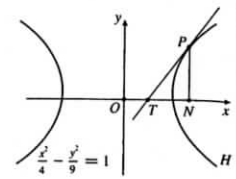
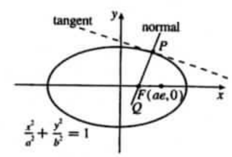
4. The perpendicular from a point P on the hyperbola H with parametric equations $x = 2\sec t$, $y = 3\tan t$, to the x-axis meets the x-axis at the point N. The tangent at P to H meets the x-axis at the point T.
   (a) Write down the coordinates of N.
   (b) Find the coordinates of T.
   (c) Prove that $ON \cdot OT = 4$, where O is the origin.



5. Let P be a point on the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b > 0$, $b^2 = a^2(1 - e^2)$, and $0 < e < 1$.
   (a) If P has coordinates $(a\cos t, b\sin t)$, determine the equation of the normal at P to the ellipse.
   (b) Determine the coordinates of the point Q where the normal in part (a) meets the axis $y = 0$.
   (c) Let F be the focus with coordinates $(ae, 0)$. Prove that $QF = e \cdot PF$.



6. Let $\mathscr{F}$ denote the family of parabolas $\{(x, y): y^2 = 4a(x + a)\}$ as a takes all positive values, and $\mathscr{G}$ denote the family of parabolas $\{(x, y): y^2 = 4a(-x + a)\}$ as a takes all positive values. Use the reflection property of the parabola to prove that, if $F \in \mathscr{F}$ and $G \in \mathscr{G}$, then, at each point of intersection, F and G cross at right angles.

Section 1.3

1. Classify the conics in $\mathbb{R}^2$ with the following equations. Determine the centre/vertex and axis of each.
   (a) $x^2 - 3xy + y^2 + 4x - 5y + 2 = 0$
   (b) $x^2 + 3xy + 4y^2 - 7 = 0$
   (c) $x^2 + xy + 4y^2 + 3x - 9 = 0$
   (d) $x^2 + 2xy + y^2 - 7x + 3 = 0$
   (e) $2x^2 - xy - 2y^2 - 2 = 0$

Section 1.4

1. Classify the quadrics in $\mathbb{R}^3$ with the following equations. Determine the centre of each.
   (a) $2x^2 + 5y^2 - z^2 + xy - 3yz - 2xz - 2x - 6y + 10z - 12 = 0$
   (b) $xy - y + yz = xz$
   (c) $4x^2 + y^2 + 4z^2 - 4xy + 8xz - 4yz - 12x - 12y + 6z = 0$
   (d) $-3x^2 + 7y^2 + 72x + 126y + z + 95 = 0$
2. Determine the equations of the generators of the hyperboloid of one sheet E with equation $2x^2 - 3y^2 + 4z^2 = 3$ through the point $(1, 1, 1)$.

2 Affine Geometry

In Chapter 1 we studied conics in Euclidean geometry. In the rest of the book we prove a whole range of results about figures such as lines and conics, in geometries other than Euclidean geometry. In the process of doing this, we meet two particular features of our approach to geometry which may be new to you.

The first feature is the use of transformations in geometry to simplify problems and bring out their essential character. You may have met some of these transformations previously in courses on Group Theory or on Linear Algebra.

For example, we shall often use matrices to simplify our work.

The second feature arises from the fact that the transformations we introduce form groups. Generally, we restrict our attention to geometry in the plane, $\mathbb{R}^2$, but even in this familiar setting there may be more than one group of transformations at our disposal. This leads to the exciting new idea that there are many different geometries!

Each geometry consists of a space, some properties possessed by figures in that space, and a group of transformations of the space that preserve these properties. For example, Euclidean plane geometry uses the space $\mathbb{R}^2$, and is concerned with those properties of figures that depend on the notion of distance. The group associated with Euclidean geometry is the group of isometries of the plane.

This is because isometries of the plane preserve distances.

This idea, that geometry can be thought of in terms of a space and a group acting on it, is called the *Kleinian view of geometry*, after the 19th-century German mathematician Felix Klein who proposed it first. It has the virtue of enabling us to generate many geometries, while seeing how they are related.

For instance, we can take $\mathbb{R}^2$ as our space and use the group of all transformations of the form $t(\mathbf{x}) = \mathbf{Ax} + \mathbf{a}$, where $\mathbf{a} \in \mathbb{R}^2$ and $\mathbf{A}$ is a 2×2 invertible matrix. These are the so-called *affine transformations* of $\mathbb{R}^2$. But what properties of figures in $\mathbb{R}^2$ are preserved by such transformations, and what is the corresponding geometry? The geometry is called *affine geometry*, and it is the subject of this chapter. As you will see, it has some features in common with Euclidean geometry, but also some very different features.

In Section 2.1 we examine Euclidean geometry from the Kleinian point of view, and explain why geometries other than Euclidean geometry exist.

In Sections 2.2 and 2.3 we introduce affine geometry and consider its properties. In particular, we show that affine transformations map straight lines to straight lines, map parallel lines to parallel lines, and preserve ratios of lengths along a given line. We also discover that in affine geometry all triangles are congruent, in the sense that any triangle can be mapped onto any other triangle by an affine transformation. This result is known as the *Fundamental Theorem of Affine Geometry*.

In Section 2.4 we establish two important theorems, due to Ceva and Menelaus, which involve ratios of lengths along the sides of a triangle.

Finally, in Section 2.5 we investigate the effect of affine transformations on conics, and discover that we can use the methods of affine geometry to obtain very simple proofs of certain types of theorems about conics.

2.1 Geometry and Transformations

Before embarking on a study of various other geometries, it is useful first to look back at our familiar Euclidean geometry.

2.1.1 What is Euclidean Geometry?

To help us answer this question, we begin by considering the following well-known result.

Example 1 Let ΔABC be a triangle in which $\angle ABC = \angle ACB$. Prove that $AB = AC$.

Solution First, reflect the triangle in the perpendicular bisector of BC, so that the points B and C change places and the point A moves to some point A', say. Since reflection preserves angles, it follows that $\angle A'BC = \angle ACB$.

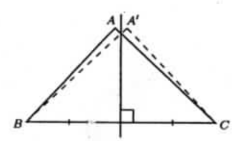
Also, we are given that $\angle ACB = \angle ABC$, so

$$\angle A'BC = \angle ABC.$$

But this can happen only if A' lies on the line through A and B. Similarly,

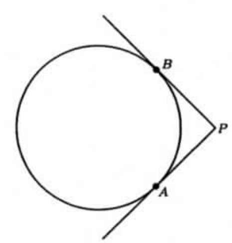
$$\angle A'CB = \angle ABC = \angle ACB,$$

so A' must also lie on the line through A and C. This means that A' and A must coincide. Hence the line segment AB reflects to the line segment AC, and vice versa. Since reflection preserves lengths, it follows that $AB = AC$. □



Problem 1 Let A and B be two points on a circle, and let the tangents to the circle at A and B meet at P. Prove that $AP = BP$.

Hint: Consider a reflection in the line which passes through P and the centre of the circle.



The result in Example 1 is concerned with the properties of length and angle associated with the triangle ABC. To investigate these properties, we introduced a reflection that enabled us to compare various lengths and angles. We were able to do this because reflections leave lengths and angles unchanged.

Of course, reflections are not the only transformations that preserve lengths and angles: other examples include rotations and translations. In general, any transformation that preserves lengths and angles can be used to tackle problems which involve these properties. In fact, we need worry only about leaving distances unchanged, since any transformation from $\mathbb{R}^2$ onto $\mathbb{R}^2$ that changes angles must also change lengths. Transformations that leave distances unchanged are called *isometries*.

This is because, once we know the lengths of the sides of a triangle, the angles are uniquely determined.

Definition An **isometry** of $\mathbb{R}^2$ is a function which maps $\mathbb{R}^2$ onto $\mathbb{R}^2$ and preserves distances.

In fact, every isometry has one of the following forms:

a translation along a line in $\mathbb{R}^2$;
a reflection in a line in $\mathbb{R}^2$;
a rotation about a point in $\mathbb{R}^2$;
a glide reflection in $\mathbb{R}^2$.

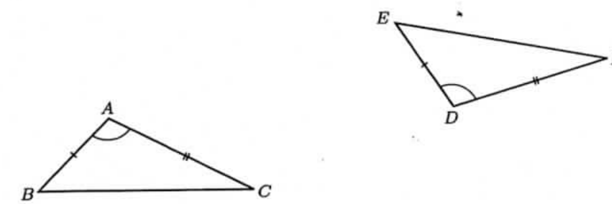
The identity isometry can be regarded as a rotation through an angle that is a multiple of 2π.

Also, the set $S(\mathbb{R}^2)$ of isometries forms a group under composition of functions, so the composite of two isometries is also an isometry. These observations can be used to build up the transformations we need in order to prove Euclidean results.

Example 2 Prove that if ΔABC and ΔDEF are two triangles such that

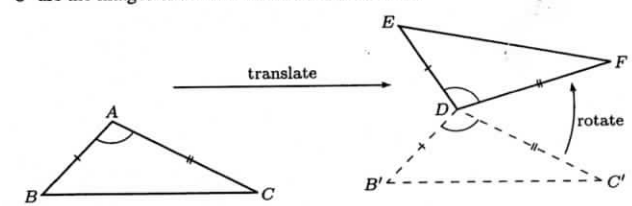
$$AB = DE, \quad AC = DF \quad \text{and} \quad \angle BAC = \angle EDF,$$

then $BC = EF$, $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$.



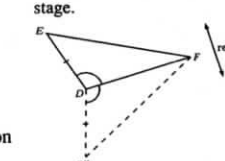
Solution It is sufficient to show that there is an isometry which maps ΔABC onto ΔDEF. We construct this isometry in stages, starting with the translation which maps A to D. This translation maps ΔABC onto $\Delta DB'C'$, where B' and C' are the images of B and C under the translation.

Of course, A and D may already coincide, in which case we omit the translation stage.



Since we are given that $DF = AC$, and since the translation maps AC onto DC', it follows that $DF = DC'$. We can therefore rotate the point C', about D, until it coincides with the point F. This rotation maps $\Delta DB'C'$ onto $\Delta DB''F$, as shown in the margin, where B'' is the image of B' under the rotation.

If C' already coincides with F, then we omit the rotation stage.



Finally, notice that

$$\begin{aligned}\angle FDE &= \angle CAB \quad \text{(given)}\\ &= \angle C'DB' \quad \text{(translation)}\\ &= \angle FDB'' \quad \text{(rotation)},\end{aligned}$$

so B'' (or its reflection in the line FD) must lie on DE. In fact, B'' (or its reflection in the line FD) must actually coincide with E, because

$$\begin{aligned}DE &= AB \quad \text{(given)}\\ &= DB' \quad \text{(translation)}\\ &= DB'' \quad \text{(rotation)}.\end{aligned}$$

So, composing the translation, the rotation, and (if necessary) the reflection, we obtain the required isometry that maps ΔABC onto ΔDEF. Since isometries preserve length and angle, it follows that $BC = EF$, $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$. □

Problem 2 Prove that if ΔABC and ΔDEF are two triangles such that

$$AC = DF, \quad \angle BAC = \angle EDF \quad \text{and} \quad \angle ACB = \angle DFE,$$

then $BC = EF$, $AB = DE$ and $\angle ABC = \angle DEF$.

We can now answer the question 'What is Euclidean geometry?'. **Euclidean geometry** is the study of those properties of figures that are unchanged by the group of isometries. We call these properties **Euclidean properties**. Roughly speaking, a Euclidean property is one that is preserved by a rigid figure as it moves around the plane. Of course, these properties include distance and angle, but they also include other properties such as collinearity of points and concurrence of lines.

This idea, that geometry can be thought of in terms of a group of transformations acting on a space, is known as the *Kleinian view of geometry*. It enables us to generate many geometries, without losing sight of the relationship between them.

When we consider geometries in this way, it is often convenient to have an algebraic representation for the transformations involved. This not only enables us to solve problems in the geometry algebraically, but also provides us with formulas that can be used to compare different geometries.

In the case of Euclidean geometry, perhaps the easiest way to represent isometries algebraically is to use matrices. For example, the function defined by

$$t : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \quad ((x, y) \in \mathbb{R}^2) \qquad (1)$$

is an isometry because it is the composite of an anticlockwise rotation through an angle θ about the origin, followed by a translation through the vector (e, f).

Similarly, the function

$$t : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \quad ((x, y) \in \mathbb{R}^2) \qquad (2)$$

is an isometry because it is the composite of a reflection in the line through the origin that makes an angle $\theta/2$ with the x-axis, followed by a translation through the vector (e, f).

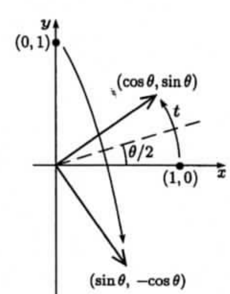
Remarkably, we can represent *any* isometry by one or other of the forms given in (1) and (2). To see this, notice that any isometry t can be written in the form

$$t(\mathbf{x}) = t_0(\mathbf{x}) + (e, f) \quad (\mathbf{x} \in \mathbb{R}^2), \qquad (3)$$

where t_0 is an isometry which fixes the origin. Indeed, if we let $(e, f) = t(\mathbf{0})$, then we can let t_0 be the transformation defined by $t_0(\mathbf{x}) = t(\mathbf{x}) - (e, f)$. This is an isometry because it is the composite of the isometry t and the translation through the vector $-(e, f)$. It fixes the origin since $t_0(\mathbf{0}) = t(\mathbf{0}) - (e, f) = \mathbf{0}$.

Now an isometry that fixes the origin must be either a rotation about the origin, or a reflection in a line through the origin. If t_0 is a rotation about the origin, then (3) can be written in the matrix form given in (1), whereas if t_0 is a reflection in a line through the origin, then (3) can be written in the matrix form given in (2).

Recall that the effect of the matrix multiplication in (1) and (2) can be interpreted geometrically by examining what happens to the vectors (1, 0) and (0, 1). For example, in (2) the matrix multiplication sends (1, 0) and (0, 1) to $(\cos\theta, \sin\theta)$ and $(\sin\theta, -\cos\theta)$ respectively, so it corresponds to the reflection shown in the figure.



So together, equations (1) and (2) provide us with an algebraic representation of all possible isometries of the plane. The next problem indicates how we can obtain a more concise description of this algebraic representation by using orthogonal matrices to combine equations (1) and (2).

Problem 3 Show that both the matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

which appear in (1) and (2), are orthogonal for each real number θ.

Recall that a matrix $\mathbf{U}$ is *orthogonal* if $\mathbf{U}^{-1} = \mathbf{U}^T$, that is, if $\mathbf{U}^T\mathbf{U} = \mathbf{I}$. This is equivalent to saying that the columns of $\mathbf{U}$ are orthonormal.

By applying the solution of Problem 3 to equations (1) and (2), we see that every isometry t has an algebraic representation of the form

$$t(\mathbf{x}) = \mathbf{U}\mathbf{x} + \mathbf{a},$$

where $\mathbf{U}$ is an orthogonal 2×2 matrix, and $\mathbf{a}$ is a vector in $\mathbb{R}^2$.

This equation shows the matrix $\mathbf{U}$ acting on the vector $\mathbf{x} = (x, y)$. Strictly speaking, $\mathbf{U}$ acts on the coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$ of $\mathbf{x}$ with respect to the standard basis, as in equations (1) and (2). However, since the numbers x and y are the same for the vector and its standard coordinates, no confusion should arise.

Definition A **Euclidean transformation** of $\mathbb{R}^2$ is a function $t : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$t(\mathbf{x}) = \mathbf{U}\mathbf{x} + \mathbf{a},$$

where $\mathbf{U}$ is an orthogonal 2×2 matrix and $\mathbf{a} \in \mathbb{R}^2$. The set of all Euclidean transformations of $\mathbb{R}^2$ is denoted by $E(2)$.

We may summarize the discussion above by saying that every isometry of the plane is a Euclidean transformation of $\mathbb{R}^2$.

