

Lecture 14 and 15: Algebraic approach to the SHO

B. Zwiebach
April 5, 2016

Contents

1 Algebraic Solution of the Oscillator	1
2 Operator manipulation and the spectrum	4

1 Algebraic Solution of the Oscillator

We have already seen how to calculate energy eigenstates for the simple harmonic oscillator by solving a second-order differential equation, the time-independent Schrödinger equation.

Let us now try to factorize the harmonic oscillator Hamiltonian. By this we mean, roughly, writing the Hamiltonian as the product of an operator times its *Hermitian conjugate*. As a first step we rewrite the Hamiltonian as

$$\hat{H} = \frac{1}{2}m\omega^2\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right). \quad (1.1)$$

Motivated by the identity $a^2 + b^2 = (a - ib)(a + ib)$, holding for numbers a and b , we examine if the expression in parenthesis can be written as a product

$$\begin{aligned} \left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right) &= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} + \frac{i}{m\omega}(\hat{x}\hat{p} - \hat{p}\hat{x}), \\ &= \hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} - \frac{\hbar}{m\omega}\mathbf{1}, \end{aligned} \quad (1.2)$$

where the extra terms arise because \hat{x} and \hat{p} , as opposed to numbers, do not commute. We now define the right-most factor in the above product to be V :

$$V \equiv \hat{x} + \frac{i\hat{p}}{m\omega}, \quad (1.3)$$

Since \hat{x} and \hat{p} are Hermitian operators, we then have

$$V^\dagger = \hat{x} - \frac{i\hat{p}}{m\omega}, \quad (1.4)$$

and this is the left-most factor in the product! We can therefore rewrite (1.2) as

$$\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2} = V^\dagger V + \frac{\hbar}{m\omega}\mathbf{1}, \quad (1.5)$$

and therefore back in the Hamiltonian (1.1) we find,

$$\hat{H} = \frac{1}{2}m\omega^2 V^\dagger V + \frac{1}{2}\hbar\omega\mathbf{1}. \quad (1.6)$$

This is a factorized form of the Hamiltonian: up to an additive constant E_0 , \hat{H} is the product of a positive constant times the operator product $V^\dagger V$. We note that the commutator of V and V^\dagger is simple

$$[V, V^\dagger] = \left[\hat{x} + \frac{i\hat{p}}{m\omega}, \hat{x} - \frac{i\hat{p}}{m\omega} \right] = -\frac{i}{m\omega}[\hat{x}, \hat{p}] + \frac{i}{m\omega}[\hat{p}, \hat{x}] = \frac{2\hbar}{m\omega}\mathbf{1}. \quad (1.7)$$

This implies that

$$\left[\sqrt{\frac{m\omega}{2\hbar}} V, \sqrt{\frac{m\omega}{2\hbar}} V^\dagger \right] = \mathbf{1}. \quad (1.8)$$

This suggests the definition of unit-free operator operators \hat{a} and \hat{a}^\dagger :

$$\begin{aligned} \hat{a} &\equiv \sqrt{\frac{m\omega}{2\hbar}} V, \\ \hat{a}^\dagger &\equiv \sqrt{\frac{m\omega}{2\hbar}} V^\dagger. \end{aligned} \quad (1.9)$$

Due to the scaling we have

$$\boxed{[\hat{a}, \hat{a}^\dagger] = \mathbf{1}.} \quad (1.10)$$

The operator \hat{a} is called *annihilation* operator and \hat{a}^\dagger is called a *creation* operator. The justification for these names will be seen below. From the above definitions we read the relations between $(\hat{a}, \hat{a}^\dagger)$ and (\hat{x}, \hat{p}) :

$$\boxed{\begin{aligned} \hat{a} &= \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}}{m\omega} \right), \\ \hat{a}^\dagger &= \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i\hat{p}}{m\omega} \right). \end{aligned}} \quad (1.11)$$

