation, and represents the ratio of characteristic diffusion and oscillation times. The problem is now

$$\frac{\partial \overline{T}}{\partial \bar{t}} = \frac{\partial^2 \overline{T}}{\partial \overline{x}^2} \tag{6.42}$$

$$\overline{T}(0, \bar{t}) = 0 \tag{6.43}$$

$$\overline{T}(1, \bar{t}) = \cos(\omega \bar{t}) \tag{6.44}$$

Solving such a problem is immensely simplified if we use complex arithmetic. Whenever we have a time–harmonic BC of the form in Eq. (6.44), we assume that the solution for $\overline{T}$ can be represented by the real part of the complex variable:

$$\overline{T} = \text{Re}\left(s(\overline{x}) e^{-i\omega \bar{t}}\right) \tag{6.45}$$

where $s(\overline{x})$ is a complex–valued function that depends only on $\overline{x}$, and Re denotes the real part. If we replace this form into the DE, we will find that

$$\text{Re}\left(-i\omega s(\overline{x}) e^{-i\omega \bar{t}} - s''(\overline{x}) e^{-i\omega \bar{t}}\right) = 0$$

Note that we can factor out the $\exp(-i\omega \bar{t})$ part to the above. Consequently, if we let s satisfy

$$s''(\overline{x}) + \mathcal{K}^2 s(\overline{x}) = 0 \tag{6.46}$$

in which $\mathcal{K}$ is the complex *wavenumber* of the medium, given by

$$\mathcal{K} = \sqrt{i\,\omega} \tag{6.47}$$

then the form in Eq. (6.45) will satisfy the DE. Also, by letting

$$s(0) = 0 \tag{6.48}$$

we will recover the BC in Eq. (6.43). Finally, to formulate the BC for s at $\overline{x} = 1$, note that

$$\cos(\omega \bar{t}) = \text{Re}\, e^{-i\omega \bar{t}} \tag{6.49}$$

Using Eq. (6.45) in Eq. (6.44), we would then have

$$\text{Re}\left(s(1)e^{-i\omega\bar{t}} - e^{-i\omega t}\right) = 0 \tag{6.50}$$

Accordingly, we can let $s(1)$ be given by

$$s(1) = 1 \tag{6.51}$$

The approach used here attempts to anticipate the time dependence of the solution – which will be proportional to $\exp(-i\omega\bar{t})$. By doing so, we can separate out the time dependence from the spacial dependence and obtain an ordinary differential equation for the spacial part of the problem.

The general solution to Eq. (6.46) is

$$s = C_1\cos(\mathcal{K}\overline{x}) + C_2\sin(\mathcal{K}\overline{x})$$

To satisfy Eq. (6.48) we set $C_1 = 0$. Using Eq. (6.44), the complete solution for s is

$$s = \frac{\sin(\mathcal{K}\overline{x})}{\sin(\mathcal{K})}$$

and our solution for the temperature in the wall is

$$\overline{T} = \text{Re}\left(\frac{\sin(\mathcal{K}\overline{x})}{\sin(\mathcal{K})}e^{-i\omega\bar{t}}\right) \tag{6.52}$$

Algebraically obtaining the real part to the above is not easy. In practice, you would not want to try to separate out the real part. Rather, you would code up the above equation using complex number types and functions (which are standard on most FORTRAN and C compilers) or use *Mathematica*, and let the code pull the real part out.

The wavenumber $\mathcal{K}$ plays an important role in the behavior of the solution. This number is complex, i.e., it has real and imaginary parts. To identify these parts, note first that $\mathcal{K}$ is, again,

$$\mathcal{K} = \sqrt{\omega}\cdot\sqrt{i}$$

If we use the fact that

$$e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$$

then

$$\sqrt{i} = e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = \frac{1}{\sqrt{2}}(1+i)$$

or,

$$\mathcal{K} = \mathcal{K}' + i\mathcal{K}'' = \sqrt{\frac{\omega}{2}} + i\sqrt{\frac{\omega}{2}} \tag{6.53}$$

So it turns out that the real and imaginary parts of $\mathcal{K}$ have the same magnitude. The effect of these different parts on the solution will now be explored for two limiting cases.

Consider first the case in which $\omega \gg 1$ (and, equivalently, $|\mathcal{K}| \gg 1$). Physically this would correspond to a period of oscillation much greater than the diffusion time of the slab. The sin function can be expanded into the complex representation of

$$\sin(\mathcal{K}) = -\frac{i}{2}\left(e^{i\mathcal{K}} - e^{-i\mathcal{K}}\right)$$

Now

$$e^{i\mathcal{K}} = e^{i(\mathcal{K}'+i\mathcal{K}'')} = e^{i\mathcal{K}'-\mathcal{K}''} \to 0, \quad |\mathcal{K}| \gg 1$$

Carrying on with this approach, it turns out that

$$\frac{\sin(\mathcal{K}\overline{x})}{\sin(\mathcal{K})} \to e^{i\mathcal{K}'(1-\overline{x})} \cdot e^{-\mathcal{K}''(1-\overline{x})}, \quad |\mathcal{K}| \gg 1$$

It then follows that, in the limit of $|\mathcal{K}| \gg 1$, the solution for T in Eq. (6.45) reduces to

$$\begin{aligned} T &= \mathrm{Re}\left(e^{i(\mathcal{K}'(1-\overline{x})-\omega\overline{t})}\right) \cdot e^{-\mathcal{K}''(1-\overline{x})} \\ &= \cos\left[(\mathcal{K}'(1-\overline{x}) - \omega\overline{t})\right] \cdot e^{-\mathcal{K}''(1-\overline{x})} \end{aligned} \tag{6.54}$$

This result would be recognized by people familiar with acoustics and electromagnetics as a *wave equation*. The first part of the solution gives the *phase* of the wave as it propagates into the medium. From inspection, all combinations of $\overline{x}$ and $\overline{t}$ in which $\mathcal{K}'(1-\overline{x}) - \omega\overline{t}$ is a constant will have the same phase. By differentiating this quantity and setting it to zero, and using the definitions of our dimensionless quantities, it follows that

$$-\left.\frac{dx}{dt}\right|_{\text{phase=C}} = \sqrt{2}\left(\frac{2\pi\alpha}{t_1}\right)^{1/2} = V \tag{6.55}$$

which identifies the velocity of the wavefront as it moves into the medium. Note that this velocity is not a function of the slab thickness.

The second term in Eq. (6.54) identifies the *attenuation* of the amplitude of the wave. The amplitude will exponentially decay with distance into the slab, with a exponential factor of $\mathcal{K}'' = \sqrt{\omega/2}$. Again using the definitions for the dimensionless quantities, the distance into the slab at which the amplitude has decayed by a factor of $1/e$ is $\sqrt{(t_1\alpha/\pi)}$. This quantity is independent of the slab thickness and is proportional to $\sqrt{t_1}$. As the frequency of oscillation increases (t_1 decreases), the depth of penetration of the wave will decrease.

To put another physical interpretation to the $|\mathcal{K}| \gg 1$ limit, recall that ω is defined as

$$\omega = \frac{2\pi L^2}{t_1\alpha} = \frac{t_{dif}}{t_{oscil}} \tag{6.56}$$

The dimensionless frequency ω can therefore be viewed as a ratio of the characteristic time scales for thermal diffusion and wall temperature oscillation. When $\overline{t}_{diff} \gg t_{oscil}$ the wall will act to

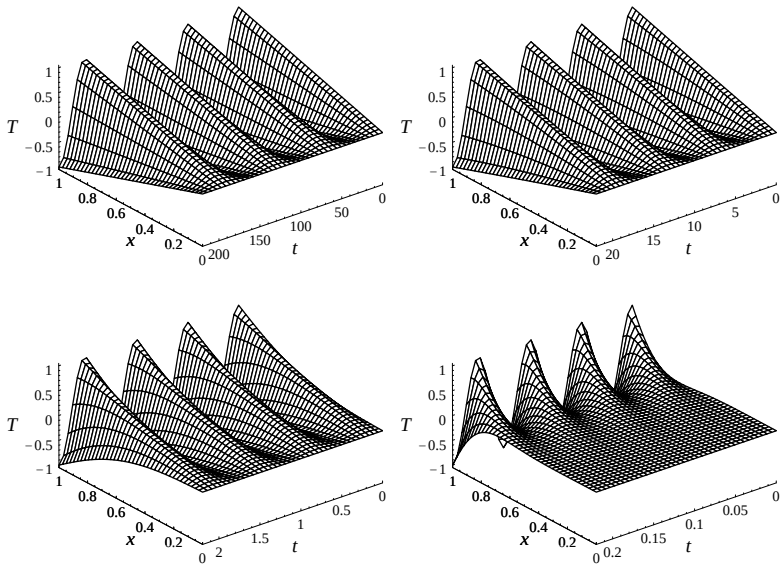


Figure 6.4: periodic surface temperature: $\omega = 0.1$(TL), 1(TR), 10(BL), 100(BR)

'damp' out the temperature disturbances – and consequently the regions that are removed from the surface do not experience the fluctuation in the surface temperature. This situation is analogous to a damped mechanical system that is being forced at a frequency that is greater than the characteristic frequency of the system[1].

In the opposite limit, i.e., $\omega \ll 1$, we can apply L'Hospitals rule to the solution for s and find that

$$\overline{T}(\omega \to 0) \approx \mathrm{Re}\, x\, e^{-i\omega t} = \overline{x}\cos(\omega \bar{t}) \tag{6.57}$$

Here, the characteristic time of the surface oscillation is considerably greater than the diffusion time. In this limit the wall, at any instant, will essentially be in a steady–state condition. The resulting solution would therefore be $\overline{x}$ times the surface temperature, and the surface temperature is given by $\cos(\omega \bar{t})$.

Surface plots of the dimensionless temperature in the wall, for ω values of 0.1, 1, 10, and 100, are given in Fig. 6.4. The results show the temperature distribution that would exist during $\sim$ 4 cycles of the surface temperature. Again, for small ω (top plots) the wall maintains a nearly linear temperature profile at any time – the profile simply oscillates up and down in phase with

[1]It is important to recognize that the analogy with a mechanical system can only go so far. It is physically impossible for the temperature 'waves' within the system to reach an amplitude that exceeds those at the surface. That is, the thermal system has no analog to inertia (or mass) in the mechanical system – and thus cannot have 'resonant' behavior

the surface temperature. On the other hand, for $\omega = 10$ and 100 (bottom plots) the temperature disturbance is confined to an increasingly smaller region near the surface. Regions farther from the surface are essentially unaffected by the wall temperature oscillation. Note also, for this case, that the temperature in regions slightly in from the surface (say $\overline{x} \sim 0.6 - 0.8$) is somewhat retarded in phase behind the surface temperature. This reflects the finite amount of time in takes the temperature 'wave' to propagate from the surface into the inner regions.

6.3.1 Periodic BCs/sources of arbitrary form

The previous section dealt with a specific (and fundamental) form of the periodic BC and/or source, i.e., the time–harmonic form of $\cos(2\pi\, t/t_1)$. In general, the oscillating function that is driving the system could take on an arbitrary form – with the only restriction being that it be periodic with interval t_1. To illustrate, the plane–wall problem examined in the previous section could have a time–dependent surface temperature given by

$$T(L,t) - T_1 = \Delta T\, F(2\pi t/t_1)$$

where the function F is periodic (i.e., $F(2\pi(t+t_1)/t_1) = F(2\pi t/t_1)$) and has unit amplitude and zero mean. This function could represent, for example, a square wave, a saw–tooth wave, or any arbitrary periodic function.

The solution approach to such problems is a relatively simple extension of that developed for the simple, time–harmonic case. The first step is to represent F by a general Fourier expansion, of the form

$$F(\omega\bar{t}) = \mathrm{Re} \sum_{n=-\infty}^{\infty} f_n\, e^{-in\omega\bar{t}} \tag{6.58}$$

in which $\omega = 2\pi L^2/\alpha t_1$ is the dimensionless fundamental frequency. The expansion coefficients f_n are given by

$$\begin{aligned} f_n &= \frac{\displaystyle\int_0^{2\pi/\omega} F(\omega\bar{t})\, e^{in\omega\bar{t}}\, d\bar{t}}{\displaystyle\int_0^{2\pi/\omega} d\bar{t}} \\ &= \frac{\omega}{2\pi} \int_0^{t_1} F(\omega\bar{t})\, e^{in\omega\bar{t}}\, d\bar{t} \end{aligned} \tag{6.59}$$

It should now be easy to see that a general solution for the temperature field is obtained from

$$\overline{T}(\overline{x},\bar{t}) = \mathrm{Re} \sum_{n=-\infty}^{\infty} f_n\, s_n(\overline{x})\, e^{-in\omega\bar{t}} \tag{6.60}$$

where s_n is the solution to the wave equation (Eq. (6.46)) for frequency $n\omega$.

This shows that the solution for an arbitrary oscillation F is obtained from a rather simple superposition of the solutions for time–harmonic oscillations, each solution representing an integer multiple of the fundamental frequency ω. As we saw in the previous section, the higher frequencies are increasingly damped by diffusion as one extends into the medium. Because of this, the temperature fluctuation in the medium will approach the time–harmonic form, of fundamental frequency ω, for regions well removed from the oscillating surface – regardless of the form of F.

An application of the formulation presented in this section is left as a homework exercise.

6.4 The semi–infinite medium

A semi–infinite medium is, in a mathematical sense, one in which the system extends in one direction to infinity. Although there are situations in which this condition is literally met – such as the ground – it applies to all cases in which the system has a finite yet sufficient depth so that the temperature at the inner boundary (the boundary removed from the surface) is unaffected by a perturbation in temperature at the outer boundary (the surface).

If the perturbation at the surface takes the form of a step change from one constant temperature to another constant temperature (such as the plane wall problem examined first in Ch. 3), it will be only a matter of time before any point within the medium experiences the effect of the surface temperature change. We have seen in numerous examples that a change in surface temperature sets up thermal 'wave' which propagates into the medium and that the characteristic time of propagation of the wave across a distance L is on the order of $t_d \sim 0.1L^2/\alpha$. In this sense, a finite–thickness medium (such as a wall) can be considered semi–infinite for times up to t_d following a change in surface temperature.

A system which undergoes a finite–time–length change in surface temperature – e.g., the surface temperature increases at $t = 0$ and then returns to its initial temperature after a time of t_c – can be considered semi–infinite providing that $t_c \ll t_d$. We observed this behavior when examining, using Duhamel's theorem, the effect of a ramp in surface temperature from zero to $\overline{T}_1$ over a period $\bar{t}_1$ followed by a sudden drop back to zero. When $\bar{t}_1$ was less than around 0.1 the temperature change at the surface had negligible effect on the temperature at the inner boundary. Likewise, the inner boundary was unaffected by a periodic change in surface temperature providing that the period of oscillation was significantly less than the diffusion time of the wall.

Semi–infinite problems involving periodic boundary conditions can be analyzed using the same methods as those for finite regions, i.e., time–harmonic substitution. However, semi–infinite problems in which the surface temperature change is not periodic must use completely different methods than those for finite systems. It is easy to see why this is the case. The SOV method, when applied to transient problems, requires that the boundary conditions in all spacial direction be homogeneous. It will be impossible to identify an eigenfunction and eigencondition which can satisfy the boundary condition at $x \to \infty$. For example, $\phi_n(x) = \sin(\lambda_n x)$ obviously satisfies the homogeneous condition of $\phi_n(0) = 0$, yet the second BC $\phi_n(x \to \infty) = 0$ could only be satisfied by $\lambda_n = 0$ for

all n – and this would give the trivial (and useless) solution of $\phi_n = 0$ for all n. The implication is that time–dependent SOV can only be applied to regions that have a finite length.