In fact, the converse is also true, for if $\mathbf{U}$ is any orthogonal matrix, then its columns are orthonormal. In particular, its first column has unit length and can therefore be written in the form $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ for some real θ. To be orthonormal to this, the second column must be

$$\begin{pmatrix} \cos(\theta + \pi/2) \\ \sin(\theta + \pi/2) \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \quad \text{or}$$

$$\begin{pmatrix} \cos(\theta - \pi/2) \\ \sin(\theta - \pi/2) \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}.$$

So

$$\mathbf{U} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \mathbf{U} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

It follows that every Euclidean transformation $t(\mathbf{x}) = \mathbf{U}\mathbf{x} + \mathbf{a}$ of $\mathbb{R}^2$ has one of the forms given in equations (1) and (2). Since both of these forms represent isometries of the plane,we have the following theorem.

Theorem 1 Every isometry of $\mathbb{R}^2$ is a Euclidean transformation of $\mathbb{R}^2$, and vice versa.

Now the set of all isometries of $\mathbb{R}^2$ forms a group under composition of functions, so it follows from Theorem 1 that the same must be true of the set of all Euclidean transformations of $\mathbb{R}^2$. We therefore have the following theorem.

Theorem 2 The set of Euclidean transformations of $\mathbb{R}^2$ forms a group under the operation of composition of functions.

It is instructive to check the group axioms algebraically, for in the process of doing so we obtain formulas for the composites and inverses of Euclidean transformations.

We start by considering closure. Suppose that t_1 and t_2 are two Euclidean transformations given by

$$t_1(\mathbf{x}) = \mathbf{U}_1\mathbf{x} + \mathbf{a}_1 \quad \text{and} \quad t_2(\mathbf{x}) = \mathbf{U}_2\mathbf{x} + \mathbf{a}_2,$$

where $\mathbf{U}_1$ and $\mathbf{U}_2$ are orthogonal 2×2 matrices. Then the composite $t_1 \circ t_2$ is given by

$$\begin{aligned} t_1 \circ t_2(\mathbf{x}) &= t_1(\mathbf{U}_2\mathbf{x} + \mathbf{a}_2) \\ &= \mathbf{U}_1(\mathbf{U}_2\mathbf{x} + \mathbf{a}_2) + \mathbf{a}_1 \\ &= \mathbf{U}_1\mathbf{U}_2\mathbf{x} + (\mathbf{U}_1\mathbf{a}_2 + \mathbf{a}_1). \end{aligned}$$

This is a Euclidean transformation since $\mathbf{U}_1\mathbf{U}_2$ is orthogonal. Indeed,

$$(\mathbf{U}_1\mathbf{U}_2)^T = \mathbf{U}_2^T\mathbf{U}_1^T = \mathbf{U}_2^{-1}\mathbf{U}_1^{-1} = (\mathbf{U}_1\mathbf{U}_2)^{-1}.$$

Here we are using the result that $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

So the set of Euclidean transformations is closed under composition of functions.

Problem 4 Let the Euclidean transformations t_1 and t_2 of $\mathbb{R}^2$ be given by

$$t_1(\mathbf{x}) = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

and

$$t_2(\mathbf{x}) = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}\mathbf{x} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Determine $t_1 \circ t_2$ and $t_2 \circ t_1$.

Next recall that under composition of functions the identity is the transformation given by $i(\mathbf{x}) = \mathbf{x}$. This is a Euclidean transformation since it can be written in the form

$$i(\mathbf{x}) = \mathbf{Ix} + \mathbf{0},$$

where $\mathbf{I}$ is the 2×2 identity matrix, which is orthogonal.

The next problem asks you to show that inverses exist.

Problem 5 Prove that if t_1 is a Euclidean transformation of $\mathbb{R}^2$ given by

$$t_1(\mathbf{x}) = \mathbf{Ux} + \mathbf{a} \qquad (\mathbf{x} \in \mathbb{R}^2),$$

then:

(a) the transformation of $\mathbb{R}^2$ given by

$$t_2(\mathbf{x}) = \mathbf{U}^{-1}\mathbf{x} - \mathbf{U}^{-1}\mathbf{a} \qquad (\mathbf{x} \in \mathbb{R}^2)$$

is also a Euclidean transformation;

(b) the transformation t_2 is the inverse of t_1.

The solution of Problem (5) shows that we can calculate the inverse of a Euclidean transformation by using the following result.

> The inverse of the Euclidean transformation $t(\mathbf{x}) = \mathbf{Ux} + \mathbf{a}$ is given by
> $$t^{-1}(\mathbf{x}) = \mathbf{U}^{-1}\mathbf{x} - \mathbf{U}^{-1}\mathbf{a}.$$

Problem 6 Determine the inverse of the Euclidean transformation given by

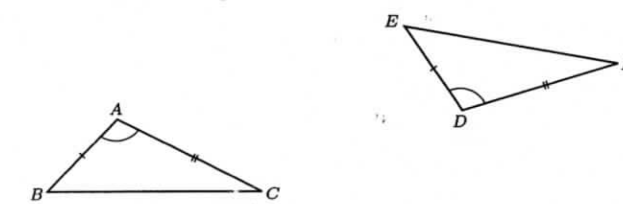
$$t(\mathbf{x}) = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Finally, composition of functions is always associative. So all four group properties hold, as we expected.

Earlier, we described Euclidean geometry as the study of those properties of figures that are preserved by isometries. Having identified these isometries with the group of Euclidean transformations, we can now give the equivalent algebraic description of Euclidean geometry. **Euclidean geometry** is the study of those properties of figures that are preserved by Euclidean transformations of $\mathbb{R}^2$.

2.1.2 Euclidean-Congruence

In the solution to Example 2 we showed that if two triangles ΔABC and ΔDEF are such that $AB = DE$, $AC = DF$ and $\angle BAC = \angle EDF$, then there is a Euclidean transformation which maps ΔABC onto ΔDEF.



The existence of this transformation enabled us to deduce that both triangles have the same Euclidean properties. In particular, we were able to deduce that $BC = EF$, $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$.

In order to formalize this way of relating two figures, we say that two figures are *congruent* if one can be moved to fill exactly the position of the other by means of a Euclidean transformation. Loosely speaking, two figures are congruent if they have the same size and shape.

Later we consider congruence with respect to other groups of transformations (that is, congruence in other geometries), so if there is any danger of confusion we sometimes say that two figures are *Euclidean-congruent*.

> **Definition** A figure F_1 is **Euclidean-congruent** to a figure F_2 if there is a Euclidean transformation which maps F_1 onto F_2.

For example, any two circles of *unit* radius are Euclidean-congruent to each other because we can map one of the circles onto the other by means of a translation that makes their centres coincide.

Of course, there are many other Euclidean transformations which map one of the circles onto the other.

Problem 7 Which of the following sets consist of figures that are Euclidean-congruent to each other?

(a) The set of all ellipses

(b) The set of all line segments of length 1

(c) The set of all triangles

(d) The set of all squares that have sides of length 2

Earlier, we emphasized that the Euclidean transformations form a group. This is important because it ensures that Euclidean-congruence has the kind of properties that we should expect. For example, we should expect every figure to be congruent to itself. Also, if a figure F_1 is congruent to a figure F_2, then we should expect F_2 to be congruent to F_1. We can, in fact, establish the following result.

Theorem 3 Euclidean-congruence is an equivalence relation.

Proof We show that the three equivalence relation axioms E1, E2 and E3 hold.

E1 REFLEXIVE For all figures F in $\mathbb{R}^2$, the identity transformation maps F onto itself; so Euclidean-congruence is reflexive.

This uses the existence of an *identity* transformation.

E2 SYMMETRIC Let a figure F_1 in $\mathbb{R}^2$ be congruent to a figure F_2, and let t be a Euclidean transformation which maps F_1 onto F_2. Then the inverse Euclidean transformation t^{-1} maps F_2 onto F_1, so that F_2 is congruent to F_1. Thus Euclidean-congruence is symmetric.

This uses the existence of *inverse* transformations.

E3 TRANSITIVE Let a figure F_1 in $\mathbb{R}^2$ be congruent to a figure F_2, and let F_2 be congruent to a figure F_3. Then there exist Euclidean transformations t_1 mapping F_1 onto F_2 and t_2 mapping F_2 onto F_3. Thus the Euclidean transformation $t_2 \circ t_1$ maps F_1 onto F_3, so that F_1 is congruent to F_3. Hence Euclidean-congruence is transitive.

This uses the closure axiom.

It follows that Euclidean-congruence is an equivalence relation, because it satisfies the axioms E1, E2 and E3. ■

Problem 8 Prove that if two figures in $\mathbb{R}^2$ are each Euclidean-congruent to a third figure, then they are Euclidean-congruent to each other.

Since Euclidean-congruence is an equivalence relation, it partitions the set of all figures into disjoint equivalence classes. Each class consists of figures which are Euclidean-congruent to each other, and hence share the same Euclidean properties (for example, one class consists of all circles of unit radius, another class consists of all equilateral triangles with sides of length 3, and so on). If we wish to show that two figures have the same Euclidean properties, then it is sufficient to show that they are Euclidean-congruent.

For example, to show that two triangles $\triangle ABC$ and $\triangle DEF$ have the same Euclidean properties, it is sufficient to show that $AB = DE$, $AC = DF$ and $\angle BAC = \angle EDF$, as mentioned above. This congruence condition is frequently used in Euclidean geometry. It is known as the 'side angle side' (SAS) condition for congruence.

Now Euclidean geometry is just one of several different geometries. Each geometry is defined by a group G of transformations that act on a space. In general, we say that two figures are ***G*-congruent** if there is a transformation in G which maps one of the figures onto the other. Since the only properties used in the proof of Theorem 3 are the group properties of Euclidean transformations, the theorem holds also with 'G-congruent' in place of 'Euclidean-congruent'. Thus, like Euclidean-congruence, G-congruence is an equivalence relation that partitions the set of all figures into disjoint equivalence classes.

This idea of partitioning figures into equivalence classes is central to geometry. It enables us to distinguish between figures in different equivalence classes, without having to worry about the differences between figures in the same equivalence class. For example, if we are interested in whether a conic is an ellipse rather than a hyperbola or a parabola, but do not care about its shape (that is, the ratio of the lengths of the axes), we might choose to work with some geometry whose group of transformations makes all ellipses congruent to each other – but not to any hyperbola or parabola. We describe a group of transformations which defines such a geometry in Section 2.2.

2.2 Affine Transformations and Parallel Projections

2.2.1 Affine Transformations

In Section 2.1 you met a new approach to Euclidean geometry in $\mathbb{R}^2$ – namely, the idea that Euclidean geometry of $\mathbb{R}^2$ can be interpreted as a set, $\mathbb{R}^2$, together with the group of Euclidean transformations which act on that set. Recall that a Euclidean transformation is a function $t : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$t(\mathbf{x}) = \mathbf{Ux} + \mathbf{a} \quad (\mathbf{x} \in \mathbb{R}^2),$$

where $\mathbf{U}$ is an orthogonal 2×2 matrix. Euclidean properties of figures are those, like distance and angle, that are preserved by these transformations.

In this section we meet the first of our new geometries in $\mathbb{R}^2$ – *affine geometry*. This geometry consists of the set $\mathbb{R}^2$ together with a group of transformations, the *affine transformations*, acting on $\mathbb{R}^2$.

Affine geometry can be defined in $\mathbb{R}^n$, for any $n \geq 2$; we restrict our attention here to the case when $n = 2$.

Definition An **affine transformation** of $\mathbb{R}^2$ is a function $t : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$t(\mathbf{x}) = \mathbf{Ax} + \mathbf{b},$$

where $\mathbf{A}$ is an invertible 2×2 matrix and $\mathbf{b} \in \mathbb{R}^2$. The set of all affine transformations of $\mathbb{R}^2$ is denoted by $A(2)$.

Remark

Note that every Euclidean transformation of $\mathbb{R}^2$ is an affine transformation of $\mathbb{R}^2$ since every orthogonal matrix is invertible. This means that all properties of figures that are preserved by affine transformations must be preserved also by Euclidean transformations.

Problem 1 Determine whether or not each of the following transformations of $\mathbb{R}^2$ is an affine transformation.

(a) $t_1(\mathbf{x}) = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ (b) $t_2(\mathbf{x}) = \begin{pmatrix} -6 & 5 \\ 3 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(c) $t_3(\mathbf{x}) = \begin{pmatrix} -2 & -1 \\ 8 & 4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ (d) $t_4(\mathbf{x}) = \begin{pmatrix} 5 & -3 \\ -2 & 2 \end{pmatrix}\mathbf{x}$

The algebra required to compose affine transformations is similar to the algebra that we used to compose Euclidean transformations.

Problem 2 For the transformations of $\mathbb{R}^2$ given in Problem 1, determine formulas for the following composites. In each case, state whether or not the composite is an affine transformation.

(a) $t_1 \circ t_2$ (b) $t_2 \circ t_4$

We now verify our assertion above that the set of affine transformations forms a group.

Theorem 1 The set of affine transformations $A(2)$ forms a group under the operation of composition of functions.

Proof We check that the four group axioms hold.

G1 CLOSURE Let t_1 and t_2 be affine transformations given by

$$t_1(\mathbf{x}) = \mathbf{A}_1\mathbf{x} + \mathbf{b}_1 \quad \text{and} \quad t_2(\mathbf{x}) = \mathbf{A}_2\mathbf{x} + \mathbf{b}_2,$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are invertible 2×2 matrices. Then, for each $\mathbf{x} \in \mathbb{R}^2$,

$$\begin{aligned}(t_1 \circ t_2)(\mathbf{x}) &= t_1(\mathbf{A}_2\mathbf{x} + \mathbf{b}_2)\\ &= \mathbf{A}_1(\mathbf{A}_2\mathbf{x} + \mathbf{b}_2) + \mathbf{b}_1\\ &= (\mathbf{A}_1\mathbf{A}_2)\mathbf{x} + (\mathbf{A}_1\mathbf{b}_2 + \mathbf{b}_1).\end{aligned}$$

Since $\mathbf{A}_1$ and $\mathbf{A}_2$ are invertible, it follows that $\mathbf{A}_1\mathbf{A}_2$ is also invertible. So by definition $t_1 \circ t_2$ is an affine transformation.

G2 IDENTITY Let i be the affine transformation given by

$$i(\mathbf{x}) = \mathbf{I}\mathbf{x} + \mathbf{0} \quad (\mathbf{x} \in \mathbb{R}^2),$$

where $\mathbf{I}$ is the 2×2 identity matrix. If t is an affine transformation given by

$$t(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b} \quad (\mathbf{x} \in \mathbb{R}^2),$$

then, for each $\mathbf{x} \in \mathbb{R}^2$,

$$(t \circ i)(\mathbf{x}) = \mathbf{A}(\mathbf{I}\mathbf{x} + \mathbf{0}) + \mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{b} = t(\mathbf{x})$$

and

$$(i \circ t)(\mathbf{x}) = \mathbf{I}(\mathbf{A}\mathbf{x} + \mathbf{b}) + \mathbf{0} = \mathbf{A}\mathbf{x} + \mathbf{b} = t(\mathbf{x}).$$

Thus $t \circ i = i \circ t = t$. Hence i is the identity transformation.

