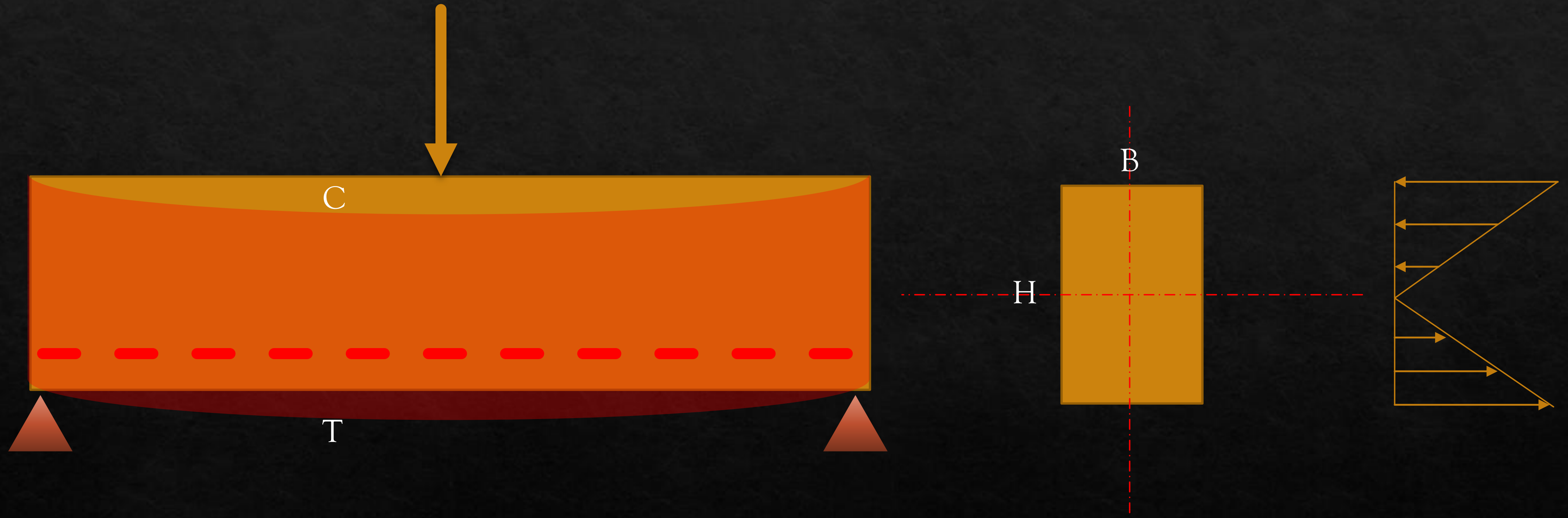
جامعة بيرزيت
BIRZEIT UNIVERSITY

Reinforced Concrete Design I
ENCE 335

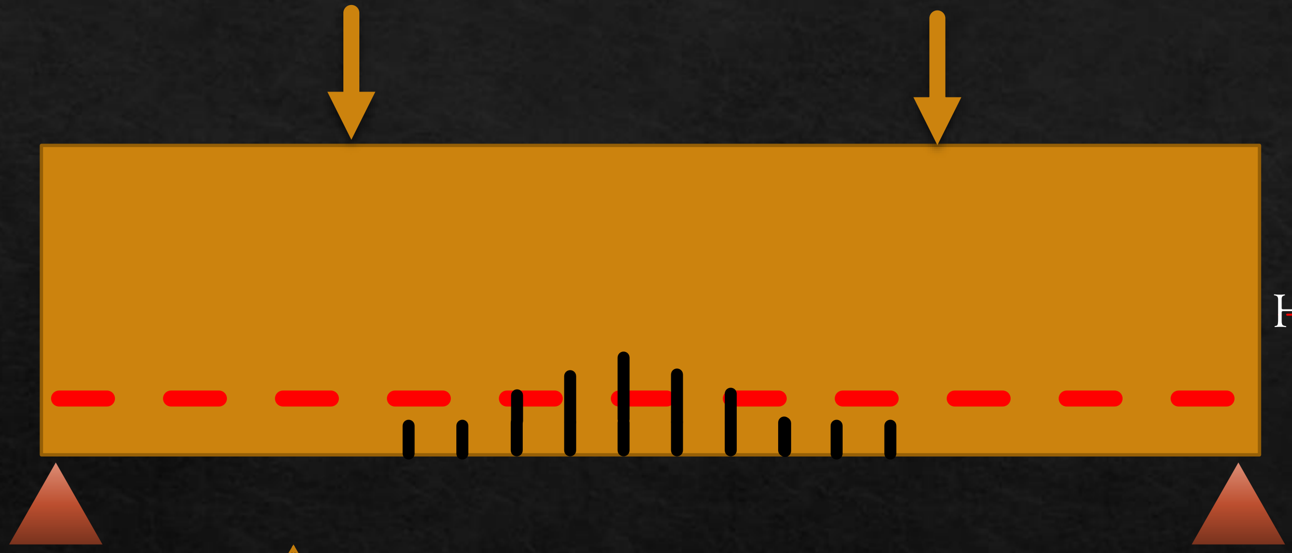
Flexural analysis and design of beams

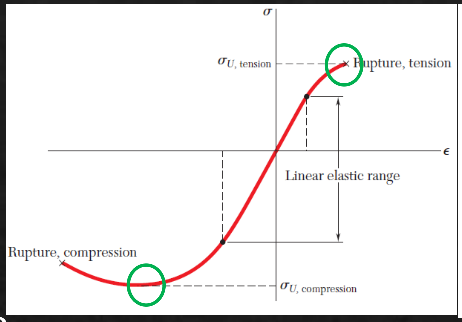
Dr. Khalil M. Qatu

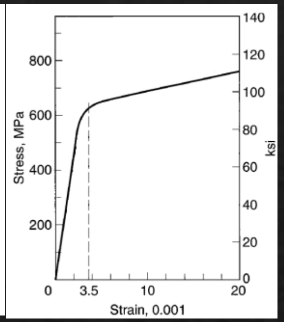
General Beam behavior

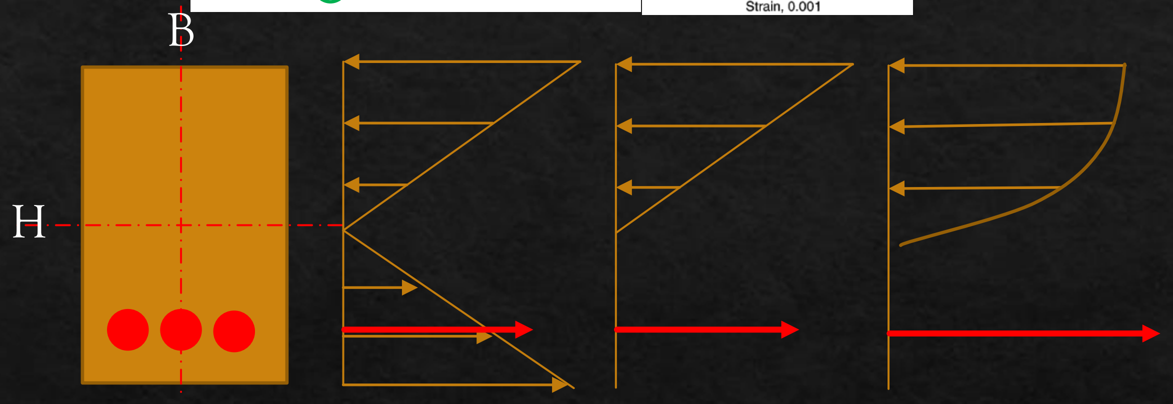


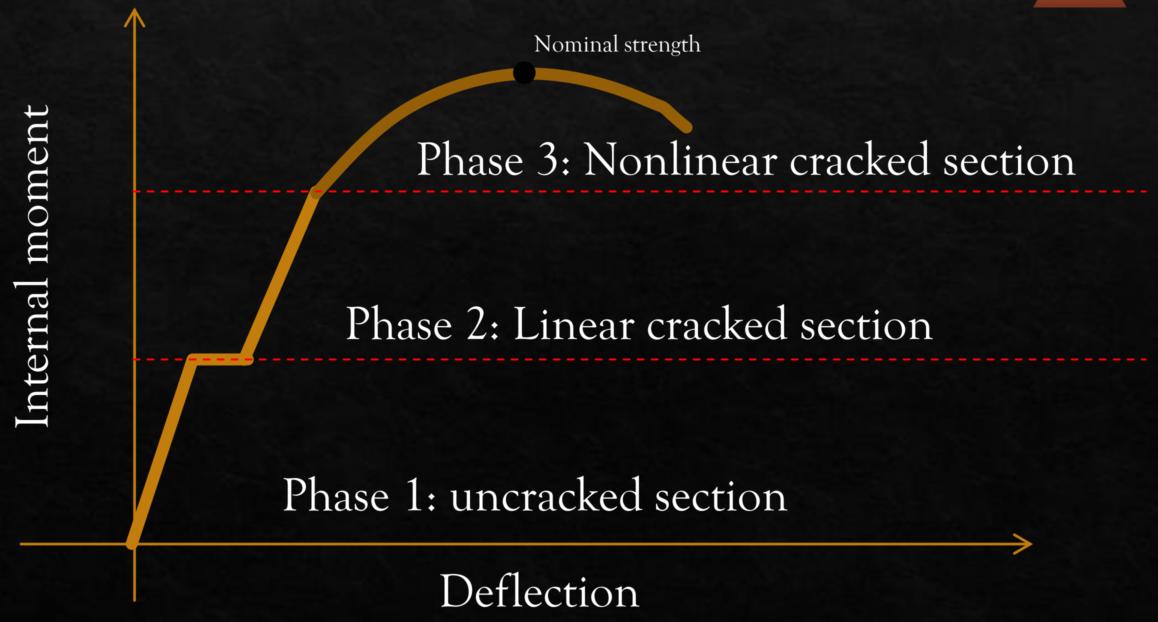
RC Beam Behavior

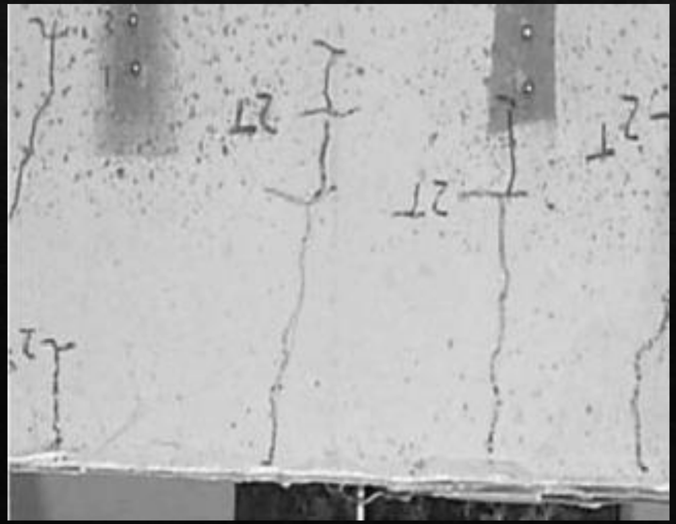












RC Beam Behavior

- Phase 1: Linear elastic uncracked section
  - Tension stress in concrete is less than $f_r = 0.62\,\lambda\sqrt{f_c'}$
  - The whole section need to be transformed using modular ratio to calculate the stresses

$$n = \frac{E_s}{E_c}$$

E_s: *Modulus of elasticity of steel* (200 *GPa*)