A couple of methods can be used to examine the semi–infinite medium, transient impulse problem – those being a *similarity transformation* and *Laplace transform*. Extension to forced time–dependent boundary conditions can utilize Duhamel's theorem as given in the previous section, in which the fundamental solution corresponds to that obtained, for the semi–infinite medium, for the unit impulse problem.

6.4.1 The step change in temperature: similarity solution

The most mathematically simple semi–infinite problem occurs when the medium (taken to be a slab) is initially at temperature T_1, and at $t = 0$ the surface of the slab is instantaneously brought to T_∞. We want to find solutions for the temperature distribution in the slab that are valid for $t/t_d \sim 10t\alpha/L^2 \ll 1$ where L is the characteristic length of the system. For this we can treat the far boundary as being at infinity, and pose the dimensional problem as

$$\begin{gathered}\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \\ T(0,t) = T_\infty \\ T(x \to \infty, t) \to T_1\end{gathered}$$

The non–dimensional temperature is obviously

$$\overline{T} = \frac{T - T_\infty}{T_1 - T_\infty}$$

However, when we attempt to nondimensionalize the length x, we find that there is no characteristic length – other that $\sqrt{\alpha t}$! Likewise, when we try to nondimensionalize the time, we see that the only characteristic time scale is x^2/α. The only way in which the problem can be made nondimensional – and *all* problems must have a valid nondimensional form – is if time and length coordinates become combined into the same variable. Indeed, if we go through a formal nondimensionalization procedure, we would find that

$$\overline{T} = \overline{T}(\frac{x}{\sqrt{\alpha t}})$$

i.e., the dimensionless temperature must be a function of one variable. For reasons which will become obvious below, define the dimensionless variable as

$$\eta = \frac{x}{2\sqrt{\alpha t}} \tag{6.61}$$

If we now use

$$\frac{\partial \overline{T}}{\partial t} = \frac{d\overline{T}}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{\eta}{2t}\overline{T}'$$
$$\frac{\partial \overline{T}}{\partial x} = \frac{d\overline{T}}{d\eta}\frac{\partial \eta}{\partial x} = \frac{\eta}{x}\overline{T}'$$
$$\frac{\partial^2 \overline{T}}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial \overline{T}}{\partial x} = \frac{d^2\overline{T}}{d\eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 = \left(\frac{\eta}{x}\right)^2 \overline{T}''$$

our differential equation will become

$$-\frac{\eta}{2\alpha t}\overline{T}' = \frac{\eta^2}{x^2}\overline{T}''$$

or

$$\overline{T}'' = -2\eta\overline{T}'$$

Separating out the DE, we get

$$\frac{d\overline{T}'}{\overline{T}'} = -2\eta\, d\eta$$

and integrating;

$$\ln(\overline{T}') = -\eta^2 + C_1$$

or

$$\overline{T}' = C' e^{-\eta^2}$$

Integrating once more:

$$\overline{T} = C' \int_0^\eta e^{-z^2}\, dz + T(0)$$

where z is a dummy variable of integration. For our choice of $\overline{T}$ we have $\overline{T}(0) = 0$. The remaining constant is obtained from the second BC (or, equivalently, the IC) for which $\overline{T}(\eta \to \infty) \to 1$. This gives

$$C' = \left[\int_0^\infty e^{-z^2}\, dz\right]^{-1} = \frac{2}{\sqrt{\pi}}$$

And the formula for $\overline{T}$ is

$$\overline{T} = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-z^2}\, dz = \mathrm{erf}(\eta) \tag{6.62}$$

where $\mathrm{erf}(\eta)$ is the Gaussian Error Function. This function is tablulated in the standard mathematical and heat transfer texts and appears as an intrinsic function in *Mathematica*.

The concept of similarity suggests that the temperature profiles in the slab, for different values of t, are mathematically similar. Because of this, a single coordinate η, which 'collapses' the profiles

for different t onto a single curve, can be identified. The specific solution in Eq. (6.62) provides also a more quantitative measure of semi–infinite criterion. The condition $\overline{T} \geq 0.99$ (i.e., $T(x,t) - T_\infty$ will be 99% of $T_1 - T_\infty$) will be met for $\eta \geq 1.821$. Accordingly, a slab of length L can be modeled as semi–infinite providing that $L \geq 3.642\sqrt{\alpha t}$. Alternatively, if we use the common definition of dimensionless time as $\bar{t} = \alpha\, t/L^2$, it follows that a semi–infinite analysis would be accurate in a finite–thickness slab providing that $\bar{t} \leq 0.075 \sim 0.1$ – which is consistent with our previous observations for the characteristic diffusion time.

6.4.2 Laplace transform methods

If the same configuration is examined – with the exception that the boundary condition at $x = 0$ becomes an imposed uniform heat flux of q_0'' starting at $t = 0$ – we will find that a similarity solution will no longer be possible. This is because the parameters in the problem now provide a characteristic length scale, that being $L_c = k\, T_1/q_0''$. If we now define the temperature scale per the usual approach for imposed flux conditions as $\Delta T_c = q_0'' L_c/k$, we find simply that $\Delta T_c = T_1$. This is not a temperature difference yet it is the only temperature scale that can make the temperature dimensionless. Consequently, the nondimensional variables become

$$T \to \frac{T - T_1}{T_1}, \quad x \to \frac{x q_0''}{k T_1}, \quad t \to \alpha t \left(\frac{q_0''}{k T_1}\right)^2$$

and the problem statement is

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \tag{6.63}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = -1 \tag{6.64}$$

$$T(x \to \infty, t) \to 0 \tag{6.65}$$

$$T(x, 0) = 0 \tag{6.66}$$

The solution method for this problem is the *Laplace transform*. This may be a mathematical technique that you learned in a differential equations class – one of many that were promptly forgotten. The Laplace transformation of a function $f(t)$ is defined by

$$\hat{f}(s) = \int_0^\infty f(t)\, e^{-s\, t}\, dt$$

in which s is a transform variable. The inverse transform involves a contour integration of $\hat{f}(s)$ over the complex plane; details of which need not be presented here. Suffice to say that the inverse transform of $\hat{f}(s)$ yields the original function $f(t)$, i.e.,

$$\mathcal{L}^{-1}\left(\hat{f}(s)\right) = \mathcal{L}^{-1}\left(\mathcal{L}\left(f(t)\right)\right) = f(t)$$

The usefulness of the Laplace transform is that it can remove the time dependence from a problem. In particular, the partial DE in t and x (Eq. (6.63)) when transformed becomes an ordinary DE in x with s appearing as a parameter, and this DE can usually be solved easily. We then take the inverse transformation of this solution to obtain the full solution of the partial DE.

Only a few basic properties of the Laplace transform need to be presented for our purposes. The transform of a function of x and t (i.e., $T(x,t)$) simply becomes $\hat{T}(x,s)$. The transform of a derivative of the function with respect to t is given by

$$\mathcal{L}\left(\frac{\partial T}{\partial t}\right) = s\,\hat{T}(x,s) - T(x,0)$$

Likewise, the transform of the derivative with respect to x is simply the derivative of the transformed function – since the order of differentiation and integration can be exchanged. That is

$$\mathcal{L}\left(\frac{\partial^2 T}{\partial x^2}\right) = \hat{T}''(x,s)$$

in which the primes denote differentiation with respect to x; we don't have to use the partial differential notation because (in view of the equation proceeding the above one) differentials with respect to s will not occur in the transformed problem. Finally, the transform of a constant is given by

$$\mathcal{L}\,(C) = \frac{C}{s}$$

Using the above tools, we apply the Laplace transform to the system of equations in Eqs. (6.63–6.65). This gives

$$s\,\hat{T}(x,s) - T(x,0) = \hat{T}''(x,s) \tag{6.67}$$

$$\hat{T}'(0,s) = -\frac{1}{s} \tag{6.68}$$

$$\hat{T}(x\to\infty,s) = 0 \tag{6.69}$$

The initial condition eliminates the $T(x,0)$ appearing in the DE, and the solution for $\hat{T}$ is obtained directly as

$$\hat{T}(x,s) = s^{-3/2}\,\exp\left(-\sqrt{s}\,x\right) \tag{6.70}$$

So far the problem has been simple. The hard part is obtaining the inverse transform, i.e.,

$$T(x,t) = \mathcal{L}^{-1}\left(\hat{T}(x,s)\right)$$

in which $\hat{T}(x,s)$ is explicitly given by Eq. (6.70). Most of the material on the application of Laplace transforms to solution of DEs is devoted to the problem of the inverse transform. We, however,

will not go into these details, because the black–box provided by *Mathematica* can perform this chore for us.

On earlier versions of *Mathematica*, the user (you) needs to load a special (or add–on) package, which contains functions for performing transforms and inverse transforms. This is not required for Version 4.2. The following code, which is based on an earlier version, shows how the package was put to use. For those of you using the latest version, simply ignore the `<<Calculus‘LaplaceTransform‘` step.

```
<<Calculus‘LaplaceTransform‘

InverseLaplaceTransform[E^(-s^(1/2)x)/s^(3/2),s,t]
Out[41]=
1/Sqrt[Pi]*(2*Sqrt[t]*(E^(-(x^2/(8*t))) -
  (E^(x^2/(8*t))*Sqrt[Pi]*x)/
  (2*Sqrt[t]) +(E^(x^2/(8*t))*Sqrt[Pi]*
  Sqrt[x^2]*Erf[Sqrt[x^2]/(2*Sqrt[t])])/
  (2*Sqrt[t])))/E^(x^2/(8*t))

Simplify[%]
Out[42]=
(2*Sqrt[t])/(E^(x^2/(4*t))*Sqrt[Pi]) - x + Sqrt[x^2]*
   Erf[Sqrt[x^2]/(2*Sqrt[t])]

temp[x_, t_] :=(2*Sqrt[t])/(E^(x^2/(4*t))*Sqrt[Pi]) -
  x +Sqrt[x^2]*Erf[Sqrt[x^2]/(2*Sqrt[t])]
```

The *Mathematica* instructions used in the code should be self–explanatory.

The results of the solution are shown in Fig. 6.5, in which T is plotted vs. x with $t = 0.01$, 0.1, and 1. The unit flux BC at $x = 0$ is clearly evident. This problem would obviously have no steady state, as the surface temperature would continuously increase with time.

The associated problem involving instantaneous convection conditions at $x = 0$ would be solved using the same methods. This problem is left as an exercise.

Recognize that the solution for the instantaneous, unit heat flux in a semi–infinite medium, given in the *Mathematica* output, corresponds to the a fundamental solution $U(x, t)$ that could be used, in conjunction with Duhamel's theorem, to obtain the temperature response in a semi–infinite medium for time–dependent flux boundary conditions. The basic equations presented in the section on Duhamel's theorem, i.e., Eqs. (6.7) and (6.16), would apply directly to the fundamental solutions for the semi–infinite medium.

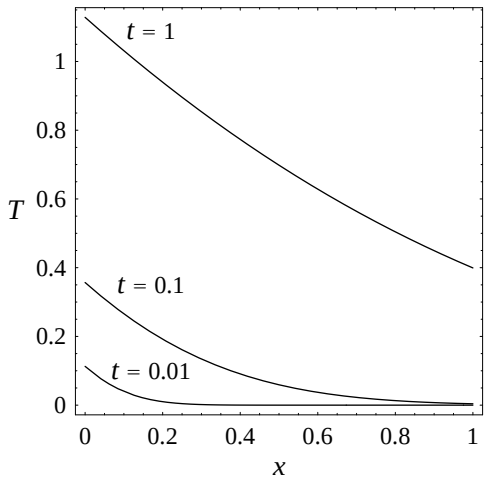


Figure 6.5: temperature response in the semi–infinite medium: unit flux BC

A Laplace transform method could also be used to solve time–dependent problems in media of finite extent – such as the plane wall and the cylinder. However, the method offers no real advantange over the SOV approach for such problems, and the solution that it would provide would, of course, be completely equivalent to that obtained from SOV.

6.4.3 Periodic BCs in semi–infinite media

As mentioned above, the analytical methods developed to examine periodic effects in finite systems will apply directly to semi–infinite media. Consider, for example, the periodic BC problem which was examined originally except now applied to an infinite thickness slab. The x origin is (obviously) placed now at the surface, and the problem become

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$
$$T(0,t) = T_1 + \Delta T \cos(2\pi t/t_1)$$
$$T(x \to \infty, t) \to T_1$$

If we nondimensionalize the problem we find that the characteristic length becomes proportional to $\sqrt{\alpha\, t_1}$. Therefore, let

$$\overline{T} = \frac{T - T_1}{\Delta T}, \quad \overline{x} = x\sqrt{\frac{2\pi}{\alpha\, t_1}}, \quad \overline{t} = \frac{2\pi t}{t_1}$$

The dimensionless problem is then

$$\frac{\partial \overline{T}}{\partial \bar{t}} = \frac{\partial^2 \overline{T}}{\partial \overline{x}^2}$$
$$\overline{T}(0, \bar{t}) = \cos(\bar{t})$$
$$\overline{T}(\overline{x} \to \infty, \bar{t}) \to 0$$

As before, the solution is taken to be in a time harmonic form,

$$\overline{T} = \text{Re}\left(s(\overline{x}) e^{-i\,\bar{t}}\right) \tag{6.71}$$

Observe that there is no longer an ω in this problem – because the dimensionless time is defined with respect to the period of oscillation. Our characteristic problem for s is

$$s'' - is = 0$$
$$s(0) = 1$$
$$s(\overline{x} \to \infty) = 0$$

which has the simple solution

$$s = e^{-\sqrt{i}\overline{x}} = \exp\left[-\frac{1}{\sqrt{2}}(1+i)\overline{x})\right]$$
$$= \exp\left(-\frac{\overline{x}}{\sqrt{2}}\right)\left[\cos\left(\frac{\overline{x}}{\sqrt{2}}\right) + i\sin\left(\frac{\overline{x}}{\sqrt{2}}\right)\right]$$

Using some trigonometric identities, our complete solution from Eq. (6.71) is

$$\overline{T} = \exp\left(-\frac{\overline{x}}{\sqrt{2}}\right)\cos\left(\bar{t} + \frac{\overline{x}}{\sqrt{2}}\right) \tag{6.72}$$

The above solution shows that the temperature at $\overline{x}$ lags in phase an amount $\overline{x}/\sqrt{2}$ behind the surface temperature oscillation. In addition, the magnitude of the temperature oscillations exponentially decrease with $\overline{x}/\sqrt{2}$.

Exercises

1. A semi–infinite medium is initially at temperature T_∞. At time $t = 0$ the medium is subjected to a radiative flux which results in a position–dependent heat generation rate within the medium. The form of the generation rate is given by

   $$q'''(x) = q''_R \kappa\, e^{-\kappa\, x}$$

   where q''_R and κ are the incident flux (W/m^2) and absorption coefficient (1/m), respectively, and x is measured from the surface inwards. Throughout the heating process the surface is cooled by convection to T_∞, characterized by a heat transfer coefficient h.

(a) How many characteristic length and temperature scales occur in this problem? Which ones do you think are most appropriate to nondimensionalize the problem? Note that there is no one 'correct' answer to this question. Choose an appropriate pair of scales and formulate the problem in dimensionless variables.