G3 INVERSES If t is an arbitrary affine transformation given by

$$t(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b} \quad (\mathbf{x} \in \mathbb{R}^2),$$

then we can define another affine transformation t' by

$$t'(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{x} - \mathbf{A}^{-1}\mathbf{b}.$$

Now for each $\mathbf{x} \in \mathbb{R}^2$, we have

$$\begin{aligned}(t \circ t')(\mathbf{x}) &= t\left(\mathbf{A}^{-1}\mathbf{x} - \mathbf{A}^{-1}\mathbf{b}\right)\\ &= \mathbf{A}\left(\mathbf{A}^{-1}\mathbf{x} - \mathbf{A}^{-1}\mathbf{b}\right) + \mathbf{b}\\ &= \left(\mathbf{A}\mathbf{A}^{-1}\mathbf{x} - \mathbf{A}\mathbf{A}^{-1}\mathbf{b}\right) + \mathbf{b}\\ &= (\mathbf{x} - \mathbf{b}) + \mathbf{b}\\ &= \mathbf{x}.\end{aligned}$$

Also,

$$\begin{aligned}(t' \circ t)(\mathbf{x}) &= t'(\mathbf{A}\mathbf{x} + \mathbf{b})\\ &= \mathbf{A}^{-1}(\mathbf{A}\mathbf{x} + \mathbf{b}) - \mathbf{A}^{-1}\mathbf{b}\\ &= \left(\mathbf{A}^{-1}\mathbf{A}\mathbf{x} + \mathbf{A}^{-1}\mathbf{b}\right) - \mathbf{A}^{-1}\mathbf{b}\\ &= \left(\mathbf{x} + \mathbf{A}^{-1}\mathbf{b}\right) - \mathbf{A}^{-1}\mathbf{b}\\ &= \mathbf{x}.\end{aligned}$$

Thus $t \circ t' = t' \circ t = i$. Hence t' is an inverse for t.

G4 ASSOCIATIVITY Composition of functions is always associative.

It follows that the set of affine transformations $A(2)$ forms a group under composition of functions. ■

The above proof shows that we can calculate the inverse of an affine transformation by using the following result.

> The inverse of the affine transformation $t(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ is given by
> $$t^{-1}(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{x} - \mathbf{A}^{-1}\mathbf{b}.$$

Problem 3 Find the inverse of the affine transformation

$$t(\mathbf{x}) = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

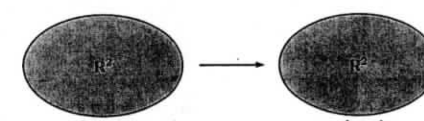
Having shown that the set of affine transformations forms a group under composition of functions, we now define **affine geometry** to be the study of those properties of figures in the plane $\mathbb{R}^2$ that are preserved by affine transformations. These are the so-called **affine properties** of figures. We begin our investigation of affine geometry by considering the three affine properties listed below.

> **Basic Properties of Affine Transformations**
>
> Affine transformations:
>
> 1. map straight lines to straight lines;
> 2. map parallel straight lines to parallel straight lines;
> 3. preserve ratios of lengths along a given straight line.

There are two approaches that we shall use to investigate these properties. One approach is to use the definition of an affine transformation to investigate the properties algebraically; we do this in Section 2.3. First, however, we investigate the properties geometrically. We begin to do this in the next Subsection by introducing a special type of affine transformation for which there is a simple geometric interpretation.

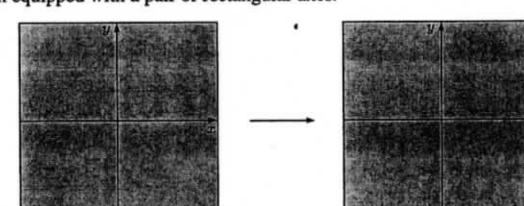
2.2.2 Parallel Projections

A *parallel projection* is a one–one mapping from $\mathbb{R}^2$ onto itself, defined in the following way. First, we think of its domain and codomain as two separate copies of $\mathbb{R}^2$.



domain codomain

Geometrically, we can represent these copies of $\mathbb{R}^2$ by two separate planes, each equipped with a pair of rectangular axes.

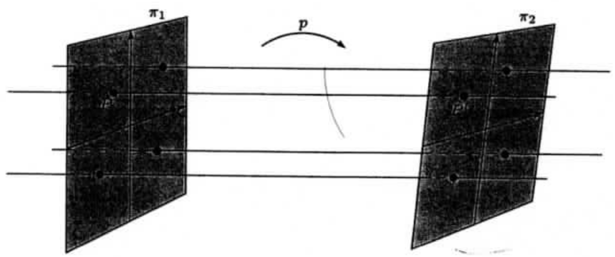


domain copy of $\mathbb{R}^2$ codomain copy of $\mathbb{R}^2$

Next we place these planes into three-dimensional space; we denote the domain plane by π_1 and the codomain plane by π_2.

Now imagine parallel rays of light shining through π_1 and π_2. Each point P in the plane π_1 has a (unique) ray passing through it, that also passes through a point P', say, in the plane π_2. This provides us with a one–one correspondence between points in the two planes π_1 and π_2. We call the function p which maps each point P in π_1 to the corresponding point P' in π_2 a **parallel projection from π_1 onto π_2**.

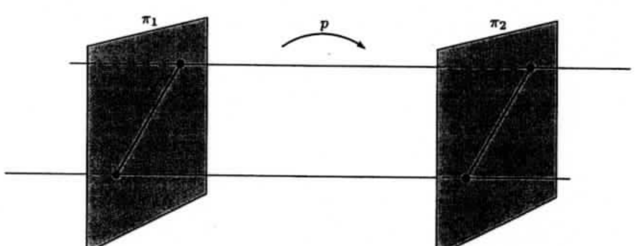
Of course, since π_1 and π_2 represent copies of $\mathbb{R}^2$, a parallel projection is really a function from $\mathbb{R}^2$ onto itself. In Subsection 2.2.3 we show that parallel projections are affine transformations.



If the roles of the planes π_1 and π_2 are reversed, so that π_2 becomes the domain plane and π_1 becomes the codomain plane, then we obtain the inverse function p^{-1} which maps points P' in π_2 back to the corresponding points P in π_1. Clearly, p^{-1} is a parallel projection of π_2 onto π_1.

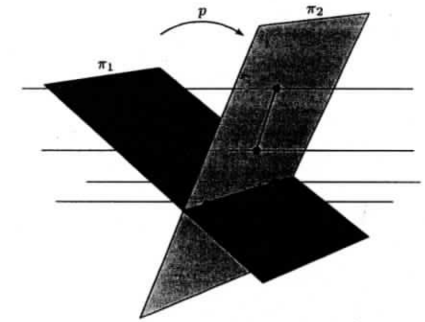
Each choice of location for the domain plane π_1, and the codomain plane π_2, and each choice of direction for the rays of light, yields a parallel projection. The only constraint is that the rays of light must not be parallel to either plane.

If the planes π_1 and π_2 are parallel to each other, then any parallel projection p from π_1 onto π_2 is an isometry, since the distance between any two points is unaltered.



You can envisage the mapping p from π_1 onto π_2 as 'sliding π_1 parallel to itself along the family of rays'.

On the other hand, if the planes are not parallel to each other, then some distances are changed under the projection, and so the parallel projection is not an isometry; notice, however, that distances along the line of intersection of the planes π_1 and π_2 do remain unchanged by the parallel projection.



Although distances are not always preserved by a parallel projection, there are some basic properties that are preserved; three of these are listed below. As you will see, these are the same as the basic affine properties that we mentioned at the end of Subsection 2.2.1.

Basic Properties of Parallel Projections

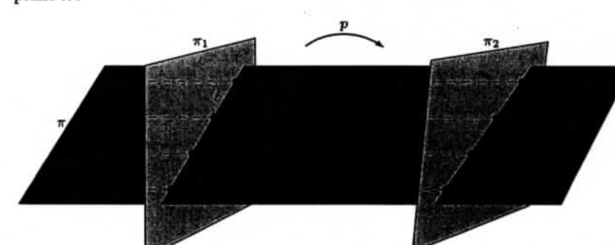
Parallel projections:

1. map straight lines to straight lines;
2. map parallel straight lines to parallel straight lines;
3. preserve ratios of lengths along a given straight line.

Later, we show that each basic affine property follows directly from the corresponding property for parallel projections. In anticipation of this, we first show that the properties hold for parallel projections.

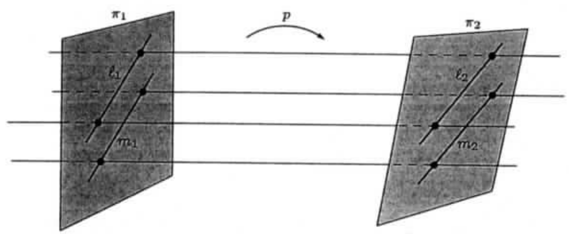
Property 1 A parallel projection maps straight lines to straight lines.

Proof Let ℓ be a line in the plane π_1, and let p be a parallel projection mapping π_1 onto the plane π_2. Now consider all the rays associated with p that pass through ℓ. Since these rays are parallel, they must fill a plane. Call this plane π.



The image of ℓ under p consists of those points where the rays that pass through ℓ meet π_2. But these points are simply the points of intersection of π with π_2. Since any two intersecting planes in $\mathbb{R}^3$ meet in a line, it follows that the image of ℓ under p is a straight line. ■

Property 2 A parallel projection maps parallel straight lines to parallel straight lines.



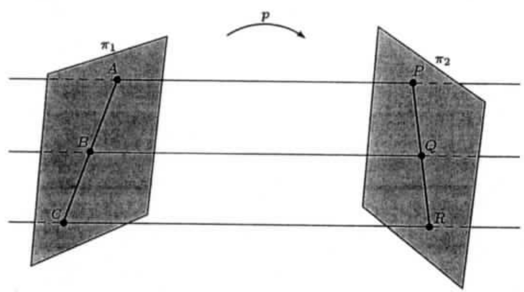
Proof Let ℓ_1 and m_1 be parallel lines in the plane π_1, and let p be a parallel projection mapping π_1 onto the plane π_2. Let ℓ_2 and m_2 be the lines in π_2 that are the images under p of ℓ_1 and m_1.

If ℓ_2 and m_2 are not parallel, they meet at some point, P_2 say. Let P_1 be the point of π_1 which maps to P_2. Then P_1 must lie on both ℓ_1 and m_1. Since ℓ_1 and m_1 are parallel, no such point of intersection can exist, which is a contradiction. It follows that ℓ_2 and m_2 must indeed be parallel. ■

P_1 is the point $p^{-1}(P_2)$.

Property 3 A parallel projection preserves ratios of lengths along a given straight line.

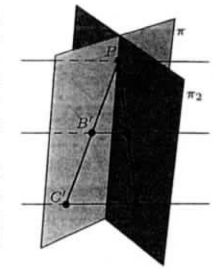
We shall give a slightly more general form of this property in Section 2.3.



Proof Let A, B, C be three points on a line in the plane π_1, and let p be a parallel projection mapping π_1 onto the plane π_2. Let P, Q, R be the points in π_2 that are the images under p of A, B, C. We know from Property 1 that P, Q, R lie on a line; we have to show that the ratio $AB:AC$ is equal to the ratio $PQ:PR$.

If the planes π_1 and π_2 are parallel, then the parallel projection p is an isometry, and so the ratios $AB:AC$ and $PQ:PR$ are equal, as required. On the other hand, if π_1 and π_2 are not parallel, then we can construct a plane π through the point P which is parallel to π_1, as shown in the margin. This plane intersects the ray through B and Q at some point B', and the ray through C and R at some point C'. So in this case the ratios $AB:AC$ and $PB':PC'$ are equal.

Now consider $\Delta PC'R$. The lines $B'Q$ and $C'R$ are parallel, since they are rays from the parallel projection. Hence $B'Q$ meets the sides PR and PC' in equal ratios. Thus $PQ:PR=PB':PC'$. It follows that $PQ:PR=AB:AC$, as required. ■



Notice, in particular, that if a point is the midpoint of a line segment, then under a parallel projection the image of the point is the midpoint of the image of the line segment.

We make use of this fact in Subsection 2.2.3.

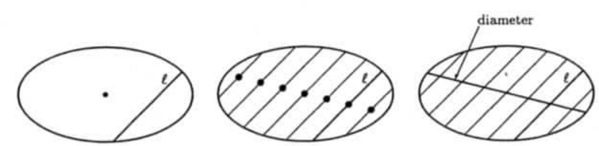
In Subsection 2.2.3 you will see why the basic properties of affine transformations and of parallel projections are the same, and you will meet some further properties of each.

2.2.3 Affine Geometry

In this subsection we explore further the ideas of affine geometry and of parallel projection in order to prove two attractive and unexpected results about ellipses. Also, we examine the relationship between affine transformations and parallel projections.

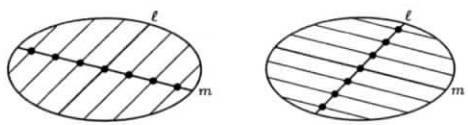
Two Results about Ellipses

First, starting with any chord ℓ of an ellipse, draw all the chords parallel to ℓ and construct their midpoints. We claim that these midpoints lie on a chord through the centre of the ellipse – that is, on a *diameter* of the ellipse.



Theorem 2 Midpoint Theorem
Let ℓ be a chord of an ellipse. Then the midpoints of the chords parallel to ℓ lie on a diameter of the ellipse.

Next, start with any diameter ℓ of an ellipse and construct a second diameter m by following the construction used in Theorem 2. Then repeat the construction starting this time with the diameter m; this might reasonably be expected to give us a third diameter of the ellipse – but, surprisingly, it gives us the diameter ℓ with which we started.



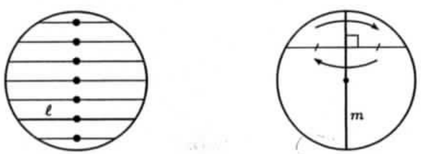
Theorem 3 Conjugate Diameters Theorem
Let ℓ be a diameter of an ellipse. Then there is another diameter m of the ellipse such that:
(a) the midpoints of all chords parallel to ℓ lie on m;
(b) the midpoints of all chords parallel to m lie on ℓ.

The directions of these two diameters are called *conjugate directions*, and the diameters are called *conjugate diameters*.

Proofs for the Special Case of a Circle

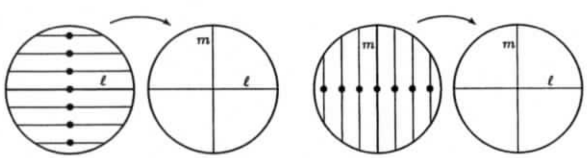
We now investigate these theorems for the special case when the ellipse is a circle. To prove the Midpoint Theorem in this case, start with a chord ℓ. If necessary, rotate the circle to ensure that ℓ is horizontal. It is then sufficient to prove that every horizontal chord is bisected by the vertical diameter, m.

Recall that a circle is an ellipse with eccentricity zero.



To do this note that the circle is symmetrical about m; so, reflection in m maps that part of every horizontal chord to the left of m exactly onto the part to the right of m. Since reflection preserves length, these two parts must be the same length; in other words, m bisects each horizontal chord, as required.

What about the Conjugate Diameters Theorem for the special case of the circle?



Start with the horizontal diameter ℓ, and carry out the construction of another diameter as in Theorem 2; this yields the vertical diameter m. If we then start with the vertical diameter m and repeat the construction, we obtain ℓ, the horizontal diameter of the circle. So Theorem 3 certainly holds when the ellipse is a circle.

Generalizing the Proof

We now investigate how the proofs of Theorems 2 and 3 for the circle can be turned into proofs for any kind of ellipse. The crucial fact is as follows.

Theorem 4 Given any ellipse, there is a parallel projection which maps the ellipse onto a circle.

An algebraic proof of a related theorem is given in Section 2.5.

A suitable parallel projection is illustrated below. Here the plane π_1 (initially parallel to π_2) has been tilted about the minor axis of the ellipse. Under the projection distances which are parallel to the minor axis remain unchanged, but distances parallel to the major axis are scaled by a factor which depends on the 'angle of tilt'. By choosing just the right amount of tilt we can ensure that the image of the major axis is equal in length to the image of the minor axis, thereby ensuring that the image of the ellipse is a circle.