E_c: *Modulus of elasticity of concrete* $\left(E_c = 4700\sqrt{f_c'}\ MPa\right)$

  - Stresses in concrete can be found using the flexure formula

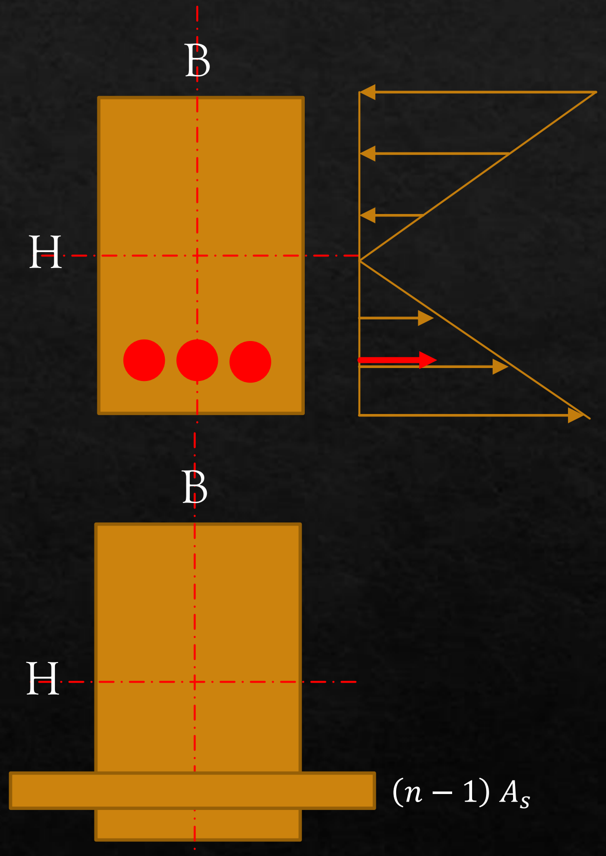
$$\sigma = \frac{M\,y}{I}$$

  - Actual stress in the steel bars equals the stress in the concrete at the same level multiplied by n

$$f_s = n\,\sigma_c$$

  - The cracking moment M_{cr} is the minimum moment that causes the first hairline cracks in the tension side

$$M_{cr} = \frac{f_r\,I}{(H - \bar{y})}$$



RC Beam Behavior

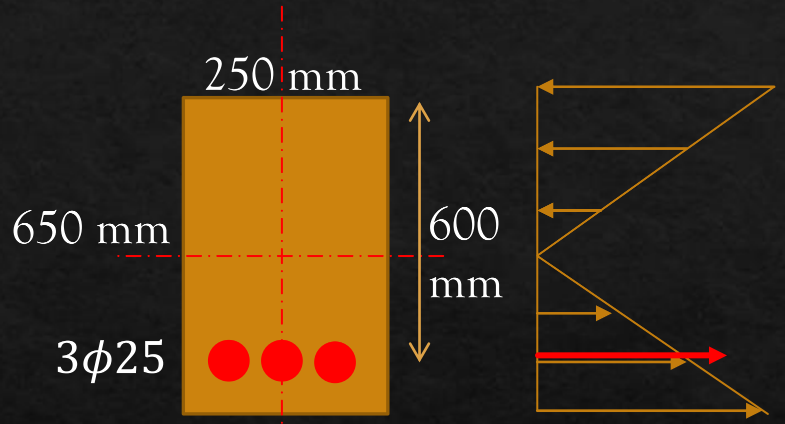
- Phase 1: Linear elastic uncracked section

  Example: $f_c' = 28\ MPa\ , F_y = 420\ MPa$

  If the applied positive moment in the section is 60 kN.m,

  Calculate the maximum tension and compression stress in concrete the stress in the steel bars.

