MATH 1321
MIDTERM EXAM

Student Name:__________ Student Number:__________

Discussion Instructor: Rash shadeed Discussion Section:__________

98/100

Excellent :)

Question 1 (3 points each) Circle the most correct answer:

1. The integral $\int_2^\infty \frac{2}{x^3 - x}\,dx$

   (a) converges by the limit comparison test with $\int_2^\infty \frac{1}{x^3}\,dx$

   (b) diverges by the limit comparison test with $\int_2^\infty \frac{1}{x^3}\,dx$

   (c) converges by the direct comparison test with $\int_2^\infty \frac{1}{x^3}\,dx$

   (d) converges by the limit comparison test with $\int_2^\infty \frac{1}{x}\,dx$

$\frac{2}{x^3-x}$ $\frac{1}{x^3}$ conv

$\frac{2x^3}{x^3-x}$ 2

$\frac{1}{\frac{1}{n}} = n \to \infty$

2. The sequence whose nth term is $a_n = \frac{n}{\ln n}$

   (a) converges to 0
   (b) converges to 1
   (c) converges to 2
   (d) diverges

$\sum_{n=1}^{\infty}(-1)^n \frac{2n^2+1}{n^2-5}$

$\frac{1}{\frac{1}{n}} = n$

3. The series $\sum_{n=1}^{\infty}(-1)^n \frac{2n^2+1}{n^2-5}$

   (a) diverges by the nth term test
   (b) converges by the nth term test
   (c) converges absolutely
   (d) converges conditionally

4. If we use S_4 to approximate the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ then the error satisfies

   (a) the error is negative and $|\text{error}| < 0.25$
   (b) the error is negative and $|\text{error}| < 0.2$
   (c) the error is positive and $|\text{error}| < 0.1$
   (d) the error is positive and $|\text{error}| < 0.2$

$\underbrace{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}}, \frac{1}{5}$

$|E| < \frac{1}{5}$

conv by A.S.T

5. The series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

(a) diverges by the nth term test

(b) converges by the nth term test

(c) converges absolutely

(d) converges conditionally

$\frac{1}{\ln n}$

$\frac{1}{n} \quad \frac{n}{\ln n} = \frac{1}{\frac{1}{n}} = \infty$

$\frac{1}{\ln n} > \frac{1}{n}$

6. The series $\sum_{n=0}^{\infty} e^{-n}$

(a) converges to $\frac{e}{e-1}$

(b) converges to $\frac{1}{1-e}$

(c) converges to $\frac{e-1}{e}$

(d) diverges

$\frac{1}{e^0} + \frac{1}{e^1} + \frac{1}{e^2}$ $\quad \left(\frac{1}{e}\right)^n \quad |r|<1$

$\frac{1}{e^n}$

$\frac{1}{1-\frac{1}{e}} = \frac{1}{\frac{e-1}{e}} = \frac{e}{e-1}$

7. The series $\sum_{n=1}^{\infty} \frac{n^5}{5^n}$

(a) diverges by the root test

(b) converges by the root test

(c) converges by the direct comparison test with $\sum_{n=1}^{\infty} \frac{1}{5^n}$

(d) diverges by the direct comparison test with $\sum_{n=1}^{\infty} \frac{1}{5^n}$

$\frac{n^5}{5^n} > \frac{1}{5^n}$

$\lim_{n\to\infty} \frac{n^{\frac{5}{n}}}{5} = \boxed{\frac{1}{5}}$

8. One of the following is true

(a) If $\sum_{n=1}^{\infty} |a_n|$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges

(b) If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges

(c) If $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} |a_n|$ converges

(d) $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge

9. The series $\sum_{n=1}^{\infty} \frac{\frac{1}{2}\tan^{-1} n}{n^3+1}$

$\frac{\tan^{-1} n}{2n^3+2}$

(a) diverges by the direct comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$\frac{\tan^{-1}(n)}{2(n^3+1)}$ $\frac{2}{2n^3}$

(b) converges by the direct comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(c) converges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$

$\frac{1}{n^3+1}$

(d) diverges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

10. The sequence whose nth term is $a_n = n\tan^{-1} n$

$\lim_{n\to\infty} a_n = \infty \cdot \frac{\pi}{2} = \infty$

(a) converges to 0

(b) converges to $\frac{\pi}{2}$

(c) converges to $-\frac{\pi}{2}$

(d) diverges

$f(x) = \frac{1}{\sqrt{x}}$

11. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(a) converges by the integral test

(b) diverges by the integral test

(c) diverges by the nth term test

(d) converges by the nth term test

12. The series $\sum_{n=2}^{\infty} \frac{(\ln n)^{35}}{n!}$

$\frac{\frac{(\ln n)^{35}}{\cancel{n!}}}{\frac{1}{\cancel{n!}}} = \boxed{\infty}$

(a) diverges by the limit comparison test with $\sum_{n=2}^{\infty} \frac{1}{n!}$

(b) diverges by the nth term test

$\frac{(\ln n)^{35}}{n!} > \frac{1}{n!}$ conv.

