Energy Methods

We will show how to apply energy methods to solve problems involving deflection. Work and strain energy will be discussed, followed by a development of the principle of conservation of energy. The method of virtual work and Castigliano's theorem are then developed, and these methods are used to determine the slope and deflection at points on structural members.

External work and strain energy

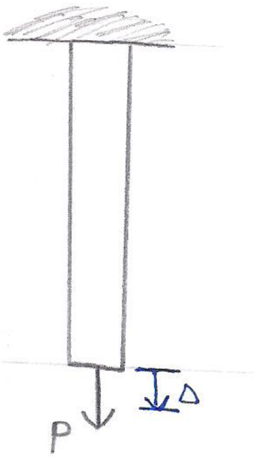
We will define the work caused by an external force and couple moment and show how to express this work in terms of a body's strain energy.

Work of a force

A force does work when it undergoes a displacement dx that is in the same direction as the force. The work done is a scalar, defined as $dU_e = F\,dx$, If the total displacement is Δ, the work becomes

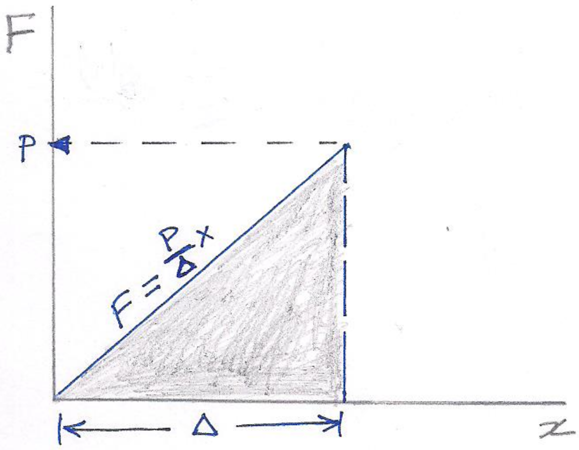
$$U_e = \int_0^{\Delta} F\,dx$$

* We will calculate the work done by an axial force applied to the end of the bar shown below. As the magnitude of the force is gradually increased from zero to some limiting value $F = P$, the final displacement of the end of the bar becomes Δ.



- F is gradually applied (from 0 to P)
- If the material has a linear elastic response, then $F = \frac{P}{\Delta} x$

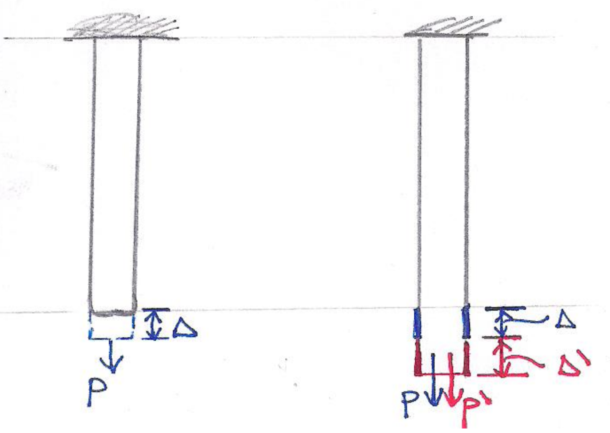
$$U_e = \int_0^{\Delta} F\,dx = \int_0^{\Delta} \frac{P}{\Delta} x \cdot dx = \frac{P}{\Delta}\left(\frac{\Delta^2}{2}\right)$$

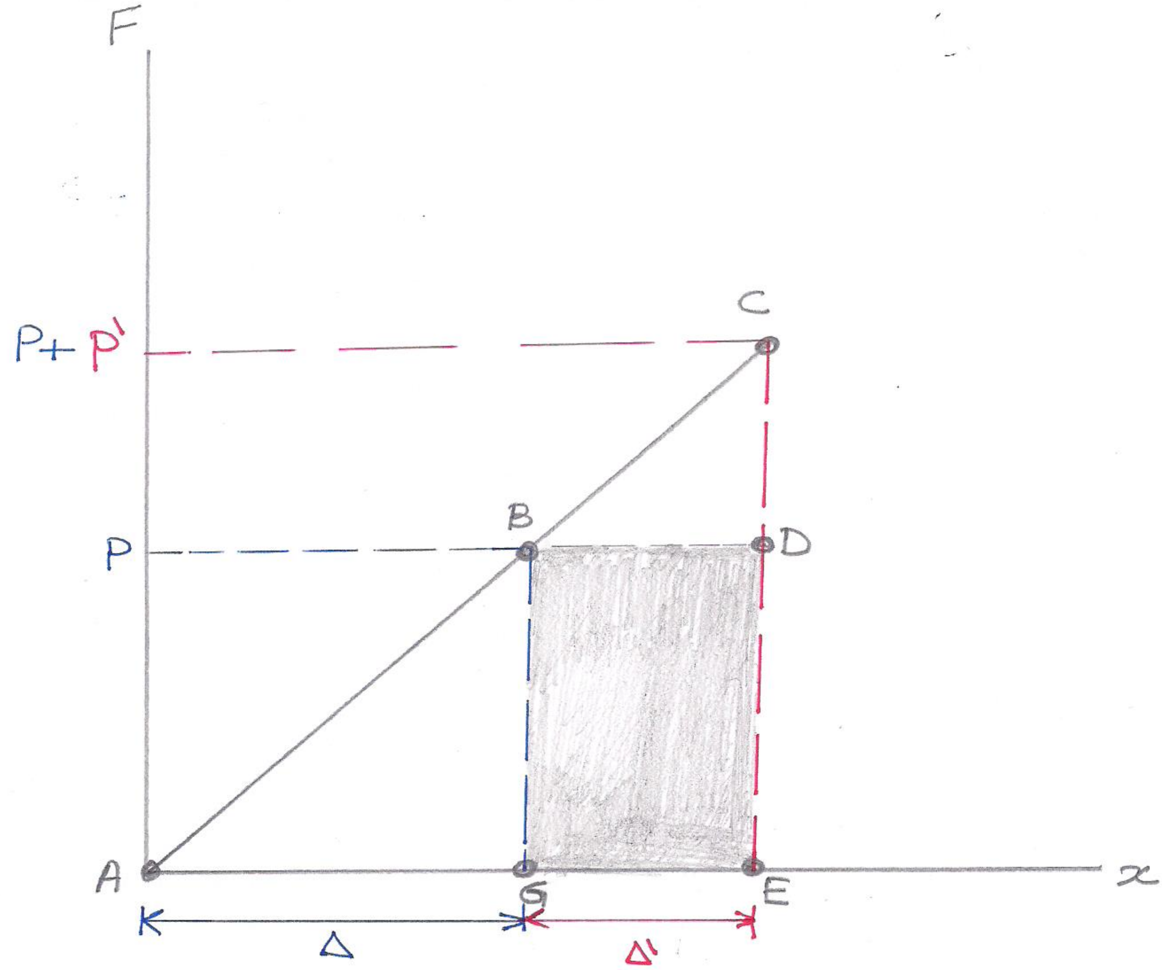


$$U_e = \frac{1}{2} P \Delta$$

which is the shaded area under the line $(F = \frac{P}{\Delta} x)$

* Suppose P is already applied and another force P' is applied so that the bar deflects further by Δ'.





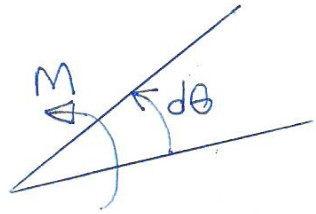
- Triangular area ACE = total work done by P and P'
- Triangular area ABG = work done by P due to Δ.
- Triangular area BCD = work done by P' due to Δ'.
- The additional work done by P is $P.\Delta'$ = shaded area BDEG.

Work of a Couple moment

- When the moment M undergoes a rotation $d\theta$ in the same direction as the moment, the external work done is $dU_e = M\,d\theta$

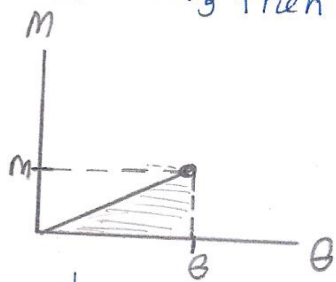
- IF the total angle of rotation is θ^{rad}, the work becomes:

$$U_e = \int_0^\theta M \cdot d\theta$$

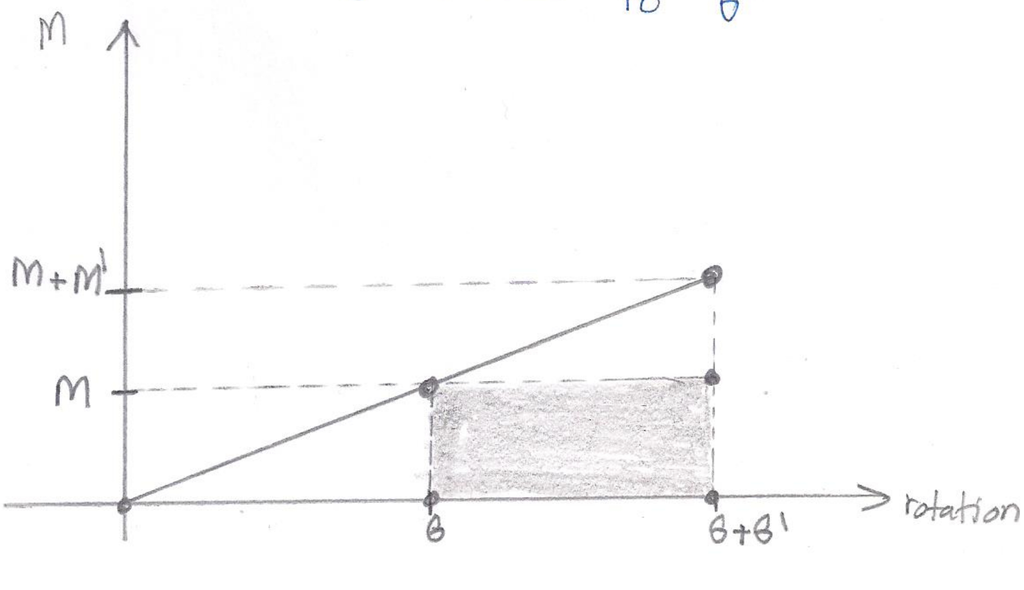


- Moment is gradually applied from 0 to M, then the work done is:

$$U_e = \frac{1}{2} M \theta$$



- If moment M is already applied and another moment M' further distort the structure by θ', then additional work done by M due to θ'



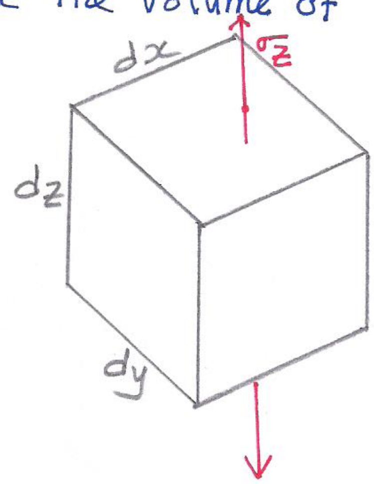
Strain Energy

When loads are applied to a body, they will deform the material. Provided no energy is lost in the form of heat, the external work done by the loads will be converted into internal work called strain energy. This energy, which is positive, is stored in the body and is caused by the action of either normal or shear stress.