(b) Solve for the temperature distribution in the medium by application of the Laplace transform method.

2. Consider the semi–infinite example discussed in Sec. 6.4.2. The flux applied at the surface is now a function of time, given by

$$q''(t) = \begin{cases} q_0'' & , \quad 0 < t \leq t_1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

Use Duhamel's theorem along with the fundamental solution obtained from the Laplace transform to derive the analytical solution for the temperature field in the medium. Plot a surface plot (i.e., dimensionless T as a function of dimensionless x and t) for dimensionless $t_1 = 1$.

3. A slab, of thickness L, has one surface insulated and the other exposed to a periodic, pulsed laser beam. The irradiance of the beam, as a function of time, is given by

$$G(t) = \begin{cases} G_0 & 0 \leq t < t_1 \\ 0 & t_1 \leq t \leq t_2 \end{cases}$$

Observe that the pulse is on for $t = 0$ to t_1, and off for $t = t_1$ to t_2. The sequence is periodic, with period t_2 (i.e., $G(t + t_2) = G(t)$). Assuming that all of the incident flux is absorbed by the surface, and assuming that the surface also exchanges heat with the environment by convection, derive the solution for the periodic temperature distribution in the slab.

4. An infinite–length circular wire, of radius R, is embedded within an infinite medium. The wire and medium have thermal conductivities and thermal diffusivities of k_w, k_m and α_w, α_m. The wire carries alternating current and is uniformly and periodically generating heat at the rate

$$q'(t) = q_0' \cos(\omega t)$$

where q_0' (W per m of length) and ω (1/s) are constants. Far from the wire the temperature in the medium is T_∞.

(a) Cast the problem in dimensionless form; use the properties of the medium (k_m and α_m) along with R and T_∞ to define the dimensionless variables. Recall that the temperature field must be split into two parts; one in the wire and another in the medium.

(b) Use the time–harmonic procedure to determine the temperature distribution in the wire and the medium as a function of dimensionless r and t. You can leave the solutions in a complex form: do not attempt to factor out the real part.

Chapter 7

Moving Interface Problems

7.1 Introduction

Up to this point we have taken our domains to be homogeneous – that is, comprised of a single substance with uniform values of thermophysical properties. The boundaries of the domain have also been fixed in time and in space. A class of problems which go beyond these assumptions are the *moving interface* problems, the typical example being the melting (or freezing) of a liquid/solid system.

For such systems, the interface (i.e., the boundary between the liquid/solid phases) moves as a result of heat transfer. It is relatively simple to apply heat and mass conservation principles at the interface – and we will do so shortly – yet what complicates the analysis of heat transfer for such problems is the fact that the interface is moving. Physically the interface represents a boundary of the system (although the boundary is, in a sense, internal to the system) and as such boundary conditions are applied to it to 'close' the analytical problem. However, the boundary is not stationary, and in this respect the boundary conditions implicitly become a function of time.

It turns out that analytical solutions to moving interface problems are possible only for some simplified problems, as we will see below.

7.2 The Interface Continuity Conditions

Consider the diagram of an interface in Fig. 7.1. This particular drawing represents the melting of a solid to a liquid, and for such the interface moves into the solid region. To apply conservation principles, it is convenient to fix the coordinate system to the moving interface – and in this representation the interface becomes stationary, and the solid and liquid phases move into and out of it, respectively.

Because the system is in steady state – and no mass is accumulated within the system – the

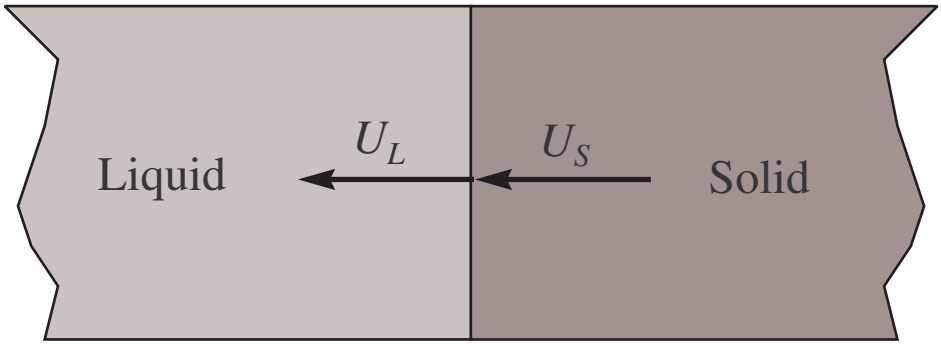


Figure 7.1: Melting of solid to liquid. The interface is stationary

mass flux into the interface must balance the flux out. Accordingly,

$$\rho_L U_L = \rho_S U_S \tag{7.1}$$

in which U_S and U_L denote the solid and liquid phase velocities.

There is no temperature jump at the interface (that is, temperature is continuous), so

$$T_{L,int} = T_{S,int} \tag{7.2}$$

There are two modes of heat transfer to/from the interface: conduction and convection. Considering both, conservation of energy has

$$-k_L \left.\frac{\partial T_L}{\partial x}\right|_{int} - \rho_L U_L h_L = -k_S \left.\frac{\partial T_S}{\partial x}\right|_{int} - \rho_S U_S h_S \tag{7.3}$$

Note that the terms on the left represent the net heat flux into the interface from the liquid side, in which h_L is the enthalpy of the liquid at the interface. The negative sign in front of the convection term is needed because U is directed in the negative x direction. Likewise, the terms on the right represent the net heat flux from the interface into the solid. By using Eq. (7.1) and introducing the latent heat of fusion, $h_{SL} = h_L - h_S$, we get

$$-k_L \left.\frac{\partial T_L}{\partial x}\right|_{int} + k_S \left.\frac{\partial T_S}{\partial x}\right|_{int} = -\rho_S U_S h_{SL} \tag{7.4}$$

Now switch the coordinate origin so that it becomes fixed to a system boundary. Also assume that the densities of the liquid and solid phases are the same ($\rho_S = \rho_L = \rho$), so that $U_S = U_L = U$. In

this representation the interface becomes located at $x = x_{SL}$, and the speed at which the interface moves will be

$$U = -\frac{dx_{SL}}{dt} \tag{7.5}$$

Putting this in the previous equation gives

$$-k_L \left.\frac{\partial T_L}{\partial x}\right|_{x=x_{SL}} + k_S \left.\frac{\partial T_S}{\partial x}\right|_{x=x_{SL}} = \rho\, h_{SL} \frac{dx_{SL}}{dt} \tag{7.6}$$

This represents the interface condition in the system–fixed coordinate system. Because density of both phases is assumed equal, there will be no motion of the liquid or solid phases due to expansion/compression during the phase transition. That is, both phases will be at rest, and conduction will be the sole energy transfer mechanism. One point to make is that the melting (or freezing) of the interface results in a heat flux boundary condition at the interface; the heat flux arising from the latent heat associated with the phase change. Indeed, another way to interpret (or derive) the interface condition is to recognize that ρU represents the mass rate of phase transition per unit area of interface, and multiplying this by h_{SL} gives the heat release, per unit area, at the interface due to the phase transition.

We can also apply thermodynamic principles to fix the temperature of the interface, T_{SL}, which would be equal to the liquid/solid equilibrium temperature (i.e., the fusion temperature) of the substance at the given pressure.

7.3 The Neumann problem

An exact solution to a moving interface problem can be obtained for the 1–D, semi–infinite domain as derived by Neumann in 1912. The problem is illustrated in Fig. 7.2. The system consists of a semi–infinite domain, initially in the solid phase at temperature $T_0 < T_{SL}$. At $t = 0$, the temperature at the $x = 0$ boundary is suddenly brought to $T = T_1 > T_{SL}$. The solid melts at the surface, and as time progresses the interface recedes in the positive x direction. We want to determine the rate at which the interface moves.

The problem has two distinct regions – liquid and solid – and we need to pose the conduction problem for each region. The DEs are

$$\frac{1}{\alpha_L}\frac{\partial T_L}{\partial t} = \frac{\partial^2 T_L}{\partial x^2}, \qquad 0 \le x < x_{SL} \tag{7.7}$$

$$\frac{1}{\alpha_S}\frac{\partial T_S}{\partial t} = \frac{\partial^2 T_S}{\partial x^2}, \qquad x_{SL} \le x < \infty \tag{7.8}$$

in which x_{SL} is the location of the interface. The initial condition for the solid phase is

$$T_S(x, t = 0) = T_0 \tag{7.9}$$

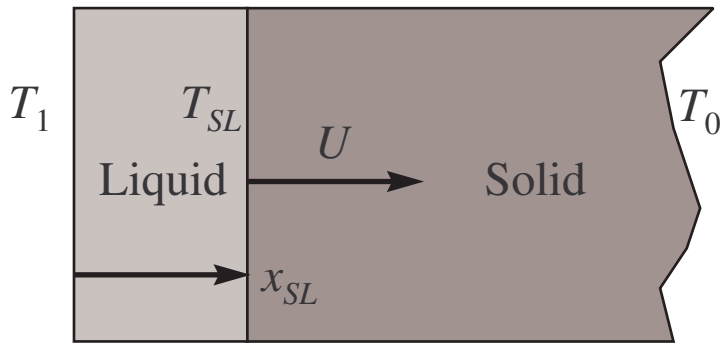


Figure 7.2: The semi–infinite moving interface problem

The IC for the liquid phase cannot be formally posed because at $t = 0$ the liquid region does not exist. This problem, though, will work out in the end. The boundary conditions have

$$T_L(x = 0, t) = T_1 \tag{7.10}$$

$$T_S(x \to \infty, t) = T_0 \tag{7.11}$$

$$T_L(x = x_{SL}, t) = T_{SL} \tag{7.12}$$

$$T_S(x = x_{SL}, t) = T_{SL} \tag{7.13}$$

$$-k_L \left.\frac{\partial T_L}{\partial x}\right|_{x=x_{SL}} + k_S \left.\frac{\partial T_S}{\partial x}\right|_{x=x_{SL}} = \rho\, h_{SL} \frac{dx_{SL}}{dt} \tag{7.14}$$

It may seem that the boundary conditions are over–constrained, in that we have five conditions given whereas only four are needed to close the problem (two for each phase). However, the rate at which the interface moves, dx_{SL}/dt, is itself an unknown and is sought from the solution. In this sense the heat flux condition at the interface does not provide a proper boundary condition. Rather, we will use this condition to obtain the speed of the interface.

As was the case with the semi–infinite domain with specified temperature boundary conditions, the form of the solution to this problem will reveal itself when we attempt to make the problem dimensionless. Specifically, the problem does not present a fixed characteristic length. You may think that x_{SL} is an appropriate length, yet this is an unknown quantity in itself. Using x_{SL} as the characteristic length would fix the interface at position $\overline{x} = 1$; this may be elegant yet it does little to tell us *the actual location of the interface as a function of time.*

In view of this rationale, it follows that the solution will be a function of the variables

$$\eta_L = \frac{x}{2\sqrt{\alpha_L t}}, \quad \eta_S = \frac{x}{2\sqrt{\alpha_S t}} \tag{7.15}$$

We can define our dimensionless temperatures by

$$\overline{T}_L = \frac{T_L - T_1}{T_{SL} - T_1} = \text{func } \eta_L \tag{7.16}$$

$$\overline{T}_S = \frac{T_S - T_0}{T_{SL} - T_0} = \text{func } \eta_S \tag{7.17}$$

By transforming the PDEs for T_L and T_S, one obtains the usual similarity form of the conduction equation for semi–infinite media;

$$\overline{T}''_L + 2\eta_L \overline{T}'_L = 0 \tag{7.18}$$

$$\overline{T}''_S + 2\eta_S \overline{T}'_S = 0 \tag{7.19}$$

Boundary conditions are

$$\overline{T}_L(\eta_L = 0) = 0 \tag{7.20}$$

$$\overline{T}_L(\eta_L = \eta_{SL}) = 1 \tag{7.21}$$

$$\overline{T}_S(\eta_S = \beta\,\eta_{SL}) = 1 \tag{7.22}$$

$$\overline{T}_S(\eta_S \longrightarrow \infty) = 0 \tag{7.23}$$

in which η_{SL} is the dimensionless position of the interface, defined as

$$\eta_{SL} = \frac{x_{SL}}{2\sqrt{\alpha_L t}} \tag{7.24}$$

The solid diffusivity α_S could have been used to define η_{SL}; the choice is arbitrary. The quantity β in Eq. (7.22) is

$$\beta = \sqrt{\frac{\alpha_L}{\alpha_S}} \tag{7.25}$$

which is a fixed parameter of the system. It is needed in Eq. (7.22) because the solid phase variable η_S is defined using α_S, whereas η_{SL} is defined using α_L.

The solutions to the liquid and solid phase problems are easily obtained using the methods learned for transient, semi–infinite problems, and are

$$\overline{T}_L = \frac{\text{erf}(\eta_L)}{\text{erf}(\eta_{SL})} \tag{7.26}$$

$$\overline{T}_S = \frac{1 - \text{erf}(\eta_S)}{1 - \text{erf}(\beta\,\eta_{SL})} \tag{7.27}$$

In obtaining the solution, it was implicitly assumed that the dimensionless interface position η_{SL} is constant, i.e., not a function of time or position. Indeed, if our similarity solution is to be valid,

then η_{SL} *must be a constant*. That is, the solution is assumed to be only a function of η (evaluated in the appropriate liquid or solid phase). The value of η at the interface will be η_{SL} – *by definition* – and this quantity could not depend on time because the solution is not a function of time, only of η. A result of this logic is that the time variation of the interface velocity can be deduced immediately. The interface speed is given by $U = dx_{SL}/dt$, and using $x_{SL} = 2\sqrt{\alpha_L\, t}\,\eta_{SL}$ it follows that

$$U = \sqrt{\frac{\alpha_L}{t}}\,\eta_{SL} \tag{7.28}$$

Recognize again that η_{SL} is a constant – the value of which is sought from the solution. The above formula shows that the interface speed will vary as $1/\sqrt{t}$ – a result which is obtained *solely from dimensional analysis*.

