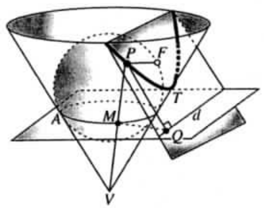
GROUP C

14. Find a way to exhibit the focus-directrix property of a parabola using one sphere tangent to a cone and a plane parallel to one of the cone's line generators, $\overleftrightarrow{AV}$.

THE PARABOLA



15. Prove that the intersection of a right circular cone and sphere whose center lies on the axis of the cone, if nonempty, is either a circle or the union of two circles. (See Problem 11.)

7.4 Volume in E^3

An understanding of the basic concepts in E^3 as introduced in the preceding sections enables us to work with the important topic of volume. We present a modern, axiomatic treatment, where the existence of volume is assumed, obeying desirable properties. Then we will use these properties to derive some of the standard volume formulas. (It might be helpful at this time to review the discussion of area and volume in Section 4.6.)

AXIOMS FOR VOLUME

We assume the same axioms for volume as before, except we shall replace Axiom 4 by the actual formula for the volume of a parallelepiped (height times area of base). As noted earlier (in the problems) we can prove this from the previous list of axioms;[1] assuming this property merely shortens the development.

We consider the following class of three-dimensional objects, whose volumes will be assumed to exist (i.e., **measurable**), to be called **solid regions** (as in Section 4.6):

[1]This is a corollary of Cavalier's Principle and the volume formula $V = Bh$ established in Problem 18, Section 4.6.

(1) Any bounded, convex set in E^3
(2) The complement of a convex set inside a bounded convex set (that is, all the points in a bounded convex set not lying in a convex subset of that convex set)
(3) A finite union or intersection of such sets (1) or (2).

The axioms on volume may now be stated.

1. EXISTENCE POSTULATE To each region T, there corresponds a real number $\text{Vol } T \geq 0$, called its volume.

2. DOMINANCE POSTULATE If $T_1 \subseteq T_2$ then $\text{Vol } T_1 \leq \text{Vol } T_2$.

3. POSTULATE OF ADDITIVITY If $\text{Vol}(T_1 \cap T_2) = 0$, then $\text{Vol}(T_1 \cup T_2) = \text{Vol } T_1 + \text{Vol } T_2$.

4. UNIT OF MEASURE The volume of a parallelepiped is the product of its altitude and the area of its base.

5. CAVALIERI'S PRINCIPLE If all the planes parallel to some fixed plane that meet the regions T_1 and T_2 do so in plane sections having equal areas, whose boundaries are part of the boundaries of the given regions, then $\text{Vol } T_1 = \text{Vol } T_2$ (Figure 7.35).

CAVALIERI'S PRINCIPLE FOR E^3

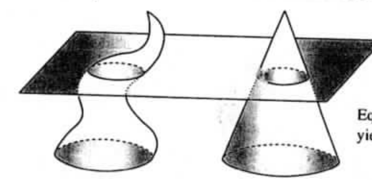


Figure 7.35

Note that there are only five axioms, as compared with six axioms originally given in Section 4.6. The Congruence Postulate was omitted because although it is true, it is simply not very useful in E^3.

VOLUMES OF PRISMS AND CIRCULAR CYLINDERS

We must first take care of a technical detail.

LEMMA: The volume of any planar cross section of a finite union of bounded, convex sets in E^3 is zero.

PROOF

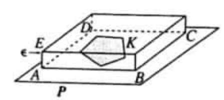


Figure 7.36

Let T be such a finite union, and consider any plane $\boldsymbol{P}$ that cuts T in a set K (Figure 7.36). Since T is bounded, so is K. In fact, K, like T, is also a finite union of bounded, convex sets—all lying in the *same plane*. Thus K is a region and $\text{Vol } K$ exists and is nonnegative. We can find a rectangle $\square ABCD$

in $\boldsymbol{P}$ that contains K in its interior, since K is bounded, and Area $\square ABCD = B > 0$. If we construct a segment AE perpendicular to $\boldsymbol{P}$ of arbitrary (small) length ϵ and consider the box $ABCDE$, then

$$K \subseteq \square ABCD \subseteq \text{Box } ABCDE,$$

and by the Dominance Postulate and Axiom 4,

$$0 \le \text{Vol } K \le \text{Vol(Box } ABCDE) = B\epsilon.$$

Since ϵ is arbitrary and B and Vol K are constant, it follows that Vol $K = 0$.