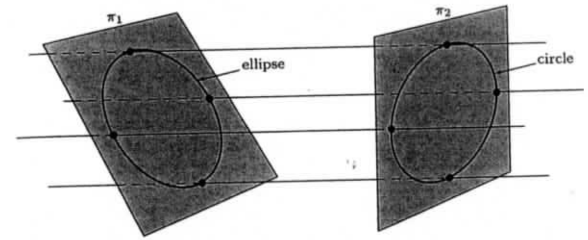
Algebraically, in terms of a suitable coordinate system, the mapping

$$x \mapsto \frac{b}{a}x, \quad y \mapsto y,$$

maps the ellipse

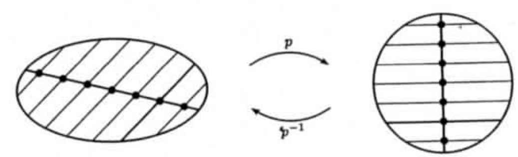
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad 0 < b < a,$$

to the circle $x^2 + y^2 = a^2$.



Both Theorems 2 and 3 may now be proved using the following technique. First, map the given ellipse onto a circle, using a suitable parallel projection p. Since we have seen that the theorems hold in the case of the circle, we then map the circle back to the ellipse, using the inverse parallel projection p^{-1}. Now collinearity and parallelism are preserved under a parallel projection, as is the property of being the midpoint of a line segment, so the above two theorems, which hold for a circle, must hold also for the ellipse.

Here we are using Properties 1, 2 and 3 of parallel projections, in a crucial way.



Notice that certain properties of figures, such as length and angle, are not preserved under a parallel projection. This is one difference between Euclidean geometry and affine geometry. The difference arises because the group of affine transformations is larger than the group of Euclidean transformations. In general, the larger the group that is used to define a geometry, the fewer properties the geometry has.

Affine Transformations and Parallel Projections

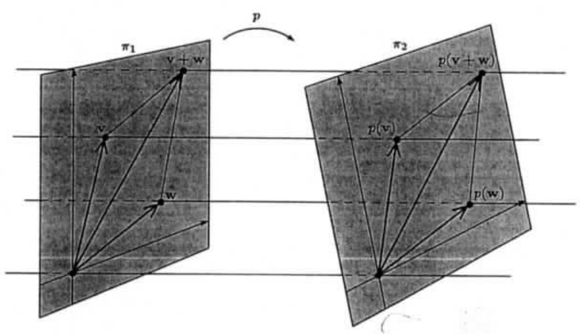
Earlier we mentioned that a parallel projection is a special type of affine transformation. We now show why this is indeed the case.

First, consider a parallel projection p of a plane π_1 onto a plane π_2. For the moment, suppose that the planes are aligned so that the origin in π_1 is mapped to the origin in π_2. Since ratios of lengths are preserved along a straight line, we must have, for any vector $\mathbf{v} \in \mathbb{R}^2$ and any $\lambda \in \mathbb{R}$,

Property 3, Subsection 2.2.2

$$p(\lambda \mathbf{v}) = \lambda p(\mathbf{v}). \qquad (1)$$

Next, let $\mathbf{v}$ and $\mathbf{w}$ be two position vectors in π_1. Their sum, $\mathbf{v}+\mathbf{w}$, is found from the Parallelogram Law for addition of vectors, as shown in the diagram below. The images under p in π_2 are $p(\mathbf{v})$ and $p(\mathbf{w})$, and the sum of these two vectors is $p(\mathbf{v}) + p(\mathbf{w})$.



But a parallel projection maps parallel lines onto parallel lines, so it must map parallelograms onto parallelograms. Hence it must map the parallelogram in π_1 onto the parallelogram in π_2, and, in particular, it must map $\mathbf{v}+\mathbf{w}$ to $p(\mathbf{v}) + p(\mathbf{w})$. We may write this as

$$p(\mathbf{v}+\mathbf{w}) = p(\mathbf{v}) + p(\mathbf{w}). \qquad (2)$$

It follows from equations (1) and (2) that p must be a *linear* transformation of $\mathbb{R}^2$ onto itself.

Remember that π_1 and π_2 represent copies of $\mathbb{R}^2$.

Hence there exists some matrix $\mathbf{A}$ such that for each $\mathbf{v} \in \mathbb{R}^2$,

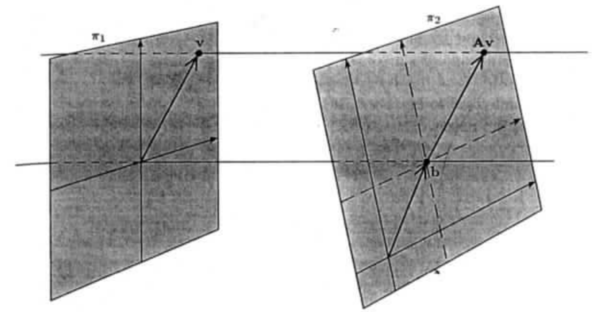
$$p(\mathbf{v}) = \mathbf{A}\mathbf{v}. \qquad (3)$$

Since the linear transformation p is invertible, it follows that $\mathbf{A}$ is invertible.

Now suppose that the parallel projection maps the origin in π_1 to some point B with position vector $\mathbf{b}$ in π_2, as shown below. If we temporarily construct a new set of axes in π_2 that are parallel to the original axes, but which intersect at the point B, then with respect to these new axes $p(\mathbf{v}) = \mathbf{A}\mathbf{v}$ for some invertible matrix $\mathbf{A}$, as before. To express $p(\mathbf{v})$ with respect to the original axes, we simply add on the vector $\mathbf{b}$ to obtain

$$p(\mathbf{v}) = \mathbf{A}\mathbf{v} + \mathbf{b} \qquad (4)$$

for some invertible 2×2 matrix $\mathbf{A}$.



It follows from equation (4) that p must be an affine transformation.

Theorem 5 Each parallel projection is an affine transformation.

The converse is false, for it is *not* true that every affine transformation can be represented as a parallel projection.

For example, consider the so-called 'doubling map' of $\mathbb{R}^2$ to itself given by

$$t(\mathbf{v}) = 2\mathbf{v} \qquad (\mathbf{v} \in \mathbb{R}^2). \qquad (5)$$

This is an affine transformation, since it can be written in the form $t(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ with $\mathbf{A} = 2\mathbf{I}$ and $\mathbf{b} = \mathbf{0}$. However, a parallel projection is *either* between two parallel planes, in which case all lengths are unchanged, *or* between two intersecting planes, in which case distances along the line of intersection are unchanged. The doubling map has neither of these properties and so is not a parallel projection.

Observation An affine transformation is not necessarily a parallel projection.

Although the doubling map is not a parallel projection, it *is* possible to double lengths in $\mathbb{R}^2$ by following one parallel projection by another: the first doubles all horizontal lengths, and the second doubles all vertical lengths. Thus the doubling map (5) can be represented as the *composition* of two parallel projections.

We end this subsection by showing that *every* affine transformation can be expressed as a composition of two parallel projections.

Recall that any affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$ has the form

$$t(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b} \qquad (\mathbf{x} \in \mathbb{R}^2), \qquad (6)$$

where $\mathbf{A}$ is an invertible 2×2 matrix. Now, t is not a linear transformation unless $\mathbf{b} = \mathbf{0}$, but we can use methods similar to those for linear transformations to determine $\mathbf{A}$ and $\mathbf{b}$.

First, it follows from equation (6) that $t(\mathbf{0}) = \mathbf{b}$; so $\mathbf{b}$ is the image of the origin under t. If we let e and f be the coordinates of $t(\mathbf{0})$, then we can write

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix},$$

where a, b, c, d are real numbers that have yet to be found. It follows from equation (6) that the images under t of the points $(1, 0)$ and $(0, 1)$ are given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}.$$

So if, in addition to $t(\mathbf{0}) = (e, f)$, we know the points onto which $(1, 0)$ and $(0, 1)$ are mapped by t, then we can determine the values of a, b, c and d. Indeed, we have

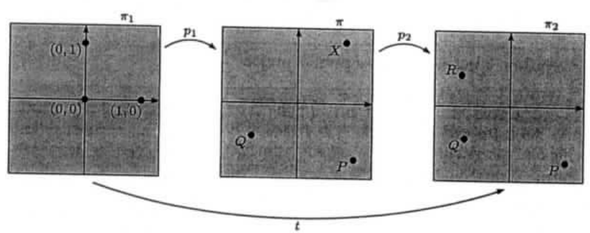
$$(a, c) = t(1, 0) - (e, f) \quad \text{and} \quad (b, d) = t(0, 1) - (e, f).$$

It follows that an affine transformation is *uniquely* determined by its effect on the three non-collinear points $(0, 0)$, $(1, 0)$ and $(0, 1)$. We shall return to this method of determining affine transformations in Section 2.3.

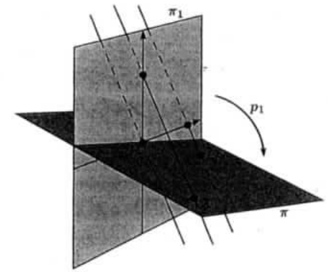
So suppose that a given affine transformation t maps the points $(0, 0)$, $(1, 0)$ and $(0, 1)$ to three non-collinear points P, Q and R, respectively. In order to express t as the composition of two parallel projections p_1 and p_2, we need to define p_1 and p_2 in such a way that $p_2 \circ p_1$ has the same effect as t on $(0, 0)$, $(1, 0)$ and $(0, 1)$. To do this, we first define p_1 so that it maps $(0, 0)$ to P, $(1, 0)$ to Q, and $(0, 1)$ to some point X, say, and then define p_2 so that it maps X to R while leaving P and Q fixed.

Notice that for an affine transformation t, the images $t(1, 0)$, $t(0, 1)$ and $t(0, 0) = (e, f)$ cannot be collinear, for if they were, then (a, c) and (b, d) would be linearly dependent, and $\mathbf{A}$ would not be invertible.

Uniqueness then guarantees that $t = p_2 \circ p_1$.

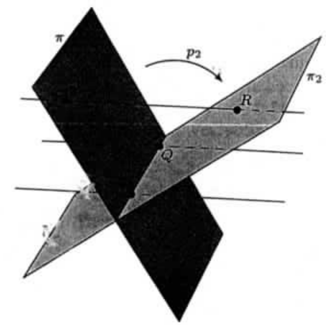


To construct p_1 we embed its domain plane π_1, and its codomain plane π, into $\mathbb{R}^3$ so that the point $(0, 0)$ in π_1 coincides with the point P in π, as shown below. It does not matter how this is done, provided that $(1, 0)$ does not lie in π. We then define p_1 by the family of rays that are parallel to the ray through the point $(1, 0)$ in π_1 and the point Q in π. When defined in this way, p_1 maps $(0, 0)$ to P, $(1, 0)$ to Q, and $(0, 1)$ to some point X, as required.



For clarity, we have omitted the axes from the plane π.

To construct p_2 we embed its domain plane π, and its codomain plane π_2, into $\mathbb{R}^3$ so that the points P and Q in π coincide with the points P and Q in π_2, as shown below. Again it does not matter how this is done, provided that X does not lie in π_2. We then define p_2 by the family of rays that are parallel to the ray through the point X in π and the point R in π_2. Then p_2 leaves P and Q fixed and maps X to R.



Overall, the composite $p_2 \circ p_1$ of the two parallel projections maps $(0, 0)$, $(1, 0)$ and $(0, 1)$ to P, Q and R, respectively. Now p_1 and p_2 are affine transformations, so $p_2 \circ p_1$ is also an affine transformation. Furthermore, $p_2 \circ p_1$ maps $(0, 0)$, $(1, 0)$ and $(0, 1)$ to the same points as does t. Since such affine transformations are unique, it follows that $t = p_2 \circ p_1$. We have therefore demonstrated the following result.

Theorem 6 An affine transformation can be expressed as the composite of two parallel projections.

An important consequence of this theorem is that all properties of figures that are unchanged by parallel projections must also be unchanged by affine transformations. In particular, the three properties of parallel projections that we met in Subsection 2.2.2 must, in fact, be affine properties.

2.3 Properties of Affine Transformations

In the previous section you saw how parallel projections can be used to explore affine geometry from a visual point of view. In this section we explore some of the same ideas from an algebraic point of view.

2.3.1 Images of Sets Under Affine Transformations

We begin by describing how to find the image of a line under an affine transformation. To do this, recall that an affine transformation is a mapping $t : \mathbb{R}^2 \to \mathbb{R}^2$ given by a formula of the form

$$t(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}, \qquad (1)$$

where $\mathbf{A}$ is an invertible 2×2 matrix. The set of such transformations forms a group, in which the transformation inverse to t is given by

$$t^{-1}(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{x} - \mathbf{A}^{-1}\mathbf{b}. \qquad (2)$$

Subsection 2.2.1

When equations (1) and (2) are used to find images under t, it is easy to confuse points in the domain plane with points in the codomain plane, as both planes are copies of $\mathbb{R}^2$. To avoid such confusion, we often reserve the symbol $\mathbf{x}$ and the coordinates (x, y) for points in the domain of t, and use the symbol $\mathbf{x}'$ and the coordinates (x', y') to denote the image of $\mathbf{x}$ under t.

With this notation, we may rewrite equations (1) and (2) in the form

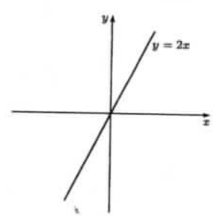
$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}, \qquad (3)$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}' - \mathbf{A}^{-1}\mathbf{b}. \qquad (4)$$

The next example illustrates how these equations can be used to find the image of a line under an affine transformation.

Example 1 Determine the image of the line $y = 2x$ under the affine transformation

$$t(\mathbf{x}) = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (\mathbf{x} \in \mathbb{R}^2). \qquad (5)$$



Solution Let (x, y) be an arbitrary point on the line $y = 2x$, and let (x', y') be the image of (x, y) under t. Then

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Next we use equation (4) to express (x, y) in terms of (x', y'). We have

$$\begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -4 \end{pmatrix},$$

so

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} -\frac{3}{2} \\ 4 \end{pmatrix}.$$

Recall that the inverse of the invertible matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

It follows that under the inverse mapping t^{-1} we have

$$x = \frac{1}{2}x' - \frac{1}{2}y' - \frac{3}{2} \quad \text{and} \quad y = -x' + 2y' + 4.$$

Since x and y are related by the equation $y = 2x$, it follows that x' and y' are related by the equation

$$-x' + 2y' + 4 = 2\left(\frac{1}{2}x' - \frac{1}{2}y' - \frac{3}{2}\right),$$

which simplifies to

$$2x' - 3y' = 7.$$

Dropping the dashes, we see that the image of the line $y = 2x$ under t is the line

$$2x - 3y = 7.$$ □

Problem 1 Determine the image of the line $y = 2x$ under the affine transformation

$$t(\mathbf{x}) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -\frac{3}{2} \\ 4 \end{pmatrix} \quad (\mathbf{x} \in \mathbb{R}^2).$$

Problem 2 Determine the image of the line $3x - y + 1 = 0$ under the affine transformation

$$t(\mathbf{x}) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -\frac{3}{2} \\ 4 \end{pmatrix} \quad (\mathbf{x} \in \mathbb{R}^2).$$

Although we have concentrated on how to find the image of a line under an affine transformation, exactly the same technique can be used to find the images of other types of figure, such as conics. You will meet some examples of this in Section 2.5.

2.3.2 The Fundamental Theorem of Affine Geometry

The algebraic approach can also be used to investigate whether there is an affine transformation which maps one given figure onto another. Recall that if there is such a transformation, then the two figures are said to be **affine-congruent**. This concept of congruence is important because, as we explained in Section 2.1, figures that are affine-congruent to each other share the same affine properties.

In this subsection we prove the remarkable result that *all* triangles are affine-congruent and therefore share the same affine properties. In fact, since a triangle is completely determined by its three vertices, the congruence of triangles follows from the so-called *Fundamental Theorem of Affine Geometry* which states that any three non-collinear points can be mapped to any other three non-collinear points by an affine transformation.

This is very different to Euclidean geometry, where two triangles are congruent only if they have the same shape and size.

First, recall that in Subsection 2.2.3 we described how the points (0, 0), (1, 0) and (0, 1) in $\mathbb{R}^2$ can be mapped to any three non-collinear points P, Q and R by an affine transformation. This transformation is unique in the sense that it is completely determined by the choice of P, Q and R. The following example should remind you of how such transformations are constructed.

