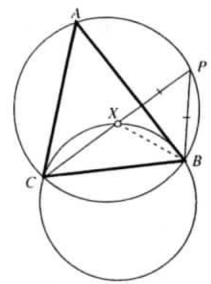
27. A point P varies on the circumcircle of $\triangle ABC$, and for each position of P a point X is located on segment $\overline{PC}$ such that $PX = PB$.

(a) Find the locus of point X.

(b) Do you notice anything peculiar about the configuration when $m\angle A = 60$?

(c) Study this phenomenon using *Sketchpad.*



28. **Problem of Appolonius** Let three circles having equal radii be given. Find a Euclidean construction for a fourth circle that is tangent to the other three.

Note: The general problem of constructing a circle tangent to three given circles is the famous **Problem of Appolonius**. Appolonius was a Greek geometer who lived around 250 B.C.

29. Using *Sketchpad*, explore the following locus problem: Find the locus of the center C of a circle $\mathcal{C}$ that is tangent (externally or internally) to two other circles $\mathcal{C}_1$ and $\mathcal{C}_2$, which are themselves tangent externally. Try to prove your conjecture using properties of circles and tangents.

*4.6 Euclid's Concept of Area and Volume

An enduring topic for geometry and applied mathematics, yet one that is as troublesome as it is obvious, is that of area and volume. We begin with a question. Which do you think has the greater area: a 1-in. tall isosceles triangle with a 2-in. base, or a rectangle 1/64,000 of an inch wide and a mile long? Could you rely on your intuition to answer this question conclusively? Does the area even *exist* for this very long and human-hair-width rectangle?

Our intuitive concept of area and volume no doubt involves the idea of simple counting—counting the number of unit squares, or fractional parts thereof, (or unit cubes) which are contained by the region concerned. This works fine for an integer-sided or even rational-sided rectangle, which involves only a finite number of unit squares or fractional parts (like the first two examples shown in Figure 4.68), and the formula $K = bh$ is pretty obvious. But it quickly becomes problematical in the case of a rectangle with irrational sides, or a triangle, even though these figures in geometry are quite elementary (by contrast, finding the area of a circle would certainly seem insurmountable). The ancient Greeks struggled with the problem of area for the so-called **incommensurable case** of the rectangle (when the ratio of length to width is irrational)—and ultimately conquered that problem, as well as that of the circle.

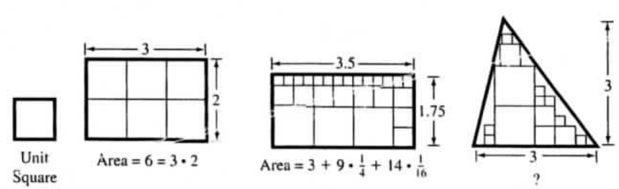


Figure 4.68

The modern approach to this idea of counting squares or cubes involves **measure theory**. This topic is devoted essentially to the study of area and volume as applied to arbitrary sets in the plane or in three-dimensional space. For a region in the plane, one essentially covers the given region by a grid of squares, "counts" the number of squares that just cover the region, and then multiplies by the area of each square in the grid (as shown in Figure 4.69). By making the grid finer and finer, a converging sequence is obtained whose limit is taken as the area of the region. A similar idea is used for the volume of a solid in three-dimensional space.

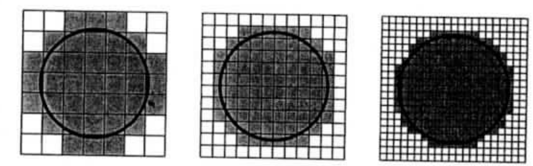


Figure 4.69

Although perhaps appealing to our basic instincts, the idea of "counting" squares remains elusive, to say the least. Since it requires using a complicated process involving limits, actual calculations are unwieldy, and useful area and volume formulas are difficult to develop. Thus we come face to face with the recurring problem in mathematics of trying to formulate a natural and workable definition, and then using the definition to derive the basic theory.

To further illustrate the dilemma facing us, we propose a short drama. It is predicated on the assumption that the only part you remember about area in geometry are the basic formulas for rectangles and triangles. As a promotion, suppose a large firm has decided to sponsor a contest on mathematics. The rules involve a \$500 entry fee, coming to the site of the contest to participate, and working on the problem without recourse to textbooks or calculators. The prize is a million dollars. Thus, the motivation for you to derive from scratch the formulas you have forgotten is guaranteed. Now suppose the problem is to find which of the following geometric figures (Figure 4.70)—a trapezoid, circle, regular octagon, and a square with the four corners cut out along the curve indicated in the coordinate system, as shown in the figure—has the largest area. Tie breakers consist of the degree of accuracy of the answers (exact answers are obviously desirable), and the inclusion of a mathematical development for the formulas used. No doubt the contestants will try very hard to capture the essence of the concept of area in order to derive the formulas they need.

