Birzeit University
Faculty of Science-Department of Physics
Quantum Mechanics I, Phys433
Fall 2020
Problems ch3

1. Consider the Hamiltonian of a particle in one-dimensional problem defined by:

$$H = \frac{1}{2m}P^2 + V(X)$$

where **X** and **P** are the position and linear momentum operators, and they satisfy the commutation relation:

$$[X, P] = i\hbar$$

The eigenvectors of H are denoted by $|\phi_n>$; where n is a discrete index

$$H|\phi_n >= E_n|\phi_n >$$

(a) Show that

$$<\phi_n|P|\phi_m >= \alpha < \phi_n|X|\phi_m >$$

and find α. Hint: Consider the commutator $[X, H]$

(b) Using the result from the previous part, and with the aid of the closure relation show that:

$$\sum_m (E_n - E_m)^2| < \phi_n|X|\phi_m > |^2 = \frac{\hbar^2}{m^2} < \phi_n|P^2|\phi_n >$$

(c) For an arbitrary operator A, prove the relation:

$$<\phi_n|[A, H]|\phi_n >= 0$$

(d) In terms of **P**, **X** and V(**X**), find the commutators: $[H, P], [H, X], [H, XP]$

(e) Show that

$$<\phi_n|P|\phi_n >= 0$$

(f) Establish a relation between the mean value of the kinetic energy E_k in state $|\phi_n>$

$$E_k =< \phi_n|\frac{P^2}{2m}|\phi_n >$$

and

$$<\phi_n|X\frac{dV}{dX}|\phi_n >$$

(g) if $V(\mathbf{X}) = V_0X^\lambda$, λ = 2,4,6,...., what is the relation between the mean value of the kinetic energy and the mean value of the potential

2. Consider The problem of one dimensional harmonic oscillator, where the Hamiltonian is give be:

$$H = \hbar\omega(a^\dagger a + \frac{1}{2})$$

and the time evolution operator to be:

$$U(t,0) = e^{-iHt/\hbar}$$

(a) Consider the following operators:

$$\tilde{a}(t) = U^\dagger(t,0) a U(t,0)$$
$$\tilde{a}^\dagger(t) = U^\dagger(t,0) a^\dagger U(t,0)$$

By calculating their action on the base kets of the hamiltonian Find an expression of $\tilde{a}(t)$ and $\tilde{a}^\dagger(t)$ in terms of a and $a^\dagger$

(b) Show that the position and momentum operators at any time t can be written as:

$$\tilde{X}(t) = U^\dagger(t,0) X U(t,0)$$
$$\tilde{P}(t) = U^\dagger(t,0) P U(t,0)$$

(c) Show that $U^\dagger(\frac{\pi}{2\omega},0)|x>$ is an eigenket of P, and what is the eigenvalue

(d) Show that $U^\dagger(\frac{\pi}{2\omega},0)|p>$ is an eigenket of X, and what is the eigenvalue

3. If **A**, **B** are Hermitian operators:

(a) Under what conditions **AB** is Hermitian

(b) Given that **A** and **B** are compatible operators show that $(A+B)^n$ is Hermitian, where n is a positive integer.

4. Vectors $|a>$ and $|b>$ belong to a certain abstract vector space such that:

$$|a><a| + |b><b| = 1$$

(a) What is the dimension of the space

(b) Find $\mathrm{Tr}(e^{|a><a|})$

(c) Find $\left[e^{|a><a|}, e^{|a><b|}\right]$

5. An operator A is represented by:

$$\hat{A} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

in the basis $|a_i>$ (where i=1,2).

(a) Write the operator A as an outer product of the basis $|a_i>$

(b) If a new basis $|b_i>$ is defined by:

$$<a_1|b_1> = <a_2|b_1> = \frac{1}{\sqrt{2}}$$
$$<a_1|b_2> = -<a_2|b_2> = \frac{-i}{\sqrt{2}}$$

Write a transformation matrix between the two basis, discuss the properties of this matrix

(c) Write a matrix representation of operator A in the new basis $|b_i>$

6. Let $|\phi_n>$ be the eigenstate of the Hamiltonian of an arbitrary physical system. Assume the the states $|\phi_n>$ form a discrete orthonormal basis. The operator U(m,n) is defined by:

$$U(m,n) = |\phi_m><\phi_n|$$

   (a) Calculate $U^\dagger(m,n)$
   (b) Calculate the commutator [H, U(m,n)]
   (c) Show that:

$$U(m,n)U^\dagger(p,q) = \delta_{nq}U(m,p)$$

   (d) Calculate Tr(U(,m,n))
   (e) If A is an operator and $A_{mn} = <\phi_m|A|\phi_n>$ show that

$$A = \sum_{n,m} A_{mn}U(m,n)$$

   (f) Show that $A_{pq} = Tr(AU^\dagger(p,q))$

7. Two observables A_1 and A_2, which do not involve time explicitly, are known not to commute:

$$[A_1, A_2] \neq 0$$

yet we also know that A_1 and A_2 both commute with the Hamiltonian:

$$[A_1, H] = [A_2, H] = 0$$

Prove that the energy eigenstates are, in general, degenerate.