The inverse relations are many times useful as well,

$$\boxed{\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger), \\ \hat{p} &= i\sqrt{\frac{m\omega\hbar}{2}} (\hat{a}^\dagger - \hat{a}). \end{aligned}} \quad (1.12)$$

While neither \hat{a} nor \hat{a}^\dagger is hermitian (they are hermitian conjugates of each other), the above equations are consistent with the hermiticity of \hat{x} and \hat{p} . We can now write the Hamiltonian in terms of the \hat{a} and \hat{a}^\dagger operators. Using (1.9) we have

$$V^\dagger V = \frac{2\hbar}{m\omega} \hat{a}^\dagger \hat{a}, \quad (1.13)$$

and therefore back in (1.6) we get

$$\boxed{\hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) = \hbar\omega \left(\hat{N} + \frac{1}{2} \right), \quad \hat{N} \equiv \hat{a}^\dagger \hat{a}.} \quad (1.14)$$

The above form of the Hamiltonian is factorized: up to an additive constant \hat{H} is the product of a positive constant times the operator product $\hat{a}^\dagger \hat{a}$. In here we have dropped the identity operator, which

is usually understood. We have also introduced the *number* operator \hat{N} . This is, by construction, a hermitian operator and it is, up to a scale and an additive constant, equal to the Hamiltonian. An eigenstate of \hat{H} is also an eigenstate of \hat{N} and it follows from the above relation that the respective eigenvalues E and N are related by

$$E = \hbar\omega \left(N + \frac{1}{2} \right). \quad (1.15)$$

Let us now show the powerful conclusions that arise from the factorized Hamiltonian. On any state ψ that is normalized we have

$$\langle \hat{H} \rangle_\psi = (\psi, \hat{H}\psi) = \hbar\omega(\psi, \hat{a}^\dagger \hat{a}\psi) + \frac{1}{2}\hbar\omega(\psi, \psi), \quad (1.16)$$

and moving the \hat{a}^\dagger to the first input, we get

$$\langle \hat{H} \rangle_\psi = \hbar\omega(\hat{a}\psi, \hat{a}\psi) + \frac{1}{2}\hbar\omega \geq \frac{1}{2}\hbar\omega. \quad (1.17)$$

The inequality follows because any expression of the form (φ, φ) is greater than or equal to zero. This shows that for any energy eigenstate with energy E : $\hat{H}\psi = E\psi$ we have

$$\text{Energy eigenstates: } E \geq \frac{1}{2}\hbar\omega. \quad (1.18)$$

This important result about the spectrum followed directly from the factorization of the Hamiltonian. But we also get the information required to find the ground state wavefunction. The minimum energy $\frac{1}{2}\hbar\omega$ will be realized for a state ψ if the term $(\hat{a}\psi, \hat{a}\psi)$ in (1.17) vanishes. For this to vanish $\hat{a}\psi$ must vanish. Therefore, the ground state wavefunction φ_0 must satisfy

$$\hat{a}\varphi_0 = 0. \quad (1.19)$$

The operator \hat{a} annihilates the ground state and this why \hat{a} is called the annihilation operator. Using the definition of \hat{a} in (1.11) and the position space representation of \hat{p} , this becomes

$$\left(x + \frac{i}{m\omega} \frac{\hbar}{i} \frac{d}{dx} \right) \varphi_0(x) = 0 \quad \rightarrow \quad \left(x + \frac{\hbar}{m\omega} \frac{d}{dx} \right) \varphi_0(x) = 0. \quad (1.20)$$

Remarkably, this is a **first order** differential equation for the ground state. Not a second order equation, like the Schrödinger equation that determines the general energy eigenstates. This is a dramatic simplification afforded by the factorization of the Hamiltonian into a product of first-order differential operators. The above equation is rearranged as

$$\frac{d\varphi_0}{dx} = -\frac{m\omega}{\hbar} x\varphi_0. \quad (1.21)$$

Solving this differential equation yields

$$\boxed{\varphi_0(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2},} \quad (1.22)$$

where we included a normalization constant to guarantee that $(\varphi_0, \varphi_0) = 1$. Note that φ_0 is indeed an energy eigenstate with energy E_0 :

$$\hat{H}\varphi_0 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\varphi_0 = \frac{1}{2}\hbar\omega\varphi_0 \quad \rightarrow \quad E_0 = \frac{1}{2}\hbar\omega. \quad (1.23)$$

