$\boxed{4.3}$ Angular Momentum
$\boxed{4.4}$ Spin

$$f_\ell^m = Y_\ell^m$$

$$L^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m \qquad L_z f_\ell^m = \hbar m f_\ell^m$$

$$S^2 |s\ m\rangle = \hbar^2 s(s+1) |s\ m\rangle \qquad S_z |s\ m\rangle = \hbar m |s\ m\rangle$$

$$S_\pm |s\ m\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\ |s\ (m\pm1)\rangle$$

$\boxed{\text{Spin } 1/2}$ $\quad s = \frac{1}{2} \rightarrow m = \frac{-1}{2}, \frac{+1}{2}$

Let $|\frac{1}{2}\ \frac{1}{2}\rangle = |\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$\quad\ \ |\frac{1}{2}\ \frac{-1}{2}\rangle = |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$$S^2 |s\ m\rangle = \hbar^2 s(s+1) |s\ m\rangle$$

$s = \frac{1}{2}$, $m = \frac{1}{2}$ $\longrightarrow$ $S^2 |\uparrow\rangle = \frac{3}{4}\hbar^2 |\uparrow\rangle$

$s = 1/2$, $m = \frac{-1}{2}$ $\rightarrow$ $S^2 |\downarrow\rangle = \frac{3}{4}\hbar^2 |\downarrow\rangle$

$$\begin{pmatrix}c & d\\ e & f\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix}1\\0\end{pmatrix} \longrightarrow \begin{pmatrix}c\\e\end{pmatrix} = \begin{pmatrix}\frac{3}{4}\hbar^2\\0\end{pmatrix}$$

$$\begin{pmatrix}c & d\\ e & f\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{3}{4}\hbar^2\begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \begin{pmatrix}d\\f\end{pmatrix} = \begin{pmatrix}0\\\frac{3}{4}\hbar^2\end{pmatrix}$$

$$\Rightarrow S^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$S_z |s\ m\rangle = \hbar m |s\ m\rangle$$

$s = 1/2$, $m = 1/2$ $\quad S_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle$

$s = 1/2$, $m = -1/2$ $\quad S_z |\downarrow\rangle = \frac{-\hbar}{2} |\downarrow\rangle$

$\Big\} \rightarrow S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

$S_+ |\downarrow\rangle = \hbar |\uparrow\rangle \qquad S_- |\uparrow\rangle = \hbar |\downarrow\rangle$

$S_+ |\uparrow\rangle = 0 \qquad S_- |\downarrow\rangle = 0$

$\downarrow \qquad\qquad \downarrow$

$S_+ = \hbar\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \qquad S_- = \hbar\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$

S_+ and S_- are not hermitian

4.4.2 Electron in a Magnetic field

$$\vec{\mu} = \gamma \vec{S}$$

$$H = -\vec{\mu} \cdot \vec{B}$$

$$= -\gamma \vec{B} \cdot \vec{S}$$

Ex: Uniform magnetic field $\longrightarrow$ along z-direction:

$$\vec{B} = B_0 \hat{k}$$

$$\Rightarrow H = -\gamma B_0 S_z = -\gamma B_0 \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\downarrow$$

Eigenstates $= \left\{ \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$|\Psi(0)\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ Eigen values $= \left\{ E_+ = -\gamma \frac{B_0 \hbar}{2}, \ E_- = \gamma B_0 \frac{\hbar}{2} \right\}$

$$\Rightarrow |\Psi(t)\rangle = a \chi_+ e^{-E_+ t/\hbar} + b \chi_- e^{-E_- t/\hbar}$$

$$= \begin{pmatrix} a\, e^{i\gamma B_0 t/2} \\ b\, e^{-i\gamma B_0 t/2} \end{pmatrix}$$