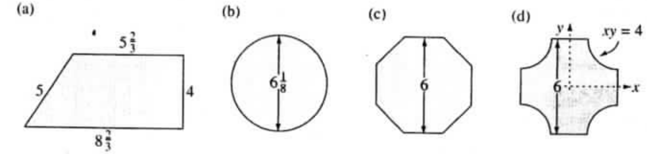


Figure 4.70

This example points up the nature of the area problem confronting ancient scholars. They, too, were trying to find formulas for the area and volume of certain geometric objects, along with proofs or plausibility arguments for those formulas. They had no books in which to look up the answer, nobody who could give them a hint. They had only their intellect and ingenuity. It is sometimes surprising to learn how extensive and ingenious those early developments of area and volume were, particularly since the tools of calculus and the theory of limits had not yet been invented. A prime example is the marvelous development given by Archimedes in his monumental derivation of the volume for a sphere.

We might note that the particular regions chosen for our imaginary contest in Figure 4.70 actually represent major plateaus of achievement in the history of mathematics. The Babylonians (2000–1600 B.C.) could have found the correct formula for the area of the trapezoid [region (a)], but they did not have the general formula for the area of a circle or regular polygon. Euclid (300 B.C.) conquered the circle and regular polygon [regions (b) and (c)], but it requires some form of calculus (A.D. 1790) to find the area of region (d), which, in terms of the usual coordinate system, requires calculating the area under the curve $y = 4/x$.

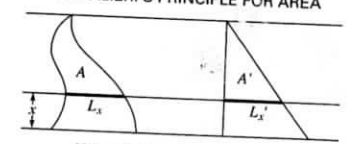
It is strangely illuminating, in modern times, to examine Euclid's ancient approach to the problem of area and volume. In the *Elements*, Euclid never bothered to define area and volume. Rather, he *identified* each object in the plane with its area and each object in space with its volume; area and volume were not regarded as real numbers, separate from the objects themselves. When he stated that two objects in the plane were "equal," he meant that they had *equal areas or volumes*. For example, his Proposition 35, Book I, states: "Parallelograms which are on the same base and in the same parallels are equal to one another."

Indeed, what Euclid did is logically equivalent to our modern approach, a development that appears in many high school geometry textbooks. One assumes the *existence* of the area and volume of all regions under consideration, and then states certain desirable laws that area and volume should obey (the *axioms*).

We present the axioms usually assumed, and show how to derive some of the basic formulas from them. In this axiom system, we take as an undefined term **region.** We may think of a region as just a set of points (like a circle or triangle and their interiors), whose boundary is not too pathological. (There do exist bounded regions in the plane whose areas do not exist, called **nonmeasurable sets** in mathematics.)

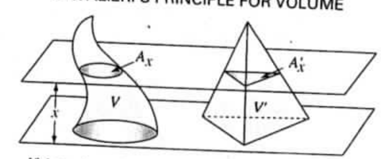
	AREA	VOLUME
1. Existence Postulate	To each region M in the plane, there corresponds a real number Area $M \geq 0$, called its area.	To each region T in space, there corresponds a real number Vol $T \geq 0$, called its volume.
2. Dominance Postulate	For regions M_1 and M_2 in a plane, if $M_1 \subseteq M_2$, then Area $M_1 \leq$ Area M_2.	For regions T_1 and T_2 in space, if $T_1 \subseteq T_2$, then Vol $T_1 \leq$ Vol T_2.
3. Postulate of Additivity	For any two planar regions M_1 and M_2 such that Area $(M_1 \cap M_2) = 0$ then Area $(M_1 \cup M_2) =$ Area M_1 + Area M_2.	For any two regions T_1 and T_2 in space such that Vol $(T_1 \cap T_2) = 0$ then Vol $(T_1 \cup T_2) =$ Vol T_1 + Vol T_2.
4. Congruence Postulate	Congruent regions in a plane have equal areas.	Congruent regions in space have equal volumes.
5. Unit of Measure	The area of the unit square is one.	The volume of the unit cube is one.
6. Cavalieri's Principle[4]	If all the lines parallel to some fixed line that meet the plane regions M_1 and M_2 do so in line segments having equal lengths, whose end points lie in the boundaries of the two regions, then Area M_1 = Area M_2. (See Figure 4.71.)	If all the planes parallel to some fixed plane that meet the solid regions T_1 and T_2 do so in plane sections having equal areas, whose boundaries lie in the boundaries of the two regions, then Vol T_1 = Vol T_2.

CAVALIERI'S PRINCIPLE FOR AREA



If $L_x = L_x'$ then Area A = Area A'

CAVALIERI'S PRINCIPLE FOR VOLUME



If Area A_x = Area A_x' then Volume V = Volume V'

Figure 4.71

[4]This ingenious axiom is due to one of the early pioneers of calculus, B. Cavalieri (1598–1647). Euclid and Archimedes obviously had to do without this labor-saving concept.

NOTE: Cavalieri's principle may actually be proven from the axioms that precede it using the theory of integration.

