Element level

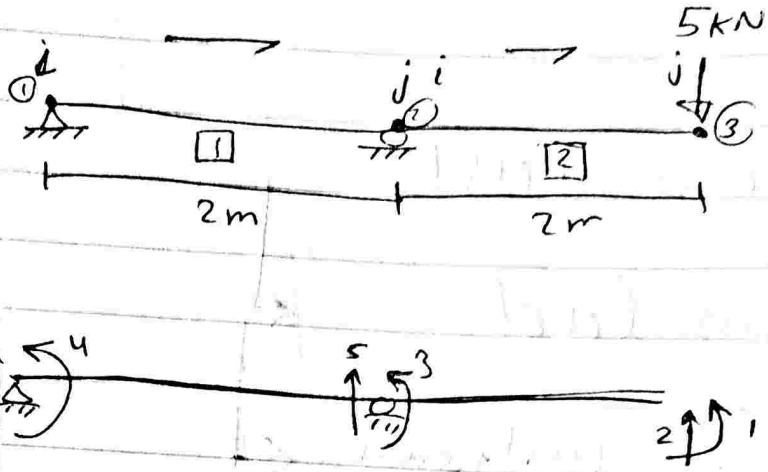
system level

solve for unknown disp.

Element Internal forces.

⇒ Example :-

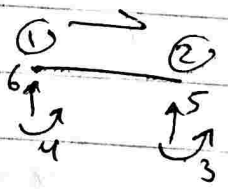
- Elements : [1] , [2]
- Nodes : ①, ②, ③
- DoFs : $\theta_1, \theta_2, \theta_3, \Delta_3$



- Element Stiffness matrices :-

[1] $k = EI$

	6	4	5	3	
	1.5	1.5	-1.5	1.5	6
	1.5	2	-1.5	1	4
	-1.5	-1.5	1.5	-1.5	5
	1.5	1	-1.5	2	3



(for Element 1)

$12EI/L^3 = 1.5EI$

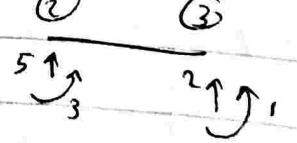
$6EI/L^2 = 1.5EI$

$4EI/L = 2EI$

$2EI/L = EI$

[2] $k = EI$

	5	3	2	1	
	1.5	1.5	-1.5	1.5	5
	1.5	2	-1.5	1	3
	-1.5	-1.5	1.5	-1.5	2
	1.5	1	-1.5	2	1



$$Q = KD$$

K → Stiffness matrix of the system

Q → Nodal forces on system

D → Nodal displacement of the system

$$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}\begin{bmatrix} 0 \\ -5 \\ 0 \\ 0 \\ Q_5 \\ Q_6 \end{bmatrix} = EI \begin{bmatrix} 2 & -1.5 & 1 & 0 & 1.5 & 0 \\ -1.5 & 1.5 & -1.5 & 0 & -1.5 & 0 \\ 1 & -1.5 & 4 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 2 & -1.5 & 1.5 \\ 1.5 & -1.5 & 0 & -1.5 & 3 & -1.5 \\ 0 & 0 & 1.5 & 1.5 & -1.5 & 1.5 \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ 0 \\ 0 \end{bmatrix}$$

(Columns of the stiffness matrix numbered 1, 2, 3, 4, 5, 6.)

Equations:-

$$0 = 2D_1 - 1.5D_2 + D_3 + 0 \quad \text{---} (1)$$

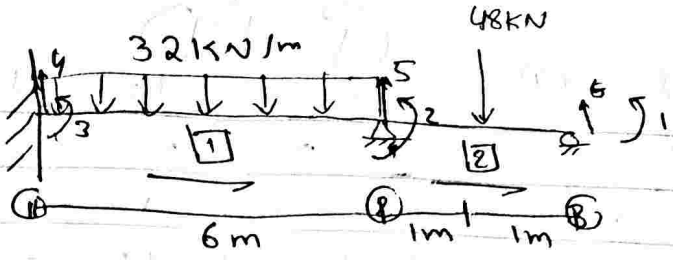
solving

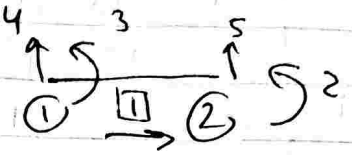
$$D_1 = -16.67 / EI$$
$$D_2 = -26.67 / EI$$
$$D_3 = -6.67 / EI$$
$$D_4 = 3.33 / EI$$

✱ Example :-

$E = 200\,GPa$

$I = 216 \times 10^6\,mm^4$





⇒ member (1) :-

$$\frac{12EI}{L^3} = \frac{12 \times 200 \times 216}{6^3} = 2400$$

$$\frac{6EI}{L^2} = 7200$$

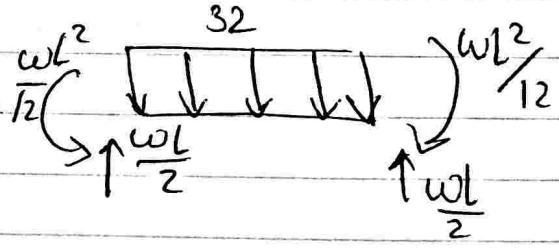
$$\frac{4EI}{L} = 28800$$

$$\frac{2EI}{L} = 14400$$

element Level

$$K_1 = \begin{bmatrix} 4 & 3 & 5 & 2 & \\ 2400 & 7200 & -2400 & 7200 & 4 \\ 7200 & 28800 & -7200 & 14400 & 3 \\ -2400 & -7200 & 2400 & -7200 & 5 \\ 7200 & 14400 & -7200 & 28800 & 2 \end{bmatrix}$$