$$\langle S_x \rangle = \langle \Psi(t) | S_x | \Psi(t) \rangle$$

$$= \begin{pmatrix} a\, e^{-i\gamma B_0 t/2} & b\, e^{+i\gamma B_0 t/2} \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a\, e^{+i\gamma B_0 t/2} \\ b\, e^{-i\gamma B_0 t/2} \end{pmatrix}$$

$$= \hbar (ab) \cos(\gamma B_0 t)$$

$$\langle S_y \rangle = \langle \Psi(t) | S_y | \Psi(t) \rangle$$

$$= \begin{pmatrix} a\, e^{-i\gamma B_0 t/2} & b\, e^{+i\gamma B_0 t/2} \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a\, e^{i\gamma B_0 t/2} \\ b\, e^{-i\gamma B_0 t/2} \end{pmatrix}$$

$$= -\hbar (ab) \sin(\gamma B_0 t)$$

$$\langle S_z \rangle = \begin{pmatrix} a\, e^{-i\gamma B_0 t/2} & b\, e^{+i\gamma B_0 t/2} \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a\, e^{i\gamma B_0 t/2} \\ b\, e^{-i\gamma B_0 t/2} \end{pmatrix}$$

$$= \frac{\hbar}{2} (a^2 - b^2)$$

4.4.3 Addition of Angular Momenta :

- Two particles with spins $s_1 = s_2 = \frac{1}{2}$

$s_1 = \frac{1}{2}$: $|s_1 \; m_1\rangle$
$\rightarrow |\frac{1}{2} \; \frac{1}{2}\rangle$
$\rightarrow |\frac{1}{2} \; \frac{-1}{2}\rangle$ $\Big\}$ dim = 2

$s_2 = \frac{1}{2}$: $|s_2 \; m_2\rangle$
$\rightarrow |\frac{1}{2} \; \frac{1}{2}\rangle$
$\rightarrow |\frac{1}{2} \; \frac{-1}{2}\rangle$ $\Big\}$ dim = 2

- Construct composite states and bases :

number of bases = dim (s_1) + dim (s_2)

$\Rightarrow$ 4 bases

$|s_1 \; m_1 ; s_2 \; m_2\rangle$

$|\frac{1}{2} \; \frac{1}{2} ; \frac{1}{2} \; \frac{1}{2}\rangle \equiv |1\rangle$

$|\frac{1}{2} \; \frac{1}{2} ; \frac{1}{2} \; \frac{-1}{2}\rangle \equiv |2\rangle$

$|\frac{1}{2} \; \frac{-1}{2} ; \frac{1}{2} \; \frac{1}{2}\rangle \equiv |3\rangle$

$|\frac{1}{2} \; \frac{-1}{2} ; \frac{1}{2} \; \frac{-1}{2}\rangle \equiv |4\rangle$

- Construct S_z : $S_z |s \; m\rangle = \hbar m |s \; m\rangle$

$$S_z |s_1 \, m_1 ; s_2 \, m_2\rangle = \hbar (m_1 + m_2) |s_1 \, m_1 ; s_2 \, m_2\rangle$$

$S_z |1\rangle = \hbar |1\rangle$
$S_z |2\rangle = 0$
$S_z |3\rangle = 0$
$S_z |4\rangle = -\hbar |4\rangle$

$$\Rightarrow S_z = \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Construct $S_+ = S_+^{(1)} + S_+^{(2)}$

$$S_+ |s \; m\rangle = \hbar \sqrt{s(s+1) - m(m+1)} \; |s \; (m+1)\rangle$$

$S_+^{(1)} |1\rangle = 0$
$S_+^{(1)} |2\rangle = 0$
$S_+^{(1)} |3\rangle = \hbar |1\rangle$
$S_+^{(1)} |4\rangle = \hbar |2\rangle$

$$\Rightarrow S_+^{(1)} = \hbar \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$S_+^{(2)} |1\rangle = 0$
$S_+^{(2)} |2\rangle = \hbar |1\rangle$
$S_+^{(2)} |3\rangle = 0$
$S_+^{(2)} |4\rangle = \hbar |3\rangle$

$$\Rightarrow S_+^{(2)} = \hbar \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow S_+ = S_+^{(1)} + S_+^{(2)} = \hbar \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