  Calculate the maximum moment the beam can support before cracking



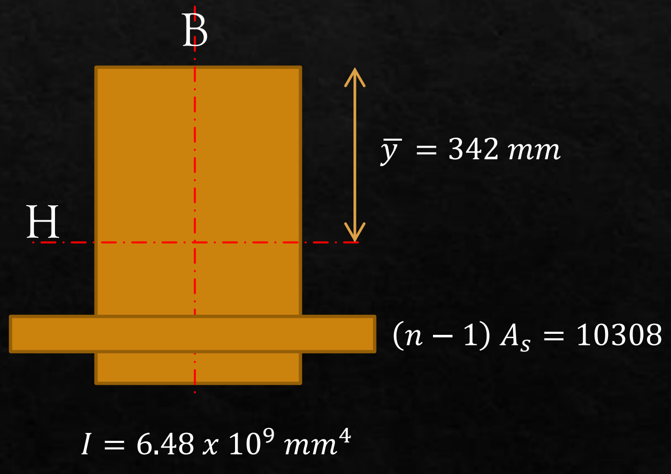
Solution:

Transform the section and find geometric properties ($\bar{y}$, I)

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A}$$

$$I = \Sigma[I'' + AD^2]$$

Part #	A (mm^2)	$\bar{y}$ (mm)	$A\bar{y}$ (mm^3)
1	250x650=162500	650/2=325	52812500
2	10308	600	6185010
sum	172808		58997510



Part #	I'' (mm^4)	A (mm^2)	D (mm)	AD^2 (mm^3)	$I'' + AD^2$ (mm^4)
1	$\frac{1}{12} 250 * 650^3 = 5.72 * 10^9$	162500	342-325=17	$2.8 * 10^6$	$5.72 * 10^9$
2	0	10308	600-342=258	$0.69 * 10^9$	$0.69 * 10^9$
sum					$6.48 * 10^9$

RC Beam Behavior

- Phase 1: Linear elastic uncracked section

Example: $f_c' = 28\ MPa\,, F_y = 420\ MPa$

If the applied positive moment in the section is 60 kN.m,

Calculate the maximum tension and compression stress in concrete the stress in the steel bars.

Calculate the maximum moment the beam can support before cracking

Solution:

Calculate the stresses in concrete

- Maximum compressive stress: Where ???

$$\sigma = \frac{M\,y}{I} = \frac{60 * 10^6 * 342}{6.48 * 10^9} = 3.17\ MPa$$

- Maximum tension stress: Where ???

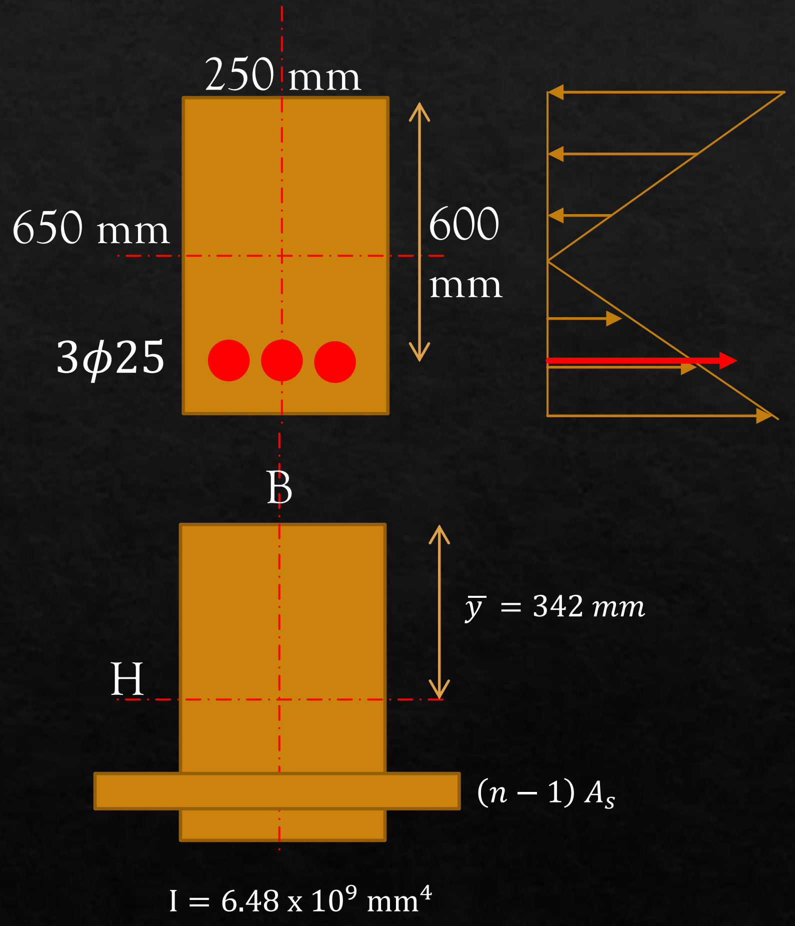
$$\sigma = \frac{M\,y}{I} = \frac{60 * 10^6 * (650 - 342)}{6.48 * 10^9} = 2.85\ MPa\ < f_r = 0.62\sqrt{f_c'} = 3.28$$

Calculate the stress in the steel bars

$$f_s = n\,\sigma_c = n\frac{M\,y}{I} = 8 * \frac{60 * 10^6 * (600 - 342)}{6.48 * 10^9} = 19.1\ MPa$$

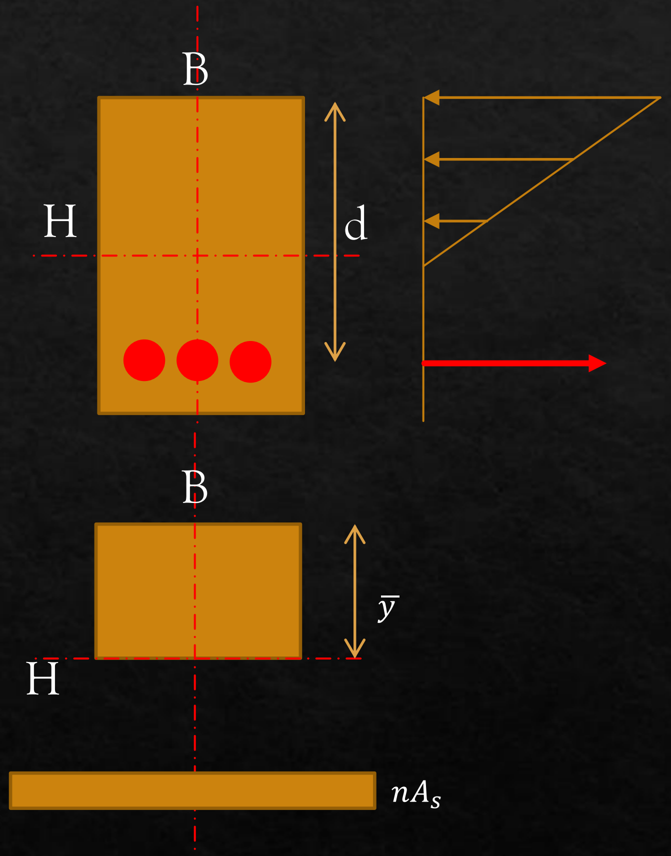
Calculate the cracking moment

$$M_{cr} = \frac{f_r I}{y} = \frac{3.28 * 6.48 * 10^9}{650 - 342} = 69\ kN.m$$



RC Beam Behavior

- Phase 2: Linear cracked section ($M > M_{cr}$)
  - All the concrete in the tension side is ignored
  - The N.A moves upward due to crack propagation
  - Stresses are found using the flexure formula with new geometric properties
  - Tension the section is only carried by the steel