OUR GEOMETRIC WORLD

Designer scratch pads with a twist provide a perfect illustration of Cavalieri's Principle. A cubical pile of square pages is twisted to form the more interesting three-dimensional figure below. If we started with this solid and asked for its volume, the problem might seem unmanageable. But, since, page for page, the solids have equal cross-sectional area, the volume of the twisted solid equals that of the cube, whose volume is quite elementary.

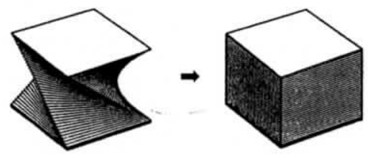


Figure 4.72

It is common in elementary treatments to assume as axioms the basic formulas for the area of a rectangle and the volume of a "box." But these may be derived by logic, using the previous axioms. To illustrate, we show this for a rectangle. Let two rectangles R and R' be given, with bases of length b and b', and altitudes h and h'.

We show first that if $h = h'$,

$$\frac{\text{Area } R'}{\text{Area } R} = \frac{b'}{b}$$

Let m/n be a rational approximation of b'/b in positive integers m and n, such that

(1) $$\frac{m}{n} < \frac{b'}{b} \le \frac{m+1}{n}$$

If we multiply throughout by the positive quantity nb, then

(2) $$mb < nb' \le (m+1)b$$

Now consider three rectangles R_1, R_2, R_3 (Figure 4.73) each having altitude h and bases of length mb, nb', and $(m+1)b$, respectively. By the Dominance Postulate (since the rectangle plus interior having the smaller length would fit inside the one having the greater length by congruence, etc.), the inequality **(2)** implies that

(3) $$\text{Area } R_1 < \text{Area } R_2 \le \text{Area } R_3$$

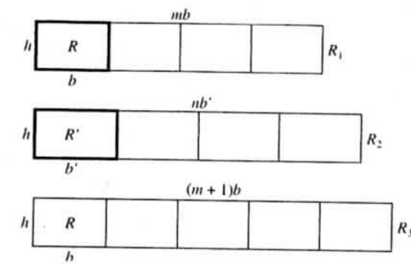


Figure 4.73

By the Postulate of Additivity, since exactly m rectangles congruent to R make up R_1, n congruent to R' make up R_2, and $m + 1$ congruent to R make up R_3, we get

$$m \text{ Area } R < n \text{ Area } R' \le (m+1) \text{ Area } R$$

or

(4) $$\frac{m}{n} < \frac{\text{Area } R'}{\text{Area } R} \le \frac{m+1}{n}$$

Therefore, by comparing the inequalities **(1)** and **(4)** (as $n \to \infty$)

(5) $$\frac{\text{Area } R'}{\text{Area } R} = \frac{b'}{b}$$

Similarly, if the original rectangles R and R' have $b = b'$, then

(6) $$\frac{\text{Area } R'}{\text{Area } R} = \frac{h'}{h}$$

From **(5)** and **(6)** it now follows that if R' is a rectangle with height h and base 1, and U is the unit square, then

$$\frac{\text{Area } R}{\text{Area } R'} = \frac{b}{1} = b \quad \text{and} \quad \frac{\text{Area } R'}{\text{Area } U} = \frac{h}{1} = h$$

Simply multiply these two equations, and use Postulate 5 to obtain the following:

THEOREM 1: If R is a rectangle with base of length b units and height of length h units, then

(7) $$\text{Area } R = bh$$

The formula Vol $P = Bh$ for a rectangular box P having base area B and height h may be derived similarly. (See Problem 18.) The formula for the general parallelogram then follows from Cavalieri's Principle. Triangles, trapezoids and regular polygons come next.

We do not attempt a rigorous axiomatic treatment for the area of a circle. That may be found in sources such as Moise, *Elementary Geometry from an Advanced Viewpoint*. Instead, we present two *plausibility* arguments any prospective teacher of geometry should know about. The first shows how to come up with the following relationship between the *area* of a circle and its *circumference*:

(8) $$2A = rC$$

where A is the area of a circle of radius r, and C is its circumference. Using the standard formula $C = 2\pi r$ already introduced (in Section 1.2), the relation **(8)** may then be readily converted into the desired formula

(9) $$A = \pi r^2.$$

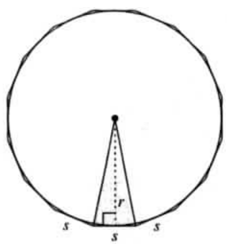


Figure 4.74

To obtain **(8)**, consider a regular polygon with many sides circumscribing the circle, as shown in Figure 4.74 for a 16-sided regular polygon. The interior of the polygon subdivides into 16 congruent isosceles triangles with altitude r and base s. Each of these has area $\frac{1}{2}rs$. The total area of the polygon is therefore

$$K = 16 \cdot \frac{1}{2}rs = \frac{1}{2}r(16s) = \frac{1}{2}rP$$

where P is the perimeter of the polygon. Thus

(10) $$K = \frac{1}{2}rP \quad \text{or} \quad 2K = rP$$

But for a polygon having 1000 or more sides, K would be very nearly the area of the circle, A, and P would be nearly equal to C, the circumference. Hence,

$$2A = rC.$$

The second plausibility argument is quite interesting and presents a very unique type of construction often found in high school geometry texts. It will be left as a discovery unit at the end of this section.