The remaining part is to determine the value of η_{SL}. To do this, the remaining condition in Eq. (7.14) is put to use. The liquid phase temperature derivative is given by

$$\left.\frac{\partial T_L}{\partial x}\right|_{x_{SL}} = (T_{SL} - T_1)\,\left.\frac{d\overline{T}_L}{d\eta_L}\right|_{\eta_{SL}} \cdot \frac{1}{2\sqrt{\alpha_L t}} \tag{7.29}$$

and likewise for the solid phase. Performing the operations on Eq. (7.14) and simplifying, the result is

$$\left.\frac{d\overline{T}_L}{d\eta_L}\right|_{\eta_{SL}} - \frac{k_S\sqrt{\alpha_L}\,(T_{SL} - T_0)}{k_L\sqrt{\alpha_S}\,(T_1 - T_{SL})}\,\left.\frac{d\overline{T}_S}{d\eta_S}\right|_{\beta\,\eta_{SL}} = 2\frac{h_{SL}}{c_{P,L}\,(T_1 - T_{SL})}\,\eta_{SL} \tag{7.30}$$

Now use the solution for $\overline{T}_L$ and $\overline{T}_S$ to evaluate the derivatives;

$$\overline{T}'_L(\eta_{SL}) = \frac{2}{\sqrt{\pi}}\frac{\exp\left(-\eta_{SL}^2\right)}{\mathrm{erf}\,(\eta_{SL})} \tag{7.31}$$

$$\overline{T}'_S(\beta\,\eta_{SL}) = -\frac{2}{\sqrt{\pi}}\frac{\exp\left(-(\beta\,\eta_{SL})^2\right)}{1 - \mathrm{erf}\,(\beta\,\eta_{SL})} \tag{7.32}$$

and replace into Eq. (7.30) to obtain

$$\frac{\exp\left(-\eta_{SL}^2\right)}{\mathrm{erf}\,(\eta_{SL})} - \frac{k_S\sqrt{\alpha_L}\,(T_{SL} - T_0)}{k_L\sqrt{\alpha_S}\,(T_1 - T_{SL})}\,\frac{\exp\left(-(\beta\,\eta_{SL})^2\right)}{1 - \mathrm{erf}\,(\beta\,\eta_{SL})} = \sqrt{\pi}\,\frac{h_{SL}}{c_{P,L}\,(T_1 - T_{SL})}\,\eta_{SL} \tag{7.33}$$

Equation (7.33) provides a nonlinear equation for η_{SL} as a function of the dimensionless system parameters $(T_{SL} - T_0)/(T_1 - T_{SL})$, $\beta = \sqrt{\alpha_L/\alpha_S}$, k_S/k_L, and $h_{SL}/c_{P,L}(T_1 - T_{SL})$. This last parameter is especially relevant in most phase change problems and is referred to as the *Stefan number Ste*. It provides a ratio of the heat released by fusion to the heat required to change the temperature of the generated phase by $T_1 - T_{SL}$. Often, the initial phase of the system (which here is the solid yet it could obviously be the liquid for the case of solidification) is initially at the

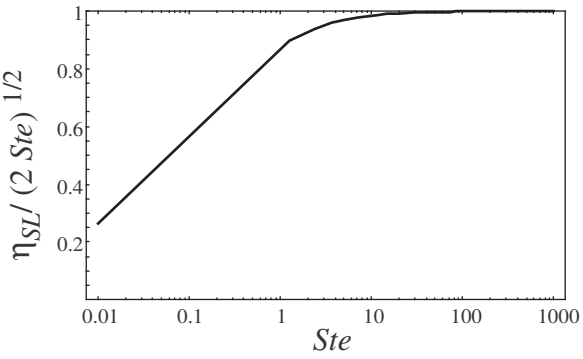


Figure 7.3: Dimensionless interface position

fusion temperature T_{SL}. For such cases the second term in the above equation disappears, and η_{SL} becomes solely dependent on Ste, i.e,

$$\eta_{SL}\,\mathrm{erf}\,(\eta_{SL})\,\exp\left(\eta_{SL}^2\right) = \frac{c_{P,L}(T_1 - T_{SL})}{\sqrt{\pi}\,h_{SL}} = \frac{1}{\sqrt{\pi}\,Ste}, \quad (T_0 = T_{SL}) \tag{7.34}$$

A plot of η_{SL}, scaled with the large–Ste result (discussed below), is presented in Fig. 7.3.

In problems involving water, the Stefan number is typically large due to the relatively large heat of fusion of water. For such cases η_{SL} will be small – which corresponds to a relatively small interface velocity U. By using the small argument expansion of the error function,

$$\mathrm{erf}(\eta) \approx \frac{2\eta}{\sqrt{\pi}}, \quad \eta \ll 1$$

Eq. (7.34) reduces to

$$\eta_{SL} \approx \left(\frac{1}{2\,Ste}\right)^{1/2} \tag{7.35}$$

The dimensional form of this result is

$$x_{SL} = \left(\frac{2k_L(T_1 - T_{SL})}{\rho\,h_{SL}}\right) \tag{7.36}$$

This result can be derived from a simpler outlook. The time scale associated with the interface motion is x_{SL}/U, and the diffusion time (the time it would take heat to diffuse from the interface to the boundary) is x_{SL}^2/α_L. When the latter is significantly smaller than the former, i.e.,

$$\frac{U\,x_{SL}}{\alpha_L} \ll 1$$

then the liquid can be assumed to be in a quasi–steady thermal state. That is, the temperature distribution would be linear, and the gradient at the interface would be

$$\left.\frac{dT_L}{dx}\right|_{x_{SL}} \approx \frac{T_{SL} - T_1}{x_{SL}}$$

If this result is now used in

$$-k_g \left.\frac{dT_L}{dx}\right|_{x_{SL}} = -\rho\, U\, h_{SL}$$

then one arrives at the simple DE

$$x_{SL} dx_{SL} = \frac{k_L(T_1 - T_{SL})}{\rho\, h_{SL}}$$

which is integrated from $x_{SL} = 0$ to x_{SL} to arrive at Eq. (7.36).

7.4 Radial Coordinates

The same procedure can be applied to cylindrical and spherical coordinates, in which the interface moves in the r direction. One example of such a situation would be that of a cylindrical pipe immersed in a frozen medium. At $t = 0$ the surface of the pipe is brought above the melting temperature, and a liquid–solid interface propagates away from the pipe. Another example is that of *nucleation*, in which a small spherical particle is immersed in a subcooled liquid. At $t = 0$ solidification occurs at the surface of the particle, and the interface moves out from the particle.

An analytical solution, however, is possible only if the initial radius of the cylinder or the sphere is infinitesimally small, so that the cylinder or sphere can be modeled as a *line* or *point* source of heat. A line source releases (or absorbs) a fixed amount of heat q' out of/into a cylinder of vanishingly small radius, per unit length of the cylinder. The temperature gradient at the surface of this cylinder goes to infinity, yet the area of the cylinder goes to zero, and the product remains constant. That is, the heat transfer in the medium surrounding the cylinder obeys the condition

$$\lim_{r\to 0} 2\pi\, k_m\, r\, \frac{dT}{dr} = -q' = \text{constant, line source} \tag{7.37}$$

where k_m is the thermal conductivity of the medium surrounding the source and T is the temperature in the medium. A similar expression is obtained for the point source,

$$\lim_{r\to 0} 4\pi\, k_m\, r^2\, \frac{dT}{dr} = -q = \text{constant, point source} \tag{7.38}$$

Note that the temperature at the line/point source would be unbounded (i.e., to infinity or negative infinity). This obviously is not physically realistic. In reality, all sources must have a finite radius

R and a finite surface temperature. The point of the line/point source model, however, is not to predict the temperature *at* the source. Rather, the method is used to model the temperature field at a sufficiently large radii r from the source; large in the sense that $r/R \gg 1$ so that, for all practical purposes, the source can be modeled as a line or a point. The line source (in cylindrical coordinates) or point source (spherical) condition is required to obtain an *analytical* solution to the phase change, moving interface problem in radial coordinates. In such a problem, an initially solid medium (or liquid) is exposed to a line or a point source (or sink) at $t = 0$, and a moving interface propagates away from the source. The reason the line/point source condition is needed is because the analytical solution relies on a similarity variable approach akin to that used in the previous section. And this similarity method (in which the independent variable becomes $\eta \sim r/\sqrt{\alpha t}$) is possible *only if the problem does not offer a characteristic geometrical length.* Such is the case with the line or point source, since the radius of the source is essentially zero.

A line source method may be used to model a melting or solidifying interface that forms from a 'thin' wire, for example. Such problems are, however, somewhat contrived because it must be assumed that the heat transfer rate to the wire is fixed

Of course, it is entirely possible (and realistic) for the interface to begin on a finite–sized cylinder or sphere (with non–negligible radius) and move away from the surface. Analysis of such problems, however, would require numerical methods.

7.4.1 Moving interface from a line source

A similar problem to that worked in the previous section will now be examined, except one that deals with a line source of heat. A medium is initially in the solid phase at temperature T_0. At time $t = 0$ a line source is turned on, resulting in a heat transfer rate (per unit length) of q'_0. The heat melts the liquid, and a circular interface propagates out from the source. The objective is to determine the speed and position of the interface as a function of time.

The problem is cast in cylindrical coordinates. Governing DEs are

$$\frac{1}{\alpha_L}\frac{\partial T_L}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T_L}{\partial r}, \qquad 0 \le r < r_{SL} \tag{7.39}$$

$$\frac{1}{\alpha_S}\frac{\partial T_S}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial T_S}{\partial r}, \qquad r_{SL} \le r < \infty \tag{7.40}$$

in which r_{SL} is the location of the interface. The initial condition for the solid phase is

$$T_S(x, t = 0) = T_0 \tag{7.41}$$

As was the case with the cartesian problem, the IC for the liquid phase cannot be formally posed.

The boundary conditions have

$$\lim_{r\to 0} 2\pi\, k_L\, r\, \frac{dT_L}{dr} = -q_0' \tag{7.42}$$

$$T_S(r \to \infty, t) = T_0 \tag{7.43}$$

$$T_L(r = r_{SL}, t) = T_{SL} \tag{7.44}$$

$$T_S(r = r_{SL}, t) = T_{SL} \tag{7.45}$$

$$-k_L \left.\frac{\partial T_L}{\partial r}\right|_{r=r_{SL}} + k_S \left.\frac{\partial T_S}{\partial r}\right|_{r=r_{SL}} = \rho\, h_{SL}\, \frac{dr_{SL}}{dt} \tag{7.46}$$

The BCs are the same as before, with the exception of the line source condition in Eq. (7.42).

The dimensionless temperatures can be defined as

$$\overline{T}_L = \frac{T_L - T_{SL}}{q_0'/(2\pi k_L)} \tag{7.47}$$

$$\overline{T}_S = \frac{T_S - T_{SL}}{q_0'/(2\pi k_L)} \tag{7.48}$$

There is no fixed characteristic length in the problem – which you should prove to yourself by trying to derive a length from the fixed parameters of the problem – and the problem must admit a similarity solution. For convenience, the similarity variable is defined as

$$\eta = \frac{r^2}{4\alpha_L t} \tag{7.49}$$

The transformed dimensionless equations and boundary conditions become

$$\overline{T}_L'' + \left(1 + \frac{1}{\eta}\right)\overline{T}_L' = 0 \tag{7.50}$$

$$\overline{T}_S'' + \left(\beta + \frac{1}{\eta}\right)\overline{T}_S' = 0 \tag{7.51}$$

$$\lim_{r\to 0} 2\eta\, \overline{T}_L' = -1 \tag{7.52}$$

$$\overline{T}_S(\eta \to \infty, t) = \overline{T}_0 = \frac{T_0 - T_{SL}}{q_0'/(2\pi k_L)} \tag{7.53}$$

$$\overline{T}_L(\eta = \eta_{SL}, t) = 0 \tag{7.54}$$

$$\overline{T}_S(\eta = \eta_{SL}, t) = 0 \tag{7.55}$$

in which the prime denotes differentiation with respect to η, and β is now defined as

$$\beta = \frac{\alpha_L}{\alpha_S} \tag{7.56}$$

As before, the similarity solution results in $\eta_{SL} = r_{SL}^2/(4\alpha_L t)$ being a constant. This implies that the interface position will increase with $t^{1/2}$, which is the same behavior seen with the cartesian case.

The solutions to $\overline{T}_L$ and $\overline{T}_S$ are

$$\overline{T}_L = \frac{1}{2}\left(\mathrm{Ei}(\eta_{SL}) - \mathrm{Ei}(\eta)\right) \tag{7.57}$$

$$\overline{T}_S = \overline{T}_0\left(1 - \frac{\mathrm{Ei}(\beta\eta)}{\mathrm{Ei}(\beta\eta_{SL})}\right) \tag{7.58}$$

In which Ei is the *exponential integral* function. It is defined here as

$$\mathrm{Ei}(\eta) = \int_{\eta}^{\infty} \frac{\exp(-x)}{x}\, dx \tag{7.59}$$

and important properties and limiting values are

$$\frac{d\,\mathrm{Ei}(x)}{dx} = \frac{\exp(-x)}{x}$$

$$\lim_{x\to 0} \mathrm{Ei}(x) \to \infty$$

$$\lim_{x\to 0}\left(r\,\frac{d\,\mathrm{Ei}(x)}{dx}\right) \to 1$$

$$\lim_{x\to \infty} \mathrm{Ei}(x) \to 0$$

The exponential integral is an intrinsic function in *Mathematica*, although it is defined differently in that the *Mathematica* version has the integral running in Eq. (7.59) from $-\eta$ to ∞. The equivalence between that used here and *Mathematica* is

$$\mathrm{Ei}(x) = \; -\texttt{ExpIntegralEi}[-\texttt{x}]$$

The interface boundary condition in Eq. (7.46) can now be evaluated. The transformed form appears as

$$\overline{T}'_L(\eta_{SL}) - \frac{k_S}{k_L}\overline{T}'_S(\eta_{SL}) = \frac{2\pi\,\rho\alpha_L\, h_{SL}}{q'_0} \tag{7.60}$$

and using the solutions for $\overline{T}_L$ and $\overline{T}_S$ and the properties of $\mathrm{Ei}(\eta)$ gives

$$\exp(-\eta_{SL}) + \frac{2\overline{T}_0\exp(\beta\,\eta_{SL})}{\mathrm{Ei}(\beta\,\eta_{SL})} = \frac{2\pi\,\rho\alpha_L\, h_{SL}}{q'_0}\,\eta_{SL} \tag{7.61}$$

This provides a nonlinear equation for the dimensionless interface position η_{SL} as a function of the dimensionless parameters of the system.

Chapter 8

Hybrid Analytical/Numerical Methods in Conduction

8.1 Introduction

In spite of the apparent power and glory of the SOV/superposition/integral methods that have been developed in the previous chapters, the methods remain applicable only to a relatively constrained (or idealized) set of problems. The domain to which the analysis is applied must conform to an orthogonal coordinate system (such as the cartesian, cylindrical, and spherical systems with which we have limited our focus) and the boundary condition on each surface of the system must be limited to a single type (fixed T, fixed gradient, or convection).

Seldom do such ideal situations occur in actual engineering practice. The domain, for example, might take the shape of a coffee cup or a gear, which would not – at first glance – correspond to the simple configurations that we have examined. Perhaps, on the other hand, the domain is in the form of a rectangle or a cylinder, yet the boundary conditions on a particular surface might be of 'mixed type', e.g., convective over one portion of the surface, and adiabatic on the remaining portion.

When faced with such situations, a common practice is to hold one's nose and apply a seemingly unrealistic model to the actual system (e.g., the cow modelled as a sphere). The desire behind such actions, of course, is to formulate a problem that admits an analytical solution – yet the formulated problem may be so removed from reality that any conclusions based on the model problem would be suspect.

The other approach to such problems, of course, is to apply a completely *numerical* method – which would provide, in priciple, an exact (to the numerical accuracy of the algorithm) solution to the differential equation and boundary conditions that are posed for the system. Numerical methods and computational technology have advanced to the point where a relative idiot can use a black–box package to obtain a vast set of numbers and color–coded, 3–D pictures of a 'solution'. As

you may detect, I am suspicious of such methods – because they can remove *analytical* reasoning from the so–called analysis of a problem.

The objective of this chapter is to present methods of extending the analytical approaches developed in the previous chapters to situations in which, formally speaking, analytical solutions do not exist. The basic procedure of the methods is to use an eigenfunction expansion of the temperature field which identically satisfies the governing differential equation and as many of the boundary conditions as possible. The boundary conditions that cannot be analytically satisfied are matched in an approximate sense, by manipulation (or fitting) of the solution to the prescribed boundary conditions. This last step will involve numerical strategies (specifically, solution of linear equations), and because of this the methods can be viewed as a hybrid analytical/numerical approach. Alternatively, such approaches are often referred to as *spectral* or *moment* methods.