There the mapping was constructed in a geometric manner. In this subsection we construct the mapping algebraically.

Example 2 Determine the affine transformation which maps the points (0, 0), (1, 0) and (0, 1) to the points (3, 2), (5, 8) and (7, 3), respectively.

Solution Let t be the affine transformation given by

$$t : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}. \qquad (6)$$

Since $t(0,0) = (3,2)$, it follows from (6) that $e = 3$ and $f = 2$.

Next, $t(1,0) = (5,8)$, so it follows from (6) that

$$\begin{pmatrix} 5 \\ 8 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The first column of the matrix is therefore

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

Finally, $t(0,1) = (7,3)$, so that

$$\begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The second column of the matrix is therefore

$$\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

Hence the desired affine transformation is given by

$$t : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 4 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$ □

In general, if we want to find an affine transformation t of the form

$$t : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \qquad (7)$$

which maps $(0,0)$ to **p**, $(1,0)$ to **q** and $(0,1)$ to **r**, then we must choose a, b, c, d, e and f so that

$$\mathbf{p} = t(0,0) = (e,f), \qquad \text{so } (e,f) = \mathbf{p};$$
$$\mathbf{q} = t(1,0) = (a,c) + (e,f), \quad \text{so } (a,c) = \mathbf{q} - \mathbf{p};$$
$$\mathbf{r} = t(0,1) = (b,d) + (e,f), \quad \text{so } (b,d) = \mathbf{r} - \mathbf{p}.$$

Notice that any three points **p**, **q** and **r** uniquely determine a transformation t of the form (7), but t is *affine* only if the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible. Since the columns of **A** correspond to the vectors $\mathbf{q} - \mathbf{p}$ and $\mathbf{r} - \mathbf{p}$, it follows that **A** is invertible only if the vectors $\mathbf{q} - \mathbf{p}$ and $\mathbf{r} - \mathbf{p}$ are linearly independent. That is, provided that **p**, **q** and **r** are *not* collinear.

So if **p**, **q** and **r** are *not* collinear, then we can use the following strategy to find an affine transformation which maps $(0,0)$ to **p**, $(1,0)$ to **q** and $(0,1)$ to **r**.

Strategy To determine the unique affine transformation $t(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ which maps $(0,0)$, $(1,0)$ and $(0,1)$ to the three non-collinear points **p**, **q** and **r**, respectively:

1. take $\mathbf{b} = \mathbf{p}$;
2. take **A** to be the matrix with columns given by $\mathbf{q} - \mathbf{p}$ and $\mathbf{r} - \mathbf{p}$.

Problem 3 Use the strategy to determine the affine transformation which maps the points $(0,0)$, $(1,0)$ and $(0,1)$ to the points $(2,3)$, $(1,6)$ and $(3,-1)$, respectively.

Problem 4 Use the strategy to determine the affine transformation which maps the points $(0,0)$, $(1,0)$ and $(0,1)$ to the points $(1,-2)$, $(2,1)$ and $(-3,5)$, respectively.

Notice that the inverse of the transformation in Problem 3 is an affine transformation which maps the points $(2,3)$, $(1,6)$ and $(3,-1)$ to the points $(0,0)$, $(1,0)$ and $(0,1)$, respectively. So if, after applying this inverse, we apply the affine transformation in Problem 4, then the overall effect is that of a composite affine transformation which sends the points $(2,3)$, $(1,6)$ and $(3,-1)$ to the points $(1,-2)$, $(2,1)$ and $(-3,5)$, respectively.

$(2,3) \mapsto (0,0) \mapsto (1,-2)$
$(1,6) \mapsto (1,0) \mapsto (2,1)$
$(3,-1) \mapsto (0,1) \mapsto (-3,5)$

In a similar way, we can find an affine transformation which sends any three non-collinear points to any other three non-collinear points.

Theorem 1 Fundamental Theorem of Affine Geometry

Let **p**, **q**, **r** and **p**′, **q**′, **r**′ be two sets of three non-collinear points in $\mathbb{R}^2$. Then:

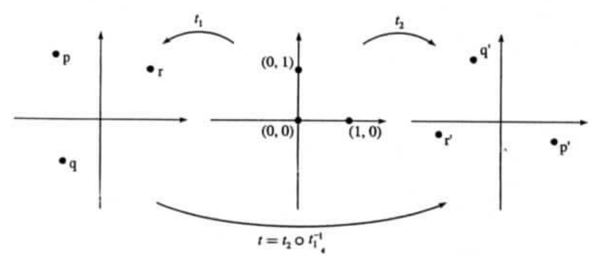
(a) there is an affine transformation t which maps **p**, **q** and **r** to **p**′, **q**′ and **r**′, respectively;

(b) the affine transformation t is unique.

Proof

(a) Let t_1 be the affine transformation which maps $(0,0)$, $(1,0)$ and $(0,1)$ to the points **p**, **q** and **r**, respectively, and let t_2 be the affine transformation which maps $(0,0)$, $(1,0)$ and $(0,1)$ to the points **p**′, **q**′ and **r**′, respectively. Then the composite $t = t_2 \circ t_1^{-1}$ is an affine transformation, and it maps **p**, **q** and **r** to **p**′, **q**′ and **r**′, respectively.

$\mathbf{p} \overset{t_1^{-1}}{\mapsto} (0,0) \overset{t_2}{\mapsto} \mathbf{p}'$
$\mathbf{q} \mapsto (1,0) \mapsto \mathbf{q}'$
$\mathbf{r} \mapsto (0,1) \mapsto \mathbf{r}'$

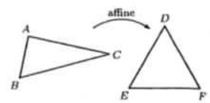


(b) Suppose that t and s are both affine transformations which map **p**, **q** and **r** to **p**′, **q**′ and **r**′, respectively, and let t_1 be the affine transformation defined in part (a). Then the composites $t \circ t_1$ and $s \circ t_1$ are both affine transformations which map $(0,0)$, $(1,0)$ and $(0,1)$ to **p**′, **q**′ and **r**′, respectively. Since an affine transformation is uniquely determined by its effect on the points $(0,0)$, $(1,0)$ and $(0,1)$, it follows that $t \circ t_1 = s \circ t_1$.

$(0,0) \overset{t_1}{\mapsto} \mathbf{p} \overset{t \text{ or } s}{\mapsto} \mathbf{p}'$
$(1,0) \mapsto \mathbf{q} \mapsto \mathbf{q}'$
$(0,1) \mapsto \mathbf{r} \mapsto \mathbf{r}'$

If we then compose both $t \circ t_1$ and $s \circ t_1$ on the right with t_1^{-1}, it follows that $t = s$. Thus the mapping t constructed in part (a) is unique. ■

Now suppose that we are given two arbitrary triangles $\triangle ABC$ and $\triangle DEF$. By the Fundamental Theorem there is an affine transformation which maps the vertices A, B, C to the vertices D, E, F, respectively. Since this transformation maps straight lines to straight lines, it must map the sides of $\triangle ABC$ to the sides of $\triangle DEF$, so we have the following important corollary. This will be used extensively in Section 2.4.



Corollary All triangles are affine-congruent.

In order to find the affine transformation which maps one triangle, vertex to vertex, onto another triangle, we follow the strategy used in part (a) of the proof of the Fundamental Theorem.

Strategy To determine the affine transformation t which maps three non-collinear points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ to another three non-collinear points $\mathbf{p}'$, $\mathbf{q}'$ and $\mathbf{r}'$, respectively:

1. determine the affine transformation t_1 which maps $(0,0)$, $(1,0)$ and $(0,1)$ to the points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, respectively;
2. determine the affine transformation t_2 which maps $(0,0)$, $(1,0)$ and $(0,1)$ to the points $\mathbf{p}'$, $\mathbf{q}'$ and $\mathbf{r}'$, respectively;
3. calculate the composite $t = t_2 \circ t_1^{-1}$.

Recall that the previous strategy explained how t_1 and t_2 can be determined.

Example 3 Determine the affine transformation which maps the points $(2,3)$, $(1,6)$ and $(3,-1)$ to the points $(1,-2)$, $(2,1)$ and $(-3,5)$, respectively.

Solution You have already seen in Problem 3 that the affine transformation t_1 which maps the points $(0,0)$, $(1,0)$ and $(0,1)$ to the points $(2,3)$, $(1,6)$ and $(3,-1)$, respectively, is given by

$$t_1(\mathbf{x}) = \begin{pmatrix} -1 & 1 \\ 3 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Also, in Problem 4 you saw that the affine transformation t_2 which maps the points $(0,0)$, $(1,0)$ and $(0,1)$ to the points $(1,-2)$, $(2,1)$ and $(-3,5)$, respectively, is given by

$$t_2(\mathbf{x}) = \begin{pmatrix} 1 & -4 \\ 3 & 7 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Following the strategy, we need to find the inverse of t_1. We have

$$\begin{pmatrix} -1 & 1 \\ 3 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} -4 & -1 \\ -3 & -1 \end{pmatrix}$$

and

$$\begin{pmatrix} -4 & -1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -11 \\ -9 \end{pmatrix},$$

so that the inverse of t_1 is given by

$$t_1^{-1}(\mathbf{x}) = \begin{pmatrix} -4 & -1 \\ -3 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 11 \\ 9 \end{pmatrix}.$$

Thus the affine transformation which maps the points $(2,3)$, $(1,6)$ and $(3,-1)$ to the points $(1,-2)$, $(2,1)$ and $(-3,5)$, respectively, is given by

$$\begin{aligned} t(\mathbf{x}) &= t_2 \circ t_1^{-1}(\mathbf{x}) \\ &= t_2\left(\begin{pmatrix} -4 & -1 \\ -3 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 11 \\ 9 \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 & -4 \\ 3 & 7 \end{pmatrix}\left(\begin{pmatrix} -4 & -1 \\ -3 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 11 \\ 9 \end{pmatrix}\right) + \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \left(\begin{pmatrix} 8 & 3 \\ -33 & -10 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -25 \\ 96 \end{pmatrix}\right) + \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 3 \\ -33 & -10 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -24 \\ 94 \end{pmatrix}. \end{aligned}$$

□

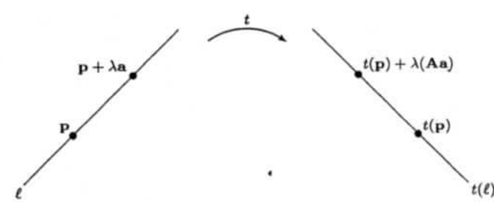
Problem 5 Determine the affine transformation which maps the points $(1,-1)$, $(2,-2)$ and $(3,-4)$ to the points $(8,13)$, $(3,4)$ and $(0,-1)$, respectively.

2.3.3 Proofs of the Basic Properties of Affine Transformations

In Subsection 2.2.2 we used parallel projections to demonstrate that affine transformations have the following basic properties: they map straight lines to straight lines, they map parallel lines to parallel lines, and they preserve ratios of lengths along a given straight line. We now give algebraic proofs of these assertions.

Theorem 2 An affine transformation maps straight lines to straight lines.

Proof



Let ℓ be a line through a point with position vector $\mathbf{p}$, and let the direction of ℓ be that of some vector $\mathbf{a}$. Then

$$\ell = \{\mathbf{p} + \lambda\mathbf{a} : \lambda \in \mathbb{R}\}.$$

Now let $t : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine transformation given by

$$t(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}.$$

We can find the image under t of an arbitrary point $\mathbf{p}+\lambda\mathbf{a}$ on ℓ as follows:

$$\begin{aligned} t(\mathbf{p}+\lambda\mathbf{a}) &= \mathbf{A}(\mathbf{p}+\lambda\mathbf{a})+\mathbf{b} \\ &= (\mathbf{A}\mathbf{p}+\mathbf{b})+\lambda\mathbf{A}\mathbf{a} \\ &= t(\mathbf{p})+\lambda\mathbf{A}\mathbf{a}. \end{aligned}$$

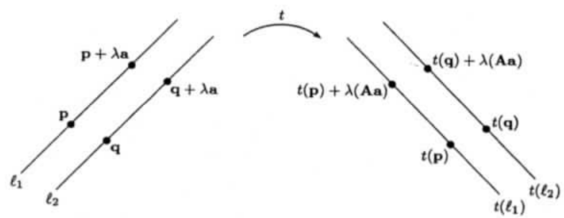
So the image of ℓ is the set

$$t(\ell)=\{t(\mathbf{p})+\lambda\mathbf{A}\mathbf{a}:\lambda\in\mathbb{R}\},$$

which is a line through $t(\mathbf{p})$ in the direction of the vector $\mathbf{Aa}$. ∎

Theorem 3 An affine transformation maps parallel straight lines to parallel straight lines.

Proof



Let ℓ_1 and ℓ_2 be parallel lines through the points with position vectors $\mathbf{p}$ and $\mathbf{q}$, respectively, and let the direction of the lines be that of the vector $\mathbf{a}$. Then

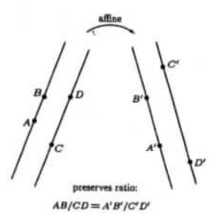
$$\ell_1=\{\mathbf{p}+\lambda\mathbf{a}:\lambda\in\mathbb{R}\} \quad\text{and}\quad \ell_2=\{\mathbf{q}+\lambda\mathbf{a}:\lambda\in\mathbb{R}\}.$$

As in the proof of Theorem 2, the images of ℓ_1 and ℓ_2 under the affine transformation $t(\mathbf{x})=\mathbf{Ax}+\mathbf{b}$ are the sets

$$t(\ell_1)=\{t(\mathbf{p})+\lambda\mathbf{Aa}:\lambda\in\mathbb{R}\} \quad\text{and}\quad t(\ell_2)=\{t(\mathbf{q})+\lambda\mathbf{Aa}:\lambda\in\mathbb{R}\}.$$

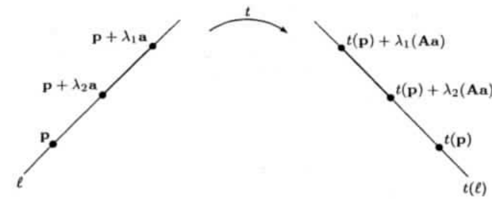
These sets are straight lines which pass through the image points $t(\mathbf{p})$ and $t(\mathbf{q})$, both in the same direction as that of the vector $\mathbf{Aa}$. Hence the two image lines under t are parallel, as claimed. ∎

Rather than prove that affine transformations preserve ratios of lengths along a given straight line, as in Section 2.2, we prove the following more general result illustrated in the margin. The original result follows because any line is parallel to itself.



Theorem 4 An affine transformation preserves ratios of lengths along parallel straight lines.

Proof We begin by examining what happens to the length of a line segment under an affine transformation.



Let ℓ be a line through a point with position vector $\mathbf{p}$, and let the direction of ℓ be that of some *unit* vector $\mathbf{a}$. Then

$$\ell=\{\mathbf{p}+\lambda\mathbf{a}:\lambda\in\mathbb{R}\}.$$

As in the proof of Theorem 2, the image of ℓ under the affine transformation $t(\mathbf{x})=\mathbf{Ax}+\mathbf{b}$ is the line

$$t(\ell)=\{t(\mathbf{p})+\lambda\mathbf{Aa}:\lambda\in\mathbb{R}\}.$$

Now consider a segment of ℓ with endpoints $\mathbf{p}+\lambda_1\mathbf{a}$ and $\mathbf{p}+\lambda_2\mathbf{a}$. Since $\mathbf{a}$ is a unit vector, the length of the segment is

Recall that $\|\mathbf{a}\|$ means the length of $\mathbf{a}$.

$$\|(\mathbf{p}+\lambda_2\mathbf{a})-(\mathbf{p}+\lambda_1\mathbf{a})\| = |\lambda_2-\lambda_1|\,\|\mathbf{a}\| = |\lambda_2-\lambda_1|.$$

The image of the segment has endpoints $t(\mathbf{p})+\lambda_1\mathbf{Aa}$ and $t(\mathbf{p})+\lambda_2\mathbf{Aa}$, so the image of the segment has length

$$\|(t(\mathbf{p})+\lambda_2\mathbf{Aa})-(t(\mathbf{p})+\lambda_1\mathbf{Aa})\| = |\lambda_2-\lambda_1|\,\|\mathbf{Aa}\|.$$

So, in the process of mapping segments along ℓ to segments along $t(\ell)$, lengths are stretched by the factor $\|\mathbf{Aa}\|$. Since this factor is the same for all segments which lie along lines parallel to $\mathbf{a}$, it follows that the *ratios* of lengths along parallel lines are unchanged by t. ∎

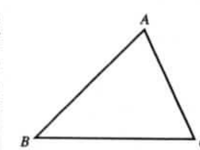
2.4 Using the Fundamental Theorem of Affine Geometry

In this section we explain how the Fundamental Theorem of Affine Geometry can be used to deduce the fact that the medians of any triangle are concurrent from the special case that the medians of an equilateral triangle are concurrent. We then use similar methods to prove the classical theorems of Ceva and Menelaus.