EXAMPLE 1 Show how Cavalieri's Principle can be used to prove the area formula $K = bh$ for the area of a parallelogram having base b and altitude h.

SOLUTION
Let the given parallelogram be $ABCD$, with base $AB = b$ and altitude h, as shown in Figure 4.75. On line $\overleftrightarrow{AB}$ construct a rectangle $\square EFGH$ with $EF = b$ and $EH = h$. If

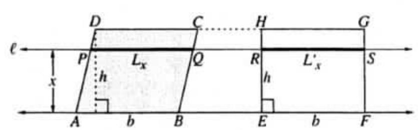


Figure 4.75

a line parallel to line $\overleftrightarrow{AB}$ lies at a distance x from $\overleftrightarrow{AB}$, that parallel will cut the sides of the parallelogram and rectangle at the points P, Q, R, and S, creating segments $\overline{PQ}$ and $\overline{RS}$ having lengths L_x and L_x'. In order to apply Cavalieri's Principle, we must show that $L_x = L_x'$. But $\overline{PQ} \parallel \overline{AB}$, so $\diamond PQBA$ is a parallelogram. Hence $PQ = AB$. Similarly, $RS = EF$, and thus we have

$$L_x = PQ = AB = b = EF = RS = L_x'$$

Since this is true for $0 < x < h$, by Cavalieri's Principle,

$$K = \text{Area } \square ABCD = \text{Area } \square EFGH = bh$$

as desired. ■

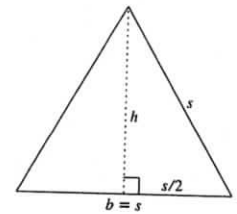


Figure 4.76

EXAMPLE 2 Establish the area formula

$$K = \frac{1}{4}\sqrt{3}\,s^2$$

for an equilateral triangle having side s. (See Figure 4.76.)

SOLUTION
Using the area formula for a triangle $K = \frac{1}{2}bh$ (which will be established in Problem 6), we have $K = \frac{1}{2}sh$. By the Pythagorean Theorem,

$$h^2 + (s/2)^2 = s^2 \quad \text{or} \quad h^2 = s^2 - \frac{1}{4}s^2 = \frac{3}{4}s^2$$
$$h = \sqrt{\frac{3}{4}s^2} = (\sqrt{3}/2)s$$

Substituting this into the preceding equation for K, we have

$$K = \frac{1}{2}s(\sqrt{3}/2)s = \frac{1}{4}\sqrt{3}s^2 \quad ■$$

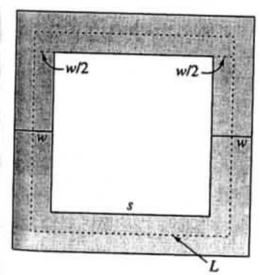


Figure 4.77

EXAMPLE 3 Show that the area of a sidewalk that borders a square (shown in Figure 4.77) is simply its width (w) times the total length of its *midline* (L):

$$K = wL$$

(The **midline** is defined as the locus of midpoints of all line segments joining two adjacent borders of the shaded region.)

SOLUTION
Let s = side of the inner square; then the side of the outer square will be $s + 2w$. The area of the sidewalk is the difference of the areas of these two squares, so

$$K = (s + 2w)^2 - s^2$$
$$= s^2 + 4sw + 4w^2 - s^2$$
$$= 4sw + 4w^2$$

On the other hand, the length of one side of the square midline is

$$w/2 + s + w/2 = s + w$$

so its total length is

$$L = 4(s + w) = 4s + 4w$$

Hence going back to the equation for K,

$$K = 4sw + 4w^2 = w(4s + 4w) = wL. \quad \blacksquare$$

Moment for Discovery

The Area of a Circle

The following diagrams depict a unique way to deduce the area of a circle of radius r.

1. A circle is cut into a sequence of pie-shaped pieces, then reassembled into rectangular shapes of dimensions r and L_1, r and L_2, r and L_3, as shown in Figure 4.78.

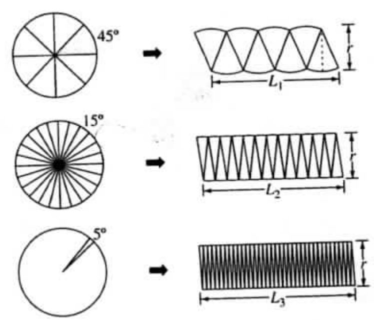


Figure 4.78

2. Find what number L the sequence $L_1, L_2, L_3, \ldots$ seems to be approaching. (Give a formula for L in terms of r.)
3. To what value, then, do the areas of these figures seem to be converging? (Give a formula.)
4. What have you deduced about the area of a circle?