⊛ Construct $S_- = S_-^{(1)} + S_-^{(2)}$

$$S_- |s\ m\rangle = \hbar \sqrt{s(s+1) - m(m-1)}\ |s\ (m-1)\rangle$$

$$S_- |1\rangle = S_-^{(1)} |1\rangle + S_-^{(2)} |1\rangle$$
$$= \hbar |3\rangle + \hbar |2\rangle$$
$$S_- |2\rangle = S_-^{(1)} |2\rangle + S_-^{(2)} |2\rangle$$
$$= \hbar |4\rangle + 0$$
$$S_- |3\rangle = S_-^{(1)} |3\rangle + S_-^{(2)} |3\rangle$$
$$= 0 + \hbar |4\rangle$$
$$S_- |4\rangle = S_-^{(1)} |4\rangle + S_-^{(2)} |4\rangle$$
$$= 0 + 0$$

$$\Rightarrow S_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

⊛ Find S_x and S_y :

$$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{i\hbar}{2} \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & +1 & 1 & 0 \end{pmatrix}$$

⊛ Find $S^2 = S_x^2 + S_y^2 + S_z^2$

$$S^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \xRightarrow{\text{This means}}$$

$$S^2 |1\rangle = 2\hbar^2 |1\rangle$$
$$S^2 |2\rangle = \hbar^2 (|2\rangle + |3\rangle)$$
$$S^2 |3\rangle = \hbar^2 (|2\rangle + |3\rangle)$$
$$S^2 |4\rangle = 2\hbar^2 |4\rangle$$

* Note:- The 4 bases ($|1\rangle, |2\rangle, |3\rangle, |4\rangle$) are eigenvectors of S_z but not eigenvectors of S^2.
  - In these bases : S_z is diagonal
    S^2 is not diagonal
  - The vectors $|1\rangle$ and $|4\rangle$ are two of the eigenvectors for S^2

* Building new bases :

3

$s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$

$S_{total} = s$ is from $(s_1 + s_2)$ to $|s_1 - s_2|$

$\Rightarrow s$ is from 1 to 0 $\Rightarrow s = 1, 0$

$$\left.\begin{array}{l} s = 1 \longrightarrow m = 1, 0, -1 \\ s = 0 \longrightarrow m = 0 \end{array}\right\} 4 \text{ bases}$$

$\|s\ m\rangle$	
$\|1\ 1\rangle$	$\equiv \|1'\rangle$
$\|1\ 0\rangle$	$\equiv \|2'\rangle$
$\|0\ 0\rangle$	$\equiv \|3'\rangle$
$\|1\ -1\rangle$	$\equiv \|4'\rangle$

* Connection between old bases and new bases :

2 methods :

① Diagonalization of S^2 :

In old bases : $S^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$\Rightarrow$ its eigenvectors $\rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$|1'\rangle \qquad |2'\rangle \qquad |3'\rangle \qquad |4'\rangle$$

$\Rightarrow$ The new bases are the same of the eigenvectors of S^2

$\Rightarrow |1'\rangle = |1\rangle$ $\longrightarrow |1\ 1\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}; \tfrac{1}{2}\ \tfrac{1}{2}\rangle$

$|2'\rangle = \frac{1}{\sqrt{2}}(|2\rangle + |3\rangle)$ $\longrightarrow |1\ 0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\ \tfrac{1}{2}; \tfrac{1}{2}\ \tfrac{-1}{2}\rangle + |\tfrac{1}{2}\ \tfrac{-1}{2}; \tfrac{1}{2}\ \tfrac{1}{2}\rangle\right)$