⇒ Because it's has a uniform load :-



Shear ⇒ $q_0 - 4 = +96$ ↷ ↑

moment ⇒ $q_0 - 3 = +96$ ccw

Shear ⇒ $q_0 - 5 = +96$ ↑

moment → $q_0 - 2 = -96$ cw

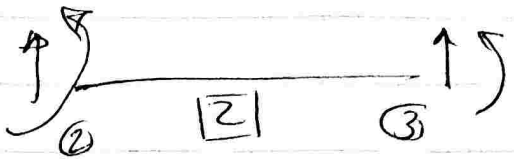
⇒ for member 2:-

$\frac{12EI}{L^3} = 64800$

$\frac{6EI}{L^2} = 64800$

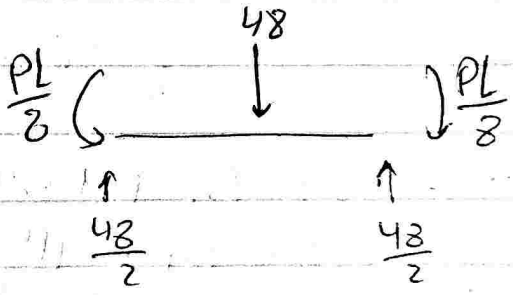
$\frac{4EI}{L} = 86400$

$\frac{2EI}{L} = 43200$



$$K_2 = \begin{bmatrix} 5 & 2 & 6 & 1 \\ 64800 & 64800 & -86400 & -86400 \\ 64800 & 86400 & -86400 & 43200 \\ -64800 & -64800 & +86400 & -64800 \\ 64800 & 43200 & -86400 & 86400 \end{bmatrix} \begin{matrix} 5 \\ 2 \\ 6 \\ 1 \end{matrix}$$

⇒ q_o for element 2:-



$q_o - 5 = 24$

$q_o - 2 = 12$

$q_o - 6 = 24$

$q_o - 1 = -12$

System level

$$Q = KD + Q_0$$

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{bmatrix} 86400 & 43200 & 0 & 0 & 64800 & -64800 \\ 43200 & 115200 & 14400 & 7200 & 57600 & -64800 \\ 0 & 14400 & 28800 & 7200 & -7200 & 0 \\ 0 & 7200 & 7200 & 2400 & -2400 & 0 \\ 64800 & 57600 & -7200 & -2400 & 67200 & -64800 \\ -64800 & -64800 & 0 & 0 & -64800 & 64800 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} Q_{0-1} = -12 \\ Q_{0-2} = -84 \\ Q_{0-3} = 96 \\ Q_{0-4} = 96 \\ Q_{0-5} = 126 \\ Q_{0-6} = 24 \end{bmatrix}$$

In General this is an external forces

$$12 = 86400 D_1 + 43200 D_2$$
$$84 = 43200 D_1 + 115200 D_2$$
$$D_1 = -0.2778 (10^{-3}) \text{ rad}$$
$$D_2 = +0.833 (10^{-3}) \text{ rad}$$

$$Q_{-3} = 1440 \times D_2$$

$$Q_{-4} = 7200 \times D_2$$

K : system stiffness matrix : 6×6

k : element stiffness matrix : 4×4

⇒ For element 1:-

End Forces on the element → (annotation on q)

Nodal displacement of the element → (annotation on d)

$$\begin{matrix} V_i \\ M_i \\ V_j \\ M_j \end{matrix}\begin{bmatrix} q_4 \\ q_3 \\ q_5 \\ q_2 \end{bmatrix} = \begin{bmatrix} \overset{4}{2400} & \overset{3}{7200} & \overset{5}{-2400} & \overset{2}{7200} \\ 7200 & 28800 & -7200 & 14400 \\ -2400 & -7200 & 67200 & 57600 \\ 7200 & 14400 & -57600 & 115200 \end{bmatrix} \begin{bmatrix} d_4 \; 0 \\ d_3 \; 0 \\ d_5 \; 0 \\ d_2 \; 0.833 \end{bmatrix} + \begin{bmatrix} 96 \\ 96 \\ 96 \\ -96 \end{bmatrix}$$

⇒ In the Element level this is an internal forces

$$\underset{4\times1}{q_1} = \underset{4\times4}{k_1}\underset{4\times1}{d} + \underset{4\times1}{q_{o,1}}$$

⇒ $q_5^{[1]} \overset{?}{=} q_5^{[2]}$ ⇒ No, Because they come from a diff. element.

[Shear left to the support] [Shear right to the support]

⇒ $q_2^{[1]} \overset{?}{=} q_2^{[2]}$ ⇒ yes, opposite direction, Because there is No external moment on the Joint

[Moment left the support] [Moment right the support]