The advantage of hybrid methods over completely numerical methods (such as finite difference and finite element) is that the resulting system of equations can be considerably smaller under the hybrid approach, and that integrated quantities such as heat transfer from a surface can be obtained analytically from the solution. Hybrid methods can also be applied to situations involving mixed coordinate systems (such as a combination of a rectangle and a cylindrical system) which are difficult to 'grid' under standard discretized–based numerical methods. In addition, the symbolic manipulation and linear equation solving capabilities of *Mathematica* make application of hybrid methods relatively simple.

There is no unique procedure for developing a hybrid method – the approach requires some mathematical imagination and an ability to anticipate the expected form of the temperature field. In view of this, the presentation on hybrid methods will be given as a series of examples.

8.2 Mixed boundary conditions

8.2.1 The rectangular enclosure

A square, 2–D region, illustrated in Fig. 8.1, has adiabatic boundaries at $x = 0$ and L. The bottom surface, at $y = 0$, is cooled by convection that is characterized by h and $T_{\infty,1}$. The top surface at $y = L$, on the other hand, has mixed boundary conditions. From $x = 0$ to x_1 the surface is heated by convection from $T_{\infty,2}$, with the same convection coefficient h as on the bottom surface, and from $x = x_1$ to L the surface is adiabatic.

Even though both the convection and flux conditions that exist on the top surface could be posed as homogeneous (by suitable definition of $\overline{T}$), the fact that they are of two distict types would prevent the characteristic form of the $v(\overline{y})$ solution from matching both BCs simultaneously. The problem therefore would not have a closed–form analytical solution.

The hybrid procedure begins by posing an eigenfunction expansion for the temperature field in the region. This expansion will automatically satisfy the DE (here, Laplace's equation) and should be defined so that it matches as many of the boundary conditions as possible.

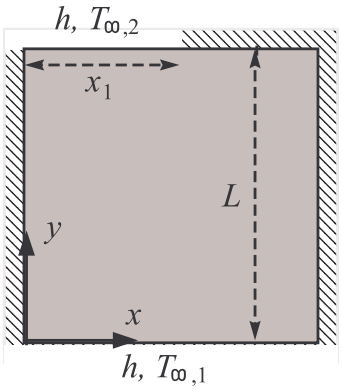


Figure 8.1: mixed boundary conditions

The boundary conditions at $x = 1$, $x = L$, and $y = 0$ are all of single–type, and the expansion can be modelled so that these conditions are identically satisfied. Specifically, if the dimensionless variables are defined by

$$\overline{T} = \frac{T - T_{\infty,1}}{T_{\infty,2} - T_{\infty,1}}, \quad \overline{x} = \frac{x}{L}, \quad \overline{y} = \frac{x}{L}$$

for which the boundary conditions become

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_0 = 0 \tag{8.1}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{x}}\right|_1 = 0 \tag{8.2}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{y}}\right|_0 = Bi\overline{T} \tag{8.3}$$

$$\left.\frac{\partial \overline{T}}{\partial \overline{y}}\right|_1 = \begin{cases} -Bi(\overline{T} - 1), & \overline{x} \le \overline{x}_1 \\ 0, & \overline{x} > x_1 \end{cases} \tag{8.4}$$

then the expansion, eigenfunction, and eigencondition of

$$\overline{T} = A_0 \left(Bi\, \overline{y} + 1\right) + \sum_{n=1}^{\infty} A_n \phi_n(\overline{x}) \left(Bi\, \sinh(\lambda_n \overline{y}) + \lambda_n \cosh(\lambda_n \overline{y})\right) \tag{8.5}$$

$$\phi_n(\overline{x}) = \cos(\lambda_n \overline{x}) \tag{8.6}$$

$$\lambda_n = n\pi \tag{8.7}$$

will satisfy Eqs. (8.1–8.3). The leading term in Eq. (8.5) is needed to account for the zeroth eigenvalue part of the solution – recognize that this term would be the only remaining part of the solution for the case of $\overline{x}_1 \rightarrow 1$ (for which the top surface would be entirely convective and the solution would become 1–D).

If we were dealing with a well–behaved problem, the expansion coefficients in Eq. (8.5) would be obtained explicitly from application of the orthogonality properties of ϕ_n on the one remaining inhomogeneous BC. Equation (8.4), however, will not provide an orthogonal relation for the coefficients – because the convective condition (which is inhomogeneous) is maintained only to $\overline{x} = \overline{x}_1$. Nevertheless, the solution will now be applied to Eq. (8.4) per the usual procedure, and we will see what happens.

Replacing the solution directly into the BC gives

$$A_0\, Bi + \sum_{n=1}^{\infty} A_n \phi_n(\overline{x})\, \lambda_n \left(Bi\, \cosh(\lambda_n) + \lambda_n \sinh(\lambda_n)\right)$$
$$= \begin{cases} Bi - Bi\left[A_0\left(Bi+1\right) + \displaystyle\sum_{n=1}^{\infty} A_n \phi_n(\overline{x}) \left(Bi\, \sinh(\lambda_n) + \lambda_n \cosh(\lambda_n)\right)\right] & \overline{x} \le \overline{x}_1 \\ 0 & \overline{x} > \overline{x}_1 \end{cases}$$

The equation can now be multiplied by the eigenfunction ϕ_m and integrated over $\overline{x}$ from 0 to 1. On the right hand side, the integral can be split into two parts, covering $\overline{x} = 0$ to $\overline{x}_1$ and $\overline{x}_1$ to 1. And the second part will be zero, per the above formula. The result is

$$\int_0^1 \left[A_0\, Bi + \sum_{n=1}^{\infty} A_n \phi_n(\overline{x})\, \lambda_n \left(Bi\, \cosh(\lambda_n) + \lambda_n \sinh(\lambda_n)\right)\right] \phi_m(\overline{x})\, d\overline{x}$$
$$= Bi \int_0^{\overline{x}_1} \left\{1 - A_0\left(Bi+1\right) - \sum_{n=1}^{\infty} A_n \phi_n(\overline{x}) \left(Bi\, \sinh(\lambda_n) + \lambda_n \cosh(\lambda_n)\right)\right\} \phi_m(\overline{x})\, d\overline{x}$$

The left hand side is orthogonal, and all terms in the expansion disappear except the one for $n = m$. This is not the case, however, on the right hand side. For $m = 0$ (i.e., the zeroth eigenvalue for which $\phi_0 = 1$), the result is (after cancelling the common Bi factor)

$$A_0 = \left[1 - A_0\left(Bi+1\right)\right] x_1 - \sum_{n=1}^{\infty} A_n \frac{\sin(\lambda_n \overline{x}_1)}{\lambda_n} \left(Bi\, \sinh(\lambda_n) + \lambda_n \cosh(\lambda_n)\right) \tag{8.8}$$

and for $m > 0$, the formula appears

$$\frac{1}{2} A_m\, \lambda_m \left(Bi\, \cosh(\lambda_m) + \lambda_m \sinh(\lambda_m)\right) = Bi\, \left[1 - A_0\left(Bi+1\right)\right] \frac{\sin(\lambda_n \overline{x}_1)}{\lambda_n}$$
$$- Bi \sum_{n=1}^{\infty} A_n\, G_{mn}\, \left(Bi\, \sinh(\lambda_n) + \lambda_n \cosh(\lambda_n)\right) \tag{8.9}$$

where G_{mn} is shorthand for the integral

$$G_{mn} = \int_0^{x_1} \phi_m(\overline{x})\phi_n(\overline{x})\, dx$$
$$= \begin{cases} \dfrac{(m+n)\,\sin((m-n)\,\pi\,\overline{x}_1) + (m-n)\,\sin((m+n)\,\pi\,\overline{x}_1)}{2\,(m-n)\,(m+n)\,\pi}, & m \neq n \\ \dfrac{2\,m\,\pi\,\overline{x}_1 + \sin(2\,m\,\pi\,\overline{x}_1)}{4\,m\,\pi}, & m = n \end{cases} \tag{8.10}$$

The previous equations shown that application of the mixed boundary condition does result in a relationship for the expansion coefficients (in Eqs. (8.8) and (8.9)) – yet for this case all of the coefficients are linearly related to each other. That is, A_m depends on A_0, A_1, $A_2, \ldots$. This relationship is not of much use if one retains the precise mathematical definition of the infinite series expansion – i.e., the summation of an infinite number of terms. However, if the upper limit on the series expansion is fixed at the onset at some number N, then the derived relationships lead to a *system of linear equations* for the coefficients. That is, Eqs. (8.8) and (8.9) can be condensed into the form

$$A_m + \sum_{n=0}^{N} F_{mn}\, A_n = f_m, \quad m = 0, 1, 2, \ldots N \tag{8.11}$$

in which the matrix F_{mn} and the vector f_m would depend on the parameters Bi and $\overline{x}_1$. For numerical values of these parameters, this system of $N+1$ equations could be solved using standard numerical methods to yield numerical values of the $N+1$ expansion coefficients.

The system of equations in Eqs. (8.11) can be viewed as *moment equations*. In a most general sense, a moment of some function $f(x)$, defined with respect to the weighting function $\phi_m(x)$, would be given by

$$\langle f\, \phi_m \rangle \equiv \int_0^1 f(x)\, \phi_m(x)\, dx$$

in which the braces $< \ldots >$ denote the *inner product* – which can be viewed as a generalized dot product[1]. The function f to which we are applying the moments corresponds, in the case here, to the normal temperature gradient at $y = 1$, and the moments would be equal to A_m times a constant. If the convective condition were maintained across the entire upper face (which is the sort of problem examined in all prior SOV applications), the moment equations in Eq. (8.11) would reduce to a *diagonal* form, i.e., the matrix G would be identically zero. Consequently, an explicit, closed–form formula could be derived for the expansion coefficients

For mixed BCs (and for other situations which will be examined in subsequent sections) the moment equations are no longer diagonal – yet this only limits our ability to obtain explicit formulas

[1] The m^{th} moment of the eigenfunction ϕ_n is, by the very property of ϕ_n, zero except for $m = n$. This is why such functions are called orthogonal. In euclidian geometry a dot product between two orthogonal vectors is zero (e.g., $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = 0$). By the same concept, the eigenfunction set $\phi_0, \phi_1, \ldots \phi_N$ can be viewed as an N^{th}–dimensional set of basis 'vectors' (or directions).

for the coefficients. Rather, we must resort to linear equation solution methods to numerically calculate the set of coefficients. The validity and accuracy of the solution is not compromised by this feature (providing that the equations are correct and the equations can be solved). As is the case with standard SOV solutions, the solution would be mathematically exact in the limit of $N \to \infty$ – and anything less is approximate.

The coding and solution procedure is straightforward to implement on *Mathematica*– since the package contains routines for solution of linear equations and for symbolic integration. The code used to obtain a solution is listed below

```
g[m_, n_] :=(2 m Pi x1 + Sin[2 m Pi x1])/
  (4 m Pi) /; n == m && m > 0
g[m_, n_] := ((m + n) Sin[(m - n) Pi x1] +
  (m - n) Sin[(m + n) Pi x1])/(2 (m - n) (m + n) Pi) /; m != n
g[m_, n_] := x1 /; n == m && m == 0
g[n_] := Which[n == 0, 1, n > 0, 1/2]

phi[n_,x_]:=Cos[n Pi x]
lam[n_]:=Pi n
term[n_, y_] :=an[0] (bi y + 1) /; n == 0
term[n_, y_] := (an[n] (Cosh[lam[n] y] lam[n]+
        bi Sinh[lam[n] y]))/Cosh[lam[n]] /; n > 0
temp[x_, y_] :=Sum[term[n, y] Cos[lam[n] x],{n, 0, ntot}]

ntot = 50; x1 = 0.5; bi = 100;
anvec = Table[an[n], {n, 0, ntot}];
eqns = Table[(D[term[m, y], y] /.y -> 1) g[m] == bi g[m, 0] -
    bi Sum[term[n, 1] g[m, n],{n, 0, ntot}], {m, 0, ntot}];

soln = NSolve[eqns, anvec][[1]];
```

The function `g[m,n]` defined in the first three lines denotes the integral G_{mn} in Eq. (8.10) (in which the case of $m = 0$ has been included into the definition), and `g[n]` gives the normalization of the eigenfunction. The formulas for G_{mn} were entered into the code by calculation of the integral using `Integrate` and subsequent pasting of the result into a function definition. Note that placing a '`/;`' following a function definition results in a conditional test. That is, the commands `/; n == m && m > 0` following the first definition indicates that the definition for `g[m,n]` holds only for $m = n$ and $m > 0$. The quantity `term[n,y]` represents the $\overline{y}$–dependent part of the n^{th} term in the series – including the unknown expansion coefficients A_n. I have divided this term in the code by $\cosh(\lambda_n)$, which eliminates excessively large elements in the coefficient matrix G_{mn} and thereby

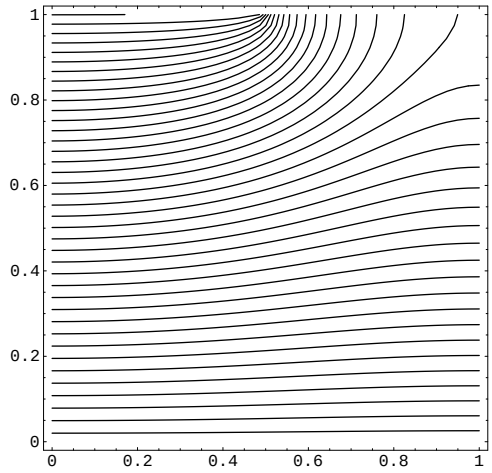


Figure 8.2: isotherms for the mixed BC problem

avoids numerical loss–of–precision errors. The limit N (=`ntot`) is assigned a value of 50, and numerical values are also given to Bi and x_1. The quantity `anvec` represents a vector (or a list in *Mathematica* parlance) of the unknown coefficients, and `eqns` represents a list of the equations for each m. Observe how the equations were formulated; I let *Mathematica* calculate the $\overline{y}$ derivative of the solution as it appears in the BC, and the use of the `term[n,y]` and `g[m,n]` functions enables the equations for $m = 0$ and $m > 0$ to be written with the same form.