Before proceeding with the analysis of excited states, let us view the properties of factorization more generally. Factorizing a Hamiltonian means finding an operator \hat{A} such that we can rewrite the Hamiltonian as $\hat{A}^\dagger \hat{A}$ up to an additive constant. Here \hat{A}^\dagger is the Hermitian conjugate of \hat{A} , an operator that is defined by

$$(\psi, \hat{A}^\dagger \varphi) = (\hat{A} \psi, \varphi). \quad (1.24)$$

We say that we have factorized a Hamiltonian \hat{H} if we can find a \hat{A} for which

$$\boxed{\hat{H} = \hat{A}^\dagger \hat{A} + E_0 \mathbf{1}}, \quad (1.25)$$

where E_0 is a constant with units of energy that multiplies the identity operator. This constant does not complicate our task of finding the eigenstates of the Hamiltonian, nor their energies: any eigenfunction of $\hat{A}^\dagger \hat{A}$ is an eigenfunction of \hat{H} . Two key properties follow from the factorization (1.25).

1. Any energy eigenstate must have energy greater than or equal to E_0 . First note that for an *arbitrary* normalized $\psi(x)$ we have

$$(\psi, \hat{H} \psi) = (\psi, \hat{A}^\dagger \hat{A} \psi) + E_0 (\psi, \psi) = (\hat{A} \psi, \hat{A} \psi) + E_0, \quad (1.26)$$

Since the overlap $(\hat{A} \psi, \hat{A} \psi)$ is greater than or equal to zero, we have shown that

$$\boxed{(\psi, \hat{H} \psi) \geq E_0}. \quad (1.27)$$

If we take ψ to be an energy eigenstate of energy E : $\hat{H} \psi = E \psi$, the above relation gives

$$E \geq E_0. \quad (1.28)$$

This shows, as claimed, that all possible energies are greater than or equal to E_0 .

2. A wavefunction ψ_0 that satisfies

$$\hat{A} \psi_0 = 0, \quad (1.29)$$

is an energy eigenstate that *saturates* the inequality (1.28). Indeed,

$$\hat{H} \psi_0 = \hat{A}^\dagger \hat{A} \psi_0 + E_0 \psi_0 = \hat{A}^\dagger (\hat{A} \psi_0) + E_0 \psi_0 = E_0 \psi_0. \quad (1.30)$$

The state ψ_0 satisfying $\hat{A} \psi_0 = 0$ is the ground state. For conventional Hamiltonians this is a first order differential equation for ψ_0 and much easier to solve than the Schrödinger equation.

2 Operator manipulation and the spectrum

We have seen that all energy eigenstates are eigenstates of the Hermitian number operator $\hat{N} = \hat{a}^\dagger \hat{a}$. This is because $\hat{H} = \hbar \omega (\hat{N} + \frac{1}{2})$. Note that since $\hat{a} \varphi_0 = 0$ we also have

$$\hat{N} \varphi_0 = 0. \quad (2.1)$$

We can quickly check that

$$\begin{aligned} [\hat{N}, \hat{a}] &= [\hat{a}^\dagger \hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}] \hat{a} = -\hat{a}, \\ [\hat{N}, \hat{a}^\dagger] &= [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger, \end{aligned} \quad (2.2)$$

which we summarize as

$$\begin{aligned} [\hat{N}, \hat{a}] &= -\hat{a}, \\ [\hat{N}, \hat{a}^\dagger] &= \hat{a}^\dagger. \end{aligned} \quad (2.3)$$

Using these identities and induction you should be able to show that:

$$\begin{aligned} [\hat{N}, (\hat{a})^k] &= -k(\hat{a})^k, \\ [\hat{N}, (\hat{a}^\dagger)^k] &= k(\hat{a}^\dagger)^k. \end{aligned} \quad (2.4)$$