RC Beam Behavior

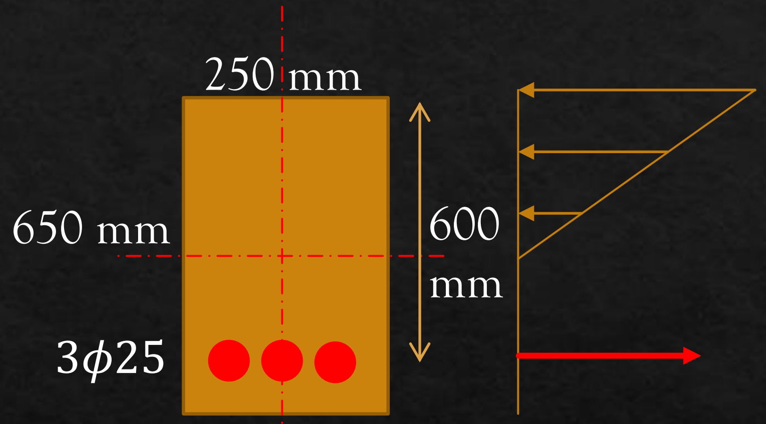
- Phase 2: Linear cracked section ($M > M_{cr}$)

Example: $f_c' = 28\ MPa\ , F_y = 420\ MPa$

If the applied positive moment in the section is 120 kN.m,

Calculate the maximum compression stress in concrete the stress in the steel bars.

Compare the stress in the steel now with the stress in steel in the previous example



Solution:

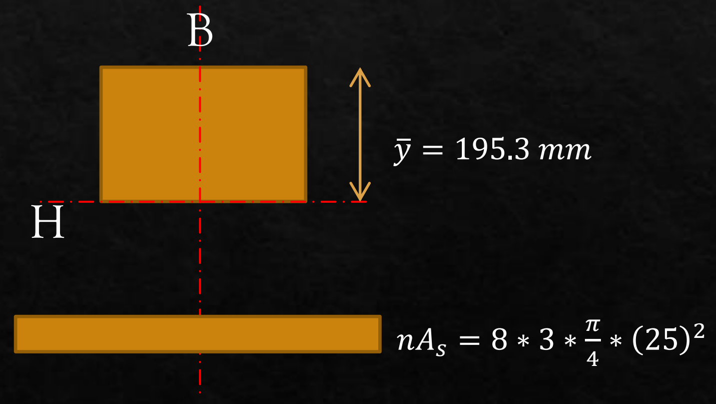
Transform the section and find geometric properties ($\bar{y}$, I)

$$\bar{y} = \frac{\Sigma Ay}{\Sigma A} = \bar{y} = \frac{125\bar{y}^2 + 7068583}{250\ \bar{y} + 11781}$$

$$\rightarrow 125\ \bar{y}^2 + 11781\bar{y} - 7068583 = 0$$

$$\rightarrow \bar{y} = 195.3, \sout{-289.5}$$

Part #	A (mm^2)	$\bar{y}_i$ (mm)	$A\bar{y}$ (mm^3)
1	250 $\bar{y}$	$\frac{\bar{y}}{2}$	$125\bar{y}^2$
2	11781	600	7068583
sum	250 $\bar{y}$+11781		$125\bar{y}^2$+7068583



$I = 2.55 \text{ x } 10^9 \text{ mm}^4$

Part #	I'' (mm^4)	A (mm^2)	D (mm)	AD^2 (mm^3)	$I'' + AD^2$ (mm^4)
1	$\frac{250 * 195.3^3}{3}$	-	-	-	$0.62 * 10^9$
2	0	11781	600-195.3	$1.93 * 10^9$	$1.93 * 10^9$
sum					$2.55 * 10^9$

RC Beam Behavior

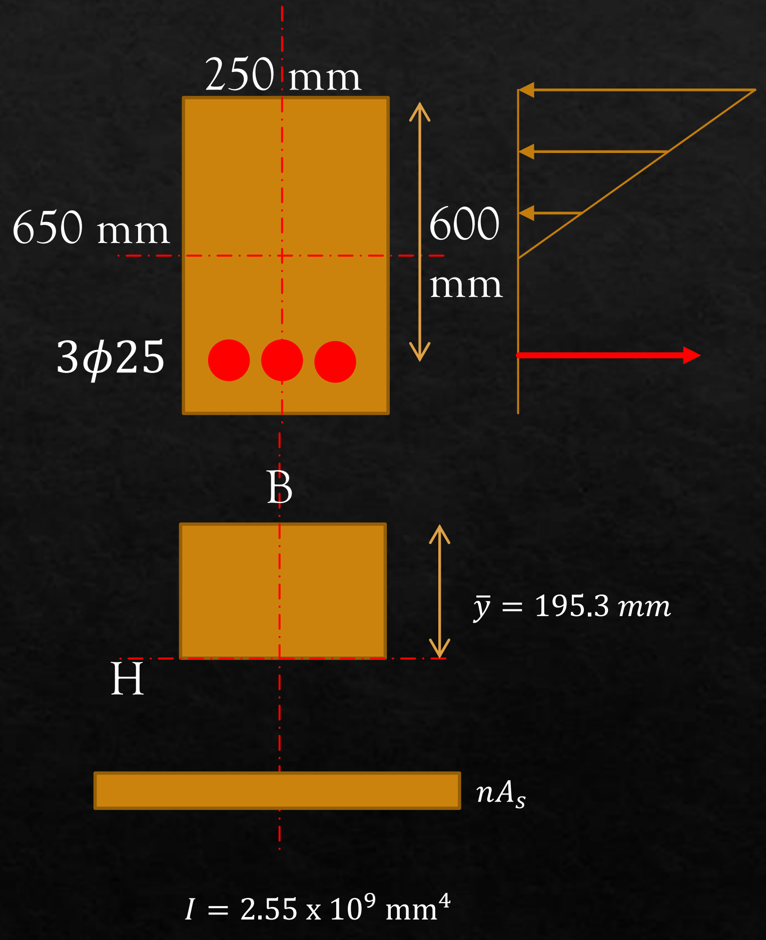
- Phase 2: Linear cracked section ($M > M_{cr}$)

Example: $f'_c = 28\ MPa$, $F_y = 420\ MPa$

If the applied positive moment in the section is 120 kN.m,

Calculate the maximum compression stress in concrete the stress in the steel bars.