Normal stress

If the volume element shown is subjected to the normal stress σ_z, then the force created on the top face is $dF_z = \sigma_z \cdot dA = \sigma_z \, dx \, dy$. If this force is applied gradually to the element (from 0 to dF_z) while the element undergoes an elongation $d\Delta_z = \varepsilon_z dz$. The work done by dF_z is therefore $dU_i = \frac{1}{2} dF_z \, d\Delta_z = \frac{1}{2}[\sigma_x \, dx \, dy] \varepsilon_z \, dz$. Since the volume of the element is $dV = dx \cdot dy \cdot dz$, we have

$$dU_i = \frac{1}{2} \sigma_z \varepsilon_z \, dV$$



In general, if the body is subjected to a uniaxial normal stress σ, the strain energy is:

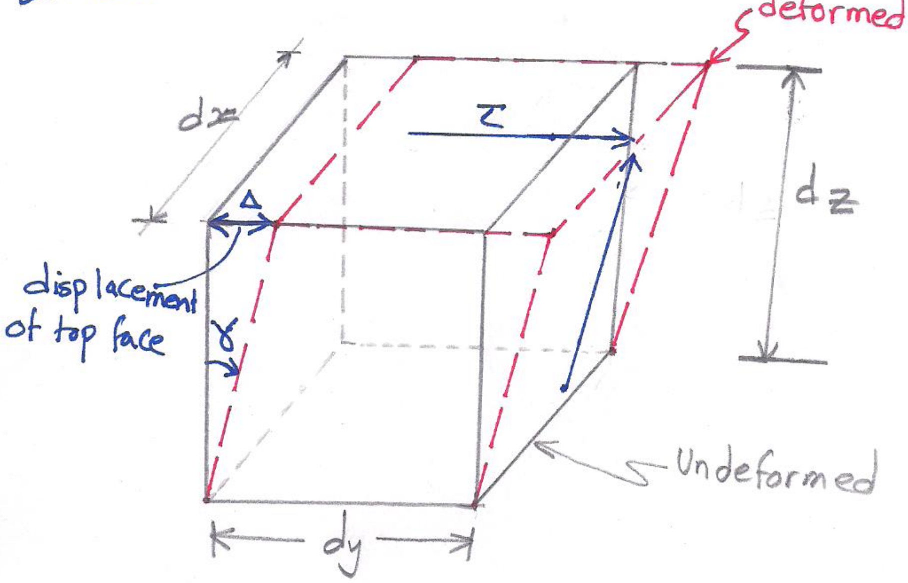
$$U_i = \int_V \frac{\sigma \varepsilon}{2} dV$$

Also, assuming linear-elastic behavior ($\sigma = E\varepsilon$):

$$U_i = \int_V \frac{\sigma^2}{2E} dV$$

Shear stress

Consider the volume element shown under pure shear stress.



The Force on top face: $dF = \tau(dx.dy)$

Displacement of top face $= \Delta = \gamma . dz$

$$dU_i = \frac{1}{2} dF\,\Delta = \frac{1}{2}(\tau . dx . dy)(\gamma . dz)$$

$$dU_i = \frac{1}{2}\tau\gamma\, dV \qquad , dV = dx.dy.dz$$

The strain energy stored in a body is therefore:

$$U_i = \int_V \frac{\tau\gamma}{2}\, dV$$

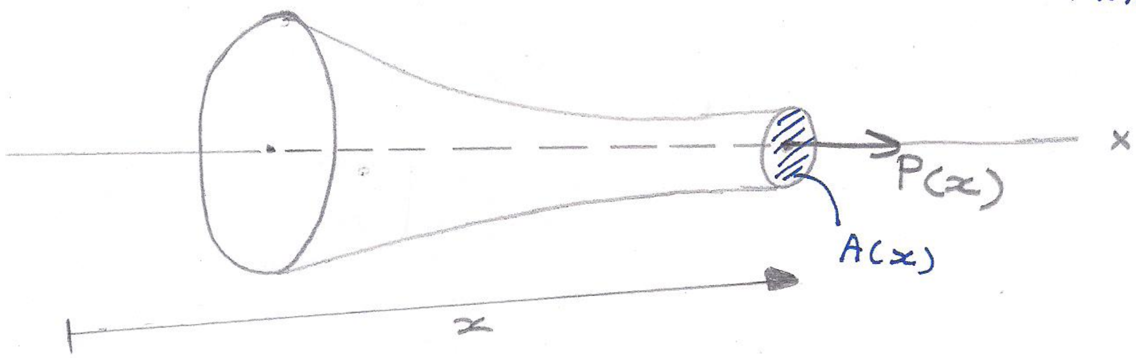
If the material is linear elastic, then, applying Hooke's law, $\gamma = \tau/G$, strain energy can be expressed as:

$$U_i = \int_V \frac{\tau^2}{2G}\, dV$$

Elastic Strain Energy for Various Types of loading

Axial Load

Consider a bar of variable cross section and subjected to an internal normal force P at a section Located a distance x.



at distance x: $\sigma_x = \dfrac{P(x)}{A(x)}$ and $dV = A(x) \cdot dx$

$$U_i = \int_V \frac{\sigma_x^2}{2E} dV = \int_0^L \frac{[P(x)]^2}{2[A(x)]^2 E} \cancel{A(x)} \cdot dx$$

$$U_i = \int_0^L \frac{[P(x)]^2}{2 A(x) \cdot E} dx$$

Forthemore common case of a prismatic bar of constant cross-sectional area A, length L, and constant axial load P,

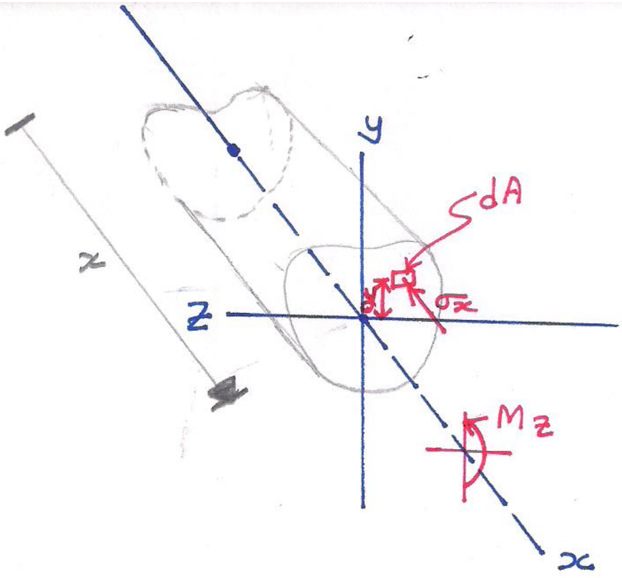
$$\boxed{U_i = \frac{P^2 L}{2AE}}$$

Bending Moment

$$U_i = \int_V \frac{\sigma^2}{2EI} dV$$

$$= \int_V \frac{\left(\frac{My}{I}\right)^2}{2EI} dA\, dx$$

$$= \int_0^L \frac{M^2}{2EI^2}\left(\int y^2 dA\right) dx \quad , \; I = \int y^2 dA$$



$$\sigma_x = \frac{M_z}{I_z} y$$

$$dV = dA\, dx$$

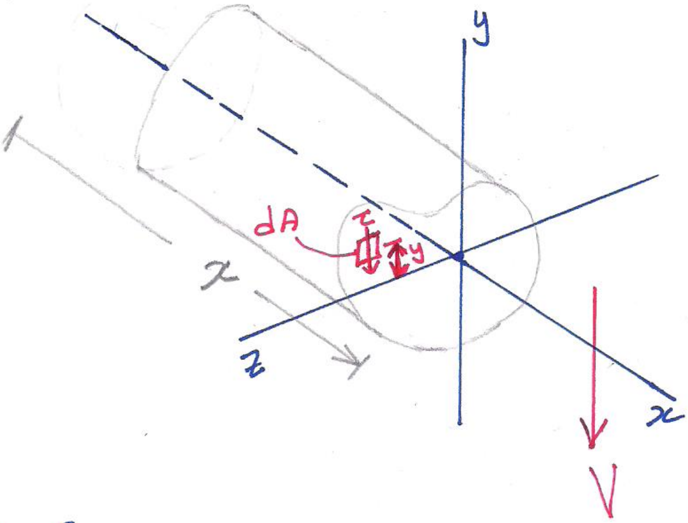
$$\boxed{U_i = \int_0^L \frac{M^2}{2EI} dx}$$

Transverse Shear

$$U_i = \int_V \frac{\tau^2}{2G} dV = \int_V \frac{1}{2G}\left(\frac{VQ}{It}\right)^2 dA\, dx$$

$$U_i = \int_0^L \frac{V^2}{2GI^2}\left(\int \frac{Q^2}{t^2} dA\right) dx$$

$$\boxed{U_i = \int_0^L \frac{f_s V^2 dx}{2GA}} \quad , \; f_s = \frac{A}{I^2}\int \frac{Q^2}{t^2} dA$$



$$dV = dA \cdot dx$$

$$\tau = \frac{VQ}{It}$$

f_s is a dimensionless number that is unique for each cross-sectional area.

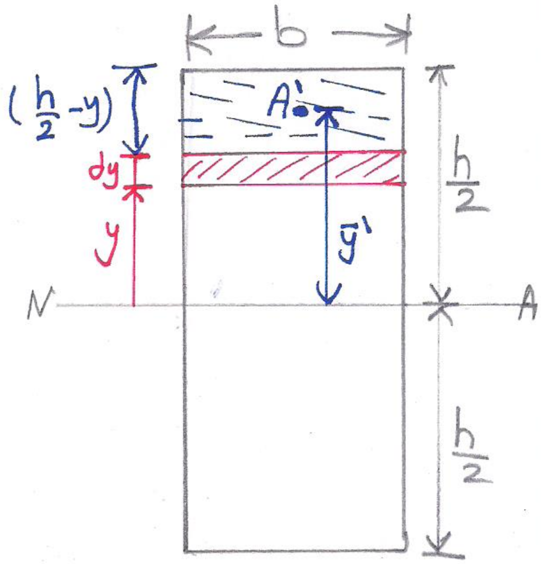
$$f_s = \frac{A}{I^2}\int \frac{Q^2}{t^2}\,dA$$

Calculate the form factor (f_s) for a rectangular cross section of width b and height h.

$t = b$

$dA = b.dy$

$I = \dfrac{bh^3}{12}$



$$Q = A' \cdot \bar{y}' = b\left(\frac{h}{2}-y\right)\left(y+\frac{\frac{h}{2}-y}{2}\right)$$

$$Q = \frac{b}{2}\left(\frac{h^2}{4}-y^2\right)$$

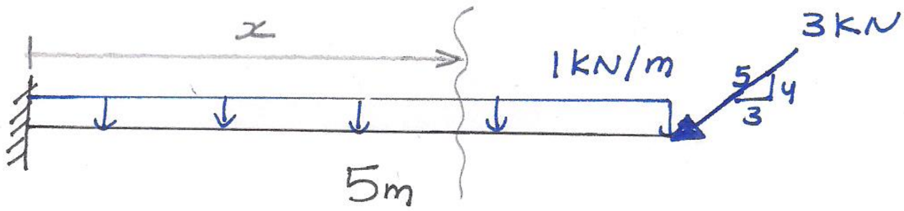
$$\bar{y}' = y + \frac{\frac{h}{2}-y}{2}$$

$$f_s = \frac{bh}{\left(\frac{bh^3}{12}\right)}\int_{-h/2}^{+h/2} \frac{b^2}{4b^2}\left(\frac{h^2}{4}-y^2\right)^2 b\,dy$$

$$f_s = \frac{6}{5}$$

Example

The cantilever beam is subjected to the loads shown, and has a rectangular cross section. Determine the total strain energy stored in the beam.



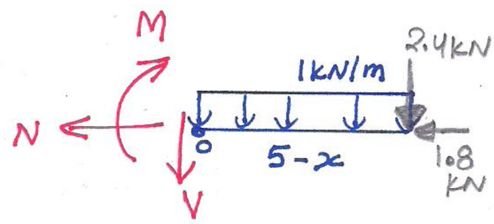
Solution

Make a section cut at a distance and determine the internal loading

$\Sigma F_x = 0 : \quad N = -1.8 \text{ kN} \quad (\text{kN})$

$+\uparrow \Sigma F_y = 0 : \quad -V - 1(5-x) - 2.4 = 0$

$V = -7.4 + x \quad (\text{kN})$



$\circlearrowleft + \Sigma M_o = 0$

$-M - \frac{1}{2}(5-x)^2 - 2.4(5-x) = 0 \qquad 0 \leq x \leq 5$

$M = -\frac{1}{2}x^2 + 7.4x - 24.5 \quad (\text{kN}\cdot\text{m})$

$N = -1.8 \text{ kN} \longrightarrow (U_i)_N = \frac{P^2 L}{2AE} = \frac{(-1.8)^2(5 \text{ m})}{2AE}$ "+ve"

$V = -7.4 + x \longrightarrow (U_i)_V = \int_0^L \frac{\left(\frac{6}{5}\right) V^2 dx}{2GA} = \int_0^{5m} \frac{\left(\frac{6}{5}\right)(-7.4+x)^2 dx}{2GA}$ "+ve"

$M = -\frac{1}{2}x^2 + 7.4x - 24.5 \rightarrow (U_i)_M = \int_0^L \frac{M^2}{2EI} dx = \int_0^L \frac{\left[-\frac{x^2}{2} + 7.4x - 24.5\right]^2}{2EI} dx$ "+ve"

$$(U_i)_{total} = (U_i)_N + (U_i)_V + (U_i)_M$$

Work and Energy Principle "Real work – real energy principle"

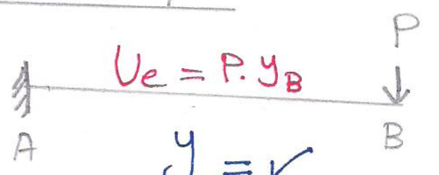
Conservation of energy principle states that the work done by all external forces acting on a structure, U_e, is transformed into internal work or strain energy, U_i, which is developed when the structure deforms.

$$U_e = U_i$$

This principle can be used directly to solve simple problems. To be precise, It can be used to solve problems involving a single force (F, M) for the displacement (y, θ) in the direction of that force (F, M).