(c) diverges by the ratio test

(d) converges by the ratio test

$\ln n$

$\frac{1}{(n+1)\cancel{n!}} \times \cancel{n!} = 0 < 1$ conv

$\frac{(\ln(n+1))^{35}}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{(\ln n)^{35}}$

$n+1$

$\frac{1}{n+1}\left(\frac{\ln n+1}{\ln n}\right)^{35}$

$|x-1| < 1$

$-1 < x-1 < 1$

$+1 \quad +1 \quad +1$

$0 < x < 2$

13. The series $\sum_{n=1}^{\infty}(x-1)^n$

(a) converges absolutely for $0 < x < 2$

(b) converges conditionally for $0 \leq x \leq 2$

(c) converges conditionally for $0 < x < 2$

(d) converges absolutely for $0 \leq x \leq 2$

$\frac{-3}{2} + \frac{2}{2}$

14. The integral $\int_1^2 \frac{dx}{(x-1)^{\frac{3}{2}}}$

$\lim_{b \to 1^+} \int_b^2 \frac{dx}{(x-1)^{\frac{3}{2}}} = \lim_{b \to 1^+} -2(x-1)^{-\frac{1}{2}} \Big|_b^2$

$= \lim_{b \to 1^+} -2 + 2(b-1)^{-\frac{1}{2}}$

$= -2 + \frac{2}{(b-1)^{\frac{1}{2}}} = \boxed{\infty}$

(a) converges to 0

(b) converges to 1

(c) converges to $-\frac{1}{2}$

(d) diverges

15. If $\sum a_n$ is a convergent series of positive terms, then the series $\sum (a_n)^n$ converges

(a) True

(b) False

16. Consider the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2+19}$. The least number of terms that are needed to estimate the sum of the series with an error of less than 0.01 is

(a) fifteen terms

(b) nine terms

(c) ten terms

(d) five terms

E

$|E| < 0.01$

$U_{n+1} < 0.01$

$\frac{1}{n^2+19} < \frac{1}{100}$

$n^2 + 19 > 100$

$-19 \quad -19$

$\sqrt{n^2} > \sqrt{81}$

$\boxed{n \geq 9}$

$19 + 81 = 100$

17. The series $\displaystyle\sum_{n=2}^{\infty} \frac{7}{n(n+1)}$

(a) converges to $\frac{7}{2}$

(b) converges to $\frac{1}{2}$

(c) converges to $-\frac{1}{2}$

(d) diverges

$$\frac{7}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \boxed{\frac{7}{n} - \frac{7}{n+1}}$$

$$\left(\frac{7}{2} - \frac{7}{3}\right) + \left(\frac{7}{3} - \frac{7}{4}\right) \quad \left(\frac{7}{2} - \frac{7}{n+1}\right)$$

18. The integral $\displaystyle\int_{2}^{\infty} \frac{dx}{\sqrt{x^2-1}}$

(a) diverges by the direct comparison test with $\displaystyle\int_{2}^{\infty} \frac{dx}{x}$

(b) diverges by the limit comparison test with $\displaystyle\int_{2}^{\infty} \frac{dx}{x^2}$

(c) converges by the direct comparison test with $\displaystyle\int_{2}^{\infty} \frac{dx}{x}$

(d) converges by the limit comparison test with $\displaystyle\int_{2}^{\infty} \frac{dx}{x^2}$

$$\frac{1}{\sqrt{x^2-1}} > \frac{1}{\sqrt{x^2}}, \qquad \frac{1}{\sqrt{x^2-1}} > \frac{1}{x}$$

19. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

(a) diverges by the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(b) converges by the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

(c) converges by the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(d) diverges by the limit comparison test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\frac{\frac{1}{n\sqrt{n^2+1}}}{\frac{1}{n^2}} = \frac{n}{\sqrt{n^2+1}} = \boxed{1}$$

