BirZeit University
Faculty of Science-Department of Physics
Quantum Mechanics Phys635
Spring 2016
First Exam, Apr. 7th 2016

1. If **A**, **B** are Hermitian operators:
   (a) Under what conditions **AB** is Hermitian
   (b) Given that **A** and **B** are combatible operators show that $(A+B)^n$ is Hermitian, where n is a positive integer.

2. Vectors $|a>$ and $|b>$ belong to a certain abstract vector space such that:

$$|a><a| + |b><b| = 1$$

   (a) What is the dimension of the space
   (b) Find $\mathrm{Tr}(e^{|a><a|})$
   (c) Find $\left[e^{|a><a|}, e^{|a><b|}\right]$

3. Consider The problem of one dimensional harmonic oscillator, where the Hamiltonian is give be:

$$H = \hbar\omega(a^\dagger a + \frac{1}{2})$$

   and the time evolution operator to be:

$$U(t,0) = e^{-iHt/\hbar}$$

   (a) Consider the following operators:

$$\tilde{a}(t) = U^\dagger(t,0) a U(t,0)$$
$$\tilde{a}^\dagger(t) = U^\dagger(t,0) a^\dagger U(t,0)$$

   By calculating their action on the base kets of the hamiltonian Find an expression of $\tilde{a}(t)$ and $\tilde{a}^\dagger(t)$ in terms of a and $a^\dagger$

   (b) Show that the position and momentum operators at any time t can be written as:

$$\tilde{X}(t) = U^\dagger(t,0) X U(t,0)$$
$$\tilde{P}(t) = U^\dagger(t,0) P U(t,0)$$

   (c) Show that $U^\dagger(\frac{2\pi}{\omega},0)|x>$ is an eigenket of P, and what is the eigenvalue
   (d) Show that $U^\dagger(\frac{2\pi}{\omega},0)|p>$ is an eigenket of X, and what is the eigenvalue

4. Three matrices M_x, M_y, M_z, each with 256 rows and columns, are known to obey the commutation rules $[M_x, M_y] = iM_Z$ (with cyclic permutations of x, y and z). The eigenvalues of the matrix M_x are ± 2, each once; $\pm 3/2$, each 8 times; ± 1, each 28 times; $\pm 1/2$, each 56 times; and 0, 70 times. State the 256 eigenvalues of the matrix $M^2 = M_x^2 + M_y^2 + M_z^2$.

5. Phase space is defined by the position and momentum. Show that:

$$T(x_0, p_0) = e^{\frac{i(Xp_0 - Px_0)}{\hbar}}$$

   is a translation operator in phase space

6. The coherent states are defined to be an eigenvector of the annihilation operator in simple harmonic oscillator.

$$a|\alpha >= \alpha|\alpha >$$

α is in general a complex number

(a) Show that $|\alpha >$ can be written as:

$$|\alpha >= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n >$$

(b) Show that

$$|\alpha, t >= U(t,0)|\alpha >= |\alpha e^{-i\omega t} > e^{-i\omega t/2}$$

ω is the angular frequency of the harmonic oscillator

(c) Calculate

$$< \alpha|X(t)|\alpha >$$
$$< \alpha|P(t)|\alpha >$$

7. A particle of mass m is allowed to move only along the circle of radius R on a plane, $x = Rcos\theta$, $y = Rsin\theta$

(a) Show that the Lagrangian is

$$L = \frac{m}{2}R^2\dot{\theta}^2$$

(b) Show that the Hamiltonian is :

$$H = \frac{1}{2mR^2}P_\theta^2$$

where P_θ is the canonical momentum

(c) What are the eigenvalues of the Hamiltonian

(d) Write down the normalized position-space wave function $\psi_k(\theta) =< \theta|k >$ for the momentum eigenstates $P_\theta|k >= k|k >$ and show that only for k is an integer are allowed.

(e) The particle is now subjected to a constant magnetic field B inside the radius $r < d < R$ but no magnetic field outside $r > d$, with the vector potential is

$$\vec{A}(x) = \begin{cases} \frac{B}{2}(-y\hat{i} + x\hat{j}) & \text{if r < d} \\ \frac{Bd^2}{2R^2}(-y\hat{i} + x\hat{j}) & \text{if r > r} \end{cases}$$

write the new Hamiltonian, and show that energy eigenvalues are influenced by the magnetic field although the magnetic field is zero in the location of the particle

8. Consider a particle in three-dimensional space, whose state vector is $|\psi >$ and whose wave function is $\psi(r) =< r|\psi >$. Let A be an observable which commutes with $\vec{L} = \vec{R} \times \vec{P}$, the orbital angular momentum of the particle. Assuming that A, L^2 and L_z form a set of commuting observables let $|n, l, m >$ their common eigenkets, whose eigenvalues are, respectively, a_n, $l(l+1)\hbar^2$ and $m\hbar$ (the index n is assumed to be discrete).

Let $U(\phi)$ be the unitary operator defined by:

$$U(\phi) = e^{\frac{-iL_z\phi}{\hbar}}$$

where ϕ is a real dimensionless parameter. For an arbitrary operator K, we call $\tilde{K}$ the transform of K by the unitary operator $U(\phi)$:

$$\tilde{K} = U(\phi)KU^\dagger(\phi)$$

(a) We set $L_+ = L_x + iL_y$, $L_- = L_x - iL_y$. Calculate $\tilde{L_+}|n, l, m >$ and show that L_+ and $\tilde{L_+}$ are proportional; calculate the proportionality constant.

(b) Express L_x terms of $\tilde{L_x}$, $\tilde{L_y}$,and $\tilde{L_z}$,. What geometrical transformation can be associated with the transformation of L into $\tilde{L}$