For example

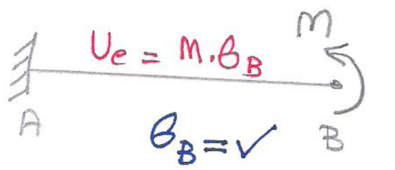
$U_e = P \cdot y_B$



$y_B = \checkmark$

" Only deflection at B " can be determined

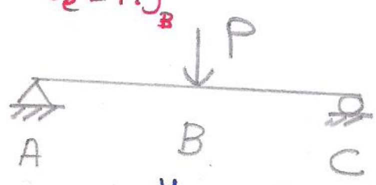
$U_e = M \cdot \theta_B$



$\theta_B = \checkmark$

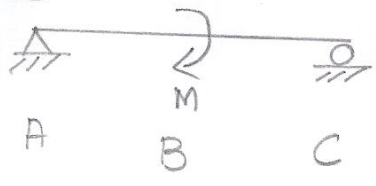
" Only slope at B " can be determined

$U_e = P \cdot y_B$



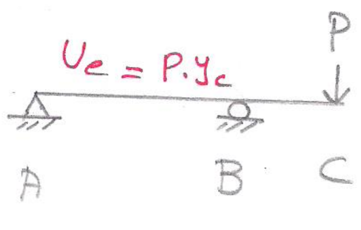
$y_B = \checkmark$

$U_e = M \cdot \theta_B$



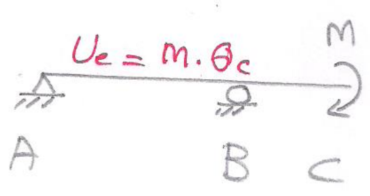
$\theta_B = \checkmark$

$U_e = P \cdot y_C$



$y_C = \checkmark$

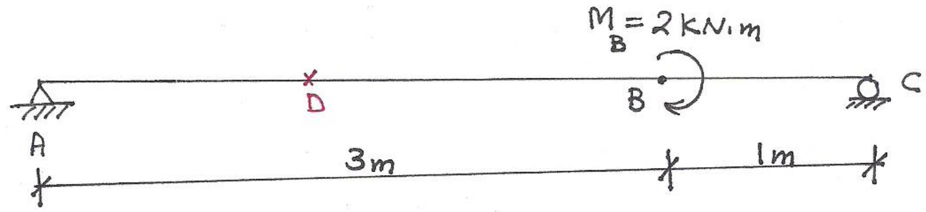
$U_e = M \cdot \theta_C$



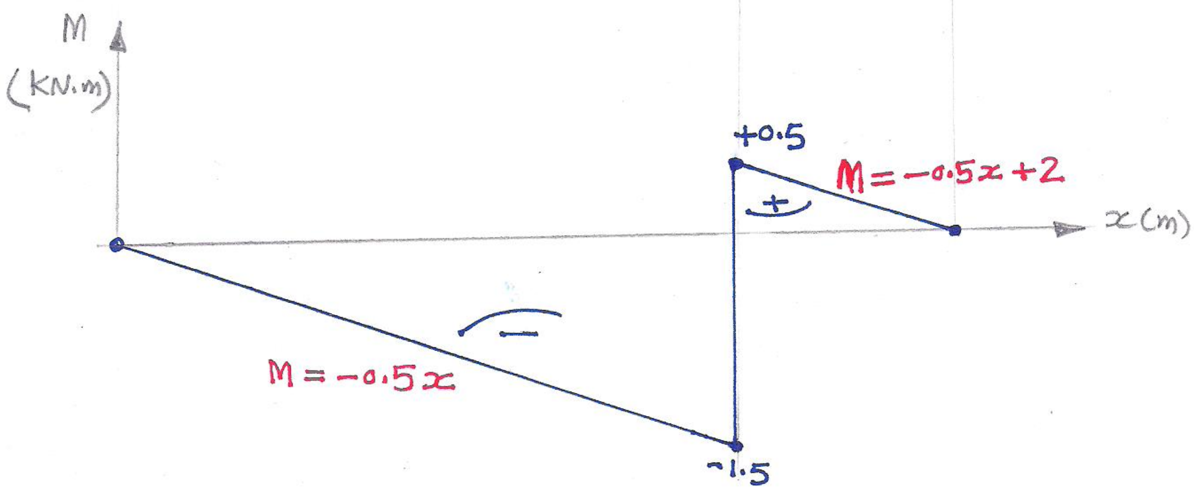
$\theta_C = \checkmark$

Example

Determine the slope at point B for the beam ABC shown below. EI = constant.



Solution



$$U_e = U_i$$

$$\frac{1}{2} M_B \theta_B = \int_0^L \frac{M^2 dx}{2EI} \Rightarrow \frac{1}{2}(2)\,\theta_B = \int_0^3 \frac{(-0.5x)^2}{2EI}dx + \int_3^4 \frac{(-0.5x+2)^2}{2EI}dx$$

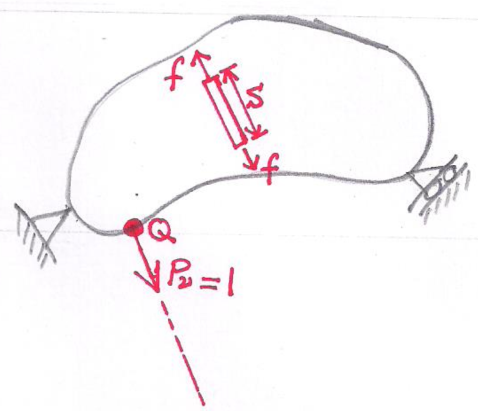
$$\theta_B = \frac{9}{8EI} + \frac{1}{24EI} = \frac{7}{6EI} \text{ rad}$$

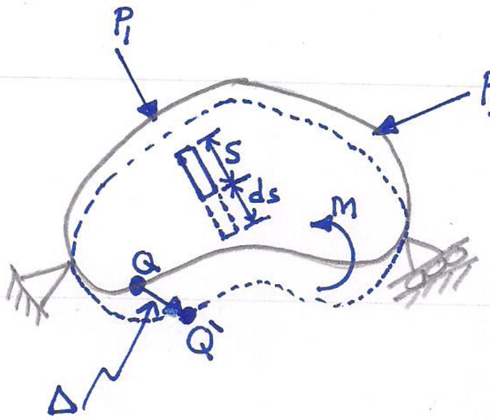
Principle of work and energy was used to determine θ_B only. Suppose that you are interested in determing θ_D, y_D, or y_B, more sophisticated energy methods are needed such as virtual work method, and Castiglian's method.

Virtual work method "Unit_Load method"

Virtual work method uses the law of conservation of energy to obtain the deflection and slope in a structure. This method was developed in 1717 by John Bernoulli.

Consider the deformable body shown below. First, applying a virtual unit load $P_v = 1$ at a point Q, where the deflection parallel to the applied load is desired, will develop internal virtual load f and will cause point Q to displace by a certain small amount. Then, placing the real external loads P_1, P_2, and M on the same body will cause an internal deformation, ds, and an external deflection of point Q to Q' by an amount Δ.





$P_v = 1$ = External virtual unit Load.

f = Internal virtual load.

Δ = External displacement caused by real loads

dS = Internal deformation caused by real loads

Upon placement of the real loads, the point of application of the virtual load also displaces by Δ, and the applied unit load performs work by traveling the distance Δ. The work done by the virtual forces are as follows:

- External work done by the unit load "P_v" $= P_v * \Delta$
- Internal work done by the virtual load "f" $= f * ds$

Using principle of conservation of energy:

External work done = Internal work done

Real displacements

$$1 * \Delta = f * ds$$

Virtual loads

Similarly, to obtain the slope at a point on a structure, apply a unit virtual moment M_v at the specified point where the slope is desired, and apply principle of conservation of energy:

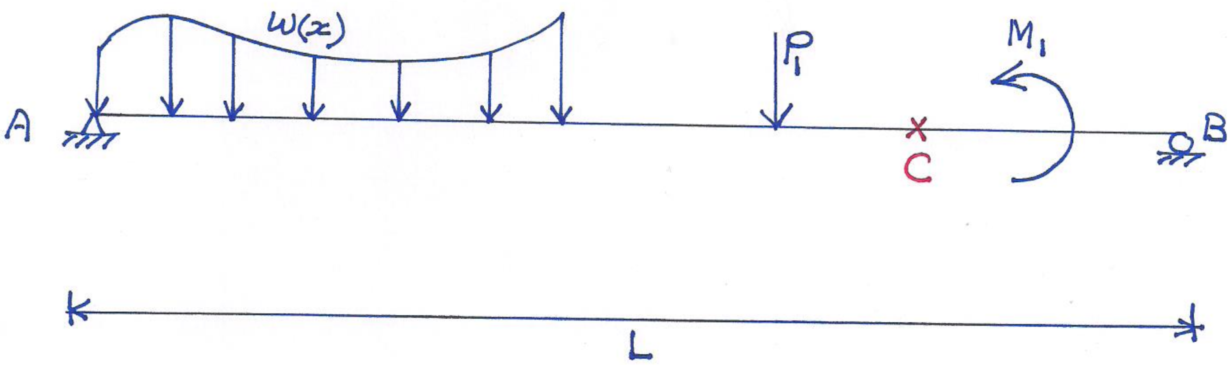
Real displacements

$$1 * \theta = f * ds$$

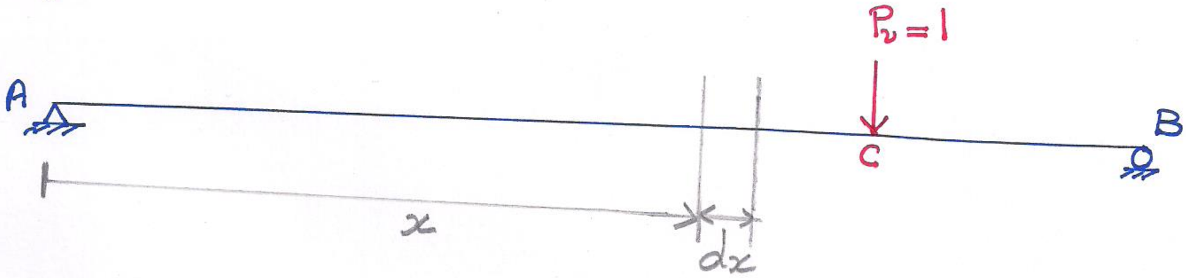
Virtual loads

Virtual Work Formulation for the Deflection and slope of Beams and Frames.

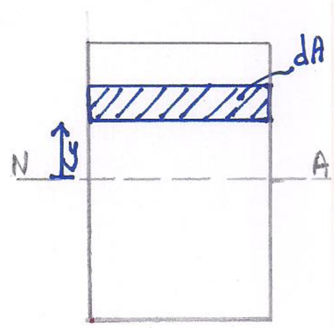
Consider beam AB shown below, the deflection at point C due to external loads is required.



First, removing all the real loads (w, P_1, M_1, ...) and applying a virtual unit load $P_v = 1$ will cause elementary forces and deformations to develop in the member, and a small deflection to occur at C, as follows:



The stress acting on the differential cross-sectional area dA at a distance x from A due to a virtual unit load is as follows:



Beam's cross section

$$\sigma' = \frac{m\,y}{I}$$ "flexure formula"

m = internal virtual moment at the section at a distance x from A due to virtual unit load.