These results are named after Giovanni Ceva (Italian mathematician, 1647/48–1734) and Menelaus of Alexandria (Greek geometer, 1st Century AD).

2.4.1 The Median Theorem

Let ΔABC be an arbitrary triangle in the plane. If you join the midpoint of each side of the triangle to the opposite vertex (these lines are called the *medians* of the triangle), these three lines appear to pass through a single point. In fact, no matter what triangle you choose, you find that its medians meet in a single point.

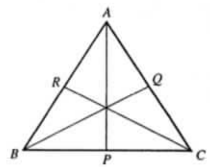


Theorem 1 Median Theorem
The medians of any triangle are concurrent.

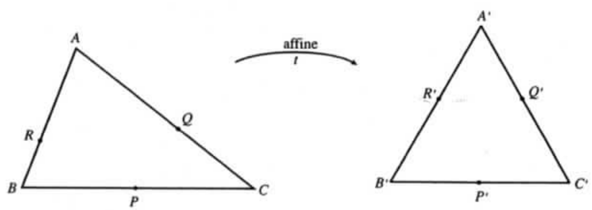
We can get some evidence that this theorem holds in general by looking first at a special case where a proof of the theorem is straight-forward – namely, when the triangle is an equilateral triangle.

This technique of looking first to see whether a result holds in a special case is often useful.

To do this, consider an equilateral triangle ΔABC, with medians AP, BQ and CR. Since ΔABC has sides of equal length, it must be symmetric about the line AP. Thus the point at which BQ meets CR must be symmetrically placed with respect to this line – that is, it must actually lie on the line AP. In other words, the lines AP, BQ and CR are concurrent if the triangle is equilateral.

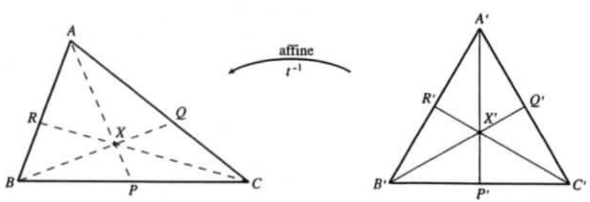


In order to show that the medians of an arbitrary triangle meet at a point, consider an arbitrary triangle ΔABC, and let P, Q and R be the midpoints of the sides BC, CA and AB, respectively. Next, choose a particular *equilateral* triangle $\Delta A'B'C'$, and let P', Q' and R' be the midpoints of the sides $B'C'$, $C'A'$ and $A'B'$, respectively.



According to the Fundamental Theorem of Affine Geometry there is an affine transformation t which maps ΔABC onto $\Delta A'B'C'$. Moreover, since affine transformations preserve ratios of lengths along lines it follows that t maps the mid-points P, Q and R to the mid-points P', Q' and R', respectively.

From the above discussion we know that the medians of any *equilateral* triangle meet at a point, so in particular we know that $A'P'$, $B'Q'$ and $C'R'$ meet at some point X', say, as shown on the right below.



The trick now is to observe that t has an inverse t^{-1} which is also an affine transformation. This inverse maps the medians $A'P'$, $B'Q'$ and $C'R'$ back to the medians AP, BQ and CR of the original triangle ΔABC. Since X' lies on all three of the lines $A'P'$, $B'Q'$ and $C'R'$ it follows that t^{-1} maps X' to some point X which lies on all three of the lines AP, BQ and CR. In other words, the medians of ΔABC are concurrent.

Since ΔABC is an arbitrary triangle we have proved the Median Theorem.

The essence of the above proof is the fact that all triangles are affine-congruent. That powerful result enables us to prove theorems concerning the affine properties of triangles (such as collinearity, lines being parallel, and ratios of lengths along a given line) following a standard pattern. First, we choose a particular type of triangle for which it is easy to prove the result. Then, by asserting the existence of an affine transformation from that triangle to an arbitrary triangle, we deduce that the result holds for all triangles.

The basic affine properties were listed in Subsection 2.2.1.

This is the approach we shall use to prove the theorems of Ceva and Menelaus later in the section.

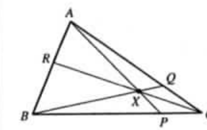
2.4.2 Ceva's Theorem

We now prove the following theorem due to Ceva.

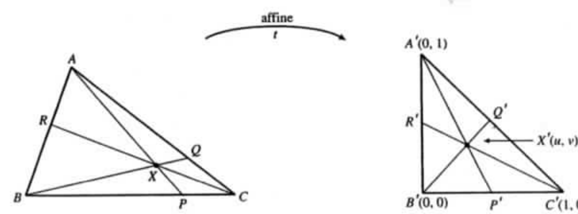
Theorem 2 Ceva's Theorem

Let ΔABC be a triangle, and let X be a point which does not lie on any of its (extended) sides. If AX meets BC at P, BX meets CA at Q and CX meets BA at R, then

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1.$$



Proof According to the Fundamental Theorem of Affine Geometry there is an affine transformation t which maps the points A, B, C to the points $A' = (0, 1)$, $B' = (0, 0)$, $C' = (1, 0)$, respectively. This transformation maps the triangle ΔABC onto the right-angled triangle $\Delta A'B'C'$, and it maps the point X to some point $X' = (u, v)$.



Using coordinate geometry we can calculate the equations of the lines $A'X'$, $B'X'$, $C'X'$, and hence find the coordinates of the point P' where $A'X'$ meets $B'C'$, of the point Q' where $B'X'$ meets $A'C'$, and of the point R' where $C'X'$ meets $A'B'$.

Starting with the point P', we note that the line $B'C'$ has equation $y = 0$.

Also, the line $A'X'$ has gradient $\frac{1-v}{0-u}$, so its equation is $y - 1 = \frac{1-v}{0-u}(x - 0)$. Hence, at the point P' where the two lines meet, we must have $y = 0$ and

$y - 1 = \frac{1-v}{0-u}(x - 0)$, so

$$P' = \left(\frac{u}{1-v}, 0\right).$$

Similarly, at the point R' we have $x = 0$, and $y - 0 = \frac{0-v}{1-u}(x - 1)$, so

$$R' = \left(0, \frac{v}{1-u}\right).$$

Finally, at Q' we have $x + y = 1$ and $y = \frac{v}{u}x$, so $x = \frac{u}{u+v}$ and $y = \frac{v}{u+v}$. Hence

$$Q' = \left(\frac{u}{u+v}, \frac{v}{u+v}\right).$$

Thus, using the coordinate formulas for calculating ratios we obtain

These formulas are given at the beginning of Appendix 2 just above the *Section Formula*.

$$\frac{A'R'}{R'B'} = \frac{\frac{v}{1-u} - 1}{0 - \frac{v}{1-u}} = \frac{u+v-1}{-v}, \quad \text{(by the } y\text{-coordinate formula)}$$

$$\frac{B'P'}{P'C'} = \frac{\frac{u}{1-v} - 0}{1 - \frac{u}{1-v}} = \frac{u}{1-u-v}, \quad \text{(by the } x\text{-coordinate formula)}$$

and

$$\frac{C'Q'}{Q'A'} = \frac{\frac{v}{u+v} - 0}{1 - \frac{v}{u+v}} = \frac{v}{u}. \quad \text{(by the } y\text{-coordinate formula)}$$

Hence

$$\frac{A'R'}{R'B'} \cdot \frac{B'P'}{P'C'} \cdot \frac{C'Q'}{Q'A'} = 1.$$

Since t^{-1} is an affine transformation, it preserves ratios along a line. It must therefore map P', Q', R' back to the points P, Q, R in such a way that

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1,$$

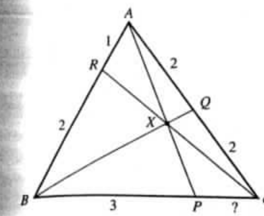
as required. ■

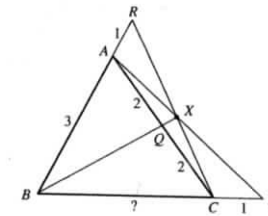
The next example illustrates how we can use Ceva's Theorem to calculate certain unknown distances along the sides of a triangle. For the method to work correctly, it is important to remember that all the ratios in Ceva's Theorem are *signed* ratios. Thus, if X lies inside the triangle, as in part (a) of the example, then all the ratios are positive. But if X lies outside the triangle, as in part (b), then two of the ratios will be negative.

Example 1

(a) In the figure on the left below, $AR=1$, $RB=2$, $BP=3$, $CQ=2$ and $QA=2$. Calculate the distance PC.

(b) For the figure on the right, $AR=1$, $AB=3$, $PC=1$, $CQ=2$ and $QA=2$. Calculate the distance BC.





Solution

(a) By Ceva's Theorem, we have

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1;$$

so,

$$\frac{1}{2} \cdot \frac{3}{PC} \cdot \frac{2}{2} = 1.$$

It follows that $PC = \frac{3}{2}$.

(b) By Ceva's Theorem, we have

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1;$$

so,

$$-\frac{1}{4} \cdot \left(-\frac{BC+1}{1}\right) \cdot \frac{2}{2} = 1.$$

It follows that $BC = 3$. □

Problem 1

(a) Determine the ratio $\frac{BP}{PC}$ in the left diagram below, given that

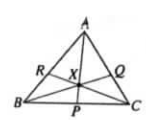
$$\frac{AR}{RB} = \frac{AQ}{QC} = \frac{3}{2}.$$

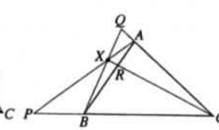
(b) Determine the ratio $\frac{CQ}{QA}$ in the middle diagram below, given that

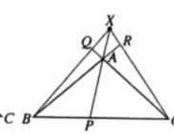
$$\frac{AR}{RB} = \frac{1}{2} \quad \text{and} \quad \frac{BP}{PC} = -\frac{2}{7}.$$

(c) Determine the ratio $\frac{AR}{RB}$ in the right diagram below, given that

$$\frac{BP}{PC} = \frac{5}{7} \quad \text{and} \quad \frac{CQ}{QA} = -7.$$







Ceva's Theorem has the following converse, which can be regarded as a generalization of the Median Theorem to configurations where P, Q, R are not all midpoints of sides.

In the Median Theorem,

$$\frac{AR}{RB}=1,\quad \frac{BP}{PC}=1,\quad \frac{CQ}{QA}=1,$$

so this result generalizes the Median Theorem.

Theorem 3 Converse to Ceva's Theorem

Let P, Q and R be points, other than vertices, on the (possibly extended) sides BC, CA and AB of a triangle ΔABC, such that

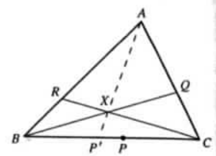
$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=1. \quad (1)$$

Then the lines AP, BQ and CR are concurrent.

Proof Let the lines BQ and CR intersect at a point X, and let the line AX meet BC at some point P'. It is sufficient to prove that $P = P'$.

It follows from Ceva's Theorem that

$$\frac{AR}{RB}\cdot\frac{BP'}{P'C}\cdot\frac{CQ}{QA}=1. \quad (2)$$



Hence, from equations (1) and (2), we have

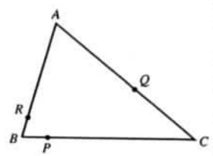
$$\frac{BP}{PC}=\frac{BP'}{P'C},$$

so that P and P' must indeed be the same point. ■

Example 2 The triangle ΔABC has vertices $A(1,3)$, $B(-1,0)$ and $C(4,0)$, and the points $P(0,0)$, $Q(8/3,4/3)$ and $R(-2/3,1/2)$ lie on BC, CA and AB, respectively.

(a) Determine the ratios in which P, Q and R divide the sides of the triangle.

(b) Determine whether the lines AP, BQ and CR are concurrent.



Solution

(a) Using the coordinate formulas for calculating ratios, we obtain

$$\frac{AR}{RB}=\frac{-\frac{2}{3}-1}{-1+\frac{2}{3}}=5,\quad \frac{BP}{PC}=\frac{0+1}{4-0}=\frac{1}{4},$$
$$\frac{CQ}{QA}=\frac{\frac{8}{3}-4}{1-\frac{8}{3}}=\frac{4}{5}, \quad (3)$$

so that P divides BC in the ratio $1:4$, Q divides CA in the ratio $4:5$ and R divides AB in the ratio $5:1$.

(b) It follows from (3) that the product

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=5\cdot\frac{1}{4}\cdot\frac{4}{5}=1;$$

so by the converse to Ceva's Theorem the lines AP, BQ and CR must be concurrent. □

Problem 2 The triangle ΔABC has vertices $A(-1,1)$, $B(2,-1)$ and $C(3,2)$, and the points $P(8/3,1)$, $Q(2,7/4)$ and $R(4/5,-1/5)$ lie on BC, CA and AB, respectively.

(a) Determine the ratios in which P, Q and R divide the sides of the triangle.

(b) Determine whether the lines AP, BQ and CR are concurrent.

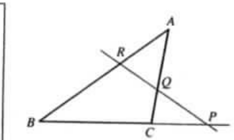
2.4.3 Menelaus' Theorem

Ceva's theorem is concerned with lines through the vertices of a triangle that meet at a point. We now use the Fundamental Theorem of Affine Geometry to prove an analogous theorem due to Menelaus which is concerned with points on the sides of a triangle that are collinear.

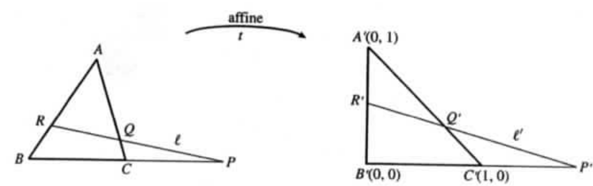
Theorem 4 Menelaus' Theorem

Let ΔABC be a triangle, and let ℓ be a line that crosses the sides BC, CA, AB at three distinct points P, Q, R, respectively. Then

$$\frac{AR}{RB}\cdot\frac{BP}{PC}\cdot\frac{CQ}{QA}=-1.$$



Proof According to the Fundamental Theorem of Affine Geometry there is an affine transformation t which maps the points A, B, C to the points $A'(0,1)$, $B'(0,0)$, $C'(1,0)$, respectively. This transformation maps the triangle ΔABC onto the right-angled triangle $\Delta A'B'C'$, and it maps the line ℓ to some line ℓ'. Let the equation of ℓ' be $y = mx + c$.



We now calculate the coordinates of the points P', Q' and R' where ℓ' meets the sides $B'C'$, $C'A'$ and $A'B'$, respectively.