These relations suggest why \hat{N} is called the number operator. Acting on powers of creation or annihilation operators by commutation it gives the same object multiplied by (plus or minus) the number of creation or annihilation operators, k in the above. Closely related commutators are also useful:

$$\begin{aligned} [\hat{a}^\dagger, (\hat{a})^k] &= -k(\hat{a})^{k-1} \\ [\hat{a}, (\hat{a}^\dagger)^k] &= k(\hat{a}^\dagger)^{k-1}. \end{aligned} \quad (2.5)$$

These commutators are analogous to $[\hat{p}, (\hat{x})^k]$ and $[\hat{x}, (\hat{p})^k]$. We will also make use of the following Lemma which helps in evaluations where we have an operator \hat{A} that kills a state ψ and we aim to simplify the action of $\hat{A}\hat{B}$, where \hat{B} is another operator, acting on ψ . Here is the result

If $\hat{A}\psi = 0$, then $\hat{A}\hat{B}\psi = [\hat{A}, \hat{B}]\psi$.

(2.6)

This is easily proved. First note that

$$\hat{A}\hat{B} = [\hat{A}, \hat{B}] + \hat{B}\hat{A}, \quad (2.7)$$

as can be quickly checked expanding the right-hand side. It then follows that

$$\hat{A}\hat{B}\psi = ([\hat{A}, \hat{B}] + \hat{B}\hat{A})\psi = [\hat{A}, \hat{B}]\psi, \quad (2.8)$$

because $\hat{B}\hat{A}\psi = \hat{B}(\hat{A}\psi) = 0$. This is what we wanted to show. This is all we need to know about commutators and we can now proceed to construct the states of the harmonic oscillator.

Since \hat{a} annihilates φ_0 consider acting on the ground state with \hat{a}^\dagger . It is clear that \hat{a}^\dagger cannot also annihilate φ_0 . If that would happen acting with both sides of the commutator identity $[\hat{a}, \hat{a}^\dagger] = 1$ on φ_0 would lead to a contradiction: the left-hand side would vanish but the right-hand side would not. Thus consider the wavefunction

$$\varphi_1 \equiv \hat{a}^\dagger \varphi_0. \quad (2.9)$$

We are going to show that this is an energy eigenstate. For this purpose we act on it with the number operator:

$$\hat{N}\varphi_1 = \hat{N}\hat{a}^\dagger \varphi_0 = [\hat{N}, \hat{a}^\dagger] \varphi_0, \quad (2.10)$$

where we noted that $\hat{N}\varphi_0 = 0$ and used Lemma (2.6). Given that $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$, we get

$$\hat{N}\varphi_1 = \hat{a}^\dagger\varphi_0 = \varphi_1. \quad (2.11)$$

Thus φ_1 is an eigenstate of the operator \hat{N} with eigenvalue $N = 1$. Since φ_0 has \hat{N} eigenvalue zero, the effect of acting on φ_0 with \hat{a}^\dagger was to increase the eigenvalue of the number operator by one unit. The operator \hat{a}^\dagger is called the *creation* operator because it creates a state out of the ground state. Alternatively, it is called the *raising* operator, because it raises (by one unit) the eigenvalue of \hat{N} . Since $N = 1$ for φ_1 it follows that φ_1 is an energy eigenstate with energy E_1 given by

$$E_1 = \hbar\omega(1 + \frac{1}{2}) = \frac{3}{2}\hbar\omega. \quad (2.12)$$

It also turns out that φ_1 is properly normalized:

$$(\varphi_1, \varphi_1) = (\hat{a}^\dagger\varphi_0, \hat{a}^\dagger\varphi_0) = (\varphi_0, \hat{a}\hat{a}^\dagger\varphi_0), \quad (2.13)$$

where we used the Hermitian conjugation property to move the \hat{a}^\dagger acting on the left input into the right input, where it goes as $(\hat{a}^\dagger)^\dagger = \hat{a}$. We then have

$$(\varphi_1, \varphi_1) = (\varphi_0, \hat{a}\hat{a}^\dagger\varphi_0) = (\varphi_0, [\hat{a}, \hat{a}^\dagger]\varphi_0) = (\varphi_0, \varphi_0) = 1, \quad (2.14)$$

where we used (2.6) in the evaluation of $\hat{a}\hat{a}^\dagger\varphi_0$. Indeed the state φ_1 is correctly normalized.