The force acting on the differential area due to the virtual unit load is :

$$f = \sigma' dA = \left(\frac{m\, y}{I}\right) dA$$

The stress due to the external (real) loads (W, P_1, M_1) on the beam is :

$$\sigma = \frac{M\, y}{I}$$

M = internal moment in the beam caused by the real load

The deformation of a differential beam length dx at a distance x from A is :

$$\delta = \varepsilon\, dx = \left(\frac{\sigma}{E}\right) dx = \left(\frac{M.y}{E.I}\right) dx$$

The work done by the force f acting on the differential area due to the deformation of the differential beam length dx is :

$$dU_i = f\, \delta = \left(\frac{m.y}{I}\right) dA * \left(\frac{M.y}{E.I}\right) dx$$

$$= \left(\frac{M m\, y^2}{E I^2}\right) dA . dx$$

The internal work done by the total force in the entire cross-sectional area of the beam due to the applied virtual unit load when the differential length of the beam dx deforms by δ can be obtained by integrating with respect to dA, as follows:

$$\int_A dU_i = \left(\int_A \left(\frac{M m y^2}{EI^2}\right) dA\right) dx = \left(\frac{M m}{EI^2} \int_A y^2 dA\right) dx$$

$$U_i = \frac{M.m}{EI} dx$$

The internal work done U_i in the entire length of the beam due to applied virtual unit load can now be obtained by integrating with respect to dx, which is written as follows

$$U_i = \int_0^L \left(\frac{M.m}{EI}\right) dx$$

The external work done U_e by the virtual unit load due to the deflection Δ at point C of the beam caused by the external load is as follows:

$$U_e = 1 * \Delta$$

The principle of conservation of energy is applied to obtain the expression of the deflection at any point in a beam or frame:

$$U_e = U_i$$

$$1 * \Delta = \int_0^L \left(\frac{Mm}{EI}\right) dx$$

$$\Delta = \int_0^L \left(\frac{Mm}{EI}\right) dx$$

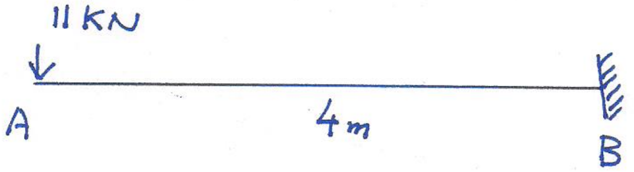
Similarly, the following expression can be obtained for the computation of the slope at a point in a beam or frame:

$$\theta = \int_0^L \left(\frac{M\ m}{EI}\right) dx$$

m = internal virtual moment in the beam or frame, expressed with respect to the horizontal distance x, caused by the external virtual unit moment applied at the point where the rotation is required.

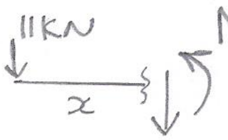
Example

Determine slope and deflection of point A of the cantilever beam shown. EI = constant



Solution

Real load



$M = -11x$

$0 \leq x \leq 4m$

$$\theta_A = \int_0^L \frac{M\, m_1}{EI}\, dx$$

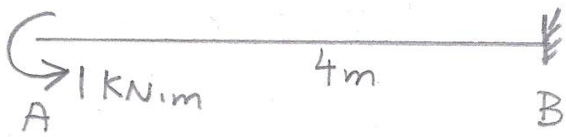
$$\theta_A = \int_0^{4m} \frac{(-11x)(-1)}{EI}\, dx = \frac{+88}{EI} \text{ rad}$$

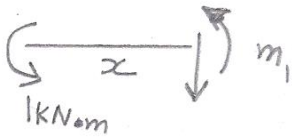
$$\Delta_A = \int_0^L \frac{M\, m_2}{EI}\, dx$$

$$\Delta_A = \int_0^4 \frac{(-11x)(-x)}{EI}\, dx = \frac{+704}{3EI} \downarrow$$

Virtual load

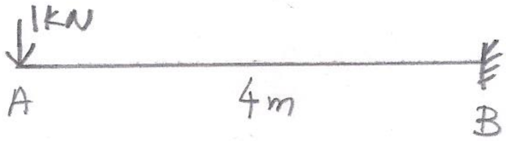
*Apply one unit moment at A to find θ_A:

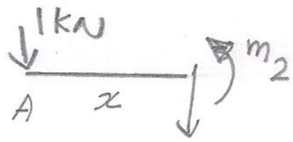




$m_1 = -1 \quad 0 \leq x \leq 4m$

*Apply one unit force at A to find v_A:

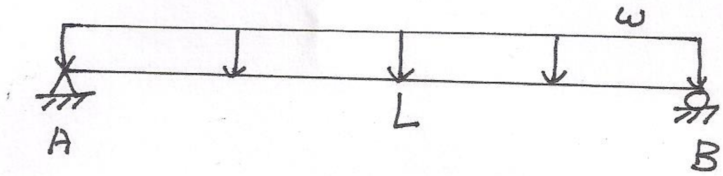




$m_2 = -x \quad 0 \leq x \leq 4m$

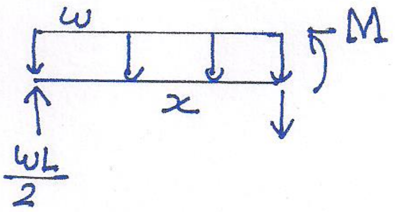
Example

Determine mid-span deflection and end slopes of a simply supported beam shown. EI = constant.



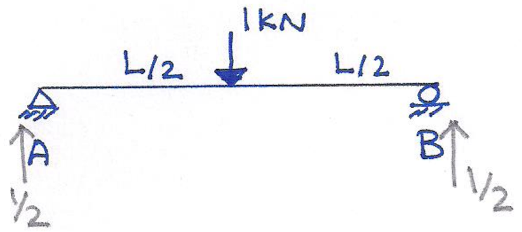
Solution

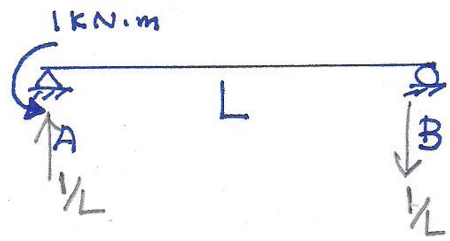
Real load:

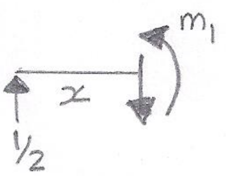


$$M = \frac{wL}{2}x - \frac{wx^2}{2}, \quad 0 \le x \le L$$

Virtual load:

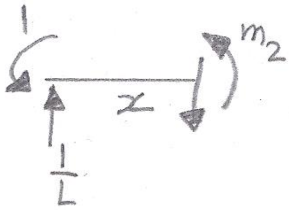






$m_1 = \frac{x}{2}$

$0 \le x \le L/2$



$m_2 = \frac{x}{L} - 1$

$0 \le x \le L$

$$v_{\text{mid-span}} = \int \frac{M m_1}{EI} dx = 2\int_0^{L/2} \frac{\left(\frac{wL}{2}x - \frac{wx^2}{2}\right)\left(\frac{x}{2}\right)}{EI} dx = \frac{2}{EI}\left(\frac{wL}{12}x^3 - \frac{w}{16}x^4\right)\Big|_0^{L/2}$$

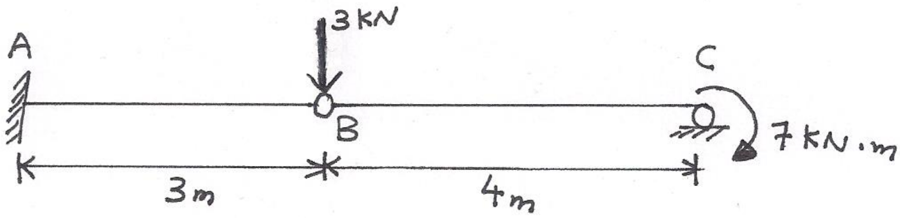
$$= \frac{5wL^4}{384EI}$$

$$\theta_A = \int \frac{M.m_2}{EI} dx = \int_0^L \frac{\left(\frac{wL}{2}x - \frac{w}{2}x^2\right)\left(\frac{x}{L} - 1\right)}{EI} dx = -\frac{wL^3}{24EI} \text{ cw}$$

Due to symmetry $\theta_B = \theta_A = \frac{wL^3}{24EI}$ ccw

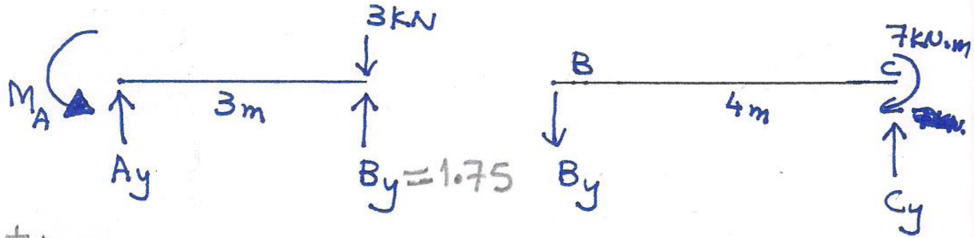
Example

Beam ABC has a fixed support at A, an internal hinge at B, and a roller support at C. EI = constant. Determine the deformations at point B.



Solution

Real loads:



$+\uparrow \Sigma F_y = 0$

$+A_y - 3 + 1.75 = 0$

$A_y = 1.25$ kN

$\circlearrowleft + \Sigma M_A = 0$

$+M_A - (\cancel{3 -}1.75)(3) = 0$

$M_A = 3.75$ kN.m

$\circlearrowleft + \Sigma M_B = 0$

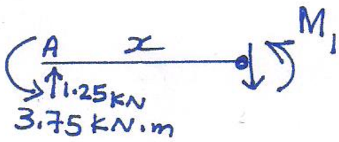
$-7 + 4\,C_y = 0$

$C_y = 1.75$ kN

$\uparrow \Sigma F_y = 0$

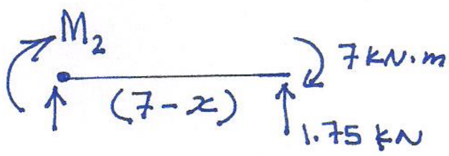
$B_y = 1.75$ kN

Part AB :



$M_1 = 1.25x - 3.75 \quad 0 \le x \le 3m$

Part BC :

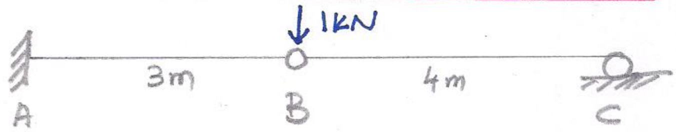


$M_2 = 1.75(7-x) - 7$

$M_2 = 5.25 - 1.75x$

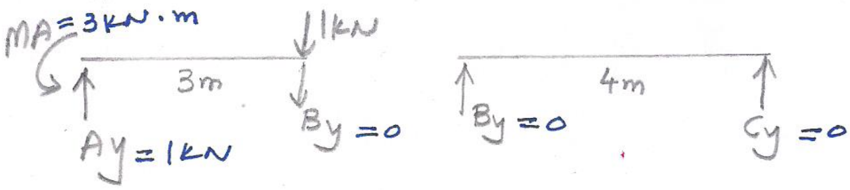
$3m \le x \le 7m$

* Virtual unit force at B :



$m_1' = x - 3 \quad 0 \le x \le 3m$

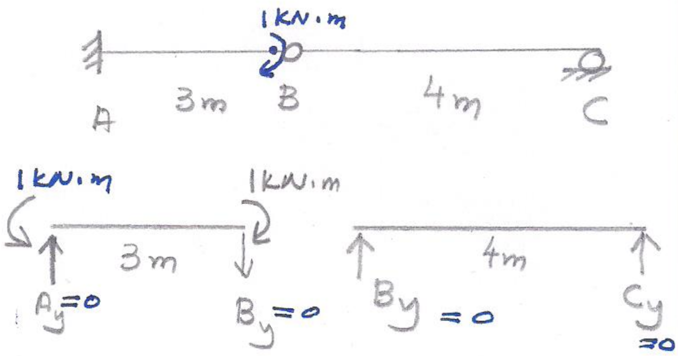
$m_2' = 0 \quad 3m \le x \le 7m$



*Virtual unit moment just before the internal hinge

$m_1'' = -1 \qquad 0 \le x \le 3m$

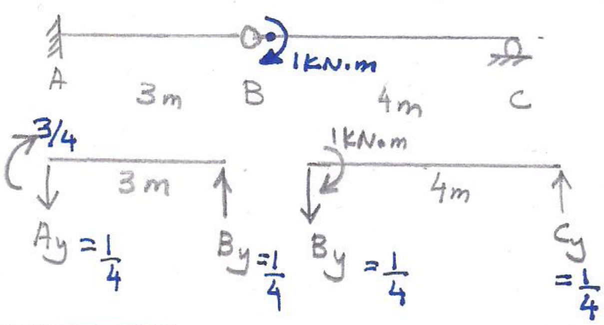
$m_2'' = 0 \qquad 3m \le x \le 7m$



*Virtual unit moment just after the internal hinge

$m_1''' = \frac{3}{4} - \frac{x}{4} \qquad 0 \le x \le 3m$

$m_2''' = \frac{7}{4} - \frac{x}{4} \qquad 3m \le x \le 7m$



$$v_B = \int_0^{3m} \frac{M_1 \cdot m_1'}{EI}\,dx + \int_{3m}^{7m} \frac{M_2 \cdot m_2'}{EI}\,dx$$

$$= \int_0^{3m} \frac{(1.25x - 3.75)(x-3)}{EI}\,dx + \int_{3m}^{7m} 0\,dx = \frac{+45}{4EI} \downarrow$$