At P' we have $y = 0$ and $y = mx + c$. This implies that $x = -\frac{c}{m}$, and hence

$$P'=\left(-\frac{c}{m},0\right).$$

At R' we have $x = 0$ and $y = mx + c$. This implies that $y = c$, and hence

$$R'=(0,c).$$

At Q' we have $x + y = 1$ and $y = mx + c$. This implies that $x=\frac{1-c}{m+1}$ and $y=\frac{m+c}{m+1}$, and hence

$$Q'=\left(\frac{1-c}{m+1},\frac{m+c}{m+1}\right).$$

Using the coordinate formulas for calculating ratios we obtain

$$\frac{A'R'}{R'B'} = \frac{c-1}{0-c} = \frac{c-1}{-c}, \quad \text{(by the } y\text{-coordinate formula)}$$

$$\frac{B'P'}{P'C'} = \frac{-\frac{c}{m}-0}{1+\frac{c}{m}} = \frac{-c}{m+c}, \quad \text{(by the } x\text{-coordinate formula)}$$

and

$$\frac{C'Q'}{Q'A'} = \frac{\frac{1-c}{m+1}-1}{0-\frac{1-c}{m+1}} = \frac{-(m+c)}{c-1}. \quad \text{(by the } x\text{-coordinate formula)}$$

Hence,

$$\frac{A'R'}{R'B'} \cdot \frac{B'P'}{P'C'} \cdot \frac{C'Q'}{Q'A'} = -1.$$

Since t^{-1} is an affine transformation, it preserves ratios along a line. It must therefore map P', Q', R' back to the points P, Q, R in such a way that

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = -1,$$

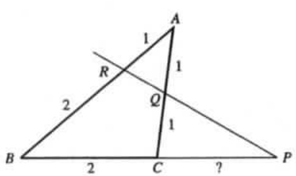
as required. ■

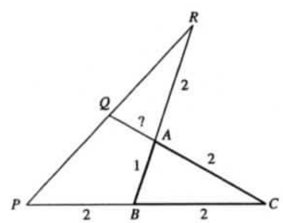
As for Ceva's Theorem, it is important to remember that all the ratios in Menelaus' Theorem are *signed* ratios. In fact if ℓ passes through the interior of the triangle, then precisely one of the ratios is negative; otherwise all the ratios are negative.

Example 3

(a) In the figure on the left below: $AR = 1, RB = 2, BC = 2, CQ = 1$ and $QA = 1$. Calculate the distance PC.

(b) In the figure on the right below: $AR = 2, AB = 1, BC = 2, CA = 2$ and $BP = 2$. Calculate the distance QA.





Solution

(a) By Menelaus' Theorem, we have

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = -1.$$

So

$$\frac{1}{2} \cdot \left(-\frac{2+PC}{PC}\right) \cdot \frac{1}{1} = -1.$$

It follows that $2 + PC = 2PC$, and hence $PC = 2$.

(b) By Menelaus' Theorem, we have

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = -1.$$

So

$$\left(-\frac{2}{3}\right) \cdot \left(-\frac{2}{4}\right) \cdot \left(-\frac{2+QA}{QA}\right) = -1.$$

It follows that $2 + QA = 3QA$, and hence $QA = 1$. □

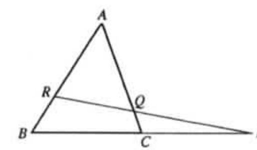
Problem 3

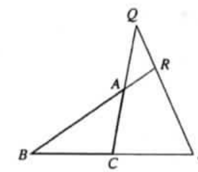
(a) Determine the ratio $\frac{CQ}{QA}$ in the left diagram below, given that

$$\frac{AR}{RB} = 2 \quad \text{and} \quad \frac{BP}{PC} = -2.$$

(b) Determine the ratio $\frac{CQ}{QA}$ in the right diagram below, given that

$$\frac{AR}{RB} = -\frac{1}{4} \quad \text{and} \quad \frac{BP}{PC} = -2.$$





Menelaus' Theorem has a converse that enables us to check whether points on the three sides of a triangle are collinear.

> **Theorem 5 Converse to Menelaus' Theorem**
>
> Let P, Q and R be points other than vertices on the (possibly extended) sides BC, CA and AB of a triangle $\triangle ABC$, such that
>
> $$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = -1. \quad (3)$$
>
> Then the points P, Q and R are collinear.

Proof Let the line ℓ that passes through Q and R meet BC at some point P'. It is sufficient to prove that $P = P'$.

The strategy of the proof is the same as that of Theorem 3.

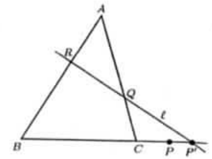
It follows from Menelaus' Theorem that

$$\frac{AR}{RB} \cdot \frac{BP'}{P'C} \cdot \frac{CQ}{QA} = -1. \quad (4)$$

Hence, from equations (3) and (4) we deduce that

$$\frac{BP}{PC} = \frac{BP'}{P'C}.$$

It follows that P and P' must indeed be the same point. ■

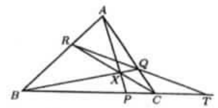


Problem 4 The triangle $\triangle ABC$ has vertices $A(2, 4)$, $B(-2, 0)$ and $C(1, 0)$, and the points $P(5/2, 0)$, $Q(3/2, 2)$ and $R(1, 3)$ lie on BC, CA and AB, respectively.

(a) Determine the ratios in which P, Q and R divide the sides of the triangle.

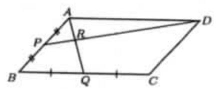
(b) Determine whether the points P, Q and R are collinear.

We end this subsection with two revision problems.



Problem 5 A point X lies inside a triangle $\triangle ABC$, and the lines AX, BX and CX meet the opposite sides of the triangle at P, Q and R, respectively. The lines QR and BC meet at T.

Given that $\frac{BP}{PC} = k$, $0 < k < 1$, determine $\frac{BT}{TC}$ in terms of k.



Problem 6 Suppose that P and Q are the midpoints of the sides AB and BC of a parallelogram $ABCD$, and that the lines DP and AQ meet at R.

(a) Determine the image of B under the affine transformation t which maps A, D and C to $(0, 1)$, $(0, 0)$ and $(1, 0)$, respectively.

(b) By considering the image of $ABCD$ under t, determine the ratios $PR : RD$ and $AR : RQ$.

2.5 Affine Transformations and Conics

2.5.1 Classifying Non-Degenerate Conics in Affine Geometry

In Section 2.2 you saw that under an affine transformation a straight line maps to a straight line. Indeed, it follows from the Fundamental Theorem of Affine Geometry that any straight line can be mapped to any other straight line by some affine transformation. We now explore the corresponding situation for conics.

Conics were discussed in Chapter 1.

Recall that a conic is a set in $\mathbb{R}^2$ given by an equation of the form

$$Ax^2 + Bxy + Cy^2 + Fx + Gy + H = 0, \qquad (1)$$

where A, B, C, F, G and H are real numbers, and A, B and C are not all zero. The three types of non-degenerate conic are *ellipses*, *parabolas* and *hyperbolas*. A non-degenerate conic is a hyperbola if $B^2 - 4AC > 0$, a parabola if $B^2 - 4AC = 0$, and an ellipse if $B^2 - 4AC < 0$.

First, consider the case where equation (1) represents an ellipse, as illustrated on the left of the figure below. We can apply a translation to move the

centre of the ellipse to the origin, and then a rotation to align its major and minor axes with the directions of the x-axis and y-axis respectively. After we have applied these two Euclidean transformations, the equation of the ellipse becomes

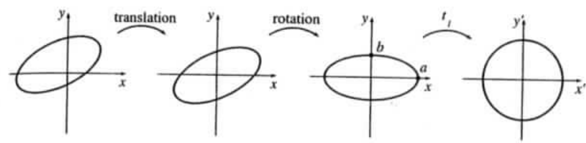
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (2)$$

If we now apply the affine transformation $t_1 : (x, y) \mapsto (x', y')$, where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

then $x' = x/a$ and $y' = y/b$, so equation (2) becomes

$$(x')^2 + (y')^2 = 1.$$



Since the translation, the rotation and the transformation t_1 are all affine, their composite must also be affine. Overall, this shows that each ellipse can be mapped onto the unit circle by an affine transformation. We therefore have the following theorem.

> **Theorem 1** Every ellipse is affine-congruent to the unit circle with equation $x^2 + y^2 = 1$.

Secondly, consider the case where equation (1) represents a hyperbola, as illustrated on the left of the figure below. Again, we can apply a translation to move the centre of the hyperbola to the origin, and then a rotation to align its major and minor axes with the directions of the x-axis and y-axis respectively. After we have applied these two transformations, the equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (3)$$

Under the affine transformation t_1 defined above, equation (3) becomes

$$(x')^2 - (y')^2 = 1,$$

that is,

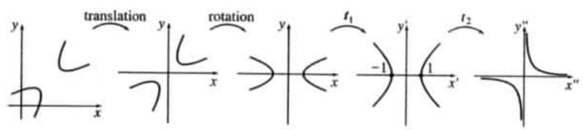
$$(x' - y')(x' + y') = 1. \qquad (4)$$

Finally, if we apply the affine transformation $t_2 : (x', y') \mapsto (x'', y'')$, where

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix},$$

then equation (4) becomes

$$x''y'' = 1.$$



Dropping the dashes from the equation $x''y'' = 1$, we obtain the following theorem.

> **Theorem 2** Every hyperbola is affine-congruent to the rectangular hyperbola with equation $xy = 1$.

Recall that 'rectangular' means that the asymptotes of the hyperbola are at right angles to each other.

Finally, consider the case where equation (1) represents a parabola, as illustrated on the left of the figure below. We can apply a translation to move the vertex of the parabola to the origin, and then a rotation to align its axis with the (positive) x-axis. After we have applied these two Euclidean transformations, the equation of the parabola becomes

$$y^2 = ax, \qquad (5)$$

Here we omit the details of the particular transformations involved, and concentrate instead on the principles underlying the successive mappings which are used.

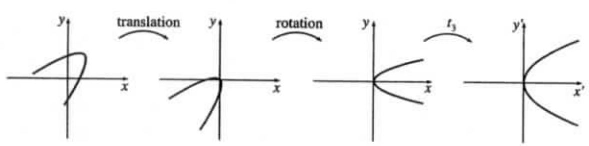
where a is some positive number which depends on the coefficients in equation (1).

Next, if we apply the affine transformation $t_3 : (x, y) \mapsto (x', y')$, where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1/a & 0 \\ 0 & 1/a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

then $x' = x/a$ and $y' = y/a$, so equation (5) becomes $(y'a)^2 = a(x'a)$, or

$$(y')^2 = x'.$$



Dropping the dashes, we obtain the following theorem.

> **Theorem 3** Every parabola is affine-congruent to the parabola with equation $y^2 = x$.

Since all parabolas are affine-congruent to $y^2 = x$, they must be affine-congruent to each other. Similarly, by Theorems 1 and 2, all ellipses must be affine-congruent to each other, and all hyperbolas must be affine-congruent to each other.

This raises the question as to whether it is possible for one type of conic (such as an ellipse) to be affine-congruent to another type of conic (such as a hyperbola). The next theorem shows that this cannot happen. In fact, since an affine transformation can be expressed as the composition of two parallel projections, this should not surprise you. After all, no parallel projection can change a bounded curve (such as an ellipse) into an unbounded one (such as a parabola or a hyperbola); nor can it change a curve with two branches (a hyperbola) into a curve with just one branch (an ellipse or a parabola).

> **Theorem 4** Affine transformations map ellipses to ellipses, parabolas to parabolas, and hyperbolas to hyperbolas.

Remember that a circle is a special type of ellipse.

Proof Consider the non-degenerate conic with equation

$$Ax^2 + Bxy + Cy^2 + Fx + Gy + H = 0, \qquad (6)$$

and its image under an affine transformation $t : \mathbf{x} \mapsto \mathbf{x}'$ given by

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b},$$

where $\mathbf{A}$ is an invertible 2×2 matrix.

The inverse affine transformation $t^{-1} : \mathbf{x}' \mapsto \mathbf{x}$ is given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}' - \mathbf{A}^{-1}\mathbf{b},$$

You met this formula for the inverse in Subsection 2.2.1.

which we may write in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix},$$

for some real numbers p, q, r, s, u and v. It follows that

$$x = px' + qy' + u \quad \text{and} \quad y = rx' + sy' + v. \qquad (7)$$

If we now substitute these expressions for x and y into equation (6), then the resulting equation is a second-degree equation in x' and y', so the image of the conic under the affine transformation t must be another conic.

We omit the details of these calculations, as they are complicated and uninformative.

Next we show that this image conic cannot be degenerate. A degenerate image would consist of a pair of lines, a single line, a point, or the empty set. Since the affine transformation t^{-1} maps lines to lines, it would map the degenerate image to another degenerate conic. But this cannot happen since t^{-1} maps the image back to the original non-degenerate conic (6). It follows that the image of (6) cannot be degenerate.

Theorem 2 of Section 2.3

Finally, if we substitute for x and y from equations (7) into equation (6), and keep careful track of the algebra involved, it turns out that the discriminant of the image conic is just

$$(ps - rq)^2(B^2 - 4AC).$$

Here $B^2 - 4AC$ is the discriminant of the original conic. Since $(ps - rq)^2 > 0$, the sign of the discriminant is not changed by an affine transformation of a conic. Hence the type of the conic is also unchanged. ■

Here $ps - rq \neq 0$ since $\mathbf{A}$ is invertible.

Recall that the sign of the discriminant of a non-degenerate conic determines the type of the conic.

We can combine the results of Theorems 1–4 to obtain the following corollary.

Corollary In affine geometry:

(a) all ellipses are congruent to each other;

(b) all hyperbolas are congruent to each other;

(c) all parabolas are congruent to each other.

No non-degenerate conic is congruent to one of a different type.

The corollary shows that affine-congruence partitions the set of non-degenerate conics into three disjoint equivalence classes. One class consists of all the ellipses, another class consists of all the hyperbolas, and the third consists of all the parabolas. Each class contains one of the so-called *standard* conics $x^2 + y^2 = 1$, $xy = 1$ and $y^2 = x$.

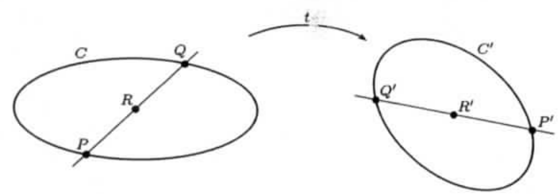
Just as the Fundamental Theorem of Affine Geometry enables us to deduce a given result about an arbitrary triangle by showing that the result holds for an equilateral triangle, so the corollary enables us to deduce a given result about an arbitrary ellipse, hyperbola or parabola by showing that the result holds for the corresponding standard conic. Of course, this works only if the result is concerned with the affine properties of the conic, so we need to be able to recognize such properties.

The following theorem shows that one such property is the property of being the centre of an ellipse or hyperbola.

Theorem 5 Let t be an affine transformation, and let C be an ellipse or hyperbola with centre R. Then $t(C)$ has centre $t(R)$.

Proof Let C' and R' be the images of C and R under t. If P' is any point on C', then it must be the image of some point P on C. Since R is the centre of C, we can rotate P about R through an angle π to a point Q which must also lie on C. Hence $Q' = t(Q)$ is a point on C'.

This uses the definition of centre given in Chapter 1.



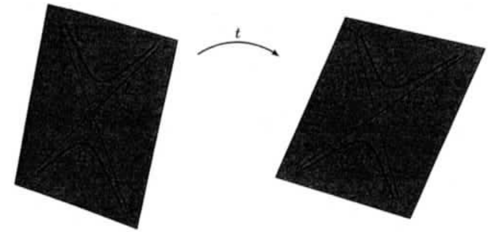
This figure illustrates the proof for an ellipse C, but the proof works equally well for a hyperbola.

Now t preserves ratios of lengths along lines, so the line segment PRQ maps onto the line segment $P'R'Q'$ with $P'R' = R'Q'$. Thus if we rotate P' about R' through an angle π, it must go to Q' on C'. Since P' is an arbitrary point on C', it follows that R' is the centre of C', as required. ■

Another affine property is the property of being an asymptote of a hyperbola.

Theorem 6 Let t be an affine transformation, and let H be a hyperbola with asymptotes ℓ_1 and ℓ_2. Then $t(H)$ has asymptotes $t(\ell_1)$ and $t(\ell_2)$.