Contour plots of the temperature field, and a plot of the normal gradient at $\overline{y} = 1$, are generated with the following commands;

```
ContourPlot[temp[x,y]/.soln,{x,0,1},{y,0,1},PlotPoints->40,
Contours->40,ContourShading->False]

Plot[(D[temp[x,y],y]/.y->1/.soln),{x,0,1}]
```

The plots from the computations appear in Fig. 8.2 and 8.3, in which $Bi = 100$ and $\overline{x}_1 = 0.5$. The contour plot shows, qualitatively, that the upper surface temperature is nearly uniform for $\overline{x} \leq 0.5$ (which would be the case for the large Bi used) and adiabatic elsewhere. A more quantitative measure of the success of the method is provided by the plot of the normal gradient. As can be seen, the gradient is not identically zero at all points along the surface for $\overline{x} > 0.5$ – rather, it oscillates about zero. This behavior is similar to that encountered in Ch. 3 when a step function was expanded into a series of eigenfunctions. The oscillations can be viewed as an *aliasing* error in the solution. This occurs because the minimum wavelength (or length scale) that can be represented

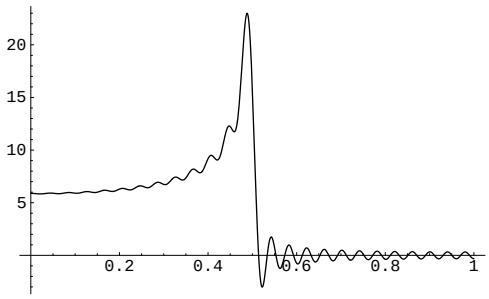


Figure 8.3: normal gradient at $\overline{y} = 1$, mixed BC problem

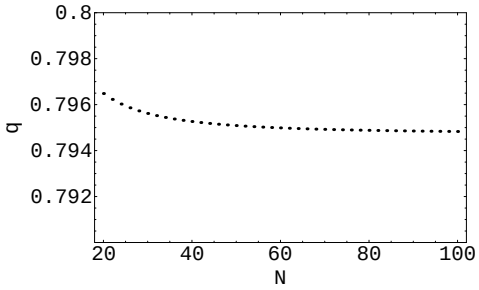


Figure 8.4: total heat transfer vs. truncation number N

by the truncated series is $\sim \pi/N$ (~ 0.06 for $N = 50$), yet the step change that occurs at $\overline{x} = \overline{x}_1$ has a wavelength of essentially zero. As you may recall from Fourier transforms, the 'information' in the BC that has a wavelength smaller than π/N will 'appear' in the series as spurious signals. If, on the other hand, the form of the BC was such that the change from convective to adiabatic conditions was continuous, then the series would have provided a much improved representation of the exact solution.

Numerical methods based on finite difference or finite element methods can resolve the temperature field to the scale of the steps size Δx used to discretize the governing equations. For the 2–D problem examined here, and for a constant mesh size, these numerical methods would require solution of $\sim 1/\Delta x^2$ equations to obtain the temperature field. A comparable level of precision between the numerical and the hybrid methods would have $\Delta x \sim \pi/N$ – which indicates that the hybrid method will require solution of significantly fewer equations (on the order N) to obtain the same level of precision as the numerical methods. This is not surprising, because the hybrid method utilizes a functional form of the solution which automatically satisfies the differential equation (and the homogeneous BCs, in this case).

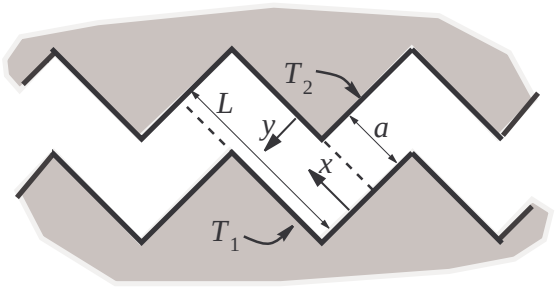


Figure 8.5: sawtooth region

Observe also that the dimensionless total heat transfer (per unit width in and out of the paper) across the region would be

$$q' = \int_0^1 \left.\frac{\partial T}{\partial y}\right|_{y=0} dx = Bi\, A_0 \tag{8.12}$$

(of course, the derivative could have been evaluated on the upper surface as well – the same result would be obtained). The point to make here is that the total heat transfer – which would be an important quantity from the solution – is obtained from a single moment of the solution. The value of the heat transfer converges fairly quickly for increasing N, which can be observed in Fig. 8.4.

8.2.2 The saw–tooth region

Prediction of conduction heat transfer across a gap is trivial if the structure of the gap corresponds to two parallel surfaces separated by a distance L. In many relevant situations – such as in the prediction of *contact resistance* – the gap surfaces are rough and irregular. As a first stab towards predicting heat transfer between two rough surfaces, the surfaces are modeled in a sawtooth pattern as illustrated in Fig. 8.5. The geometry of the pattern is characterized by the gap width a and length L. The separate surfaces are at uniform temperatures of T_1 and T_2, respectively.

The given situation obviously does not, as a whole, correspond to an orthogonal coordinate frame. However, providing that $a/L \leq 0.5$, a section of the geometry can be modeled as the 2–D rectangular region of length L and width a. This is shown in Fig. 8.5, in which the coordinate x runs in the L direction. For $a/L > 0.5$ the tangent from one 'tooth' (seen as the dotted line in Fig. 8.5) would no longer intersect with the face of the opposite tooth, and a temperature field could not be described by that in a single rectangular subregion.

Let L represent the characteristic length and T_1 the characteristic temperature. Redifine the variables as $a \to a/L$, $T \to (T - T_1)/(T_2 - T_1)$, $x \to x/L$, and $y \to y/L$.

The boundary conditions on the $x = 0$ and 1 faces are simple; T of 0 and 1, respectively. The situation is more complicated on the $y = 0$ and a surfaces – in that part of the surface (that

corresponding to the physical surface) is at a uniform T (either 0 on the $y = a$ face or 1 on the $y = 1$ face), yet the remaining part (that corresponding to the gap) is at an unknown condition. Formally speaking, the problem stands as

$$T(0, y) = 0$$
$$T(1, y) = 1$$
$$T(x, 0) = \begin{cases} ?, & x \leq a \\ 1, & x > a \end{cases} \tag{8.13}$$
$$T(x, a) = \begin{cases} 0, & x < 1 - a \\ ?, & x \geq 1 - a \end{cases} \tag{8.14}$$

For this particular problem, the needed boundary condition information must be obtained from the symmetries in the temperature field. It should be easy to see that the temperature field throughout the region has a sort of 'reversed' mirror symmetry, in that

$$T(x, y) = 1 - T(1 - x, a - y) \tag{8.15}$$

Prove this to yourself by testing several pairs of points within the region (including the surfaces). With this in mind, the analytical form of the solution could be cast as

$$T(x, y) = x + \sum_{n=1}^{N} A_n \, \phi_n(x) \, [\sinh(\lambda_n y) + (-1)^n \sinh(\lambda_n (a - y)] \tag{8.16}$$
$$\phi_n(x) = \sin(\lambda_n x) \tag{8.17}$$
$$\lambda_n = \pi n \tag{8.18}$$

The superposition of the leading term x into the solution provides a means of satisfying the BC at $x = 1$; it also would give the (trivially) exact solution for $a \to 1$, for which the problem would become 1–D. The eigenfunctions are chosen in the x direction because (with the superposition of x) this direction is homogeneous. The particular form of the solution is based on the fact that $\phi_n(1 - x) = -(-1)^n \phi_n(x)$ – which will provide the desired symmetry. The use of the hyperbolic sin function is arbitrary; cosh would have given an equivalent result in the end.

The general solution in Eq. (8.16) will automatically satisfy the condition that $T(x, 0) = T(1 - x, a)$ – which when applied to the gap ($x \leq a$ for $y = 0$) would seem to provide a boundary condition. This, however, is not the case – the temperature profile in the gaps is still unknown; the form of the solution simply states that the profile in both gaps are identical.

The information required to close the problem comes from a higher–level symmetry in the problem. Specifically, the line joining opposite corners of the hot and cold surfaces would represent a plane of symmetry (i.e., an adiabatic surface), as illustrated in Fig. 8.6. The temperature distribution in the square, $a \times a$ subregion at either end of the domain would be symmetrical about this

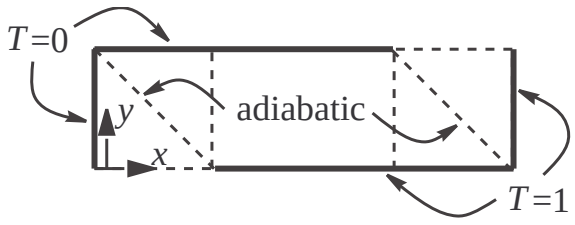


Figure 8.6: symmetries in the region

adiabatic plane. With regard to the boundary condition in the gap, it could then be stated that

$$T(x,0) = \begin{cases} T(a, a-x) = 1 - T(1-a, x) & x \le a \\ 1, & x > a \end{cases} \tag{8.19}$$

This piece of information will close the problem. The boundary condition is not of a 'traditional' form – in that it relates the temperature on the boundary to the interior temperature – yet it does provide an adequate constraint on the solution that is independent of Eq. (8.15).

Equation (8.16) is now substituted into Eq. (8.19), and the result is multipled by $\phi_m(x)$ and integrated over x from 0 to 1 to obtain

$$\int_0^1 \left(x + \sum_{n=1}^{N} A_n \, \phi_n(x) \, (-1)^n \sinh(\lambda_n a) \right) \phi_m(x) \, dx$$
$$= \int_0^a \left(1 - (1-a) - \sum_{n=1}^{N} A_n \, \phi_n(1-a) \left[\sinh(\lambda_n x) + (-1)^n \sinh(\lambda_n (a-x)) \right] \right) \phi_m(x) \, dx + \int_a^1 \phi_m(x) \, dx$$

Similar to the previous example, the left hand side can be interpreted as the m^{th} moment of the surface temperature along $y = 0$ – which will be dependent solely on the m^{th} expansion coefficient. The integrands involving A_n on the right hand side, however, are not orthogonal functions and consequently all of the expansion coefficients will exist following the integration.

In a condensed form, the above equation will appear as

$$\frac{(-1)^n}{2} \sinh(\lambda_m a) \, A_m = f_m(1) - (1-a) f_m(a) - g_m + \sum_{n=1}^{N} A_n \, \phi_n(1-a) \, G_{nm} \tag{8.20}$$

in which

$$f_m(y) = \int_0^y \phi_m(x)\,dx \tag{8.21}$$

$$g_m = \int_0^1 x\,\phi_m(x)\,dx \tag{8.22}$$

$$G_{nm} = \int_0^a \Big[\sinh(\lambda_n x) + (-1)^n \sinh(\lambda_n(a-x)\Big]\phi_m(x)\,dx \tag{8.23}$$

All of these integrals can be worked out by simple inspection or by *Mathematica*, and the explicit results need not be given here.

Optimized *Mathematica* code

The system of equations in Eq. (8.20) can be directly coded into *Mathematica*, and a set of expansion coefficients (for a given N and a) can be calculated via the `Solve` function. The *Mathematica* code which obtains the solution is given below;

```
fm[m_] := (-m Pi Cos[m Pi] + Sin[m Pi])/(m^2 Pi^2)
gm[m_, x1_] := (1 - Cos[m Pi x1])/(m Pi)
gnm[n_, m_] :=1/((m^2 + n^2) Pi) (m ((-1)^n - Cos[a m Pi]) +
  n Coth[a n Pi] Sin[a m Pi] -
  (-1)^n n Csch[a n Pi] Sin[a m Pi])
phi[n_,x_]:=Sin[lambda[n] x]
lambda[n_]:=n Pi
temp[x_,y_]:=
  x+Sum[an[n](Sinh[lambda[n] y]+(-1)^n Sinh[lambda[n](a-y)])/
  Sinh[n Pi a]phi[n,x],{n,1,ntot}]

ntot=50;a=.25;
coefvec=Table[an[n],{n,1,ntot}];
eqns=Table[fm[m]+an[m] (-1)^m /2==
           gm[m,1]-(1-a)gm[m,a]-
           Sum[an[n] phi[n,1-a] gnm[n,m],{n,1,ntot}],
      {m,1,ntot}];
soln=Solve[eqns,coefvec][[1]]
```

The formulas for the integrals, given in the first three function definitions, were again coded by first using `Integrate` to evaluate the integral, and then pasting the result into a function

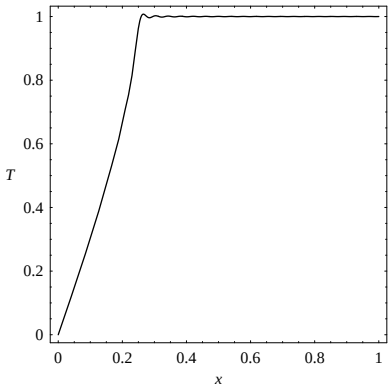


Figure 8.7: surface temperature distribution, $a = 0.25$

definition. As before, I divided each term in the series with $\sinh(\lambda_n a)$ to avoid complaints from *Mathematica* due to an ill–conditioned matrix (which is essentially the same as dividing Eq. (8.20) by $\sinh(\lambda_m a)$), and the integral definitions reflect this scaling. Realize that you would not want to define the integral functions using `Integrate` in the definition – this would be terribly inefficient as *Mathematica* would derive the integral each and every time the functions are called. Rather, the optimized code used here employs the symbolic result of the integrals – which can be generated much faster.

Inefficient (yet easy) *Mathematica* code

If you are willing to sacrifice execution speed for code simplicity, there is an easy and direct method to translate the formulas for the A_n coefficients into *Mathematica* instructions. The approach is to leave the formulas for the moment equations in the most basic form and let *Mathematica* take care of the details. In particular, the equation for the m^{th} moment is obtained directly from Eq. (8.19) as

$$\int_0^1 T(x,0)\phi_m(x)\,dx = \int_0^1 \phi_m(x)\,dx - \int_0^a T(1-a,x)\,\phi_m(x)\,dx \tag{8.24}$$

Mathematica could perform the integrations providing that $\phi_m(x)$ and $T(x,y)$ have a defined functional form – which are given in Eqs. (8.16–8.18). The A_n coefficients in Eq. (8.16), however, would *initially be undefined* – and would appear as variables in Eq. (8.24) for each m. Executing Eq. (8.24) for $m = 1, 2, \ldots N$ would therefore given N equations for the N unknown coefficients.

This strategy can be applied in a few lines of code;

```
phi[n_,x_]:=Sin[lambda[n] x]
lambda[n_]:=n Pi
temp[x_,y_]:=x+Sum[an[n](Sinh[lambda[n] y]+(-1)^n Sinh[lambda[n](a-y)])/
   Sinh[n Pi a]phi[n,x],{n,1,ntot}]
ntot=5;a=.25;
coefvec=Table[an[n],{n,1,ntot}];
eqns=Table[
   Integrate[temp[x,0] phi[m,x],{x,0,1}]==
      Integrate[phi[m,x],{x,0,1}]-
      Integrate[temp[1-a,x]phi[m,x],{x,0,a}],
   {m,1,ntot}];
soln=Solve[eqns,coefvec][[1]]
```

Observe the literal translation of the working formulas in Eq. (8.24) into *Mathematica* code. This approach works – it is entirely equivalent to the optimized code – yet it is slow. As discussed above, the integrations are performed 'from scratch' for each n and m pair. Furthermore, the approach does not recognize beforehand the orthogonality inherent in the left hand side integral. Nevertheless, this approach allows for a quick determination (using only a few terms in the series) of the correctness in the formulation.

Results

Results are presented for $a = 1/4$, in which 50 terms were retained in the series. A precise test of this limit could be obtained from, say, examination of the convergence of the total heat transfer rate across the cooled surface for increasing N. I used 50 here because it 'qualitatively' gave results that looked correct (a poor standard). Plotted in Fig. 8.7 is the temperature profile along the $y = 0$ surface as a function of x – which indicates directly that the imposed uniform temperature boundary condition is met on the physical surface ($x > a$). An additional measure of accuracy is seen in the contour plots of Fig. 8.8, in which I superimposed three contour plots together to form a plot for the sawtooth geometry. The lines line up very nicely, and the symmetry condition between adjacent corners – which provided the BC information – is well established.