$$(\theta_B)_L = \int_0^{3m} \frac{M_1 \cdot m_1''}{EI}\,dx + \int_{3m}^{7m} \frac{M_2 \cdot m_2''}{EI}\,dx = \int_0^{3m} \frac{(1.25x - 3.75)(-1)}{EI}\,dx = \frac{+45}{8EI} \circlearrowright$$

$$(\theta_B)_R = \int_0^{3m} \frac{M_1 \cdot m_1'''}{EI}\,dx + \int_{3m}^{7m} \frac{M_2 \cdot m_2'''}{EI}\,dx = \int_0^{3m} \frac{(1.25x - 3.75)\left(\frac{3}{4} - \frac{x}{4}\right)}{EI}\,dx$$

$$+ \int_{3m}^{7m} \frac{(5.25 - 1.75x)\left(\frac{7}{4} - \frac{x}{4}\right)}{EI}\,dx$$

$$= \frac{-45}{16EI} + \frac{-14}{3EI} = \frac{-359}{48EI} \circlearrowleft$$

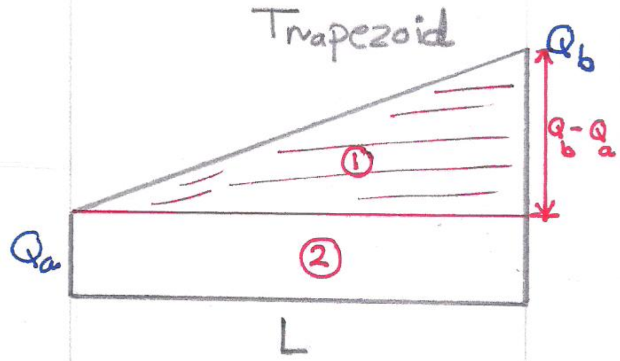
Table for: $\int_0^L M\,Q\,dx$

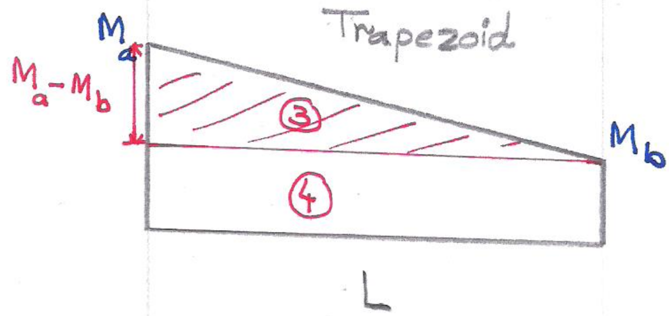
The values in the table represent the integration of the product of the two shapes with a common length L.

	Rectangle (Q, L)	Triangle (Q at right, L)	Triangle (Q at left, L)	Trapezoid (Q_a, Q_b, L)
Rectangle (M, L)	LMQ	$\frac{LMQ}{2}$	$\frac{LMQ}{2}$	$\frac{LM}{2}(Q_a + Q_b)$
Triangle (M at right, L)	$\frac{LMQ}{2}$	$\frac{LMQ}{3}$	$\frac{LMQ}{6}$	$\frac{LM}{6}(Q_a + 2Q_b)$
Triangle (M at left, L)	$\frac{LMQ}{2}$	$\frac{LMQ}{6}$	$\frac{LMQ}{3}$	$\frac{LM}{6}(2Q_a + Q_b)$
Triangle ($\leftarrow a \rightarrow \leftarrow b \rightarrow$, M, L)	$\frac{LMQ}{2}$	$\frac{MQ}{6}(L + a)$	$\frac{MQ}{6}(L + b)$	$\frac{M}{6}\left[Q_a(L+b) + Q_b(L+a)\right]$
Trapezoid (M_a, M_b, L)	$\frac{LQ}{2}(M_a + M_b)$	$\frac{LQ}{6}(M_a + 2M_b)$	$\frac{LQ}{6}(2M_a + M_b)$	$\frac{L}{6}\left[Q_a(2M_a + M_b) + Q_b(M_a + 2M_b)\right]$
Parabola (slope = 0 at right, M, L)	$\frac{2LMQ}{3}$	$\frac{5LMQ}{12}$	$\frac{LMQ}{4}$	$\frac{LM}{12}(3Q_a + 5Q_b)$
Parabola (slope = 0 at left, M, L)	$\frac{2LMQ}{3}$	$\frac{LMQ}{4}$	$\frac{5LMQ}{12}$	$\frac{LM}{12}(5Q_a + 3Q_b)$
Parabola (slope = 0 at left, M at right, L)	$\frac{LMQ}{3}$	$\frac{LMQ}{4}$	$\frac{LMQ}{12}$	$\frac{LM}{12}(Q_a + 3Q_b)$
Parabola (M at left, slope = 0 at right, L)	$\frac{LMQ}{3}$	$\frac{LMQ}{12}$	$\frac{LMQ}{4}$	$\frac{LM}{12}(3Q_a + Q_b)$

$$\frac{L(M_a - M_b)(Q_b - Q_a)}{6} + \frac{L(M_b)(Q_b - Q_a)}{2}$$

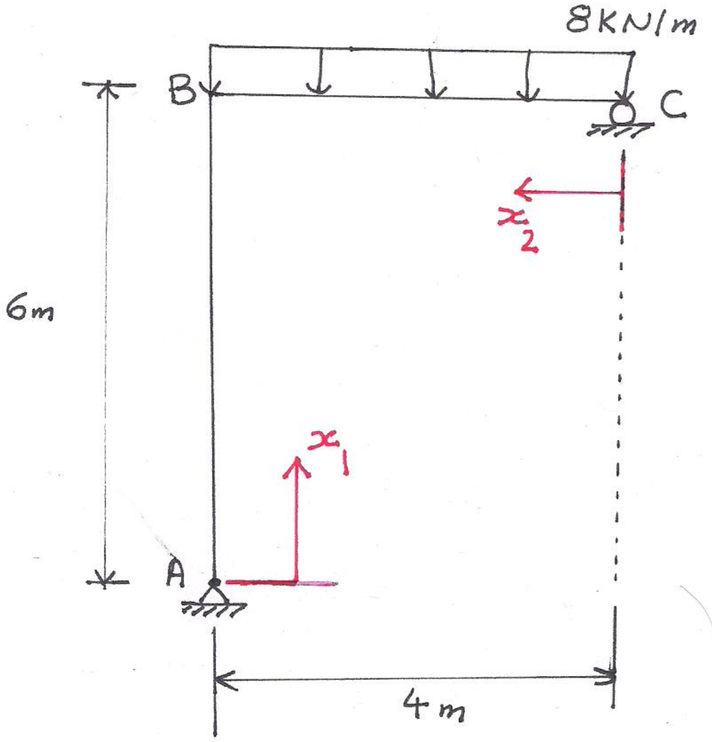
$$\frac{L(M_a - M_b)(Q_a)}{2} + L(M_b)(Q_a)$$





Example

Determine horizontal displacement of C and slope at A of a rigid-jointed plane frame shown. Both members of the frame have same flexural rigidity (EI).



Solution

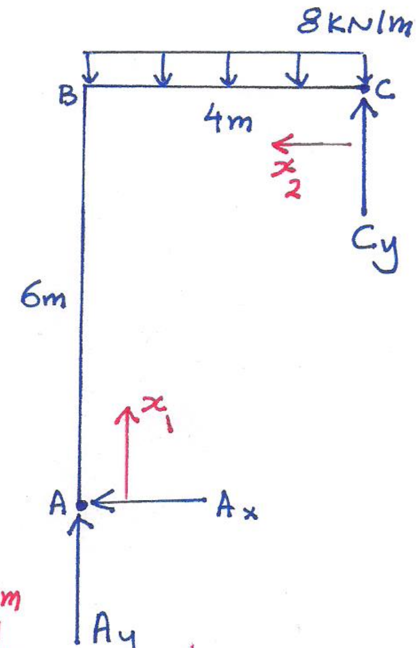
$\Sigma F_x = 0 \rightarrow A_x = 0$

$\circlearrowleft + \Sigma M_A = 0: \; +4C_y - (8*4)(2) = 0$

$C_y = 16 \text{ KN}$

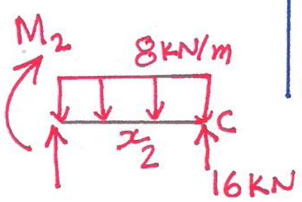
$+\uparrow \Sigma F_y = 0: \; +A_y + 16 - 8*4 = 0$

$A_y = 16 \text{ KN}$



$M_1 = 0 \qquad 0 \leq x_1 \leq 6\text{m}$

$M_2 = -4x_2^2 + 16x_2 \qquad 0 \leq x_2 \leq 4\text{m}$



Horizontal displacement at C

$\circlearrowleft^{+} \Sigma M_A = 0: +4C_y - 6(1) = 0$

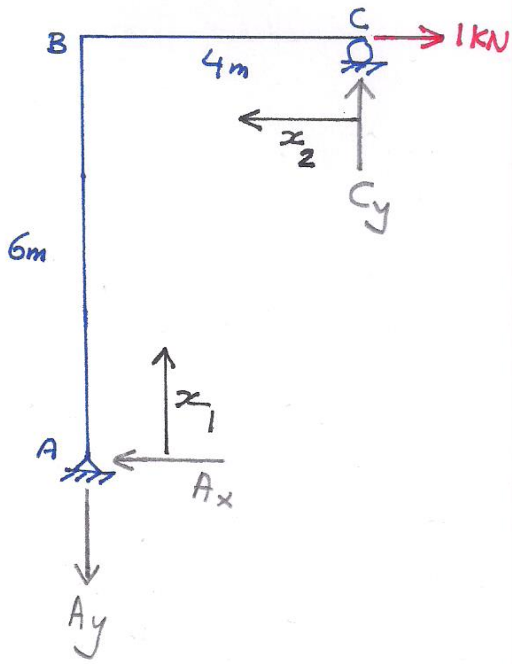
$C_y = 1.5 \text{ kN}$

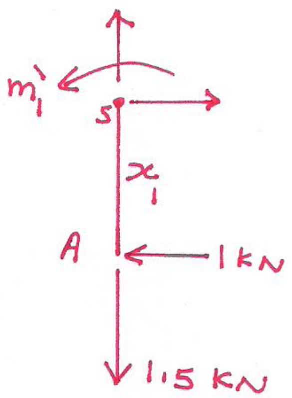
$\xrightarrow{+} \Sigma F_x = 0: -A_x + 1 = 0$

$A_x = 1 \text{ kN}$

$+\uparrow \Sigma F_y = 0: -A_y + 1.5 = 0$

$A_y = 1.5 \text{ kN}$

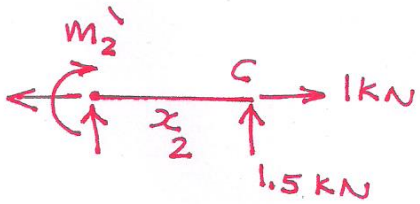




$\circlearrowleft^{+} \Sigma M_S = 0$

$+m_1' - x_1 = 0$

$m_1' = x_1 \qquad 0 \leq x_1 \leq 6 \text{ m}$



$m_2' = 1.5\, x_2 \qquad 0 \leq x_2 \leq 4 \text{ m}$

$$\Delta_C = 0 + \int_0^{4m} \frac{(-4x_2^2 + 16x_2)(1.5\, x_2)}{EI}\, dx = \frac{+128}{EI} \text{ m} \quad (\rightarrow)$$

Slope at A

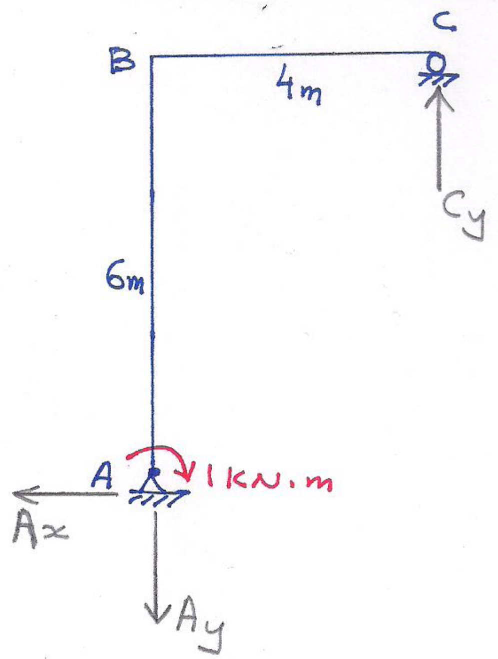
$\circlearrowleft^{+} \Sigma M_A = 0: \quad -1 + 4C_y = 0$