Moment for Discovery

Cavalieri's Principle

This *Sketchpad* experiment will provide a dynamic demonstration of Cavalieri's Principle. Follow these steps (illustrated in Figure 4.79):

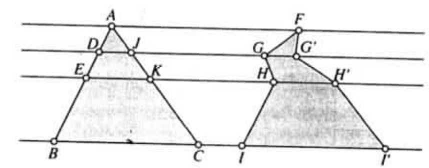


Figure 4.79

1. Construct a triangle $\triangle ABC$ and locate points D and E on side $\overline{AB}$.
2. Construct line $\overleftrightarrow{BC}$, then through points A, D, and E construct lines parallel to $\overleftrightarrow{BC}$.
3. Locate arbitrary points F, G, H, and I on the four lines of Step 2.
4. Select the points of intersection J and K of the parallels $\overleftrightarrow{GD}$ and $\overleftrightarrow{HE}$ with segment $\overline{AC}$. Hide the lines.
5. Mark Vector D to J, then Translate point G to G' By Marked Vector, so that $DJ = GG'$.
6. Repeat Step 5 for points H and I, yielding H' and I' such that $EK = HH'$ and $BC = II'$.
7. Select Interior ABC and Interior $FGHII'H'G'$ and calculate their areas.

Did anything happen? Drag points F, G, H, I before and after changing the shape of $\triangle ABC$. Also, drag D and E along segment $\overline{AB}$. Did you notice anything? Discuss, with proofs, if possible.

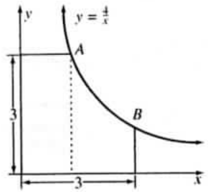
PROBLEMS (4.6)

The problems in this section that are prefixed by the letter C require calculus for their solution.

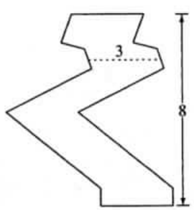
GROUP A

1. Find the areas of the first three regions proposed for the contest in Figure 4.70.

C2. One quadrant of region (d) of Figure 4.70 is shown in the figure for this problem (the shaded region). Part of this region is a rectangle, and the rest consists of the area under the curve $y = 4/x$ between A and B. Find the coordinates of points A and B and use calculus to find the area under the curve, then determine the area of region (d). Finally, find which of the regions of Figure 4.70 has the greatest area.



3. Use Cavalieri's Principle to find the area of the shaded region in the figure.



4. Use Cavalier's Principle to find the total area covered by the letters in the word *Cavalieri* if the width of the downstroke of each letter is 2 mm and the height of the letters is 8 mm. (Assume that the gap in "C" has height equal to that of the hook in the letter "L.")

5. These steps in *Sketchpad* lead to an interesting phenomenon concerning area.

[1] Construct any triangle ABC and line $\overleftrightarrow{AB}$. Locate D any point on line $\overleftrightarrow{AB}$, with an initial position between A and B.

[2] Calculate the ratio AD/AB, displayed on the screen.

[3] Select the interiors of $\triangle ACD$ and $\triangle ACB$ and display the ratio

(Area ACD) / (Area ACB)

Do you notice anything? Drag D on line $\overleftrightarrow{AB}$ to see if the phenomenon persists. Is there a conjecture you can make based on this experiment? State it, and try proving it using the results of this section.

6. **(a)** Use the axioms for area to prove the formula

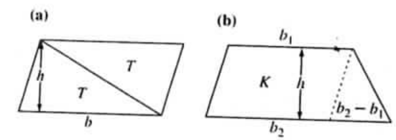
$$\text{Area } T = \frac{1}{2}bh$$

for a triangle T having base of length b and height h. (See figure below.)

(b) Prove the formula

$$K = \frac{1}{2}(b_1 + b_2)h$$

for the area of a trapezoid having bases of length b_1 and b_2, and height h.

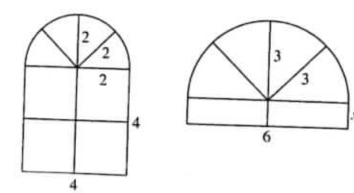


7. A window design consists of a rectangle and semicircle at the top, and is to be placed into a rectangular opening having a perimeter of 20 ft. To increase the amount of light the window will allow, a carpenter has decided that the maximal area for such a window is obtained when the rectangle is a square with 4 ft. on each side, as shown. Do you think the carpenter is correct? Base your answer on the area of the carpenter's window as compared to one having a base of

(a) 6 ft.

(b) 5 ft.

(***Hint:*** If x is the height of the rectangle in the window having a base of 6 ft., the perimeter of the opening is given by $12 + 2(x + 3) = 20$ or, solving for x, $x = 1$ ft.)



8. Use the classical area formulas to determine the ratio of the side of a square to the radius of a circle if the square and circle have the same area.