Next consider the state

$$\varphi'_2 \equiv \hat{a}^\dagger\hat{a}^\dagger\varphi_0. \quad (2.15)$$

This has

$$\hat{N}\varphi'_2 = \hat{N}\hat{a}^\dagger\hat{a}^\dagger\varphi_0 = [\hat{N}, \hat{a}^\dagger\hat{a}^\dagger]\varphi_0 = 2\hat{a}^\dagger\hat{a}^\dagger\varphi_0 = 2\varphi'_2, \quad (2.16)$$

so φ_2 is a state with number $N = 2$ and energy $E_2 = \frac{5}{2}\hbar\omega$. Is it properly normalized? We find

$$\begin{aligned} (\varphi_2, \varphi_2) &= (\hat{a}^\dagger\hat{a}^\dagger\varphi_0, \hat{a}^\dagger\hat{a}^\dagger\varphi_0) = (\varphi_0, \hat{a}\hat{a}\hat{a}^\dagger\hat{a}^\dagger\varphi_0) = (\varphi_0, \hat{a}[\hat{a}, \hat{a}^\dagger\hat{a}^\dagger]\varphi_0) \\ &= (\varphi_0, 2\hat{a}\hat{a}^\dagger\varphi_0) = (\varphi_0, \varphi_0) = 2. \end{aligned} \quad (2.17)$$

The properly normalized wavefunction is therefore

$$\varphi_2 \equiv \frac{1}{\sqrt{2}}\hat{a}^\dagger\hat{a}^\dagger\varphi_0. \quad (2.18)$$

We now claim that the n -th excited state of the simple harmonic oscillator is

$$\boxed{\varphi_n \equiv \frac{1}{\sqrt{n!}} \underbrace{\hat{a}^\dagger \cdots \hat{a}^\dagger}_n \varphi_0 = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n \varphi_0.} \quad (2.19)$$

Exercise: Verify that this state has \hat{N} eigenvalue n .

Exercise: Verify that the state φ_n is properly normalized.

Since the \hat{N} eigenvalue of φ is n , its energy E_n is given by

$$E_n = \hbar\omega(n + \frac{1}{2}). \quad (2.20)$$

Since the various states φ_n are eigenstates of a Hermitian operator (the Hamiltonian \hat{H}) with different eigenvalues, they are orthonormal

$$(\varphi_n, \varphi_m) = \delta_{m,n}. \quad (2.21)$$

We now note that $\hat{a}\varphi_n$ is a state with $n-1$ operators \hat{a}^\dagger acting on φ_0 because the \hat{a} eliminates one of the creation operators in φ_n . Thus we expect $\hat{a}\varphi_n \sim \varphi_{n-1}$. We can make this precise

$$\hat{a}\varphi_n = \hat{a} \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n \varphi_0 = \frac{1}{\sqrt{n!}} [\hat{a}, (\hat{a}^\dagger)^n] \varphi_0 = \frac{n}{\sqrt{n!}} (\hat{a}^\dagger)^{n-1} \varphi_0. \quad (2.22)$$

At this point we use (2.19) with n set equal to $n-1$ and thus get

$$\hat{a}\varphi_n = \frac{n}{\sqrt{n!}} \sqrt{(n-1)!} \varphi_{n-1} = \sqrt{n} \varphi_{n-1}. \quad (2.23)$$