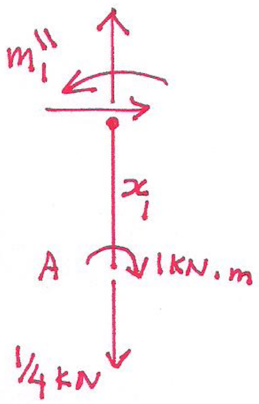
$$C_y = 1/4 \text{ kN}$$

$\xrightarrow{+} \Sigma F_x = 0: \quad A_x = 0$

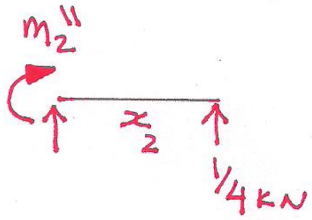
$+\uparrow \Sigma F_y = 0: \quad -A_y + 1/4 = 0$

$$A_y = 1/4 \text{ kN}$$





$m_1'' = 1 \text{ kN.m} \quad 0 \leq x_1 \leq 6 \text{ m}$



$m_2'' = x_2/4 \quad 0 \leq x_2 \leq 4 \text{ m}$

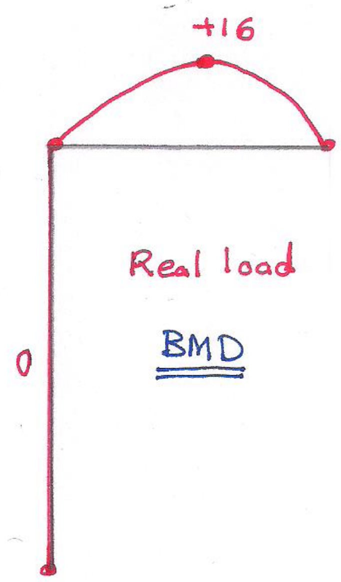
$$\theta_A = 0 + \int_0^{4m} \frac{(-4x_2^2 + 16x_2)(x_2/4)}{EI}\, dx = \frac{+64}{3EI} \text{ rad}$$

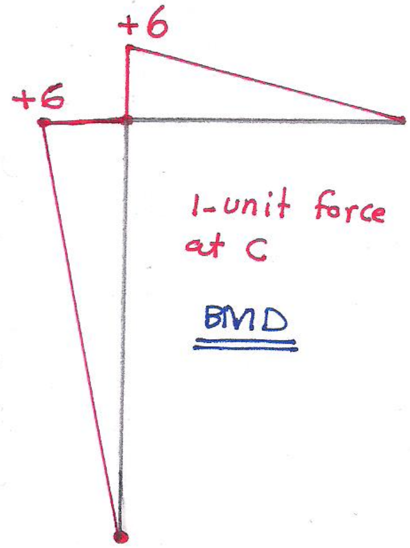
Or using graphical method:

$$\Delta_C = \frac{5(2m)(16\,kN.m)(3\,kN.m)}{12\;EI} +$$

$$+\frac{(2m)(16\,kN.m)(3*6\,kN.m+5*3\,kN.m)}{12\;EI}$$

$$= \frac{+128}{EI}\,m(\rightarrow)$$

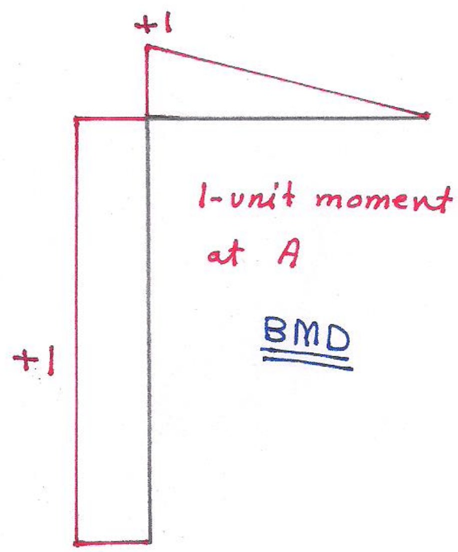




$$\theta_A = \frac{5(2m)(16\,kN.m)(0.5\,kN.m)}{12EI} +$$

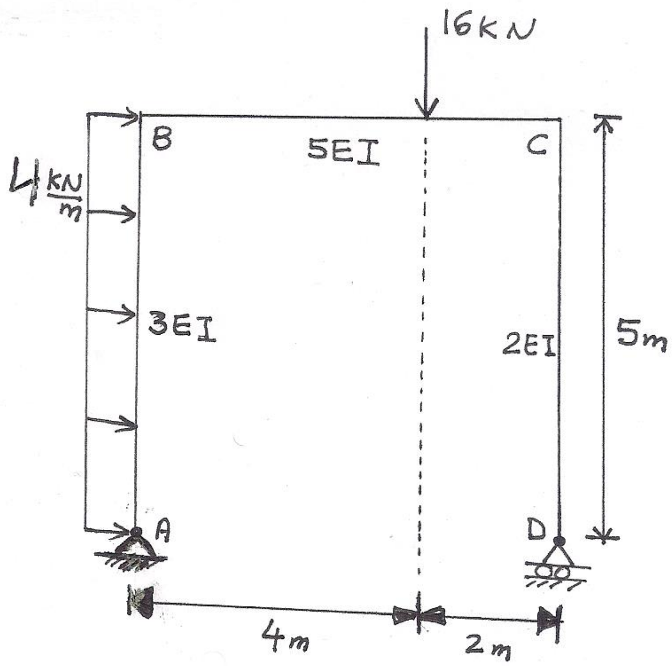
$$+\frac{(2m)(16\,kN.m)(3*1\,kN.m+5*0.5\,kN.m)}{12EI}$$

$$= +\frac{64}{3EI}\;\text{rad} \qquad (2)$$



Example

Determine the deformations at C of the frame shown.

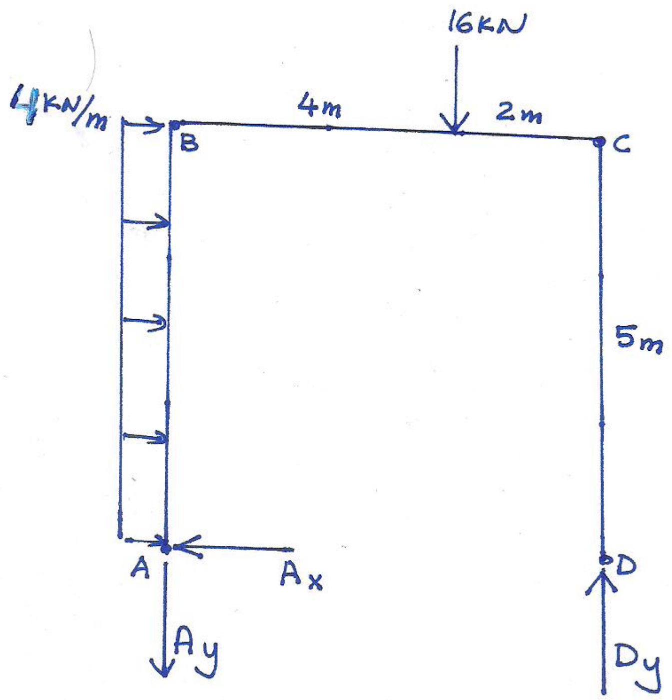


Solution

$\circlearrowleft^{+} \Sigma M_A = 0$

$-4(5)(2.5) - 16(4) + 6D_y = 0$

$D_y = 19 \text{ kN}$



$+\uparrow \Sigma F_y = 0$

$-A_y - 16 + 19 = 0$

$A_y = +3 \text{ kN}$

$\overset{+}{\rightarrow} \Sigma F_x = 0$

$+4(5) - A_x = 0$

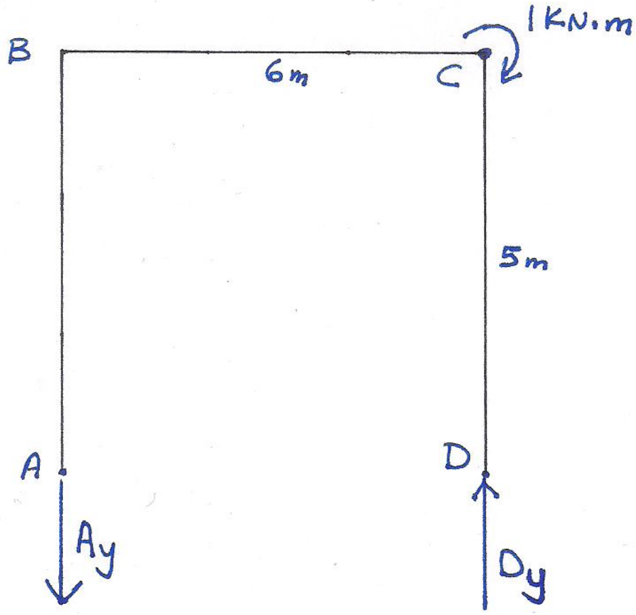
$A_x = 20 \text{ kN}$

1-unit moment at C

$$\circlearrowleft^{+} \Sigma M_A = 0$$

$$-1 + 6D_y = 0$$

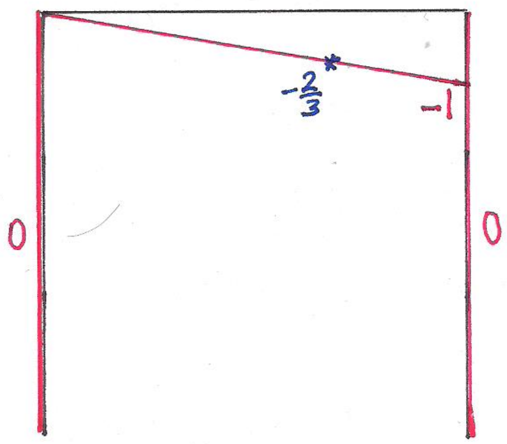
$$D_y = 1/6 \text{ KN}$$



$$+\uparrow \Sigma F_y = 0$$

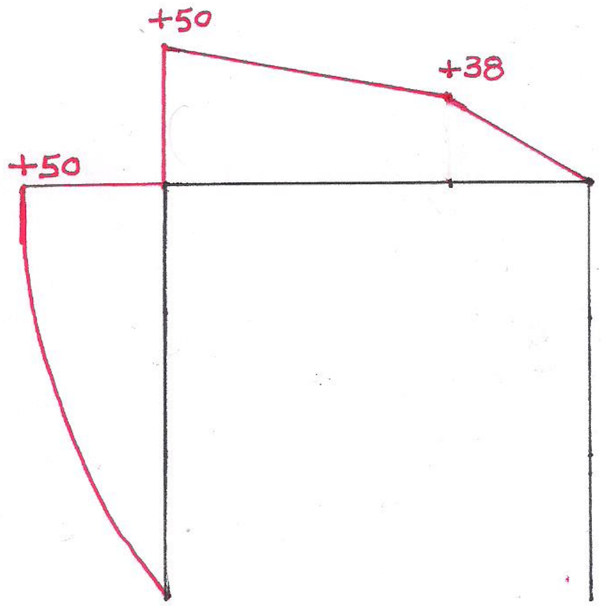
$$-A_y + 1/6 = 0$$

$$A_y = 1/6 \text{ KN}$$



$$\theta_C = \frac{(4m)\left(\frac{-2}{3}\right)(2*38+50)}{6(5EI)} + \frac{(2m)*(38)\left(2*\frac{-2}{3}-1\right)}{6(5EI)}$$

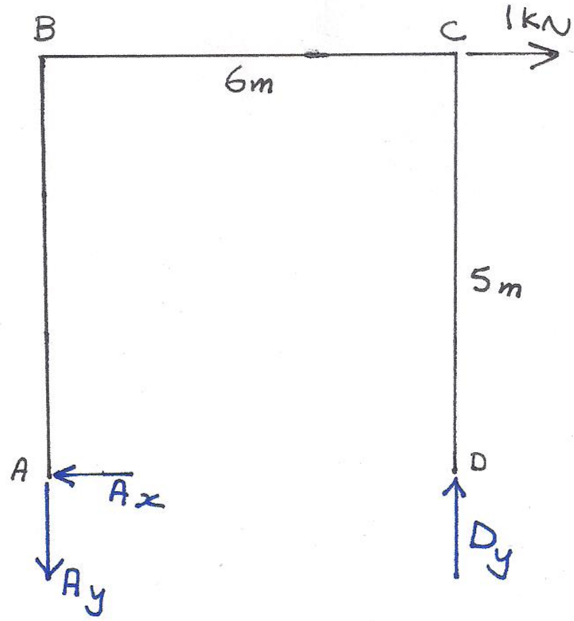
$$= -\frac{770}{9EI} \text{ rad } \circlearrowright$$