GROUP B

C9. Show by calculus that the window of Problem 7 in fact has maximal area when the rectangle has a base of approximately 4.52 ft. Find the maximal area.

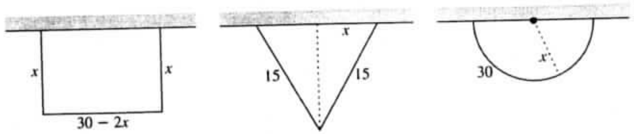
10. A farmer has 30 yd. of fencing and wants to enclose an area beside his barn. What are the dimensions of the region of maximal area, and what is that maximal area, if the region is

(a) a rectangle?

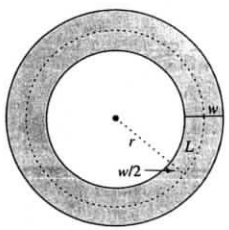
(b) an isosceles triangle with its base along the side of the barn?

(c) a semicircle with diameter along the side of the barn?

(Refer to the figure below. Calculus is not needed for this problem.)



11. Show that the area of the washer shown in the figure having width w and a midline of length L is given by $A = wL$. (***Hint:*** L = circumference of a circle of radius $r + w/2$.)



12. **Ratios of Areas of Similar Figures** Use *Sketchpad* to investigate an important phenomenon concerning area by performing these steps:

[1] Construct an arbitrary pentagon $ABCDE$. (See figure below.)

[2] Select segment $\overline{AB}$ and choose Length under MEASURE to find $m\overline{AB}$.

[3] Select points A, B, C, D, and E and choose Polygon Interior to calculate the area of the pentagon.

[4] Calculate the quantities

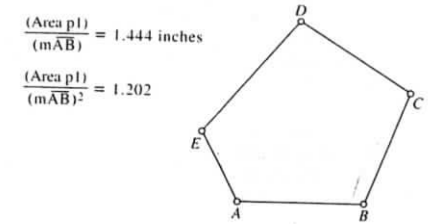
$$(\text{Area } ABCDE) / (m\overline{AB}) \equiv x \quad \text{and} \quad (\text{Area } ABCDE) / (m\overline{AB})^2 \equiv y$$

displaying these two values on the screen.

[5] Double-click on A and using the DILATE TOOL, shrink the polygon to a smaller size (select all five sides of the pentagon). As you do so, watch what happens to x and y.

[6] Return to ARROW TOOL and radically change the shape of the pentagon (for example, making it nonconvex). Then repeat Step 5 to see if a relationship prevails.

State the general relationship you discovered and apply it to state a conjecture about the ratios of the areas of any two similar polygons $\boldsymbol{P}$ and $\boldsymbol{Q}$ as related to the measures of two corresponding sides.

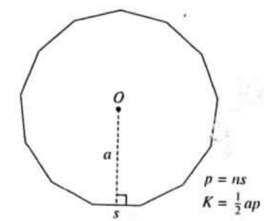


13. (a) A replica of a Model T Ford has a scale of 1:12 (1 in. on the model equals 1 ft. on the car). The front of the windshield on the model has an area 6.3 in.^{2}. What is the corresponding area of the windshield of the car in square inches?

(b) A bedroom measures 10 by 12 ft. How many square yards of carpet will be needed to cover the floor?

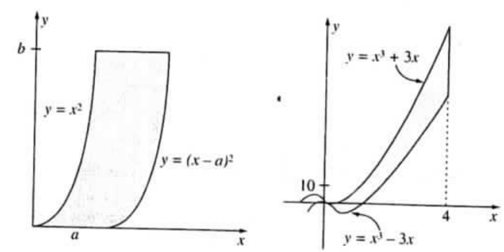
(c) Two fishermen each catch a Salmon on a lake, and the two fish have similar shapes in all respects. One fish is twice as long as the other. How do the weights compare? (***Hint:*** Weight is proportional to volume.)

14. Area of Regular Polygon If a is the **apothem** of a regular n-sided polygon (distance from center to any side), prove that its area is given by $K = \frac{1}{2}ap$, where p is the perimeter. (See the figure below.)



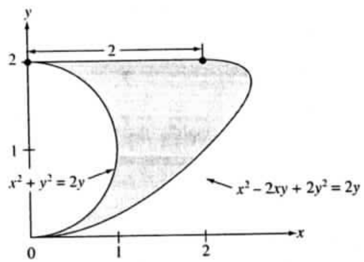
15. Find an explicit formula for the area of a regular hexagon and regular octagon in terms of the length s of a side.