$|3'\rangle = \frac{1}{\sqrt{2}}(|2\rangle - |3\rangle)$ $\longrightarrow |0\ 0\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\ \tfrac{1}{2}; \tfrac{1}{2}\ \tfrac{-1}{2}\rangle - |\tfrac{1}{2}\ \tfrac{-1}{2}; \tfrac{1}{2}\ \tfrac{1}{2}\rangle\right)$

$|4'\rangle = |4\rangle$ $\longrightarrow |1\ -1\rangle = |\tfrac{1}{2}\ \tfrac{-1}{2}; \tfrac{1}{2}\ \tfrac{-1}{2}\rangle$

* Writing S^2 in the new bases $\rightarrow$ makes S^2 diagonal

* To find S^2 in the new bases, we use : $S^2|s\ m\rangle = \hbar^2 s(s+1)|s\ m\rangle$

$$\left.\begin{array}{l} S^2|1'\rangle = S^2|1\ 1\rangle = 2\hbar^2|1'\rangle \\ S^2|2'\rangle = 2\hbar^2|2'\rangle \\ S^2|3'\rangle = 0 \\ S^2|4'\rangle = 2\hbar^2|4'\rangle \end{array}\right\} \rightarrow S^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

new bases ✓

diagonalized ✓

(2) Method 2 : From table :

$s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$

$|1\ 1\rangle = |\frac{1}{2}\ \frac{1}{2};\ \frac{1}{2}\ \frac{1}{2}\rangle$

$|1\ 0\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2}\ \frac{1}{2};\ \frac{1}{2}\ \frac{-1}{2}\rangle + |\frac{1}{2}\ \frac{-1}{2};\ \frac{1}{2}\ \frac{1}{2}\rangle\right)$

$|0\ 0\rangle = \frac{1}{\sqrt{2}}\left(|\frac{1}{2}\ \frac{1}{2};\ \frac{1}{2}\ \frac{-1}{2}\rangle - |\frac{1}{2}\ \frac{-1}{2};\ \frac{1}{2}\ \frac{1}{2}\rangle\right)$

$|1\ -1\rangle = |\frac{1}{2}\ \frac{-1}{2};\ \frac{1}{2}\ \frac{-1}{2}\rangle$

(*) How to guess the connection between them ?

if $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$, then $s = 1, 0$

$1 = \frac{1}{2} + \frac{1}{2}$
$(m = m_1 + m_2)$

case 1: $s = 1$, $m = 1$: $|s\ m\rangle = |1\ 1\rangle$

there is only <u>one</u> way to get $|1\ 1\rangle$, which is when $\boxed{m_1 = m_2 = \frac{1}{2}}$

$$\Rightarrow |1\ 1\rangle = |\overset{s_1}{\frac{1}{2}}\ \overset{m_1}{\underline{\frac{1}{2}}};\ \overset{s_2}{\frac{1}{2}}\ \overset{m_2}{\underline{\frac{1}{2}}}\rangle$$

case 2: $s = 1$, $m = 0$: $|1\ 0\rangle$

there is <u>two</u> ways : 1) $\boxed{m_1 = \frac{1}{2}\ ,\ m_2 = \frac{-1}{2}}$ $\rightarrow m = m_1 + m_2 = 0$

2) $\boxed{m_1 = \frac{-1}{2}\ ,\ m_2 = \frac{1}{2}}$ $\rightarrow m = m_1 + m_2 = 0$

$$\Rightarrow |1\ 0\rangle = c_1 |\overset{s_1}{\frac{1}{2}}\ \overset{m_1}{\underline{\frac{1}{2}}};\ \overset{s_2}{\frac{1}{2}}\ \overset{m_2}{\underline{\frac{-1}{2}}}\rangle + c_2 |\overset{s_1}{\frac{1}{2}}\ \overset{m_1}{\underline{\frac{-1}{2}}};\ \overset{s_2}{\frac{1}{2}}\ \overset{m_2}{\underline{\frac{1}{2}}}\rangle$$