1-unit force at C in the horizontal direction

$\xrightarrow{+} \Sigma F_x = 0: \; -A_x + 1 = 0$

$A_x = 1 \text{ kN}$



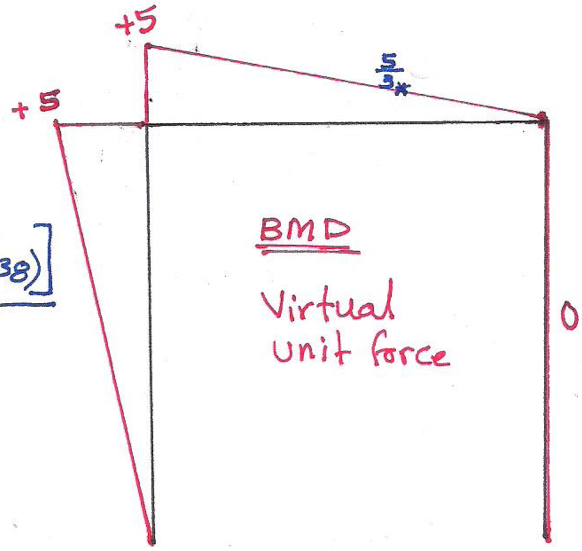
$\circlearrowleft + \Sigma M_A = 0$

$+D_y(6) - 5(1) = 0$

$D_y = 5/6 \text{ kN}$

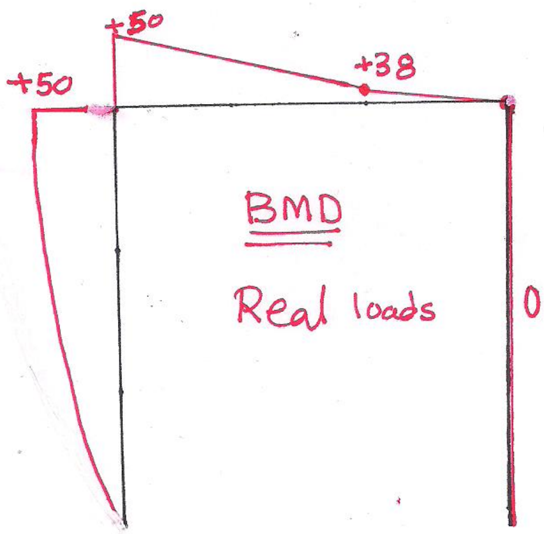
$+\uparrow \Sigma F_y = 0$

$-A_y + \frac{5}{6} = 0 \rightarrow A_y = \frac{5}{6} \text{ kN}$



$$\Delta_C = \frac{5(5m)(5 \text{ kN}\cdot m)(50 \text{ kN}\cdot m)}{12\,(3EI)} + \frac{(4m)\left[(5 \text{ kN}\cdot m)(2*50+38) + \left(\frac{5}{3}\right)(50+2*38)\right]}{6\,(5EI)} + \frac{(2m)(5/3)(38)}{3(5EI)}$$

$$= +\frac{5437}{18EI} \; m \; (\rightarrow)$$



Virtual work method – Trusses

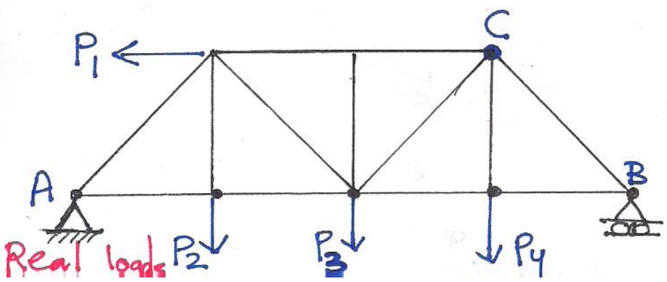
Consider a pin-jointed structure as shown below and subjected to external loads $P_1, P_2, \ldots, P_4$. Let the vertical displacement of point C (Δ_c) is required. Under the action of real external load, let the axial force in each member be N_i, and therefore, the deformation of the member $\Delta L = \frac{N_i L_i}{A_i E_i}$ (L_i = Length of member i, $A_i E_i$ = axial rigidity of member i).

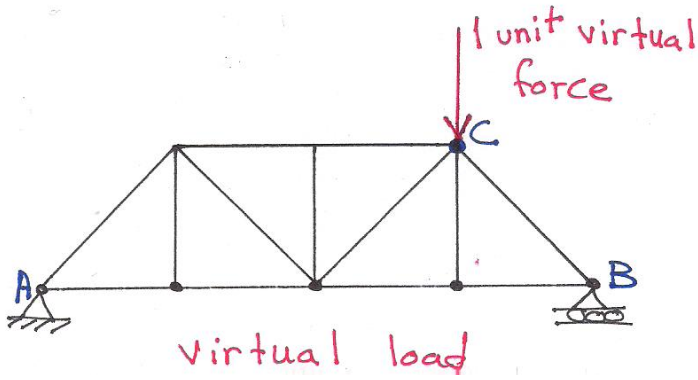
Then, apply a one-unit force at joint C in the direction of the required displacement. Under the action of the virtual unit force, let the axial force in each member be n_i.

$$1 \cdot \Delta_C = \sum n_i * (\Delta L)_i = \sum_{i=1}^{m} \left(\frac{n \, N \, L}{A \, E} \right)_i$$

m = total number of members.

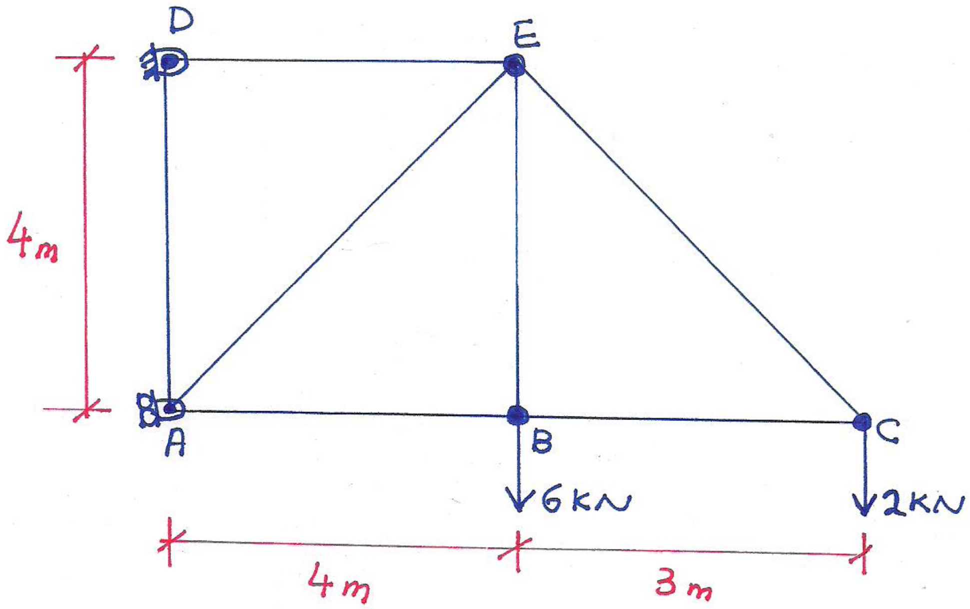
Important note: the axial force N_i or n_i shall be taken as positive if tensile and negative if compressive





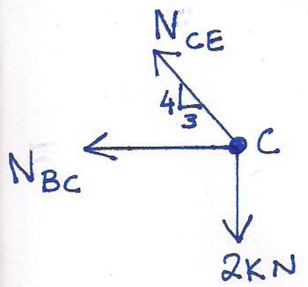
Example

For the steel truss shown, determine the vertical and horizontal displacements at joint C. Each member has a cross sectional area of $A = 350\ mm^2$ and $E = 200\ GPa$.



Solution

Real analysis using method of joints



$+\uparrow \sum F_y = 0$

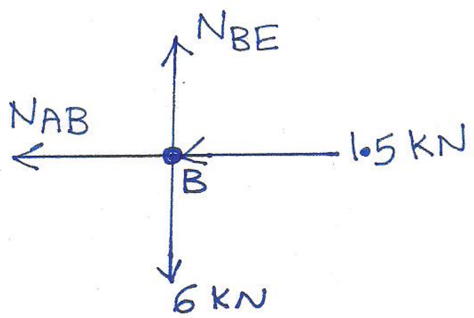
$-2 + \frac{4}{5} N_{CE} = 0$

$N_{CE} = +2.5\ kN\ (T)$

$\xrightarrow{+} \sum F_x = 0$

$-N_{BC} - \frac{3}{5}(2.5) = 0$

$N_{BC} = -1.5\ kN$ (C)



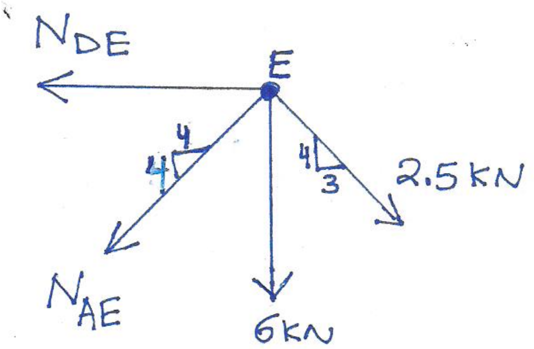
$\xrightarrow{+} \sum F_x = 0$

$-N_{AB} - 1.5 = 0 \rightarrow N_{AB} = -1.5\ kN$ (C)

$+\uparrow \sum F_y = 0$

$-6 + N_{BE} = 0$

$N_{BE} = +6\ kN\ (T)$



$$+\uparrow \Sigma F_y = 0$$

$$-N_{AE}\left(\frac{4}{4\sqrt{2}}\right) - 6 - 2.5\left(\frac{4}{5}\right) = 0$$

$$N_{AE} = -11.31 \text{ kN (C)}$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0$$

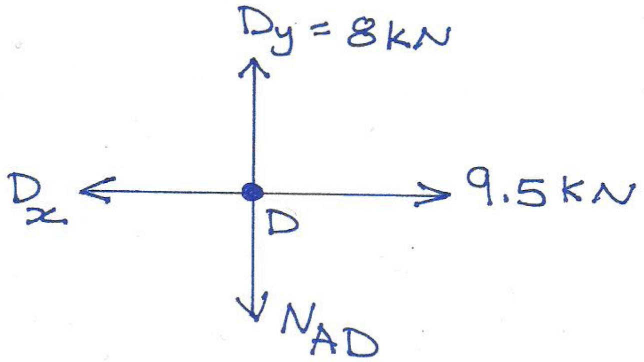
$$-N_{DE} + 11.31\left(\frac{4}{4\sqrt{2}}\right) + 2.5\left(\frac{3}{5}\right) = 0$$

$$N_{DE} = +9.5 \text{ kN (T)}$$

From the equilibrium of the entire truss:

$$+\uparrow \Sigma F_y = 0 \quad : \quad -6 - 2 + D_y = 0 \rightarrow D_y = 8 \text{ kN} (\uparrow)$$

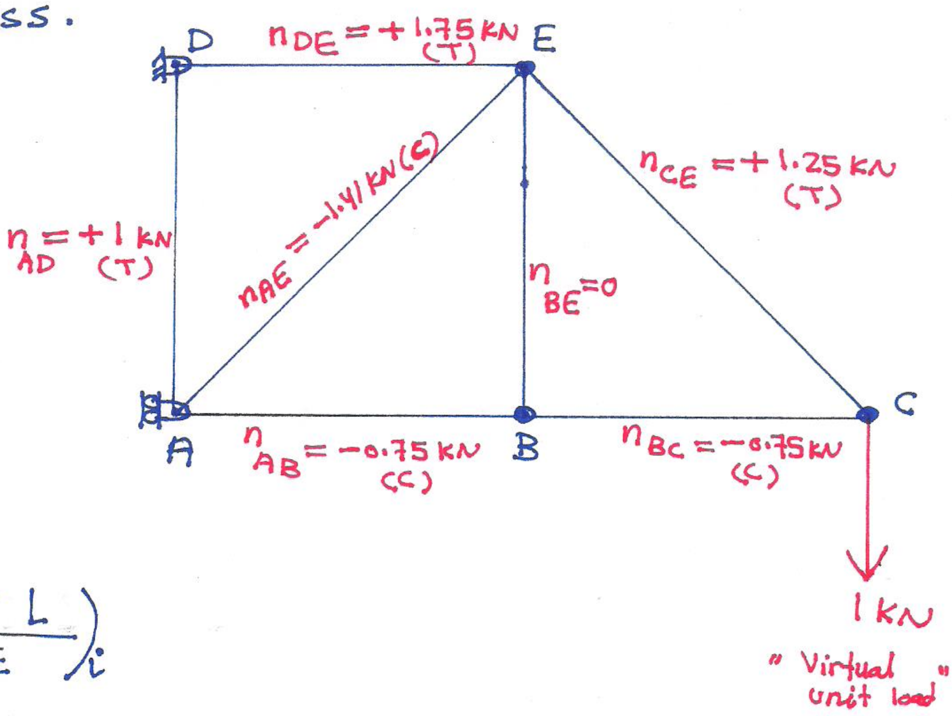
For joint D :



$$+\uparrow \Sigma F_y = 0 : +8 - N_{AD} = 0$$

$$N_{AD} = +8 \text{ kN (T)}$$

Apply 1-unit vertical <u>Vertical displacement at C</u> force at joint C and repeat the analysis of truss.