Compare the stress in the steel now with the stress in steel in the previous example



Solution:

Calculate the max stress in the concrete: TOP

$$\sigma_c = \frac{M\ y}{I} = \frac{120 * 10^6 * 195.3}{2.55\ * 10^9} = 9.2\ MPa$$

Calculate the stress in the steel:

$$f_s = n\ \sigma_c = 8 * \frac{120 * 10^6\ * (600 - 195.3)}{2.55 * 10^9} = 152.4\ MPa$$

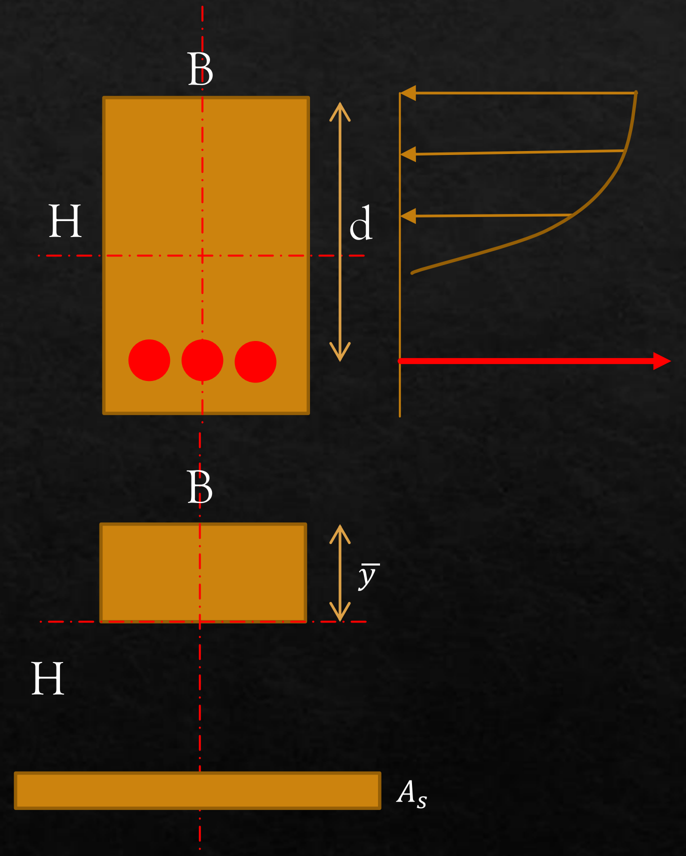
RC Beam Behavior

- Phase 3: Nonlinear cracked section (Nominal strength)
  - Under-reinforced: Steel fails (yields) before concrete crushed
  - Balanced: steel and concrete fail simultaneously
  - Over-reinforced: concrete (crushed) fails before steel yields

- Reinforcement ratio (ρ) governs failure mode

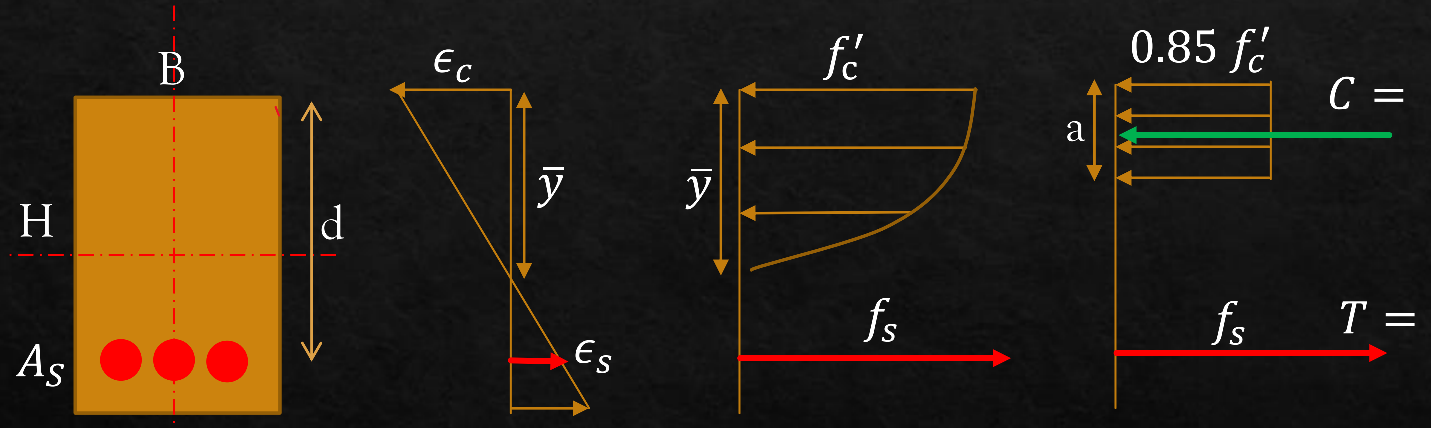
$$\rho = \frac{A_s}{Bd}$$

d: effective depth



RC Beam Behavior

- Phase 3: Nonlinear cracked section (Nominal strength)
  - Whitney Block distribution



$$a = \beta_1\ \bar{y}$$

$$\beta_1 = 0.85\ - 0.05\left[\frac{f_c' - 28}{7}\right]$$

$$0.65 \leq \beta_1 \leq 0.85$$

Can we apply the flexure formula ??