We can now obtain the following formula, where B = base area and h = altitude,

(1) VOLUME OF A PRISM $\quad V = Bh$

We start with a triangular prism whose base has area B and altitude, of length h. In Figure 7.37 is illustrated such a prism $QABC$, with base $\triangle ABC$ having area B, and altitude (length h) perpendicular to the plane $\boldsymbol{P}(ABC)$. In the base plane, construct a parallelogram $ABCD$ having adjacent sides $\overline{AB}$ and $\overline{BC}$, and in the planes $\boldsymbol{P}(QAD)$ and $\boldsymbol{P}(DCS)$ construct parallelograms $AQTD$ and $CSTD$. In this manner, we construct a parallelepiped $QABCD$, as shown in Figure 7.37. This creates a second triangular prism $QACD$ having the same volume as the original prism $QABC$ (by Cavalieri's Principle). That is,

$$V = \text{Vol}(QABC) = \text{Vol}(QACD)$$

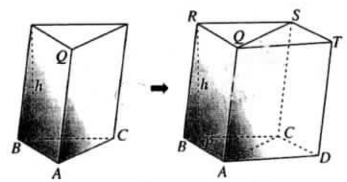


Figure 7.37

Using the theory of area, since $\triangle ABC \cong \triangle ACD$, we obtain Area $\square ABCD$ = 2 Area $\triangle ABC = 2B$.

Now by Postulate 4,

$$\text{Vol}(QABCD) = RP \cdot \text{Area } \square ABCD = 2Bh$$

and by the Postulate of Additivity for volume and the preceding lemma [which takes care of the requirement Vol $(QABC \cap QACD)$ = Vol $(\square ACSQ) = 0$],

$$2Bh = \text{Vol}(QABCD) = \text{Vol}(QABC) + \text{Vol}(QACD) = 2V$$

which gives us the desired result **(1)**.

We extend this formula to any prism in the obvious manner, by subdividing the base polygon (which is convex) into triangles having areas $B_1, B_2, \ldots, B_n$, as shown

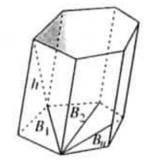


Figure 7.38

in Figure 7.38, which induces a subdivision of the prism into triangular prisms where **(1)** is valid. Thus, again by the Postulate of Additivity and the lemma,

$$V = \text{Vol (Prism)} = B_1h + B_2h + \cdots + B_nh = (B_1 + B_2 + \cdots + B_n)h = Bh$$

where B is the area of the base polygon. This completes the proof of **(1)**.

For the circular cylinder, let $SABD$ be a cylinder with the circle of radius r and center O as base, and altitude $\overline{SE} \perp \boldsymbol{P}$ where $\boldsymbol{P}$ is the plane of the base circle (Figure 7.39). Let a triangle $\triangle PQR$ be determined in plane $\boldsymbol{P}$ having the same area as the circle (i.e., $B = \pi r^2$). (Can you see how to guarantee this?) Let $\overline{PT}$ be the perpendicular to $\boldsymbol{P}$ at P with $SE = TP = h$. It is obvious that a plane section of the cylinder and triangular prism parallel to $\boldsymbol{P}$ are, respectively, congruent to the base of the cylinder and base of the prism, hence have equal areas. By Cavalieri's principle,

$$\text{Volume of prism} = \text{Volume of cylinder} = Bh = \pi r^2h$$

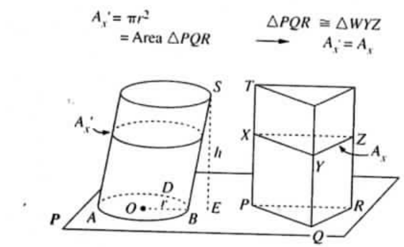


Figure 7.39

The classical result then follows:

(2) VOLUME OF A CIRCULAR CYLINDER $\quad V = \pi r^2h$

VOLUMES OF PYRAMIDS AND CIRCULAR CONES

The program for pyramids and cones can be carried out in much the same manner as for prisms and cylinders. As before, we begin with the formula for the volume of a triangular pyramid. It is evident from Cavalieri's principle that any two triangular pyramids having congruent bases and congruent altitudes have the same volume (see Figure 7.40 for a "proof by picture"). Note that by properties of ratios of parallel seg-

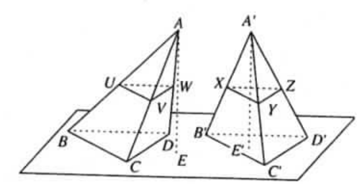


Figure 7.40

ments ($\overline{UV}$ and $\overline{BC}$, etc.), the triangular cross sections have equal areas because $\triangle BCD \cong \triangle B'C'D'$ and $\triangle UVW \cong \triangle XYZ$. This means that we can assume that the given pyramid is a right triangular pyramid, so that two lateral faces lie in planes perpendicular to the base.

From a given right triangular pyramid we can construct a right triangular prism, as shown in Figure 7.41. Two planes can be passed through vertex A that divides the prism into three triangular pyramids as shown in the sequence of diagrams in Figure 7.41. Let the volumes of these three pyramids be designated V_1, V_2, and V_3, where V_1 corresponds to the bottom—and given—pyramid. By choosing the appropriate base and altitude for each pyramid, we can show that $V_1 = V_2 = V_3$: for the top and bottom pyramids, simply use the edges designated h as altitude and the top and base of the prism as bases, which are congruent triangles (area denoted by B). Thus we obtain $V_1 = V_3$. For the top and **middle** pyramids, we use the edge designated h' as common altitude from vertex A, and the side triangles as bases with area denoted B'. Again, since the triangles are congruent, they have the same area. Thus, $V_2 = V_3$. By additivity of volume, $V_1 + V_2 + V_3 = 3V$ where $V_1 = V$ is the volume of the given right triangular pyramid. Therefore, using the formula already established for a prism, $3V = Bh$ and we have arrived at the following formula for a triangular pyramid having altitude h and base area B:

(3) VOLUME OF A TRIANGULAR PYRAMID $\quad V = \frac{1}{3}Bh$

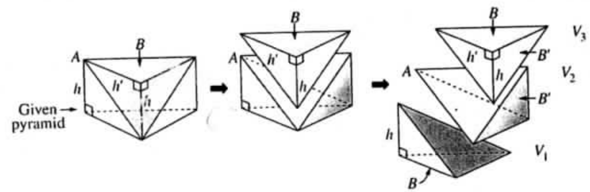


Figure 7.41

The next step is to obtain the formula for the volume V of a circular cone having height h and radius r for the circular base. For convenience, we assume the cone is a right circular cone and that a pyramid with a lateral edge perpendicular to the base and right triangle for base has been constructed so that $\frac{1}{2}b'h' = \pi r^2$, as shown in Figure 7.42. The analysis is similar to that for deducing the volume of a circular

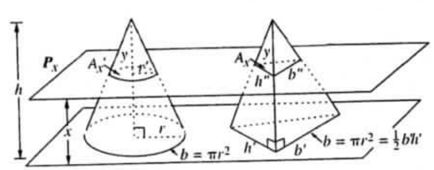


Figure 7.42

cylinder from that of a right prism, so we leave the details for you to fill in (Problem 9). The final result, for a cone having base radius r and altitude h, is:

(4) VOLUME OF A CIRCULAR CONE $\quad V = \frac{1}{3}\pi r^2 h$

THE VOLUME OF A SPHERE, SPHERICAL SEGMENT

We have had a sampling of the usefulness of Cavalieri's principle. It is so powerful that we can deduce the volume of a sphere with relative ease, compared to Archimedes' hard work on the problem. The analysis is illustrated in Figure 7.43. A solid cone with base radius r and height r is removed from a solid circular cylinder with the same radius and height. A plane $\boldsymbol{P}_x$, parallel to the base plane, x units above it, cuts the sphere in a disk of area A_x', and the cylinder minus the cone in an annular ring of area A_x. Using the Pythagorean Theorem for $\triangle OPX$, we have

$$A_x' = \pi s^2 = \pi(r^2 - x^2) = \pi r^2 - \pi x^2$$

$$A_x = \pi r^2 - \pi t^2 \qquad (t/x = r/r = 1 \text{ by similar triangles})$$

$$= \pi r^2 - \pi x^2$$

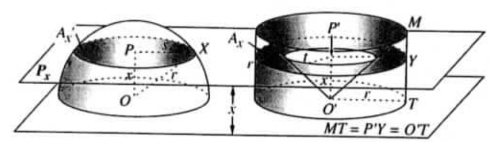


Figure 7.43

Therefore,

$$A_x' = A_x$$

By Cavalieri's principle, with V the volume of the sphere and $\frac{1}{2}V$, the hemisphere,

$$\frac{1}{2}V = \begin{Bmatrix}\text{VOLUME OF CYLINDER}\\ \text{RADIUS } r\text{, HEIGHT } r\end{Bmatrix} - \begin{Bmatrix}\text{VOLUME OF CONE}\\ \text{RADIUS } r\text{, HEIGHT } r\end{Bmatrix}$$

From **(2)** and **(4)**, with $h = r$, we obtain $\frac{1}{2}V = \pi r^2 \cdot r - \frac{1}{3}\pi r^2 \cdot r = \frac{2}{3}\pi r^3$, or

(5) VOLUME OF A SPHERE $\quad V = \frac{4}{3}\pi r^3$ (radius r)

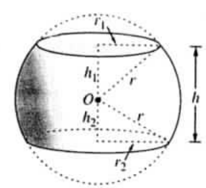


Figure 7.44

The same method can be used to derive the formula for the volume of certain parts of the sphere, such as one important type called a **spherical segment**. This solid is the part of a sphere and interior lying between two parallel planes, whose outer shell is that part of the sphere known as a **zone**, the region between two circles on the sphere in parallel planes. (See Figure 7.44.) The top and bottom of a spherical segment are circular disks—its **bases**. Let the radii of these bases be r_1 and r_2, and let the distance between the parallel planes be h, called the **height**. We first obtain a formula for the special case when one of the planes passes through the center of the

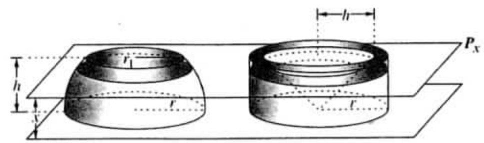


Figure 7.45

sphere (yielding a **hemispherical segment**), as shown in Figure 7.45). We can use Cavalieri's principle as before; the set-up is exactly the same as that of Figure 7.43, when x varies between 0 and h (instead of 0 and r). Since again, $A'_x = A_x$, we have

$$V = \begin{Bmatrix} \text{VOLUME OF CYLINDER} \\ \text{RADIUS } r\text{, HEIGHT } h \end{Bmatrix} - \begin{Bmatrix} \text{VOLUME OF CONE} \\ \text{RADIUS } h\text{, HEIGHT } h \end{Bmatrix}$$

which yields the formula

(6) $$V = \pi r^2 h - \frac{1}{3}\pi h^3$$

To find the formula for the more general spherical segment, having height $h = h_1 + h_2$ and radii r_1 and r_2 of the top and bottom disks, we merely use **(6)** and sum (assuming the bases lie in different hemispheres):

$$\begin{aligned} V_1 + V_2 &= (\pi r^2 h_1 - \tfrac{1}{3}\pi h_1^3) + (\pi r^2 h_2 - \tfrac{1}{3}\pi h_2^3) \\ &= \pi r^2(h_1 + h_2) - \tfrac{1}{3}\pi(h_1^3 + h_2^3) \\ &= \pi r^2 h - \tfrac{1}{3}\pi(h_1 + h_2)(h_1^2 - h_1h_2 + h_2^2) \\ &= \pi r^2 h - \tfrac{1}{3}\pi h(h_1^2 - h_1h_2 + h_2^2) \\ &= \tfrac{1}{3}\pi h(3r^2 - h_1^2 + h_1h_2 - h_2^2) \end{aligned}$$