20. The series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{5}{3^n}$

(a) converges to $-\frac{5}{4}$

(b) converges to $\frac{15}{4}$

(c) converges to $-\frac{5}{2}$

(d) diverges

$$5\left(\frac{-1}{3}\right)^n \qquad \frac{-\frac{5}{3}}{1+\frac{1}{3}} = \frac{-5}{4}$$

21. The sequence whose nth term is $a_n = 1 - \cos(\frac{1}{n})$

(a) converges to $1 - \frac{\pi}{2}$

(b) converges to 1

(c) converges to 0

(d) diverges

$1 - 1 = 0$

22. The series $\sum_{n=1}^{\infty} \frac{(\sin n)^2}{n^{\frac{5}{2}}}$

(a) converges by the nth term test

(b) diverges the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$

(c) converges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(d) converges by the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$0 < \frac{\sin^2 n}{n^{\frac{5}{2}}} < \frac{1}{n^{\frac{5}{2}}}$ conv

23. The sequence whose nth term is $a_n = \sqrt[n]{4^n n}$

(a) converges to 0

(b) converges to 4

(c) converges to 2

(d) diverges

$\boxed{4} n^{\frac{1}{n}}$

24. The series $\sum_{n=1}^{\infty} (1 - \frac{1}{2n})^n$

(a) diverges by the nth term test

(b) diverges by the root test

(c) converges by the root test

(d) converges by the nth term test

$1 - \frac{1}{2n} =$

$(1)^{\infty}$

$n \ln(1 - \frac{1}{2n})$

Question 2 (10 points) Evaluate the integral $\int_0^2 \frac{dx}{(x-1)^{\frac{2}{3}}}$.

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$\int_0^2 \frac{dx}{(x-1)^{\frac{2}{3}}}$ has a discontinuity at $\boxed{x=1}$

so $\int_0^2 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^2 \frac{dx}{(x-1)^{\frac{2}{3}}} = \boxed{\lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{\frac{2}{3}}}} + \boxed{\lim_{a\to 1^+} \int_a^2 \frac{dx}{(x-1)^{\frac{2}{3}}}}$

* $\lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{b\to 1^-} \int_0^b (x-1)^{-\frac{2}{3}}\, dx = \lim_{b\to 1^-} 3(x-1)^{\frac{1}{3}} \Big|_0^b$

$= \lim_{b\to 1^-} 3(b-1)^{\frac{1}{3}} - 3(0-1)^{\frac{1}{3}} = \lim_{b\to 1^-} 3\sqrt[3]{b-1} + 3 = \boxed{3}$

* $\lim_{a\to 1^+} \int_a^2 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{a\to 1^+} \int_a^2 (x-1)^{-\frac{2}{3}}\, dx = \lim_{a\to 1^+} 3(x-1)^{\frac{1}{3}} \Big|_a^2$

$= \lim_{a\to 1^+} 3(2-1)^{\frac{1}{3}} - 3(a-1)^{\frac{1}{3}} = \lim_{a\to 1^+} 3 - 3(a-1)^{\frac{1}{3}} = \boxed{3}$

so $\int_0^2 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^2 \frac{dx}{(x-1)^{\frac{2}{3}}}$

$= 3 + 3 = \boxed{6}$

Question 3 (14 points) Consider the power series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n\,2^n}$. Answer the following questions:

$\boxed{a=3}$ center

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1. For what values of x does the series converge absolutely?
2. Find the radius of convergence.
3. For what values of x does the series converge conditionally?
4. Find the interval of convergence.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n2^n} \xrightarrow{\text{Apply}} \text{Ratio test} \quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(x-3)}^{n+1}}{(n+1)\cancel{2^{n+1}}} \cdot \frac{\cancel{2^n}\,n}{\cancel{(x-3)^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-3)(n)}{2(n+1)}\right| = \frac{|x-3|}{2} \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \frac{|x-3|}{2}\cdot(1) = \boxed{\frac{|x-3|}{2}}$$

by the Ratio test, the series converges when $\boxed{\rho<1}$ and diverges if $\boxed{\rho>1}$

$\boxed{\text{So}}$ the series converges when $\frac{|x-3|}{2} < 1 \Rightarrow |x-3| < 2$