16. Use Cavalieri's Principle to find the areas of each of the regions indicated in the figure.



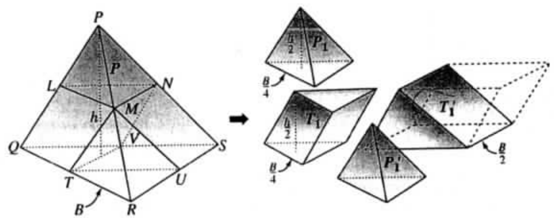
GROUP C

17. Find the area of the shaded region in the figure without calculus. (***Hint:*** Cavalieri's Principle applies—show that at height y the generating segment of the region has length y. You will need to solve the quadratic equation $x^2 - 2xy + 2y^2 - 2y = 0$ for x.)



18. Using the same procedure as that used for rectangles, prove the volume formula $V = \ell wh$ (or $V = Bh$) for a parallelepiped having dimensions ℓ, w, h, and B = area of base = ℓw. (***Hint:*** First prove that Vol P'/Vol $P = \ell'/\ell$ for boxes P' and P have the same width and height, and lengths ℓ' and ℓ, respectively.)

19. Alternate Euclidean Proof of Pyramid Formula See the figure below and fill in the details of the following analysis.



(1) Dissect the original pyramid $\boldsymbol{P}$ into two triangular prisms $\boldsymbol{T}_1$ and $\boldsymbol{T}_1{}'$ and two congruent pyramids $\boldsymbol{P}_1$ and $\boldsymbol{P}_1{}'$ similar to $\boldsymbol{P}$ and half its size. Why does Vol $\boldsymbol{T}_1 = \frac{1}{8}Bh$ and Vol $2\boldsymbol{T}_1{}' = \frac{1}{4}Bh$ (where $2\boldsymbol{T}_1{}'$ = parallelepiped shown, having base $\frac{1}{2}B$)? Thus,

$$\text{Vol } \boldsymbol{P} = \frac{1}{4}Bh + 2 \text{ Vol } \boldsymbol{P}_1$$

(2) Dissect the pyramid $\boldsymbol{P}_1$ in like manner and obtain

$$\text{Vol } \boldsymbol{P} = \frac{1}{4}Bh + 2\left(\frac{1}{4}B_1h_1 + 2 \text{ Vol } \boldsymbol{P}_2\right)$$

$$= Bh\left(\frac{1}{4} + \frac{1}{16}\right) + 4 \text{ Vol } \boldsymbol{P}_2$$

where B_1 and h_1 are the base area and altitude, respectively, of $\boldsymbol{P}_1$.

(3) Continue the process and obtain the result

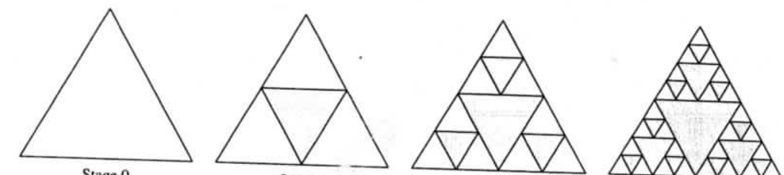
$$\text{Vol } \boldsymbol{P} = Bh\left(\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots\right) \equiv Bh \sum_{n=1}^{\infty} \frac{1}{4^n}$$

(4) Evaluate the geometric series in **(3)**.

NOTE: Euclid used this same decomposition of the pyramid to prove that two triangular pyramids having congruent bases and equal altitudes were "equal," which then enabled him to finish his proof of the relation $V = \frac{1}{3}Bh$ which appears in Section 7.4. Cavalieri's Principle was not available to Euclid, so he had to find a different method of proof.

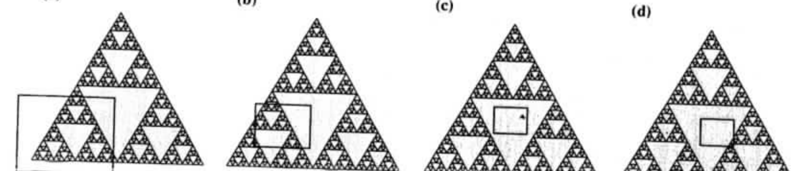
The following problems are on the topic of fractals, discussed briefly in Section 2.1. They are optional, and are not graded according to difficulty.

20. Sierpinski Triangle For us the working definition of the term **fractal** (whose Latin derivative *fractus* means "to break") is: a geometric figure or set of points in the plane having the property that a window, if properly placed, contains a replica similar to the original figure. A good example to begin with is Sierpinski's Triangle, which can be created by the following initial stages using *Sketchpad*. The process continues ad infinitum and a fractal results. Determine which of the windows shown in the figures below contains a reduced version of the entire fractal.



Stage 0 (Equilateral triangle) | Stage 1 (1 midtriangle) | Stage 2 (4 midtriangles) | Stage 3 (13 midtriangles)

(a) (b) (c) (d)



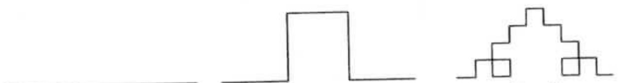
21. If the side of the equilateral triangle in Stage 0 of Sierpinski's Triangle (Problem 20) is unity, calculate the following:

(a) The total perimeters of the shaded triangles (i.e., *boundary*) in Stages 1–3.