By the action of \hat{a}^\dagger on φ_n we get

$$\hat{a}^\dagger \varphi_n = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^{n+1} \varphi_0 = \frac{1}{\sqrt{n!}} \sqrt{(n+1)!} \varphi_{n+1} = \sqrt{n+1} \varphi_{n+1}. \quad (2.24)$$

Collecting the results, we have

$$\boxed{\begin{aligned} \hat{a}\varphi_n &= \sqrt{n} \varphi_{n-1}, \\ \hat{a}^\dagger \varphi_n &= \sqrt{n+1} \varphi_{n+1}. \end{aligned}} \quad (2.25)$$

These relations make it clear that \hat{a} lowers the number of any energy eigenstate by one unit, except for the vacuum φ_0 which it kills. The raising operator \hat{a}^\dagger increases the number of any eigenstate by one unit.

Exercise: Calculate the uncertainty Δx of position in the n -th energy eigenstate.

Solution: By definition,

$$(\Delta x)_n^2 = \langle \hat{x}^2 \rangle_{\varphi_n} - \langle \hat{x} \rangle_{\varphi_n}^2. \quad (2.26)$$

The expectation value $\langle \hat{x} \rangle$ vanishes for any energy eigenstate since we are integrating x , which is odd, against $|\varphi_n(x)|^2$, which is always even. Still, it is instructive to see how this happens explicitly:

$$\langle \hat{x} \rangle_{\varphi_n} = (\varphi_n, \hat{x} \varphi_n) = \sqrt{\frac{\hbar}{2m\omega}} (\varphi_n, (\hat{a} + \hat{a}^\dagger) \varphi_n), \quad (2.27)$$

using the formula for \hat{x} in terms of \hat{a} and \hat{a}^\dagger . The above overlap vanishes because $\hat{a}\varphi_n \sim \varphi_{n-1}$ and $\hat{a}^\dagger \varphi_n \sim \varphi_{n+1}$ and both φ_{n-1} and φ_{n+1} are orthogonal to φ_n . Now we compute the expectation value of \hat{x}^2

$$\begin{aligned} \langle \hat{x}^2 \rangle_{\varphi_n} &= (\varphi_n, \hat{x}^2 \varphi_n) = \frac{\hbar}{2m\omega} (\varphi_n, (\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) \varphi_n) \\ &= \frac{\hbar}{2m\omega} (\varphi_n, (\hat{a}\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}^\dagger) \varphi_n). \end{aligned} \quad (2.28)$$

Since $\hat{a}\hat{a}\varphi_n \sim \varphi_{n-2}$ and $\hat{a}^\dagger\hat{a}^\dagger\varphi_n \sim \varphi_{n+2}$ and both φ_{n-2} and φ_{n+2} are orthogonal to φ_n , the $\hat{a}\hat{a}$ and $\hat{a}^\dagger\hat{a}^\dagger$ terms do not contribute. We are left with

$$\langle \hat{x}^2 \rangle_{\varphi_n} = \frac{\hbar}{2m\omega} (\varphi_n, (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \varphi_n). \quad (2.29)$$

At this point we recognize that $\hat{a}^\dagger \hat{a} = \hat{N}$ and that $\hat{a} \hat{a}^\dagger = [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} = 1 + \hat{N}$. As a result

$$\langle \hat{x}^2 \rangle_{\varphi_n} = \frac{\hbar}{2m\omega} (\varphi_n, (1 + 2\hat{N})\varphi_n) = \frac{\hbar}{2m\omega} (1 + 2n). \quad (2.30)$$

We therefore have

$$(\Delta x)_n^2 = \frac{\hbar}{m\omega} \left(n + \frac{1}{2} \right). \quad (2.31)$$

The position uncertainty grows linearly with the number.

Sarah Geller and Andrew Turner transcribed Zwiebach's handwritten notes to create the first LaTeX version of this document.

MIT OpenCourseWare
<https://ocw.mit.edu>

8.04 Quantum Physics I
Spring 2016

For information about citing these materials or our Terms of Use, visit: <https://ocw.mit.edu/terms>.