<u>*force equilibruim and strain distribution and Nominal strength*</u>

<u>*Force equilibrium*</u>

$$T = C$$

$$A_s f_S = 0.85 * f_c * a\ * B$$

$$a = \frac{A_s f_S}{0.85\ f_c\ B}$$

<u>*Nominal strength*</u>

$$M_n = Force\ * arm$$

$$M_n = A_s f_y\ * \left[d - \frac{a}{2}\right]$$

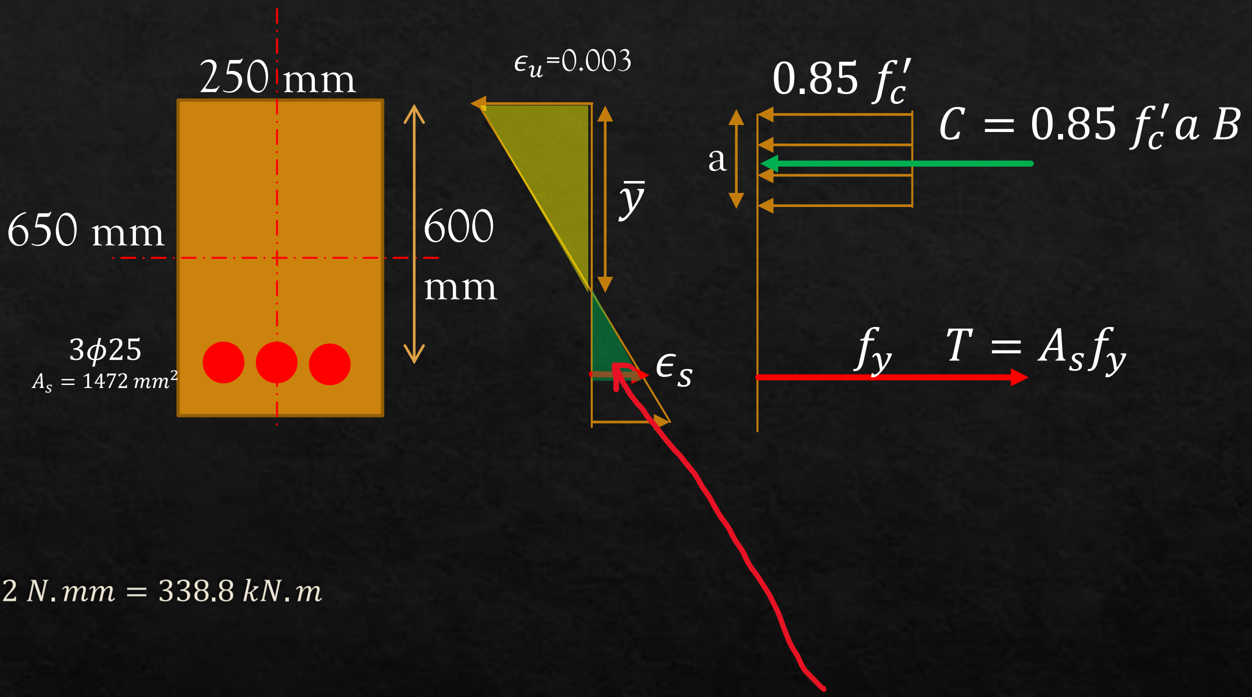
RC Beam Behavior

- Phase 3: Nonlinear cracked section (Nominal strength)

  Example: $f_c' = 28\ MPa$, $F_y = 420\ MPa$

  Calculate the Nominal moment capacity of the given section

  Calculate the strain in concrete and steel at failure



Solution:

Force equilibrium:

$$a = \frac{A_s f_y}{0.85 f_c' B} = \frac{1472 * 420}{0.85 * 28 * 250} = 103.9\ mm$$

$$M_n = A_s f_y \left(d - \frac{a}{2}\right) = 1472 * 420 * \left(600 - \frac{103.9}{2}\right) = 338{,}826{,}432\ N.mm = 338.8\ kN.m$$

At failure, the strain in concrete is ϵ_u= 0.003: the corresponding strain in steel at failure can be found by comparing the two triangles shown

$$\frac{\epsilon_u}{\bar{y}} = \frac{\epsilon_s}{d - \bar{y}} \rightarrow \epsilon_s = 0.0117 \gg \epsilon_y \ (\,warning??)$$

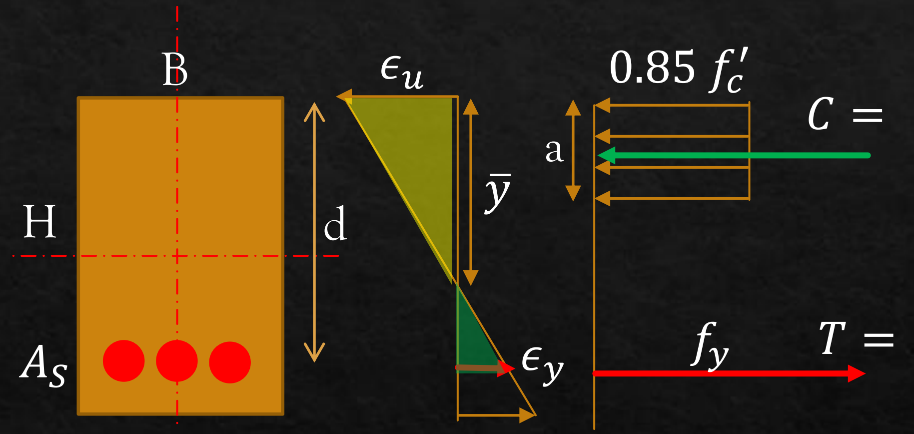
RC Beam Behavior

- Phase 3: Nonlinear cracked section (Nominal strength)
  - Balanced sections ($\epsilon_c = \epsilon_u \, , \epsilon_s = \epsilon_y$)

$$\epsilon_y = \frac{f_y}{E} \rightarrow for \; f_y = 420 \; MPa \; \rightarrow \epsilon_y = 0.0021$$

$$\epsilon_u = 0.003$$

  - Find the balanced steel ratio (ρ_b):



Force equilibrium

$$T = C \rightarrow a = \frac{A_s f_y}{0.85 \; f_c' B} \; \dots (1)$$

$$\rho = \frac{A_s}{Bd} \rightarrow A_s = \rho B d \; \dots (2)$$

Combine equation 1 to 4

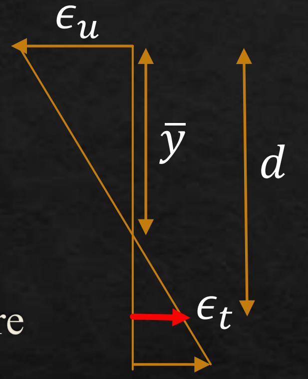
Similar triangles:

$$\frac{\epsilon_u}{\bar{y}} = \frac{\epsilon_y}{d - \bar{y}} \rightarrow \bar{y} = d \frac{\epsilon_u}{\epsilon_u + \epsilon_y} \; \dots (3)$$

$$a = \beta_1 \bar{y} \; \dots (4)$$

$$\rho_b = 0.85 \; \beta_1 \frac{f_c'}{f_y} \frac{\epsilon_u}{\epsilon_u + \epsilon_y}$$

ACI provisions for design of rectangular RC Beams



- Reinforcement limits
  - ACI requires tension-controlled design
  - Main reinforcing steel must yield well before crushing of concrete
  - ACI code specifies the maximum reinforcement ratio to be used in a rectangular section that ensures the concrete failure ($\epsilon_c = \epsilon_u = 0.003$) when the steel reaches a strain of $\epsilon_s \geq \epsilon_y = 0.004$.

$$\rho_{max} = \rho_{0.004} = 0.85\,\beta_1 \frac{f_c'}{f_y} \frac{0.003}{0.003+0.004}$$ (where in the code ??)

  - ACI code also specifies a minimum reinforcement ration for flexural members (ρ_{min})