The figure below illustrates that this theorem is plausible for parallel projections.



Proof The hyperbola H possess exactly two (distinct) families of parallel lines each of which fills the plane, with each member of each family meeting H exactly once – that is, apart from one line in each family that is an asymptote of H, and so does not meet H.

The image of H under the affine transformation t is also a hyperbola, $t(H)$. The images under t of the two families of parallel lines are also (distinct) families of parallel lines; within each family, a line that meets H once is mapped onto a line that meets $t(H)$ once, and the single line that does not meet H maps onto a line that does not meet $t(H)$. So the two exceptional lines in the image families must be the asymptotes of the hyperbola $t(H)$.

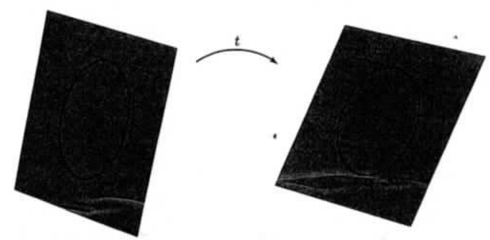
It follows that the asymptotes of H are mapped by t to the asymptotes of $t(H)$, as required. ■

Many of the problems concerning conics which are particularly amenable to solution using the methods of affine geometry involve tangents.

This is due to the following theorem, which asserts that tangency is an affine property.

Theorem 7 Let t be an affine transformation, and let ℓ be a tangent to a conic C. Then $t(\ell)$ is a tangent to the conic $t(C)$.

The figure below illustrates the theorem for parallel projections.



Solution We shall use the fact that a tangent to a conic (whether it is an ellipse, a hyperbola or a parabola) intersects the conic at exactly one point.

However we have to be a little careful. For example, many lines that are not tangents meet a parabola in just one point, such as any line parallel to its axis.

First, the image of an ellipse E under an affine transformation t is an ellipse. A tangent to E is a line that intersects E in exactly one point. These properties remain unchanged under an affine projection; hence the image of a tangent to E under an affine transformation t must be a tangent to $t(E)$.

Corollary, Subsection 2.5.1

This characterizes tangents to ellipses.

Here we use Theorem 2 of Subsection 2.3.3 and the fact that t maps the plane one–one onto itself.

Next, the image of a parabola P under an affine transformation t is a parabola. A tangent to P is a member of a family of parallel lines that fill the plane such that there are lines in the family that meet P twice, once and not at all; the tangent is the unique member of the family that meets P exactly once. The image of the family of lines under t is again a family of parallel lines that fill the plane; it contains lines that meet the parabola $t(P)$ twice and not at all, and a single line that meets P exactly once. This line is the image of the original tangent to P, and must itself be a tangent to $t(P)$. Hence, the image of a tangent to P under an affine transformation t must be a tangent to $t(P)$.

Corollary, Subsection 2.5.1

This characterizes tangents to parabolas.

Theorem 3 of Subsection 2.3.3

Finally, the image of a hyperbola H under an affine transformation t is a hyperbola. A tangent to H is a member of a family of parallel lines that fill the plane such that there are lines in the family that meet H twice, once and not at all; there are exactly two lines in the family that meet H exactly once, and these are tangents to H. The image of the family of lines under t is again a family of parallel lines that fill the plane; it contains lines that meet the parabola $t(H)$ twice and not at all, and exactly two lines that meet H exactly once. These lines are the images of the original tangents to H, and must themselves be tangents to $t(H)$. Hence, the image of a tangent to H under an affine transformation t must be a tangent to $t(H)$.

Corollary, Subsection 2.5.1

This characterizes tangents to hyperbolas.

Theorem 3 of Subsection 2.3.3

This completes the proof. ■

In applications we often use the following facts that you met earlier.

Chapter 1

Tangents to Conics in Standard Form The equation of the tangent to a standard conic at the point (x_1, y_1) is as follows.

Conic	*Tangent*
Unit circle $x^2 + y^2 = 1$	$xx_1 + yy_1 = 1$
Rectangular hyperbola $xy = 1$	$xy_1 + yx_1 = 2$
Parabola $y^2 = x$	$2yy_1 = x + x_1$

2.5.2 Applying Affine Geometry to Conics

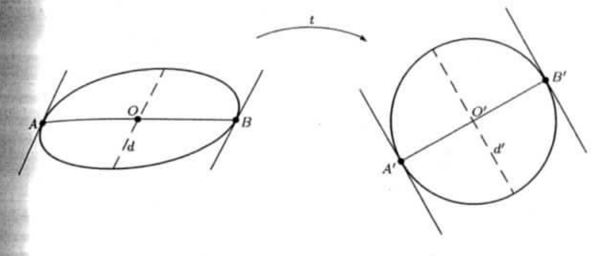
We are now in a position to apply the methods of affine geometry to the solution of problems involving conics. Of course, affine geometry can be helpful in this task only if the property being investigated is one which is preserved under affine transformations. The underlying idea is that we use an affine transformation to map the original conic onto one of our standard conics, tackle the problem in hand there, and then map back to the original conic.

We used these techniques in Subsection 2.2.3 to prove the Conjugate Diameters Theorem for the ellipse. Recall that the diameter conjugate to AB is the set of midpoints of all the chords parallel to AB (see Subsection 2.2.3).

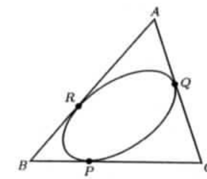
Example 1 AB is a diameter of an ellipse. Prove that the tangents to the ellipse at A and B are parallel to the diameter conjugate to AB.

Solution First, map the ellipse onto the unit circle, by an affine transformation t. Since the centre O of the ellipse maps to the centre O' of the circle, the image

of the diameter AB is a diameter $A'B'$ of the unit circle.



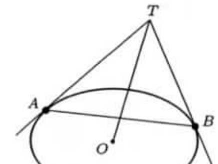
The tangents at A' and B' are both parallel to the diameter d' perpendicular to $A'B'$. But d' is the image under t of the diameter d of the ellipse conjugate to AB. Since parallel lines must map to parallel lines under the inverse affine transformation t^{-1}, the tangents at A and B are parallel to the diameter conjugate to AB. □



Problem 1 An ellipse touches the sides AB, BC and CA of $\triangle ABC$ at the points R, P and Q, respectively. Prove that

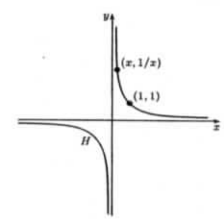
$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1,$$

and deduce that the lines AP, BQ and CR are concurrent.



Problem 2 The tangents to an ellipse at two points A and B meet at a point T. Prove that the line joining T to the centre O of the ellipse bisects the chord AB.

The rectangular hyperbola $H = \{(x, y) : xy = 1\}$ does not possess as much symmetry as does the unit circle; so the fact that every hyperbola is affine-congruent to H may not be sufficient to simplify a given problem. Fortunately, however, we can arrange for any given point on the original hyperbola to map to the point $(1, 1)$ on H.



To see this, note that for any non-zero number a, the affine transformation

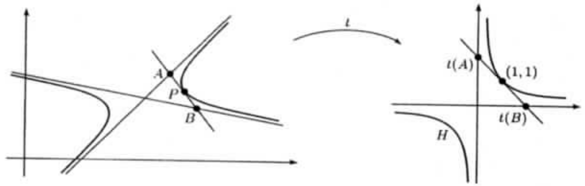
$$t_a : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

maps H to itself. Indeed, an arbitrary point on H has coordinates of the form $(x, 1/x)$, $x \neq 0$, and under t_a this is mapped to the point $(ax, 1/ax)$, which also lies on H. As x varies through $\mathbb{R} - \{0\}$, its image $(ax, 1/ax)$ varies over the whole of H, so the image of H under t_a is the whole of H.

So if we start with a given hyperbola and a point P on it, we can map the hyperbola to H by some affine transformation s. The point $s(P)$ will then have coordinates $(b, 1/b)$ for some number $b \in \mathbb{R} - \{0\}$; so if we choose $a = 1/b$, then the affine transformation t_a will map $s(P)$ to $(1,1)$. Overall, the composite $t = t_a \circ s$ is an affine transformation which maps the given hyperbola to H, and maps P to $(1, 1)$. We state this as a corollary to Theorem 2.

Corollary Given any hyperbola and a point P on it, there is an affine transformation which maps the hyperbola onto the rectangular hyperbola $xy = 1$, and the point P to $(1, 1)$.

Example 2 The tangent at the point P on a hyperbola meets the asymptotes at the points A and B. Prove that $PA = PB$.



Solution Let t be an affine transformation which maps the hyperbola onto the rectangular hyperbola $H = \{(x, y) : xy = 1\}$ in such a way that $t(P) = (1, 1)$. Then the asymptotes of the hyperbola map to the asymptotes of H, and the tangent at P maps to the tangent at $(1, 1)$.

By symmetry, $(1, 1)$ is the midpoint of the line segment from $t(A)$ to $t(B)$. Since midpoints are preserved under the affine transformation t^{-1}, it follows that P is the midpoint of AB. □

Problem 3 P is a point on a hyperbola H with centre O. Prove that there exists a line ℓ through O such that all chords of the hyperbola which are parallel to ℓ are bisected by OP.

This result is an analogue for the hyperbola of the Conjugate Diameters Theorem for the ellipse (see Subsection 2.2.3).

2.6 Exercises

Section 2.1

1. Let ΔABC be a triangle in which $AB = AC$. Prove that
$$\angle ABC = \angle ACB.$$
Hint: Consider a reflection in the bisector of $\angle BAC$.

2. Determine which of the following transformations $t : \mathbb{R}^2 \to \mathbb{R}^2$ are Euclidean transformations.

(a) $t(\mathbf{x}) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

(b) $t(\mathbf{x}) = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

(c) $t(\mathbf{x}) = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

3. The Euclidean transformations t_1 and t_2 are given by
$$t_1(\mathbf{x}) = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}\mathbf{x} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
and
$$t_2(\mathbf{x}) = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$
Determine the composite $t_1 \circ t_2$.

4. Determine the inverse of each of the following Euclidean transformations.

(a) $t(\mathbf{x}) = \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ \frac{12}{13} & \frac{5}{13} \end{pmatrix}\mathbf{x} + \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

(b) $t(\mathbf{x}) = \begin{pmatrix} -\frac{12}{13} & -\frac{5}{13} \\ -\frac{5}{13} & \frac{12}{13} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

5. The Euclidean transformations t_1 and t_2 are given by
$$t_1(\mathbf{x}) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
and
$$t_2(\mathbf{x}) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
Determine the composite $t_2^{-1} \circ t_1$.

Section 2.2

1. Determine whether or not each of the following transformations $t : \mathbb{R}^2 \to \mathbb{R}^2$ is an affine transformation.

(a) $t(\mathbf{x}) = \begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

(b) $t(\mathbf{x}) = \begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

(c) $t(\mathbf{x}) = \begin{pmatrix} -1 & 1 \\ -1 & -2 \end{pmatrix}\mathbf{x}$

2. Write down an example (if one exists) of each type of transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$ described below. In each case, justify your answer.

(a) An affine transformation t which is not a Euclidean transformation

(b) A Euclidean transformation t which is not an affine transformation

(c) A transformation t which is both Euclidean and affine

(d) A transformation t which is one–one, but is neither Euclidean nor affine

3. The affine transformations t_1 and t_2 are given by

$$t_1(\mathbf{x}) = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and

$$t_2(\mathbf{x}) = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Determine the following composites.

(a) $t_1 \circ t_2$ (b) $t_2 \circ t_1$ (c) $t_1 \circ t_1$

4. Determine the inverse of each of the following affine transformations.

(a) $t(\mathbf{x}) = \begin{pmatrix} 2 & -3 \\ 3 & -5 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ (b) $t(\mathbf{x}) = \begin{pmatrix} 3 & 2 \\ 4 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

5. Prove that the transformation

$$t(\mathbf{x}) = 3\mathbf{x} \quad (\mathbf{x} \in \mathbb{R}^2)$$

is an affine transformation, but not a parallel projection.

6. Which of the following are affine properties?

(a) distance (b) collinearity
(c) circularity (d) magnitude of angle
(e) midpoint of line segment

Section 2.3

1. The affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$t(\mathbf{x}) = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

Determine the image under t of each of the following lines.

(a) $y = -2x$ (b) $2y = 3x - 1$

2. The affine transformation $t : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$t(\mathbf{x}) = \begin{pmatrix} 4 & 5 \\ 1 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Determine the image under t of each of the following lines.

(a) $2x - 5y + 3 = 0$ (b) $3x + y - 4 = 0$

3. Determine the affine transformation which maps the points $(0, 0)$, $(1, 0)$ and $(0, 1)$ to the points:

(a) $(0, -1)$, $(1, 1)$ and $(-1, 1)$, respectively;

(b) $(-4, -5)$, $(1, 7)$ and $(2, -9)$, respectively.

4. Determine the affine transformation which maps the points $(1, 1)$, $(3, 2)$ and $(4, 1)$ to the points $(0, 1)$, $(1, 2)$ and $(3, 7)$, respectively.

5. Determine the affine transformation which maps the points $(1, -1)$, $(5, -4)$ and $(-2, 1)$ to the points $(1, 1)$, $(4, 0)$ and $(0, 2)$, respectively.

Section 2.4

1. The points P, Q, R and S lie on a line, in that order; the distances between them are 4 units, 2 units and 3 units, respectively. Determine the ratios $PR:RS$ and $PS:SQ$.

2. A point X lies inside a triangle $\triangle ABC$, and the lines AX, BX and CX meet the opposite sides of the triangle at P, Q and R, respectively. The ratios $AR:AB$ and $BP:BC$ are $1:5$ and $3:7$, respectively. Determine the ratio $AC:AQ$.

3. A line ℓ crosses the sides AB, BC and CA of a triangle $\triangle ABC$ at R, P and Q, respectively. The ratios $BC:CP$ and $CQ:QA$ are $3:2$ and $1:3$, respectively. Determine the ratio $AR:RB$.

4. $ABCD$ is a parallelogram, and the point P divides AB in the ratio $2:1$; the lines AC and DP meet at Q, and the lines BQ and AD meet at R.

(a) Determine the images of P, Q and R under the affine transformation t which maps A, D and C to $(0, 1)$, $(0, 0)$ and $(1, 0)$, respectively.

(b) By considering the image of $ABCD$ under t, determine the ratios $BQ:QR$ and $AR:RD$.

5. The triangle $\triangle ABC$ has vertices $A(-1, 2)$, $B(-3, -1)$ and $C(3, 1)$, and the points $P(1, \frac{1}{3})$, $Q(1, \frac{3}{2})$ and $R(-\frac{5}{3}, 1)$ lie on BC, CA and AB, respectively.

(a) Determine the ratios in which P, Q and R divide the sides of the triangle.

(b) Determine whether or not the lines AP, BQ and CR are concurrent.

6. The triangle $\triangle ABC$ has vertices $A(2, 0)$, $B(-3, 0)$ and $C(3, -3)$, and the points $P(-1, -1)$, $Q(1, 3)$ and $R(-\frac{1}{4}, 0)$ lie on BC, CA and AB, respectively.

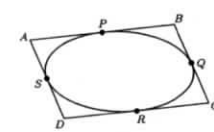
(a) Determine the ratios in which P, Q and R divide the sides of the triangle.

(b) Determine whether or not the points P, Q and R are collinear.

Section 2.5

1. An ellipse touches the sides AB, BC, CD, DA of a parallelogram $ABCD$ at the points P, Q, R, S, respectively. Prove that the lengths CQ, QB, BP and CR satisfy the equation

$$\frac{CQ}{QB} = \frac{CR}{BP}.$$



2. Determine the equation of the image of the parabola P with equation $y = x^2$ under the affine transformation $t : \mathbb{R}^2 \mapsto \mathbb{R}^2$ given by

$$t(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\mathbf{x}.$$

Show the image of the vertex of P is not the vertex of $t(P)$.

This proves that the property of 'being a vertex of a parabola' is not an affine property.