We want to express the final result in terms of the height h and the two radii r_1, r_2. Some algebra tricks and the Pythagorean relations $r^2 = r_1^2 + h_1^2 = r_2^2 + h_2^2$ are necessary (as indicated in Figure 7.44). By manipulation and substitution,

$$\begin{aligned} V &= \tfrac{1}{6}\pi h[6r^2 - 2h_1^2 + 2h_1h_2 - 2h_2^2] \qquad \text{(divided and multiplied by 2)} \\ &= \tfrac{1}{6}\pi h[3(r^2 - h_1^2) + 3(r - h_2^2) + h_1^2 + 2h_1h_2 + h_2^2] \\ &= \tfrac{1}{6}\pi h[3r_1^2 + 3r_2^2 + (h_1 + h_2)^2] \end{aligned}$$

Hence, if the spherical segment has height h and-base radii r_1 *and* r_2, we obtain

(7) VOLUME OF A SPHERICAL SEGMENT $V = \frac{1}{6}\pi h(3r_1^2 + 3r_2^2 + h^2)$

This same exact formula results when the spherical segment lies in a single hemisphere, where *subtraction* of two hemispherical segments is necessary, instead of the addition used above. In this case, r_1 and r_2 are still the radii of the top and bottom, but $h = h_1 - h_2$. We leave this as a problem for you (Problem 10).

EXAMPLE 1 A hole having diameter 0.5 cm is drilled from a ball bearing having diameter 1.5 cm. Find the volume of the material removed, and the volume remaining in the bearing. Give an approximate answer accurate to three decimals. (See Figure 7.46.)

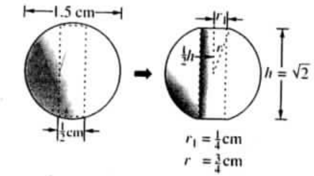


Figure 7.46

SOLUTION

We think of this in terms of removing a right circular cylinder and interior from a spherical segment, whose height would be, by the Pythagorean relation, $2\sqrt{\left(\frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2} = \sqrt{2}$. To find the volume of the part of the ball bearing left we subtract the volumes of spherical segment of radii $r_1 = r_2 = \frac{1}{4}$ and height $h = \sqrt{2}$, and a right circular cylinder of radius $\frac{1}{4}$ and height $\sqrt{2}$. Using **(2)** and **(7)**, that value is given by

$$\begin{aligned} &\frac{1}{6}\pi \cdot \sqrt{2}\left[3 \cdot \left(\frac{1}{4}\right)^2 + 3 \cdot \left(\frac{1}{4}\right)^2 + \sqrt{2}^2\right] - \pi \cdot \left(\frac{1}{4}\right)^2\sqrt{2} \\ &= 19\sqrt{2}\pi/48 - \sqrt{2}\pi/16 = \sqrt{2}\pi/3 \approx 1.481 \text{ cm}^3 \end{aligned}$$

To find the volume of the material drilled out, we merely subtract the above answer from the volume of the ball bearing itself, which is, from **(5)**, $9\pi/16 \approx 1.767$ cm^3. The volume of the part removed is therefore approximately $1.767 - 1.481 \doteq 0.286$ cm^3. ■

Moment for Discovery

The Golden Bracelet Problem .