8. Suppose we have a quantum mechanical system with an observable called **color**, and we have built a device which can measure the color of the system. We find that there are only three color states: $|red>$, $|green>$, and $|blue>$
   (a) We measure the color of the system, and then we wait a little while and measure it again. We find that the second color measurement is not always the same as the first. Based on this observation, we guess the following Hamiltonian for our system:

$$\hat{H} = E_1|red><red| + E_2|green><green| + E_3|blue><blue|$$

   Note: E_1, E_2, and E_3 are the eigenvalues of the Hamiltonian of the system. Is this a reasonable guess? In other words, is this Hamiltonian consistent or inconsistent with our observations? (Explain your answer.)
   (b) We decide to measure the color of the system over and over again, once per second. When the color is red, we immediately follow the color measurement with an energy measurement. We find that 90% of the time we get energy E_1, 10% of the time we get E_2, and we never get E_3. Write down a plausible guess for how $|red>$ appears when written in terms of the energy eigenstates: $|1>$, $|2>$, and $|3>$
   (c) What, if anything, can we say about the commutator of the color operator and the Hamiltonian for this system

9. The state of some system can be expressed using three orthonormal basis states $|1>$, $|2>$, and $|3>$. The action of the Hamiltonian $\hat{H}$ on each basis state is given is:

$$\hat{H}|1> = \frac{3i}{\sqrt{2}}\hbar\omega|2>$$
$$\hat{H}|2> = \frac{-3i}{\sqrt{2}}\hbar\omega|1> - \frac{3i}{\sqrt{2}}\hbar\omega|3>$$
$$\hat{H}|3> = \frac{3i}{\sqrt{2}}\hbar\omega|2>$$

   (a) Find the matrix representation of $\hat{H}$ in this basis.

(b) What are the possible energies of this system? The ground state $|E_0>$ is the state which corresponds to the lowest possible energy. What is the representation of $|E_0>$ in this basis.

(c) If this system is in the state

$$|\psi> = \frac{1}{\sqrt{8}} \begin{pmatrix} \sqrt{3} \\ -\sqrt{2} \\ \sqrt{3} \end{pmatrix}$$

written in the original basis. What is the probability that the system is in the ground state.

10. In the basis of states $|1>$ and $|2>$, the matrix representations of two operators $\hat{A}$ and $\hat{B}$ are:

$$\hat{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\hat{B} = \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$$

(a) For each operator determine whether it can be associated with a physical observable.

(b) Do quantum states exist for which the outcome of a measurement of both physical observables can be predicted with certainty.

(c) Determine the mean value and the variance in a measurement of

11. A certain 3-level system can have three values of energy associated with 3 stationary states:

Energy	state
0	$\|1>$
$\hbar\omega$	$\|2>$
$2\hbar\omega$	$\|3>$

(a) An operator $\hat{Q}$ is written in the following form:

$$\hat{Q} = a|1><1| + b(|2><3| + |3><2|)$$

with $0 < a < b$. What will be the outcome if we measure the operator $\hat{Q}$

(b) If at time t $= 0$, the operator $\hat{Q}$ was measured, and the largest eigenvalue was obtained, write the wavefunction at any later time t.

(c) If $\hat{H}$ was measured at any later time t, what might we get and with what probability.

12. Consider a three-level system where the Hamiltonian and observable A are given by the matrix

$$\hat{A} = \mu \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{H} = \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) What are the possible values obtained in a measurement of A

(b) Does a state exist in which both the results of a measurement of energy E and observable A can be predicted with certainty? Why or why not?

(c) Two measurements of A are carried out, separated in time by t. If the result of the first measurement is its largest possible value, determine the expectation value $<\psi(t)|A|\psi)t)>$ for the second measurement.

13. Consider a two level system, where the matrix representation of the position operator and its eigenvectors are given by:

$$\hat{X} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \qquad |1> = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |2> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and that the Hamiltonian of the system is given by:

$$\hat{H} = \begin{pmatrix} E_0 & -\epsilon \\ -\epsilon & E_0 \end{pmatrix}$$

(a) Is the Hamiltonian operator hermitian? Justify your answer!

(b) Is it possible to find common eigenstates for $\hat{H}$ and $\hat{X}$? Justify!

(c) Are states $|1>$ and $|2>$ stationary?

(d) Find eigenvalues and eigenvectors of the Hamiltonian

(e) If the electron initially at state $|2>$ write its wavefunction at any later time t

(f) Is the operator $\hat{X}$ constant of motion? Justify your answer