Heat transfer

The heat transfer rate across the gap can be obtained by integrating the normal flux over the isothermal surface. For this situation the heat transfer rate per unit width (in and out of the paper) could be given from a 'shape factor' formula via

$$q' = S\,k(T_1 - T_2) \tag{8.25}$$

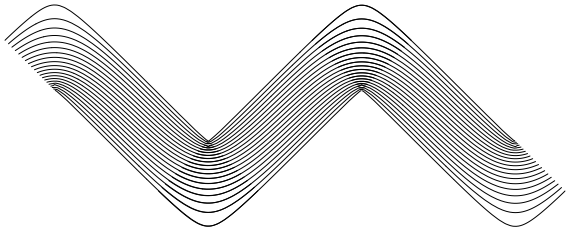


Figure 8.8: superimposed temperature profiles in the sawtooth

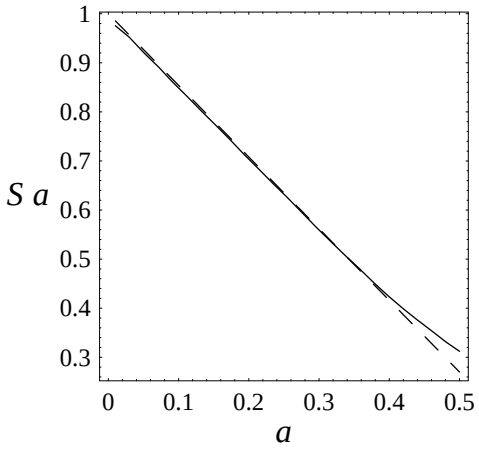


Figure 8.9: Dimensionless heat transfer in the sawtooth gap

where the dimensionless shape factor S is defined by

$$S = -\int_a^1 \left.\frac{\partial T}{\partial y}\right|_0 dx \tag{8.26}$$

Note that the integral of the flux covers only the isothermal surface along $y = 0$. Since this surface does not span the limits of the eigenfunction, the integral will not result in an orthogonal product of eigenfunctions, i.e., all the terms in the series for T will contribute to the total heat transfer.

In the limit of $a \to 0$ the flux will be dominated by 1–D conduction across the gap, and accordingly the flux will become linearly proportional to $1/a$ in this limit. The quantity Sa would therefore approach unity for small a. It is possible in this limit to use simple shape factor analyses

(from any undergraduate heat transfer text) to estimate the heat transfer rate. Specifically,

$$Sa \approx 1 - 2a + 0.54\,a \tag{8.27}$$

This approximation superimposes the 1–D conduction formula for a gap of thickness a and length $1 - 2a$ with a 2–D formula for heat transfer across a corner. Plotted in Fig. 8.9 are the exact and approximate (dotted line) formulas for the dimensionless Sa. Evidently, all the effort that went in to the exact formula is of questionable value – because the approximate formula predicts the heat transfer rate rather well. This sort of information, though, is difficult to assess without the exact 'benchmark' provided here.

8.3 Nonorthogonal domains

8.3.1 Joined rectangular regions

This example relates to the heat removal from small electronic components by conduction to a cooled boundary. The components are modelled as an array of 2–D rectangular elements, of width $2a$ and height b, and they are situated on an adiabatic boundary. The centers of the elements are a distance $2L$ apart, and the elements will be assumed to have a uniform surface temperature of T_1. The cooled boundary is located a distance H above the lower adiabatic boundary, and the surface is maintained at a uniform temperature T_2. All of this is illustrated in Fig. 8.10.

We will assume that the number of elements in the array is large (i.e., effectively infinite). Because of this, the temperature distribution about each array would be symmetrical. In particular, the vertical planes running midway between the elements and midway between the spaces would be adiabatic in view of the required symmetry of the temperature field. It is therefore possible to analyze the array as a whole by examining the temperature field about a single array, as shown in Fig. 8.11. I've flipped the original configuration, so that the cooled surface is on the bottom and the element is in the top right hand corner.

The heat transfer domain for this problem corresponds to a non–rectangular cartesian system. The SOV method will certainly not be applicable for such a system – because the location of the surfaces cannot be described simply by $x =$ constant or $y = constant$. Rather, the location of the surfaces are a function of both variables; e.g., the side of the element exists at $x = L - a$ *providing that* $y > H - b$.

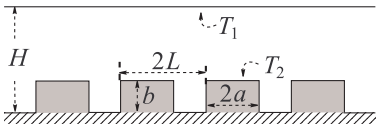


Figure 8.10: an array of heated rectangular elements

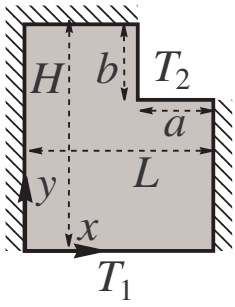


Figure 8.11: the equivalent domain

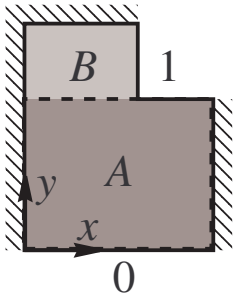


Figure 8.12: the split domain domain

In the following section we will examine a technique in which a single eigenfunction expansion is used to represent the temperature field throughout the entire domain – much as was done in the previous two examples yet one that involves more sophisticated mathematics. The problem at hand, though, lends itself well to a *split* representation of the temperature field.

In the split approach, the domain is split into two or more sub–domains (or regions), and expansions are formulated for the temperature field within each region. Obviously, each region should be of a shape that will conform to an orthogonal coordinate system (i.e., rectangular) so that analytical solutions can be formulated for the temperature field within the region. The boundary conditions on boundaries between one region and another will not be known *a priori* – yet it is possible to obtain complete sets of moment equations for each region by application of *continuity principles* at the common boundaries.

For the problem at hand, the heat transfer domain can be split into region A corresponding to $y \leq H - b$ (which would have width L), and region B corresponding to $y > H - b$ (of width $L - a$) (Fig. 8.12). From this point on the problem taken to be dimensionless, in which L is the characteristic length and T_1 the characteristic temperature. Denote as $T_A(x, y)$ and $T_B(x, y)$ the

temperature in regions. The boundary conditions along surfaces that are not common to both A and B can be posed in 'traditional' form, i.e.,

$$\begin{aligned} T_A(x,0) &= 0 & \left.\frac{\partial T_B}{\partial y}\right|_{y=H} &= 0 \\ \left.\frac{\partial T_A}{\partial x}\right|_{x=0} &= 0 & \left.\frac{\partial T_B}{\partial x}\right|_{x=0} &= 0 \\ \left.\frac{\partial T_A}{\partial x}\right|_{x=1} &= 0 & T_B(1-a,y) &= 1 \end{aligned} \tag{8.28}$$

Recognize that the same coordinate origin is used for B as for A; in this convention T_B is valid only for $0 \le x \le 1-a$, $H-b \le y \le H$ whereas T_A is limited to $0 \le x \le 1$ and $0 \le y \le H-b$. Separate coordinate origins could be used for both domains – the choice is one only of style – yet the origin(s) must be used consistently in formulating the BCs and in posing the solution.

Three BCs are provided each for T_A and for T_B in Eqs. (8.28); closure of the problem requires one additional BC for A and B. These are provided by continuity (or matching) conditions along the interface between A and B, which will correspond to continuity of *temperature* and continuity of *flux*. Specifically, these conditions will appear as

$$T_A(x,H-b) = \begin{cases} T_B(x,H-b), & 0 \le x \le 1-a \\ 1, & 1-a \le x \le 1 \end{cases} \tag{8.29}$$

$$\left.\frac{\partial T_B}{\partial y}\right|_{y=H-b} = \left.\frac{\partial T_A}{\partial y}\right|_{y=H-b}, \quad 0 \le x \le 1-a \tag{8.30}$$

The upper surface boundary condition of A appears as a mixed condition, in which part of the surface is open (and connected with B) and the other part is at the uniform element temperature of unity. Solution of the problems for A and B therefore becomes somewhat equivalent to that developed for the mixed BC case of Sec. 8.2.1. Specifically, we now formulate general solutions for T_A and T_B and develop moment equations for the unknown expansion coefficients in each solution.

To generate the moment equations, it will be absolutely necessary to formulate the general solutions so that the eigenfunctions run in the direction of the shared boundary between A and B, i.e., the x direction. This should be easy to do, because the problem for A is naturally homogeneous in the x direction, and the problem for B can be made homogeneous by a simple superposition. Specifically, let the general solutions be defined by

$$T_A(x,y) = A_0 y + \sum_{n=1}^{N_A} A_n\, \phi_n(x)\, \frac{\sinh(\lambda_n y)}{\sinh(\lambda_n (H-b))} \tag{8.31}$$

$$\phi_n(x) = \cos(\lambda_n x) \tag{8.32}$$

$$\lambda_n = n\pi \tag{8.33}$$

$$T_B(x,y) = 1 + \sum_{n=1}^{N_B} B_n\,\psi_n(x)\,\frac{\cosh(\beta_n(H-y))}{\cosh(\beta_n b)} \tag{8.34}$$

$$\psi_n(x) = \cos(\beta_n x) \tag{8.35}$$

$$\beta_n = \frac{(2n-1)\pi}{2(1-a)} \tag{8.36}$$

These solutions are formulated to represent the most general solutions to Laplace's equation which identically satisfy the BCs in Eqs. (8.28). The leading term of $A_0 y$ in Eq. (8.31) accounts for the zeroth eigenvalue contribution, and the 1 in Eq. (8.34) represents a superposition which satisfies the unit temperature condition at the right face (recognize that the eigenfunctions $\psi_n(x)$ give 0 at $x = 1-a$). The hyperbolic, y–dependent functions in the series are scaled with respect to their values at the common boundary; this is done solely for numerical reasons in order to avoid complaints from *Mathematica* during solution of the moment equations. Observe also that the series for A and B are limited to N_A and N_B terms; these numbers need not be identical. Obtaining comparable precision in the formulas for T_A and T_B will require that ϕ_{N_A} and ψ_{N_B} have similar wavelengths – which in turn will imply that $N_A/N_B \sim 1/(1-a)$.

Moment equations are now generated for A and B per the methods developed in the previous sections. Equation (8.29) is multiplied by ϕ_m and integrated over the region for A, and likewise Eq. (8.30) is multiplied by ψ_m and integrated over the B region. This gives, symbolically,

$$\int_0^1 T_A(x,1-a)\,\phi_m(x)\,dx = \int_0^{1-a} T_B(x,1-a)\,\phi_m(x)\,dx + \int_{1-a}^1 \phi_m(x)\,dx \tag{8.37}$$

$$\int_0^{1-a} \left.\frac{\partial T_B}{\partial y}\right|_{y=1-a} \psi_m(x)\,dx = \int_0^{1-a} \left.\frac{\partial T_A}{\partial y}\right|_{y=1-a} \psi_m(x)\,dx \tag{8.38}$$

The left hand side of both equations will yield an orthogonal equation and thus retain only the m^{th} coefficient term, whereas the right hand side will involve all terms. Substitution of Eqs. (8.31) and (8.34) into the moment equations results in the following system of equations for the expansion coefficients;

$$f_m\,A_0 = \delta_{m0} + \sum_{n=1}^{N_B} B_n\,g_{mn} \tag{8.39}$$

$$-\frac{1-a}{2}\beta_m\,\tanh(\beta_m b)\,B_m = A_0\,g_{0m} + \sum_{n=1}^{N_A} \frac{\lambda_n}{\tanh(\lambda_n(H-b))}\,A_n\,g_{nm} \tag{8.40}$$

in which

$$\delta_{m0} = \begin{cases} 1, & m = 0 \\ 0, & m \neq 0 \end{cases} \tag{8.41}$$

$$f_m = \begin{cases} H - b, & m = 0 \\ \frac{1}{2}, & m \neq 0 \end{cases} \tag{8.42}$$

$$\begin{aligned} g_{mn} &= \int_0^{1-a} \phi_m(x)\, \psi_m(x)\, dx \\ &= \begin{cases} \dfrac{1-a}{\pi}\left[\dfrac{\cos[(n-(1-a)m)\pi]}{1-2(n-(1-a)m)} + \dfrac{\cos[(n+(1-a)m)\pi]}{1-2(n+(1-a)m)}\right], & n - \dfrac{1}{2} \neq (1-a)m \\ \dfrac{1-a}{2}, & n - \dfrac{1}{2} = (1-a)m \end{cases} \end{aligned} \tag{8.43}$$

The first formula appearing in Eq. (8.43) is the one derived by *Mathematica*. *Mathematica* will not automatically find the limiting value of this formula when it is indeterminate (i.e., when the denominator goes to zero) – it simply complains and returns `indeterminate`. Avoiding such problems requires some intervention by the user (you) – such as finding the limit (by application of L'Hospital's rule) and defining the function for g_{mn} so that the correct form is chosen.

Optimized *Mathematica* code

The listing of the code is as follows;

```
lambda[n_]:=Pi n
beta[n_]:=(2n-1) Pi/(2(1-a))
phi[n_,x_]:=Cos[lambda[n] x]
psi[n_,x_]:=Cos[beta[n] x]

delta[n_]:=Which[n==0,1,n>0,0]
fn[n_]:=Which[n==0,h-b,n>0,1/2]
gmn[m_,n_,a_]:=Which[n-.5!=(1.-a)m,
   ((-1+a) ((1+2 (-1+a) m-2 n)
   Cos[((-1+a) m+n) Pi]+(1-2 (-1+a) m-2 n)
   Cos[(m-a m+n) Pi]))/
   ((1+2 (-1+a) m-2 n) (-1+2 (-1+a) m+2 n)Pi),
  1==1,(1-a)/2]
```

```
ta[x_,y_]:=
  an[0]y+Sum[
      an[n] phi[n,x]Sinh[lambda[n] y]/
      Sinh[lambda[n] (h-b)],{n,1,ntota}]
tb[x_,y_]:=1+Sum[bn[n] psi[n,x]Cosh[beta[n](h-y)]/
      Cosh[beta[n] b],{n,1,ntotb}]

h=.75;a=.5;b=.3;ntota=50;ntotb=25;
anvec=Flatten[{Table[an[n],{n,0,ntota}],
              Table[bn[n],{n,1,ntotb}]}];
eqns=Flatten[{
    Table[fn[m] an[m]==delta[m]+Sum[bn[n]gmn[m,n,a],
       {n,1,ntotb}],{m,0,ntota}],
    Table[-(1-a)/2 beta[m] Tanh[beta[m] b] bn[m]==
       an[0] gmn[0,m,a] +
       Sum[lambda[n]/Tanh[lambda[n](h-b)]an[n]gmn[n,m,a],
       {n,1,ntota}],{m,1,ntotb}]
      }];

soln=Solve[eqns,anvec][[1]];

temp[x_,y_]:=Which[x>=1-a&&y>=h-b,1,y>h-b,tb[x,y],
                   y<=h-b,ta[x,y]]

ContourPlot[temp[x,y]/.soln,{x,0,1},{y,0,h},
   Contours->20,PlotPoints->40,
   AspectRatio->h,ContourShading->False,
   TextStyle->{FontFamily->"Times",FontSize->12},
   FrameLabel->{StyleForm["x",FontSlant->"Italic",
   FontSize->16],StyleForm["y",FontSlant->"Italic",
   FontSize->16]},RotateLabel->False]
```

The structure of the code follows that given in previous codes; the only new feature here is the use of the `Flatten` command. This operation 'flattens' nested lists, so that

$$\texttt{Flatten}[\{\{\texttt{a},\texttt{b}\},\{\texttt{c},\texttt{d}\}\}] = \{\texttt{a},\texttt{b},\texttt{c},\texttt{d}\}$$

It is used to combine the two sets of coefficient and equation lists generated for the A and B parts into single lists, for which `Solve` can then be applied. I also define a single function `temp[x,y]`

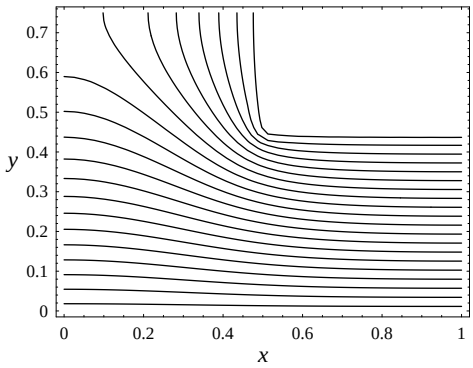


Figure 8.13: isotherms for the nonrectangular domain

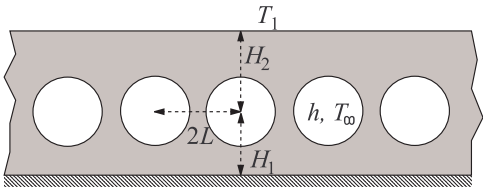


Figure 8.14: cooling pipe array

which picks out the proper part of the solution (A or B) depending on the location of x and y in the domain.