Table A4

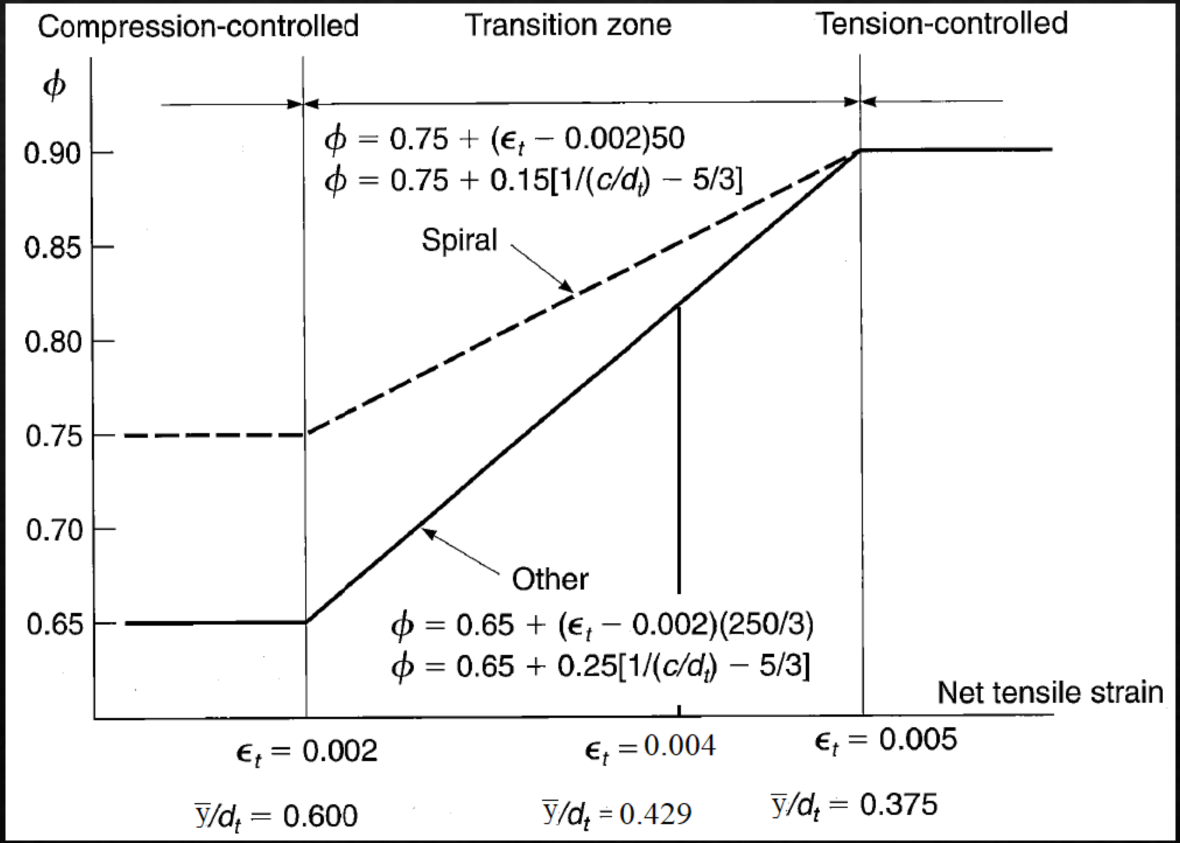
$$\max\left[\frac{1.4}{f_y}, \frac{0.25\sqrt{f_c'}}{f_y}\right]$$ (where in the code ??)

  - An easy to use parameter to check if we didn't exceed the reinforcement limit is using the ratio between the N.A location to the steel location

$$\frac{\bar{y}}{d} = \frac{\epsilon_u}{\epsilon_u + \epsilon_t}$$

ACI provisions for design of rectangular RC Beams

- Strength reduction factor (ϕ)
  - The reduction factor for flexural members depends on the tensile strain in the steel
  - For tensile strains (ϵ_t) more than 0.005 the reduction factor (ϕ) is 0.9
  - For tensile strains (ϵ_t) between 0.004 and 0.005 the reduction factor (ϕ) varies according to the shown equation
  - Although it is allowed to reach 0.004 tensile strain, it is not recommended to go below 0.005 tensile strain, why???



ACI provisions for design of rectangular RC Beams

- Serviceability requirements
  - ACI sets a limit for the minimum depth of the beam based on span continuity
  - Overall beam depth h shall satisfy the limits in Table 9.3.1.1, unless the calculated deflection limits are satisfied.
  - This minimum depth requirement usually result in oversized beam
  - We usually use the deflection as control parameter in the design

Table 9.3.1.1—Minimum depth of nonprestressed beams

Support condition	Minimum h[1]
Simply supported	$\ell/16$
One end continuous	$\ell/18.5$
Both ends continuous	$\ell/21$
Cantilever	$\ell/8$

[1]Expressions applicable for normalweight concrete and f_y = 420 MPa. For other cases, minimum h shall be modified in accordance with 9.3.1.1.1 through 9.3.1.1.3, as appropriate.

Table 24.2.2—Maximum permissible calculated deflections

Member	Condition		Deflection to be considered	Deflection limitation
Flat roofs	Not supporting or attached to nonstructural elements likely to be damaged by large deflections		Immediate deflection due to maximum of L_r, S, and R	$\ell/180$[1]
Floors	Immediate deflection due to L		$\ell/360$	
Roof or floors	Supporting or attached to nonstructural elements	Likely to be damaged by large deflections	That part of the total deflection occurring after attachment of nonstructural elements, which is the sum of the time-dependent deflection due to all sustained loads and the immediate deflection due to any additional live load[2]	$\ell/480$[3]
		Not likely to be damaged by large deflections		$\ell/240$[4]

[1]Limit not intended to safeguard against ponding. Ponding shall be checked by calculations of deflection, including added deflections due to ponded water, and considering time-dependent effects of sustained loads, camber, construction tolerances, and reliability of provisions for drainage.

[2]Time-dependent deflection shall be calculated in accordance with 24.2.4, but shall be permitted to be reduced by amount of deflection calculated to occur before attachment of nonstructural elements. This amount shall be calculated on basis of accepted engineering data relating to time-deflection characteristics of members similar to those being considered.

[3]Limit shall be permitted to be exceeded if measures are taken to prevent damage to supported or attached elements.

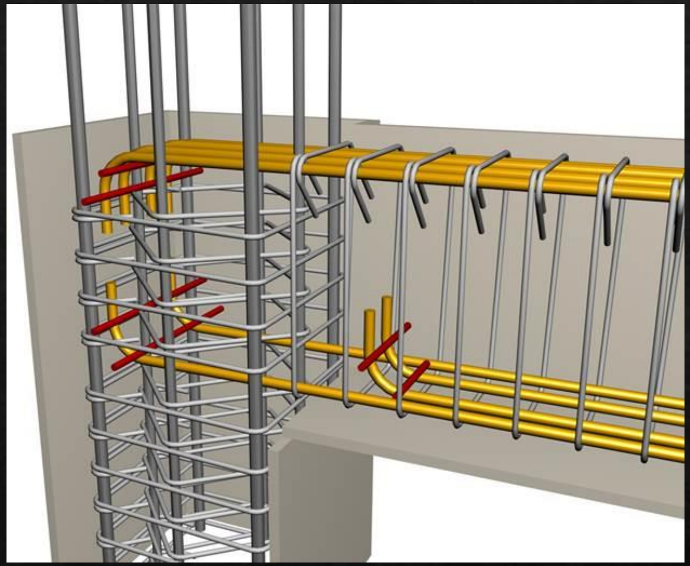
[4]Limit shall not exceed tolerance provided for nonstructural elements.