(b) The total areas of the shaded triangles in Stages 1–3.

(c) Based on your answers in **(a)** and **(b)** find the **perimeter** (= total boundary length) and **area** of Sierpinski's triangle. In terms of area, do the shaded regions "fill up" the equilateral triangle of Stage 0?

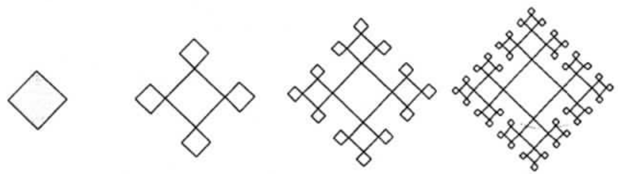
22. A Path Fractal The following stages define a one-dimensional fractal, a particular type called a **hat fractal**. Sketch the very next stage (Stage 3) and determine the total path length of each of the three stages, if the length of Stage 0 equals unity. Is the length increasing without bound? Does the fractal possess a finite length?



Stage 0 Stage 1 Stage 2

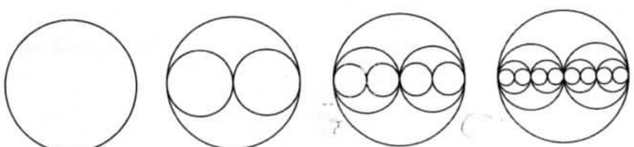
In Problems 23 and 24, Stages 0, 1, 2, and 3 of a fractal are shown. Stage 0 has an area of 1. Find the area of the shaded regions of Stages 1, 2, and 3.[5]

23.



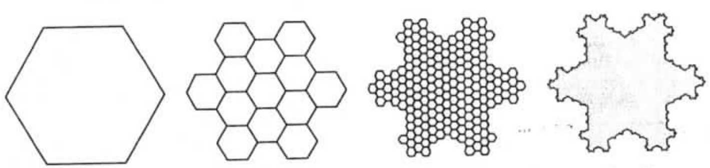
Stage 0 Stage 1 Stage 2 Stage 3

24.



Stage 0 Stage 1 Stage 2 Stage 3

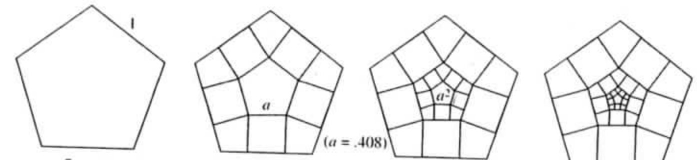
25. Stages 0, 1, 2, and 3 of a fractal are shown below. Stage 0 consists of one regular hexagon. How many regular hexagons are in Stage 1? Stage 2? Stage 3?



Stage 0 Stage 1 Stage 2 Stage 3

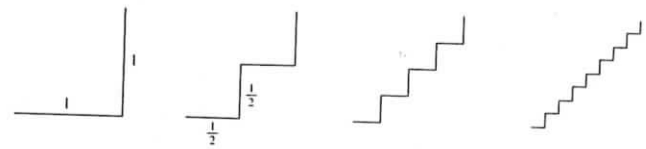
[5]Problems 23–25 are from the high school text *Geometry: An Integrated Approach* by R. E. Larson, L. Boswell, and L. Stiff, D.C. Heath and Company, 1995. They are used by permission.

26. A Pentagonal Fractal Observe the fractal defined by the stages shown in the figure. Find its area if the pentagon has unit side. [It can be shown from trigonometry that the side of the square is $a = 1/(1 + 2\tan 36°) \approx .408$.]



Stage 0 (Regular pentagon) Stage 1 (5 inner squares) Stage 2 (10 inner squares) Stage 3 (15 inner squares)

27. A Stair Step Fractal The first stages of a path fractal are shown in the figure. Show that the perimeter of Stage n of the fractal equals 2 (constant).



Stage 0 Stage 1 (1 step) Stage 2 (3 steps) Stage 3 (7 steps)

*4.7 Coordinate Geometry and Vectors

In 1637, the great mathematician/philosopher René Descartes made a discovery that would revolutionize geometry. He found that geometric configurations could be described entirely by coordinated real number pairs, and two-variable equations. From this concept ultimately emerged the entire field of real analysis, vectors, linear algebra, and matrix theory. Without it, calculus as we know it could not have developed. No doubt Descartes was one of the "giants" Newton was referring to in his famous quote.

We shall run quickly through the major steps in creating what are called **Cartesian coordinates**—a system that is also known as **coordinate geometry**—and **vectors**, or **vector geometry**. The previous development is distinguished from the present one by the term **synthetic**, versus the **analytic method** we are about to introduce. It is expected that you will be familiar with most of these ideas. *As you read through this material, you should focus on the deeper question of validity and the existence of a "linear" coordinate system for Euclidean geometry.* Non-Euclidean geometry does not have such a coordinate system.