Case 3: $s = 0$, $m = 0$: $|0\ 0\rangle$

same as <u>case 2</u> with different coefficients

$$\Rightarrow |0\ 0\rangle = c_3 |\frac{1}{2}\ \underline{\frac{1}{2}};\ \frac{1}{2}\ \underline{\frac{-1}{2}}\rangle + c_4 |\frac{1}{2}\ \underline{\frac{-1}{2}};\ \frac{1}{2}\ \underline{\frac{1}{2}}\rangle$$

case 4: $s = 1$, $m = -1$:

there is only <u>one</u> way to get $m = -1$

which is when $m_1 = m_2 = \frac{-1}{2}$

$$\Rightarrow |1\ -1\rangle = |\overset{s_1}{\frac{1}{2}}\ \overset{m_1}{\underline{\frac{-1}{2}}};\ \overset{s_2}{\frac{1}{2}}\ \overset{m_2}{\underline{\frac{-1}{2}}}\rangle$$

$\boxed{4.5}$ Electromagnetic Interactions:

4.5.1 Minimal Coupling:

Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

$$\hat{H} = \frac{1}{2m}\Big(\underbrace{-i\hbar\vec{\nabla}}_{\vec{p}} - q\underbrace{\vec{A}}_{\text{vector potential}}\Big)^2 + q\underbrace{\varphi}_{\text{scalar potential}}$$

$$i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

↓ quantum implementation of the Lorentz force (minimal coupling rule)

Problem 4.43: $\vec{A} = \frac{B_0}{2}(x\hat{j} - y\hat{i})$, $\varphi = Kz^2$

(a) $\vec{B} = \nabla \times \vec{A} = B_0\hat{k}$

$\vec{E} = -\nabla\varphi - \cancelto{0}{\frac{\partial \vec{A}}{\partial t}} = -2Kz\,\hat{k}$

(b) $\hat{H} = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\varphi$

$$= \frac{1}{2m}\left[\left(P_x + \frac{qB_0}{2}y\right)\hat{i} + \left(P_y - \frac{qB_0}{2}x\right)\hat{j} + P_z\hat{k}\right]^2 + qKz^2$$

$$= \underbrace{\frac{P_z^2}{2m} + qKz^2}_{H.O} + \underbrace{\frac{\left(P_x + \frac{qB_0 y}{2}\right)^2}{2m} + \frac{\left(P_y - \frac{qB_0}{2}x\right)^2}{2m}}_{H.O}$$

* For Any Harmonic Oscillator, there are 2 conditions:

1) $H = aq_1^2 + bq_2^2 \rightarrow \cancel{\omega^2 \neq \frac{2}{m}b}$, $\boxed{\omega^2 = 4ab}$ (الها إثبات)

2) $[q_1, q_2] = i\hbar$

$[P_z, z] = -i\hbar$ ✓

$$\left[P_x + \frac{qB_0}{2}y ,\ P_y - q\frac{B_0}{2}x\right] =$$

$$= \cancelto{0}{[P_x, P_y]} - q\frac{B_0}{2}\underbrace{[P_x, x]}_{-i\hbar} + \frac{qB_0}{2}\underbrace{[y, P_y]}_{i\hbar} + \frac{q^2B_0^2}{4}\cancelto{0}{[y, -x]}$$

$= qB_0 i\hbar$ ✓

$$\Rightarrow q_1 = \frac{1}{qB_0}(P_x + qB_0 y)$$

$$q_2 = P_y - q\frac{B_0}{2}x$$

$$\Rightarrow \hat{H} = \underbrace{\frac{P_z^2}{2m} + qKz^2}_{\omega_1^2 = \frac{2qK}{m}} + \underbrace{\frac{q^2B_0^2}{2m}q_1^2 + \frac{1}{2m}q_2^2}_{\omega_2^2 = 4\left(\frac{q^2B_0^2}{2m}\right)\left(\frac{1}{2m}\right) = \frac{q^2B_0^2}{m^2}}$$

$E(n_1, n_2) = \ldots$