A jeweler fashions a bracelet made of 14 carat gold whose exterior is a perfect spherical surface. It is constructed by taking as the center of the sphere the point O, as shown in Figure 7.47, and casting the part of the sphere lying outside the

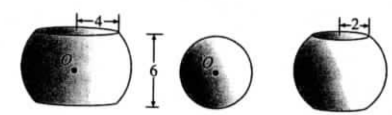


Figure 7.47

right circular cylinder whose axis passes through O. The following calculations devoted to finding how much gold will be used, if correctly computed, may surprise you.

1. Start with an example, as shown in Figure 7.47, where the inside radius $r_1 = 4$ cm, and the width of the bracelet $= h = 6$ cm. Find the volume of the spherical segment formed (you should get 132π cm^3), then subtract the volume of the cylinder. Write down your answer. [The formulas **(2)** and **(7)** are used, just as in Example 1.]
2. Find the volume of a sphere whose diameter equals the width of the bracelet (hence $r = 3$ cm). Did anything happen?
3. Now calculate the volume of gold needed for a bracelet having the same width but half the size as the first one, shown in the figure at right. (Here, $r_1 = 2$ cm and $h = 6$ cm.) Did anything happen?
4. Try to prove the phenomenon in general, using formulas **(2)**, **(5)**, and **(7)**, with arbitrary r_1, r, and h.

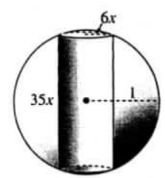
PROBLEMS (7.4)

GROUP A

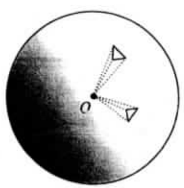
1. Find the volume of a regular tetrahedron of unit side.

2. Find the volume of a regular octahedron of unit side.

3. Find the height of the right circular cylinder inscribed in a sphere having unit radius shown in the figure.



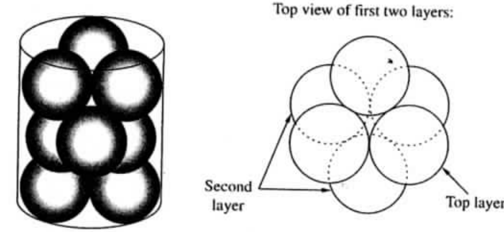
4. The formula for the area of a sphere ($S = 4\pi r^2$—the area of four great circles) can be found by observing that each spherical triangle and approximating Euclidean triangle having as sides the corresponding chords of the sphere can serve as the base of a pyramid whose apex is the center of the sphere. If we sum all these to cover the area of the sphere, we get, in the limit, a relationship between the volume of a sphere and its surface area. Find this relation, and deduce from it the desired formula, using **(5)**.



5. Nine tennis balls 3.5 in. in diameter are packed in three layers of three per layer into a cylindrical can having radius $r \approx 3.771$ in. and height $h \approx 9.215$ in. Nine tennis balls can also be packed into a long cylindrical can with the balls in a straight line, having radius $r = 1.75$ in. and height $h = 31.5$ in.

(a) What is the total space wasted using each of the two packing methods, and which wastes the most total space? (Space wasted = volume of can minus volume of balls inside.)

(b) How much material (total area in square inches) is required for the two types of cans used for packaging, and which one requires the most?



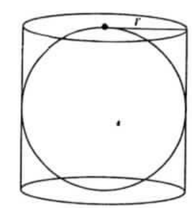
GROUP B

6. Which of the following water tanks will hold the most water when three-fourths full (in terms of the height of the water in the tank)?

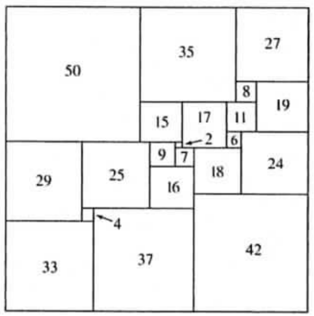
(a) A cylindrical tank with radius 20 ft and height 40 ft (height of water in tank = 30 ft); total surface area given by $S = 2\pi rh + 2\pi r^2$.