$-2 < x-3 < 2$
$+3 \quad +3 \quad +3$

$\boxed{1 < x < 5}$

conv. abs in this interval

$\boxed{\text{also}}$ the series diverges when $\frac{|x-3|}{2} > 1$

$|x-3| > 2$

$x-3 > 2$ or $x-3 < -2$

$\boxed{x > 5}$ $\boxed{x < 1}$

Alternating series test

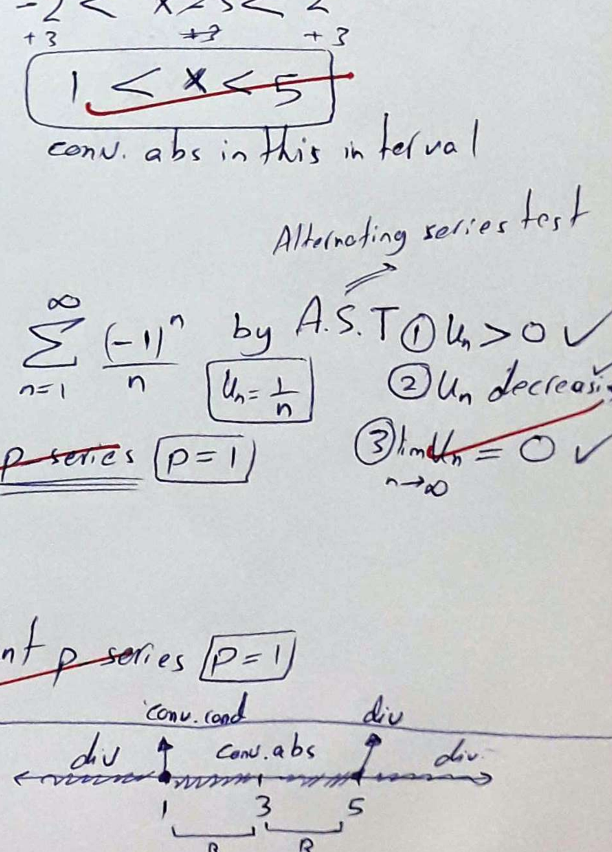
case ① when $\boxed{x=1} \Rightarrow \sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \left(\frac{-2}{2}\right)^n \cdot \frac{1}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ by A.S.T ① $u_n > 0$ ✓ $\boxed{u_n = \frac{1}{n}}$ ② u_n decreasing ✓ ③ $\lim_{n\to\infty} u_n = 0$ ✓

so by A.S.T the series converge. take $\sum_{n=1}^{\infty} \frac{1}{n} \to$ divergant p series $\boxed{p=1}$

so $\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n}$ conv. cond

case ② when $\boxed{x=5} \Rightarrow \sum_{n=1}^{\infty} \frac{\cancel{(2)^n}}{n\cancel{(2^n)}} = \sum_{n=1}^{\infty} \frac{1}{n} \to$ divergant p series $\boxed{p=1}$

so $\sum_{n=1}^{\infty} \frac{(2)^n}{n(2^n)}$ diverges



① conv. abs in $(1,5)$

② $R=2$

③ conv. cond. at $\boxed{x=1}$

④ I.C $= [1,5)$

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Question 4 (10 points) Find the Taylor series generated by $f(x) = \frac{1}{x^2}$ at $x = 2$. (Write the final answer using the sigma notation).

$f(x) = \frac{1}{x^2} \xrightarrow{f(2)} f(2) = \frac{1}{2^2} = \boxed{\frac{1}{4}}$

$f'(x) = \frac{-2}{x^3} \longrightarrow f'(2) = \frac{-2}{8} = \boxed{\frac{-1}{4}}$

$f''(x) = \frac{6}{x^4} \longrightarrow f''(2) = \frac{6}{16} = \boxed{\frac{3}{8}}$

$f'''(x) = \frac{-24}{x^5} \longrightarrow f'''(2) = \frac{-24}{32} = \boxed{\frac{-3}{4}}$

$f^{(4)}(x) = \frac{120}{x6} \longrightarrow f^{(4)}(2) = \frac{-120}{64} = \boxed{\frac{15}{8}}$

the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} - - - -$$

$$= f(2) + f'(2)(x-2) + \frac{f''(2)(x-2)^2}{2!} + \frac{f'''(2)(x-2)^3}{3!} + \frac{f^{(4)}(2)(x-2)^4}{4!}$$

$$= \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{8}\frac{(x-2)^2}{2!} - \frac{3}{4}\frac{(x-2)^3}{3!} + \frac{15}{8}\frac{(x-2)^4}{4!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{n!} \quad \frac{(n+1)!}{4 \cdot 2^k}$$

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