$$\Delta_{C_V} = \sum_{i=1}^{7} \left(\frac{n\,N\,L}{AE} \right)_i$$

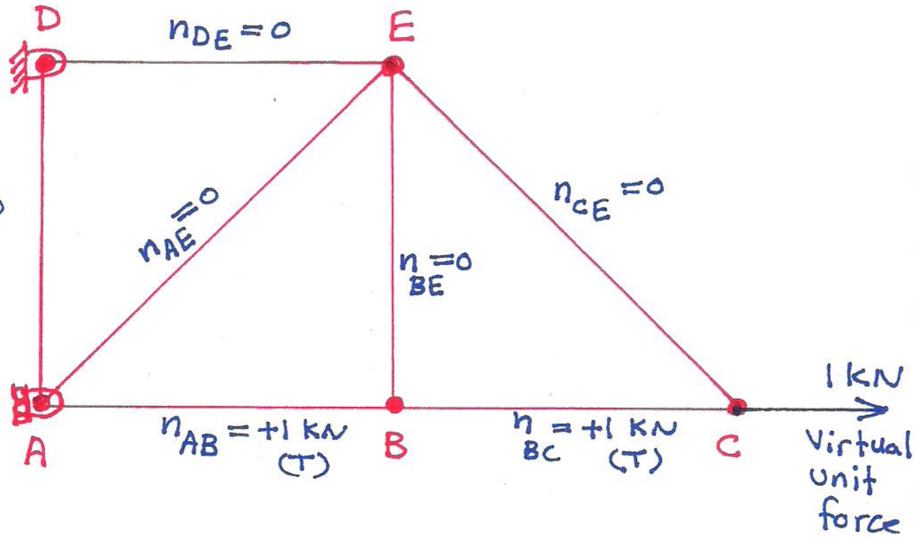
Member	n_i (kN)	N_i (kN)	L_i (m)	$n_i N_i L_i$
AB	−0.75	−1.5	4	4.5
BC	−0.75	−1.5	3	3.375
AD	+1	+8	4	32
AE	−1.41	−11.31	$4\sqrt{2}$	90.21
BE	0	+6	4	0
CE	+1.25	+2.5	5	15.625
DE	+1.75	+9.5	4	66.5
				$\Sigma = 212.21$

$$\Delta_{C_V} = \frac{212.21}{(200 \times 10^6\,\text{kPa})(350 \times 10^{-6})} = {}^{+}3.03 \times 10^{-3}\,\text{m} = {}^{+}3.03\,\text{mm}\,(\downarrow)$$

∴ Vertical displacement of joint C is 3.03 mm

Horizontal displacement at C

Apply 1-unit horizontal force at C and determine the force in each member.



Member	n_i (kN)	N_i (kN)	L_i (m)	$n_i N_i L_i$
AB	+1	−1.5	4	−6
BC	+1	−1.5	3	−4.5
AD	0	+8	4	0
AE	0	−11.31	$4\sqrt{2}$	0
BE	0	+6	4	0
CE	0	+2.5	5	0
DE	0	+9.5	4	0
				$\Sigma = -10.5$

$$\Delta_{C_h} = \frac{-10.5}{(200\times10^{6})(350\times10^{-6})} = -1.5\times10^{-4}\,\text{m} = -0.15\ \text{mm}$$

$(\leftarrow)$

Principle of virtual work for truss deflections due to temperature changes and fabrication errors

IF the temperature change happens in a structure, it affects the work done by the internal virtual loads. $U_i = \sum n_i * (\Delta L)_i$

$$\Delta L_i = \alpha_i \cdot \Delta T_i \cdot L_i$$

where α = coefficient of thermal expansion

$\Delta T = T_2 - T_1$ = temperature change

$$1 * \Delta = \sum_{i=1}^{m} \left(\frac{n \ \alpha \ \Delta T \ L}{AE} \right)_i$$

In addition, there might be an error during fabrication that might cause a virtual internal work done by the loads.

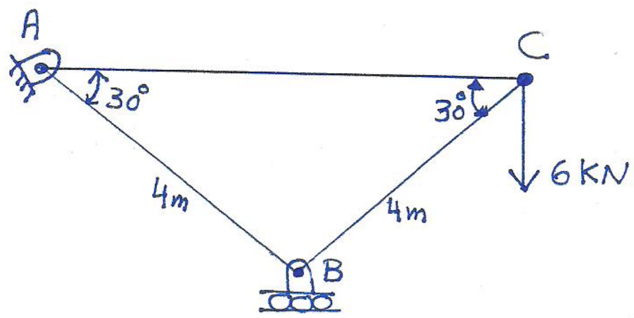
$$1 * \Delta = \sum n_i * (\Delta L)_i$$

Generally, the unit load method for the entire truss structure can be stated as:

$$1 * \Delta = \sum \left[n_i * \left(\underbrace{\frac{NL}{AE}}_{\text{Internal real loads}} + \underbrace{\alpha . \Delta T . L}_{\text{Temperature Change}} + \underbrace{\Delta L}_{\text{Fabrication error}} \right)_i \right]$$

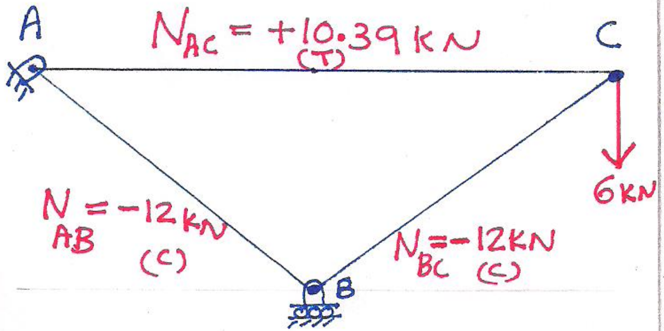
Example

Determine the horizontal displacement of the roller at B of the truss shown. Member AB is subjected to an increase in temperature of $\Delta T = +60°C$, and this member has been fabricated 3 mm too short. The members are made of steel ($E = 200\ GPa$, $\alpha = 12 \times 10^{-6}/°C$). The cross-sectional area of each member is 250 mm^2

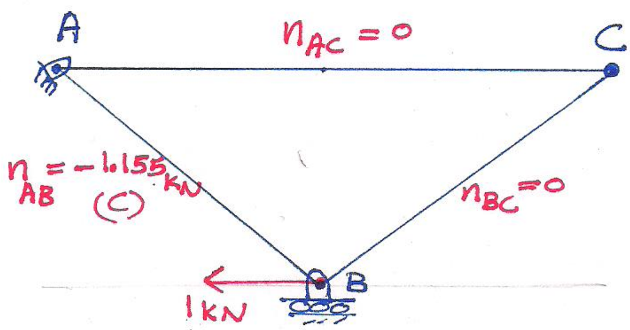


Solution

Analysis of the truss under the application of real load



Analysis of the truss under the application of a virtual unit-force at B.



$$\Delta_{B_h} = \sum_{i=1}^{3} n_i \left(\frac{NL}{AE} + \alpha.\Delta T.L + \Delta L \right)_i$$

Member	n_i (kN)	L (m)	N_i (kN)	ΔT (°C)	ΔL (m) fabrication	$n_i N_i L_i$	$n_i \alpha.\Delta T.L$	$n_i\ \Delta L$
AB	−1.155	4	−12	+60	-3×10^{-3}	+55.44	-3.3264×10^{-3}	$+3.465 \times 10^{-3}$
AC	0	$4\sqrt{3}$	+10.39	0	0	0	0	0
BC	0	4	−12	0	0	0	0	0
					Σ=	+55.44	-3.3264×10^{-3}	$+3.465 \times 10^{-3}$

$$\Delta_{B_h} = \frac{+55.44}{(200 \times 10^{6})(250 \times 10^{-6})} + -3.3264 * 10^{-3} + 3.465 * 10^{-3}$$

$$= +1.2474 \times 10^{-3} \text{ m} = 1.2474 \text{ mm} \ (\leftarrow)$$

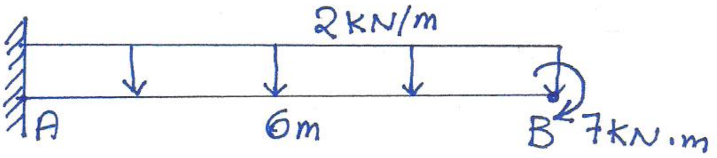
Castiglian's Theorem

The first partial derivative of the total internal energy in a structure with respect to the force (P, M) applied at any point is equal to the deflection (δ, θ) at the point of application of that force in the direction of its line of action.

$$\boxed{\begin{aligned}\theta &= \delta U_i / \delta M' \\ \delta &= \delta U_i / \delta P\end{aligned}}$$

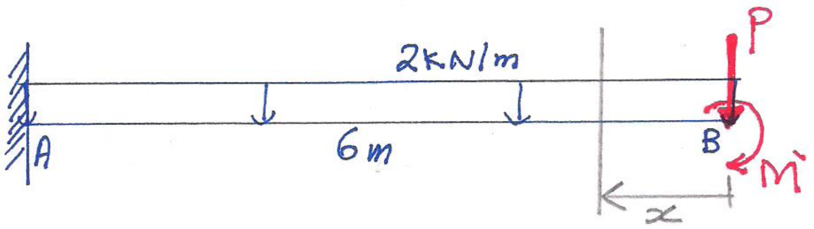
Example

Determine the slope and deflection at point B. EI = constant



Solution

Apply a dummy force "P" and a dummy moment "M'" at point B where deflection and slope is required.

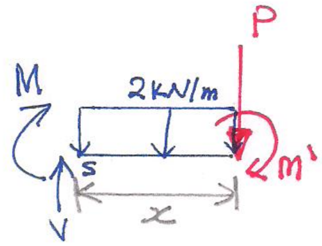


$0 \leq x \leq 6\,m$

Note:

$P = 0$

$M' = 7\ kN.m$



$$\circlearrowleft + \Sigma M_s = 0 : \quad -M - 2x\left(\frac{x}{2}\right) - Px - M' = 0$$

$$M = -x^2 - Px - M'$$

The total real internal energy :

$$U_i = \int \frac{M^2}{2EI}\,dx = \int_0^6 \frac{(-x^2 - Px - M')^2}{2EI}\,dx$$

$$= \int_0^6 \frac{x^4 + 2Px^3 + (P^2 + 2M')x^2 + 2PM'x + M'^2}{2EI}\,dx$$

$$= \frac{1}{2EI}\left(\frac{x^5}{5} + \frac{Px^4}{2} + \frac{(P^2+2M')}{3}x^3 + PM'x^2 + M'^2x\right)\Bigg|_{x=0}^{x=6\,m}$$

$$U_i = \frac{1}{2EI}\left(\left(1555.2 + 648P + 72\,(P^2 + 2M') + 36PM' + 6M'^2\right)\right)$$

Slope at B :

$$\theta_B = \frac{\partial U_i}{\partial M'}\Bigg|_{\substack{P=0 \\ M'=+7\,kN.m}} = \frac{1}{2EI}\left(72*2 + 36P + 12M'\right)\Bigg|_{\substack{P=0 \\ M'=+7\,kN.m}}$$

$$\theta_B = \frac{1}{2EI}\left(72*2 + 36*0 + 12*7\right) = \frac{+114}{EI}\ rad$$

Vertical displacement at B :

$$\delta_B = \frac{\partial U_i}{\partial P}\Bigg|_{\substack{P=0 \\ M'=+7\,kN.m}} = \frac{1}{2EI}\left[648 + 72*2P + 36M'\right]\Bigg|_{\substack{P=0 \\ M'=+7\,kN.m}}$$

$$\delta_B = \frac{1}{2EI}\left[648 + 72*2*0 + 36*7\right] = \frac{450}{EI}\ m \downarrow$$