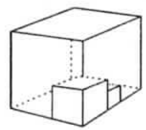
(b) A spherical tank that has the same total surface area ($S = 4\pi r^2$) as the cylindrical tank in **(a)** counting top and bottom. (***Hint:*** First show that the radius r of the spherical tank must be approximately 24.5 ft, then use **(7)** with $h = \frac{3}{4}(2r)$, $r_1 = 0$, and using the Pythagorean theorem to find r_2.)

7. Deduce Archimedes' relation (engraved on his tomb) between a sphere and its circumscribed cylinder, as shown in the figure, by finding the ratios V/V' (volume) and S/S' (total surface area) in each case.



8. Unlike the square, the cube cannot be filled with a finite number of cubes of all different sizes. In the case of a square, a finite number (of squares) is possible. In fact, the optimal number is 21, shown in the figure below. Such a configuration is called a **perfect squared square**. (This is only a recent discovery. It was long thought that a 24-piece tessellation was the optimal one for a square; a long standing theorem states that *at least* 21 squares are required.)





(a) Verify this construction for the square.

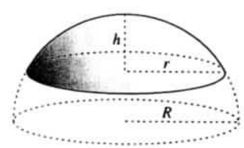
(b) Prove that *no finite tessellation is possible* for a cube in E^3 using cubes.

9. Complete the details for the volume of a circular cone, as outlined in the text preceding **(4)**. (***Hint:*** By similar triangles, $r'/r = y/h = h''/h' = b''/b'$, etc. Prove that $A_x' = A_x$.)

GROUP C

10. Complete the proof of **(7)** for the case when the bases of the segment lie in the same hemisphere. (***Hint:*** You will need the algebra rule $(h_1 - h_2)(h_1^2 + h_1h_2 + h_2^2) = h_1^3 - h_2^3$ for this case.)

11. A **spherical cap** is the part of a sphere and interior lying on one side of a plane, sometimes called a **segment of one base**. Let the radius of the base be r, and its height, h. Show that the volume is given by the formula $V = \frac{1}{6}\pi h(3r^2 + h^2) = \frac{1}{3}\pi h^2(3R - h)$, where R is the radius of the sphere. (See the figure.) (***Hint:*** From the Pythagorean Theorem can be derived the needed relation $r^2 = 2Rh - h^2$. Verify this as part of your solution.)



*7.5 Coordinates, Vectors, and Isometries in E^3

As might be expected, the construction of a model for E^3 requires the use of ordered triples of real numbers, (x, y, z), as illustrated in Figure 7.48. Since you are probably already acquainted with the mechanics of a three-dimensional coordinate system, we will just outline the basic procedure, paying close attention to the incidence axioms

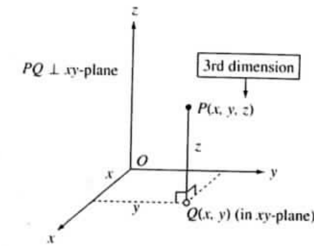


Figure 7.48

involving the interplay between lines and planes, and how all that works out in a three-dimensional coordinate system. We choose to view this as a model, where numbers and equations *represent* geometric objects, instead of rigorously constructing a coordinate system from the geometric objects—as we did for the plane in Section 4.7.

INCIDENCE AXIOMS, DISTANCE AND ANGLE MEASURE

Using coordinates, we construct representations of the undefined terms and basic objects with which the axioms of E^3 are concerned, thereby obtaining a three-dimensional model for axiomatic Euclidean geometry.

POINT (x, y, z) for real numbers x, y, and z

LINE The set of all points (x, y, z) such that

$$x = at + x_0, \qquad y = bt + y_0, \qquad z = ct + z_0$$

for real t (the **parameter**), and a certain triple of constants $[a, b, c]$ not all zero, called the **direction** of the line, and some point (x_0, y_0, z_0) on it (Figure 7.49).

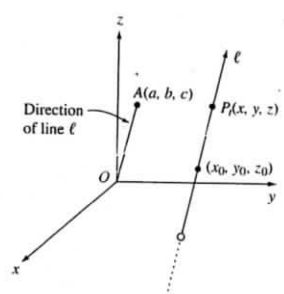


Figure 7.49