The contour plot is given in Fig. 8.13, and it clearly shows the first–order continuity in the temperature field between A and B.

8.3.2 Rectangular–cylindrical systems

It often is impossible to split the non–orthogonal domain into a set of individually–orthogonal regions; this will occur whenever the boundaries of the domain cannot be simply described by a single coordinate system. For such conditions a single expansion for the temperature must be 'fit' to match the conditions on all surfaces.

As an example, consider a somewhat similar problem to that examined in the previous section. Heat is removed from an isothermal surface to an array of circular cooling channels, which each channel having a radius R, a channel–to–channel separation distance of $2L$, and a position of H_1 below the heated surface and H_2 above an adiabatic boundary. The situation is illustrated in Fig. 8.14

Owing to the symmetry of the problem, the domain can be reduced to that existing about a

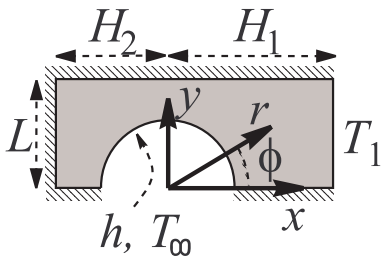


Figure 8.15: equivalent domain

single pipe as shown in Fig. 8.15. The situation has been rotated 90° relative to that in Fig. 8.14 – which is done for convenience in defining the coordinate systems. The planar surfaces on the top and bottom are adiabatic, as is the leftmost surface. The inside surface of the pipe experiences convection conditions and the right surface of the domain is at the uniform temperature of T_1.

What we have here is a mix of rectangular and cylindrical coordinates. Clearly, the domain will not allow for a simple, SOV–based solution for the temperature field. Furthermore, it is not obvious how we would proceed with the moment–based hybrid strategy. In all previous problems the moment equations were obtained by integration of the proposed solution, multiplied by the eigenfunction, over the one remaining surface in which the solution did not automatically satisfy the boundary conditions. Because of the orthogonality of the eigenfunctions, our equations so far have always been in the form of $const \cdot A_m = \sum_n G_{mn} A_n$. The situation here will not directly lend itself to this simple approach. If, for example, we pose our solution in cylindrical coordinates and have the solution automatically satisfy the boundary conditions along the cylinder surface, then this solution will certainly not be orthogonal when applied to the cartesian surfaces of the domain. And if the solution was posed in rectangular coordinates, the integration over the cylindrical surface would also not lead to 'standard' moment equations.

The way around this problem will be to utilize a mathematical relationship known as *Green's second identity*. Before we get into the details of this, it will be useful to fill in some details on the problem at hand.

Use R to scale the length coordinate, and define the dimensionless temperature as $T \rightarrow (T - T_\infty)/(T_1 - T_\infty)$. The temperature field will then satisfy the homogeneous convection condition on the cylindrical surface, will be unity on the upper surface, and be adiabatic on all other surfaces. We will formulate the solution in cylindrical coordinates – a solution in rectangular coordinates could also be used but this would, in the end, be more difficult to work with. The general form of the solution would be

$$T(r,\phi) = A_0 + B_0 \ln(r) + \sum_{n=1}^{N} \left(A_n r^n + B_n r^{-n} \right) \cos(n\phi)$$

The cosine dependence on ϕ is taken because the solution must be even in ϕ; $T(r,\phi) = T(r,-\phi)$. The convection BC along the cylinder surface is

$$\left.\frac{\partial T}{\partial r}\right|_{r=1} = Bi\, T(1,\phi)$$

Application of this to the previous equation allows B_n to be specified in terms of A_n, and the result is

$$T(r,\phi) = A_0\,(1 + Bi\,\ln(r)) + \sum_{n=1}^{N} A_n \left(r^n + \frac{n - Bi}{n + Bi}\, r^{-n}\right) \cos(n\phi) \tag{8.44}$$

Application of Green's Second Identity

Equation (8.44) provides a sufficiently general expansion for the temperature field in the domain; the problem at hand is to choose the expansion coefficients A_n so that the boundary conditions on all planar surfaces are satisfied.

To do this we will employ, as mentioned above, Green's second identity. Say that the functions $T(r,\phi)$ and $u(r,\phi)$ both satisfy Laplace's equation, i.e., $\nabla^2 T = \nabla^2 u = 0$. It can then be shown that

$$\int_A T\,\hat{\mathbf{n}}\cdot\nabla u\, dA = \int_A u\,\hat{\mathbf{n}}\cdot\nabla T\, dA \tag{8.45}$$

in which the integration is taken over all surfaces of the domain and $\hat{\mathbf{n}}$ is the outward–pointing normal. The trivial case of $u = 1$ simply gives a conservation of energy statement: the first integral would be zero and the second implies that all of the heat conducted in would equal all the heat conducted out of the domain.

Green's identity will provide the tool with which we obtain the moment equations for this problem. The first step is to apply Eq. (8.45) to the six distinct surfaces in the problem at hand. Leaving u unspecified, the integration over area would appear as

$$\begin{aligned}
&-\int_0^{\pi} \left(T\frac{\partial u}{\partial r} - u\frac{\partial T}{\partial r}\right)_{r=1} d\phi - \int_{-H_2}^{-1} \left(T\frac{\partial u}{\partial y} - u\frac{\partial T}{\partial y}\right)_{y=0} dx \\
&\quad + \int_0^{L} \left(T\frac{\partial u}{\partial x} - u\frac{\partial T}{\partial x}\right)_{x=-H_2} dy + \int_{-H_2}^{H_1} \left(T\frac{\partial u}{\partial y} - u\frac{\partial T}{\partial y}\right)_{y=L} dx \\
&\quad + \int_0^{L} \left(T\frac{\partial u}{\partial x} - u\frac{\partial T}{\partial x}\right)_{x=H_1} dy - \int_1^{H_1} \left(T\frac{\partial u}{\partial y} - u\frac{\partial T}{\partial y}\right)_{y=0} dx = 0
\end{aligned}$$

The boundary conditions on T can now be applied; all adiabatic surfaces have a normal gradient

of zero and the heated surface has a temperature of unity. This gives

$$
\begin{aligned}
&-\int_0^{\pi}\left(T\frac{\partial u}{\partial r}-u\frac{\partial T}{\partial r}\right)_{r=1}d\phi-\int_{-H_2}^{-1}\left(T\frac{\partial u}{\partial y}\right)_{y=0}dx\\
&\quad+\int_0^{L}\left(T\frac{\partial u}{\partial x}\right)_{x=-H_2}dy+\int_{-H_2}^{H_1}\left(T\frac{\partial u}{\partial y}\right)_{y=L}dx\\
&\quad+\int_0^{L}\left(\frac{\partial u}{\partial x}-u\frac{\partial T}{\partial x}\right)_{x=H_1}dy-\int_{1}^{H_1}\left(T\frac{\partial u}{\partial y}\right)_{y=0}dx=0
\end{aligned}
\tag{8.46}
$$

Finally, we need to specify the so–called *test* function u. An essentially infinite number of choices are available – the only requirement for u being that it satisfies Laplace's equation – yet we obviously want to select a test function which will yield simple and solvable moment equations. Since T is formulated in cylindrical coordinates (Eq. (8.44)), it makes sense to choose a set of test functions $u = u_m,\ m = 0,\ 1,\dots N$ so that an orthogonal integral is obtained from the surface integral over ϕ. By doing so, the integral over ϕ in Eq. (8.46) will kill every term in the series expansion for T except that for which $n = m$. With this in mind, define u by

$$
u(r,\phi)=u_m(r,\phi)=r^{-m}\cos(m\phi) \tag{8.47}
$$

Recognize that the choice of u_m will identically satisfy the adiabatic condition along $y = 0$ (which is due to the even property of the cosine function) and because of this the integrals along $y = 0$ can be removed from Eq. (8.46). In addition, the special case of $m = 0$ gives $u = 1$, for which Eq. (8.46) reduces to the conservation of energy statement.

By replacing Eqs. (8.47) and (8.44) into Eq. (8.46), we arrive at the following set of moment equations for the expansion coefficients

$$
f_m\,A_m=b_m+\sum_{n=0}^{N}g_{mn}\,A_n \tag{8.48}
$$

The various quantities appearing above are defined by

$$
f_m=\begin{cases}\pi\,Bi, & m=0\\ \pi m, & m>0\end{cases} \tag{8.49}
$$

$$
p_m=-\int_0^{L}\left(\frac{\partial u_m}{\partial x}\right)_{x=H_1}dy \tag{8.50}
$$

$$
g_{mn}=\int_0^{L}\left(u_m\frac{\partial v_n}{\partial x}\right)_{x=H_1}dy-\int_{-H_2}^{H_1}\left(v_n\frac{\partial u_m}{\partial y}\right)_{y=L}dx+\int_0^{L}\left(v_n\frac{\partial u_m}{\partial x}\right)_{x=-H_2}dy \tag{8.51}
$$

$$
v_n(r,\phi)=\begin{cases}1+Bi\,\ln(r), & n=0\\ \left(r^n+\dfrac{n-Bi}{n+Bi}\,r^{-n}\right)\cos(n\phi), & n>0\end{cases} \tag{8.52}
$$

The function v_n is the n^{th}–order 'harmonic' from the expansion of the temperature in Eq. (8.44). The surface integrals in f_m and g_{mn} involve the cartesian coordinates x and y; conversion of the cylindrical–based u_m and v_n functions to their cartesian equivalents is made by the substitions

$$r = \left(x^2 + y^2\right)^{1/2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) \tag{8.53}$$

Mathematica Coding and numerical issues

The moment equations in Eq. (8.48) are now complete insofar as the mathematical theory is concerned. Similar to the previous examples, we have a system of $N + 1$ linear equations for the expansion coefficients. The numerical problem in this example, however, is somewhat more difficult because the surface integrals (i.e., the matrix g_{mn} and the vector b_m) do not have *simple* closed–form analytic expressions. That this is the case is not surprizing, because the path of the integration (i.e., x or y = constant) does not naturally translate to an equally simple path in the cylindrical 'language' of the functions u_m and v_n.

Since the objective in solving the moment equations will be to obtain *numerical* values of the expansion coefficients for fixed values of Bi, H_1, H_2, and L, it makes sense to use a *numerical* integration scheme to calculate the integrals. *Mathematica* comes with an efficient, adaptive numerical integration scheme which is invoked via the command `NIntegrate`. The use of this command is illustrated in the following code.

```
u[n_,x_,y_]:=(x^2+y^2)^(-n/2 )Cos[n ArcTan[x,y]]/;n>0
u[n_,x_,y_]:=1/;n==0
v[n_,x_,y_]:=1+bi Log[(x^2+y^2)^(1/2)]/;n==0
v[n_,x_,y_]:=((rt^n+(n-bi)/(n+bi)rt^(-n))Cos[n pt]/.
   {rt->(x^2+y^2)^(1/2),pt->ArcTan[x,y]})/;n>0

nint[integrand_,limits_]:=
  NIntegrate[integrand,limits,AccuracyGoal->6,
  PrecisionGoal->6,MaxRecursion->12]

gmn[m_,n_]:=nint[(u[m,x,y]D[v[n,x,y],x])/.x->h1,{y,0,l}]-
       nint[(v[n,x,y] D[u[m,x,y],y])/.y->l,{x,-h2,h1}]+
       nint[(v[n,x,y] D[u[m,x,y],x])/.x->-h2,{y,0,l}]
bm[m_]:=-nint[D[u[m,x,y],x]/.x->h1,{y,0,l}]
fm[m_]:=Which[m==0,Pi bi,m>0,m Pi]

bi=100;h2=2;h1=2;l=2;ntot=20;
anvec=Table[an[n],{n,0,ntot}];
```

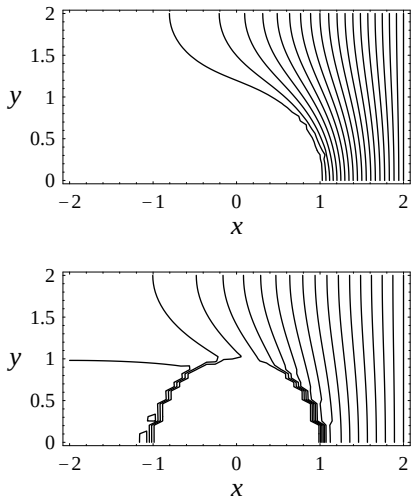


Figure 8.16: isotherms for $Bi = 100$ (top), $Bi = 1$ (bottom)

```
eqns=Table[
       fm[m]an[m]==bm[m]+Sum[gmn[m,n] an[n],{n,0,ntot}],
   {m,0,ntot}];
soln=Solve[eqns,anvec][[1]];
```

The `nint` functioned defined in the code simply calls `NIntegrate` with specified precision, accuracy, and iteration limits. As is the case with most 'black box' numerical packages – such as those for nonlinear root finding and DE solving – it is necessary to experiment with the user–adjustable parameters of the package to obtain accurate and reliable results. At the least, you need to read the *Mathematica* help material on the use and options of `NIntegrate`. As has been done in previous examples, the elements of the matrix g_{mn} need to be scaled to avoid numerical loss–of–precision errors during solution of the equations. I do this by multiplying and dividing u_n and v_n respectively by $r_{max} = (H_1^2 + L^2)^{m/2}$. The effect of this is to make the elements of g_{mn} all of order unity – which usually is enough (for these sorts of problems) to avoid a poorly conditioned matrix.

Temperature profiles within the system are shown in Fig. 8.16 for $Bi = 100$ (top) and $Bi = 1$ (bottom), in which the geometry corresponds to $H_1 = H_2 = L = 2$ (i.e., the boundary of the enclosing rectangle are located at two pipe radii). The plot shows the desired behavior: an isothermal right surface and adiabatic top and left walls. The top surface of the cylinder is also (nearly) isothermal for $Bi = 100$ – which is expected because of the minimal convection resistance for this case. Observe that essentially all of the heat flux to the cylinder occurs near the $\phi \sim 0$

region – very little heat makes its way to the opposite side of the surface. This is not the case when $Bi = 1$. Because of the smaller convection resistance, the heat flux to the cylinder is more spread out over ϕ. Indeed, in the limit of $Bi \ll 1$ the enclosure would become isothermal at a temperature of unity, and the heat flux to the cylinder would be controlled solely by the convection resistance.