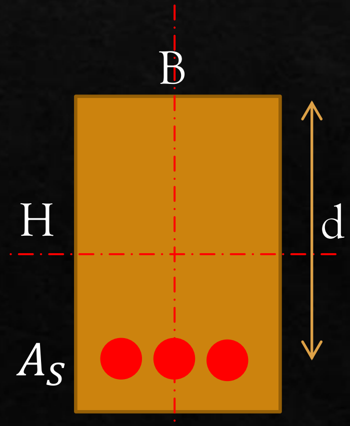
ACI provisions for design of rectangular RC Beams

- Serviceability requirements
  - ACI code requires that reinforcing bars should not be placed too close to each other NOR too far from each other
    - Placing bars too close will result in concrete not filling all voids during construction
      - Minimum spacing between bars

$$S_{min} = \max\left[25mm, d_b, \frac{4}{3} d_{agg}\right]$$

    - Placing bars too far from each other will result in larger concrete cracks
      - Will go over this limit in later chapters
      - Generally; it is better to have more bars with smaller diameter
    - Concrete clear cover of 40 mm for members not subjected to weather and soil





Design of rectangular beams

Design

- **Known beam dimensions** → Calculate the required reinforcement
  - Force equilibrium and Nominal strength
    - $a = \frac{A_s f_y}{0.85 f'_c B}$
    - $M_u = \phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right)$
  - Reinforcement ratio and Nominal strength
    - $M_u = \phi M_n = \phi \rho f_y B d^2 \left(1 - 0.59 \frac{\rho f_y}{f'_c}\right)$
- **Unknown beam dimensions** → Set the reinforcement ratio $0.5\rho_{max} < \rho < 0.75 \rho_{max}$ → Calculate $\frac{M_u}{\phi b d^2} = \phi \rho f_y \left(1 - 0.59 \frac{\rho f_y}{f'_c}\right)$ → Find an adequate section (B,d) d=[1.5-3]B

Design of rectangular beams

- Design aids
  - Flexural resistance factor R

$$R = \frac{M_n}{\phi b d^2} = \rho\, f_y \left(1 - 0.59 \frac{\rho\, f_y}{f_c'}\right)$$

TABLE A.2
Areas of groups of bars, mm^2

Bar No. SI	Bar No. Inch-Pound	Number of Bars: 1	2	3	4	5	6	7	8	9	10	11	12
10	3	71	142	213	284	355	426	497	568	639	710	781	852
13	4	129	258	387	516	645	774	903	1,032	1,161	1,290	1,419	1,548
16	5	199	398	597	796	995	1,194	1,393	1,592	1,791	1,990	2,189	2,388
19	6	284	568	852	1,136	1,420	1,704	1,988	2,272	2,556	2,840	3,124	3,408
22	7	387	774	1,161	1,548	1,935	2,322	2,709	3,096	3,483	3,870	4,257	4,644
25	8	510	1,020	1,530	2,040	2,550	3,060	3,570	4,080	4,590	5,100	5,610	6,120
29	9	645	1,290	1,935	2,580	3,225	3,870	4,515	5,160	5,805	6,450	7,095	7,740
32	10	819	1,638	2,457	3,276	4,095	4,914	5,733	6,552	7,371	8,190	9,009	9,828
36	11	1,006	2,012	3,018	4,024	5,030	6,036	7,042	8,048	9,054	10,060	11,066	12,072
43	14	1,452	2,904	4,356	5,808	7,260	8,712	10,164	11,616	13,068	14,520	15,972	17,424
57	18	2,581	5,162	7,743	10,324	12,905	15,486	18,067	20,648	23,229	25,810	28,391	30,972

TABLE A.8
Minimum number of bars as a single layer in beam stems governed by crack control requirements of the ACI Code

(*a*) 50 mm clear cover, sides and bottom

Minimum Number of Bars as a Single Layer of a Beam Stem

Bar No. SI	Bar No. Inch-Pound	Beam Stem Width b_w, mm: 200	250	300	350	400	450	500	550	600	650	700	750	800	850	900
10–43	3–14	1	1	2	2	3	3	3	3	3	4	4	4	4	4	5
57	18	1	1	2	2	2	3	3	3	3	3	4	4	4	4	4

(*b*) 40 mm clear cover, sides and bottom

Minimum Number of Bars as a Single Layer of a Beam Stem

Bar No. SI	Bar No. Inch-Pound	Beam Stem Width b_w, mm: 200	250	300	350	400	450	500	550	600	650	700	750	800	850	900
10–13	3–14	1	1	2	2	3	3	3	3	3	4	4	4	4	4	4
16–43	5–14	1	1	2	2	3	3	3	3	3	3	4	4	4	4	4
57	18	1	1	2	2	2	3	3	3	3	3	4	4	4	4	4

TABLE A.5a
Flexural resistance factor: $R = \rho f_y \left(1 - 0.588 \frac{\rho f_y}{f_c'}\right)$ MPa

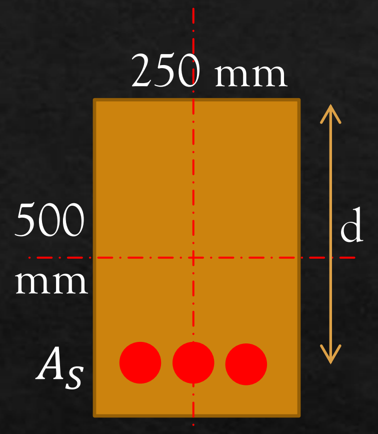
	f_y = 280 MPa					f_y = 420 MPa				
	f_c', MPa					f_c', MPa				
ρ	21	28	35	42	49	21	28	35	42	49
0.0005	0.14	0.14	0.14	0.14	0.14	0.21	0.21	0.21	0.21	0.21
0.0010	0.28	0.28	0.28	0.28	0.28	0.42	0.42	0.42	0.42	0.42
0.0015	0.42	0.42	0.42	0.42	0.42	0.62	0.62	0.62	0.62	0.63
0.0020	0.55	0.55	0.55	0.56	0.56	0.82	0.83	0.83	0.83	0.83
0.0025	0.69	0.69	0.69	0.69	0.69	1.02	1.03	1.03	1.03	1.04
0.0030	0.82	0.83	0.83	0.83	0.83	1.22	1.23	1.23	1.24	1.24
0.0035	0.95	0.96	0.96	0.97	0.97	1.41	1.42	1.43	1.44	1.44
0.0040	1.08	1.09	1.10	1.10	1.10	1.60	1.62	1.63	1.64	1.65
0.0045	1.22	1.23	1.23	1.24	1.24	1.79	1.81	1.83	1.84	1.85
0.0050	1.35	1.36	1.37	1.37	1.38	1.98	2.01	2.03	2.04	2.05
0.0055	1.47	1.49	1.50	1.51	1.51	2.16	2.20	2.22	2.24	2.25
0.0060	1.60	1.62	1.63	1.64	1.65	2.34	2.39	2.41	2.43	2.44
0.0065	1.73	1.75	1.76	1.77	1.78	2.52	2.57	2.60	2.63	2.64
0.0070	1.85	1.88	1.90	1.91	1.91	2.70	2.76	2.79	2.82	2.84
0.0075	1.98	2.01	2.03	2.04	2.05	2.87	2.94	2.98	3.01	3.03
0.0080	2.10	2.13	2.16	2.17	2.18	3.04	3.12	3.17	3.20	3.22
0.0085	2.22	2.26	2.28	2.30	2.31	3.21	3.30	3.36	3.39	3.42
0.0090	2.34	2.39	2.41	2.43	2.44	3.38	3.48	3.54	3.58	3.61
0.0095	2.46	2.51	2.54	2.56	2.58	3.54	3.66	3.72	3.77	3.80
0.0100	2.58	2.64	2.67	2.69	2.71	3.71	3.83	3.90	3.95	3.99

Design of rectangular beams

- Example: known dimensions

The given cross-section is subjected to an ultimate moment $M_u = 120\ kN.m$. What is the required area of steel

$f_c' = 28\ MPa$ & $f_y = 420\ MPa$



Design of rectangular beams

◈ Example: unknown dimensions

Design a simply supported beam with a span of 4.5m that supports a total dead load DL=20 kN/m (NOT including self-weight) and total live load LL=31 kN/m. use $\rho = 0.5\ \rho_{max}$

$f_c' = 28\ MPa\ \&\ f_y = 420\ MPa$