Reinforced Concrete Design II



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in Eq. 6-10 the sum of the normal stresses must equal the axial force $P$. Noting that in the elastic zone the stress can be expressed algebraically as $\sigma = (\sigma_{yp}/3) - [8\sigma_{yp}y/(3h)]$ and that in the plastic zone $\sigma = \sigma_{yp}$, one has

$$P_2 = \int_A \sigma\, dA = \int_{-h/4}^{+h/2} \frac{\sigma_{yp}}{3}\left(1 - \frac{8y}{h}\right) b\, dy + \int_{-h/2}^{-h/4} \sigma_{yp} b\, dy = \sigma_{yp}\frac{bh}{4}$$

$$M_2 = -\int_A \sigma y\, dA = -\int_{-h/4}^{+h/2} \frac{\sigma_{yp}}{3}\left(1 - \frac{8y}{h}\right) yb\, dy - \int_{-h/2}^{-h/4} \sigma_{yp} yb\, dy$$

$$= \frac{3}{16}\sigma_{yp}bh^2$$

Note that the axial force found above exactly equals the force acting on the plastic area of the section. The moment $M_2$ is greater than $M_{yp} = \sigma_{yp}bh^2/6$ and less than $M_{ult} = M_p = \sigma_{yp}bh^2/4$ (see Eq. 6-12).

The axial force and moment corresponding to the fully plastic case shown in Figs. 8-5(e) and (f) are simple to determine. As may be seen from Fig. 8-5(e) the axial force is developed by $\sigma_{yp}$ acting on the area $2y_1b$. Because of symmetry, these stresses make no contribution to the moment. Forces acting on the top and the bottom areas $ab = [(h/2) - y_1]b$, Fig. 8-5(d), form a couple with a moment arm of $h - a = (h/2) + y_1$. Therefore

$$P_3 = 2y_1 b\sigma_{yp} \qquad \text{or} \qquad y_1 = P_3/(2b\sigma_{yp})$$

and

$$M_3 = ab\sigma_{yp}(h - a) = \sigma_{yp}b\left(\frac{h^2}{4} - y_1^2\right) = M_p - \sigma_{yp}by_1^2$$

$$= \frac{3M_{yp}}{2} - \frac{P_3^2}{4b\sigma_{yp}}$$

Then dividing by $M_p = 3M_{yp}/2 = \sigma_{yp}bh^2/4$ and simplifying, one obtains

$$\frac{2M_3}{3M_{yp}} + \left(\frac{P_3}{P_{yp}}\right)^2 = 1 \qquad \textbf{(8-4)}$$

This is a general equation for the interaction curve for $P$ and $M$ necessary to achieve the fully plastic condition in a rectangular member (see Fig. 8-6). Unlike the equation for the elastic case, the relation is nonlinear.

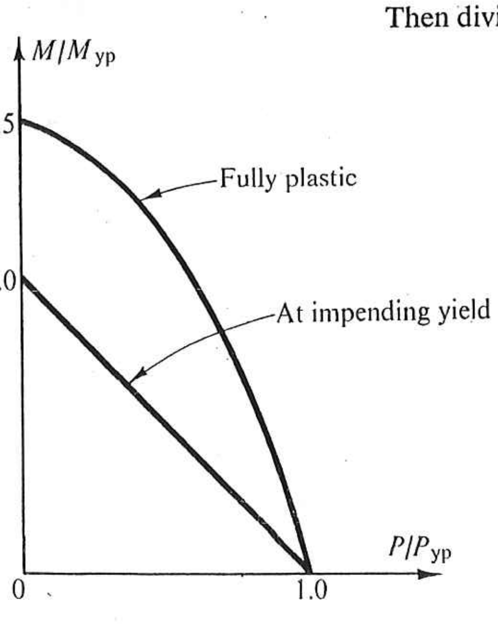


**Fig. 8-6.** Interaction curves for $P$ and $M$ for a rectangular member

## 8-3. SKEW BENDING

In Chapter 6, on the flexure of beams, it was emphasized that the derived flexure formula is applicable only if the bending moment acts around one or the other of the principal axes of the cross section. Since the plane of the applied moment $M$ may be inclined with respect to the principal axes,

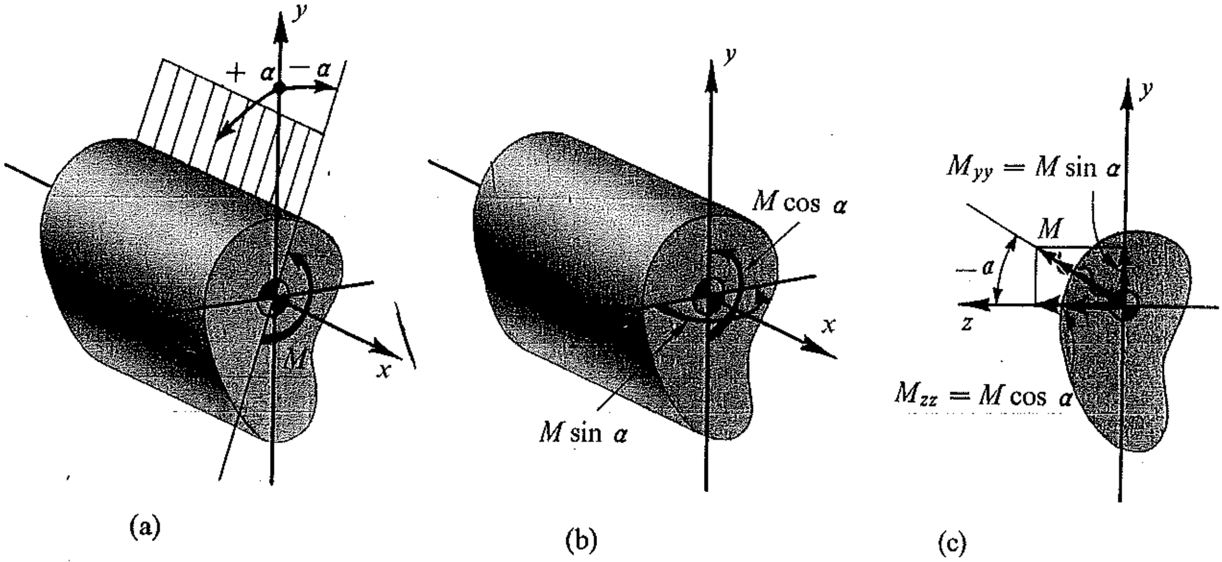


**Fig. 8-7.** (a) Bending moment in a plane which does not coincide with either principal axis; (b) and (c) bending-moment components in the planes of principal axes.

it is necessary to consider a more general case. Such a case is shown in Fig. 8-7(a) and is called *skew bending.** The bending plane of $M$ is located by an angle $\alpha$ which is positive when measured from the $y$ axis toward the $z$ axis in a counterclockwise direction.

To solve the stated problem, the applied moment $M$ is resolved into two components acting in the planes of the principal axes. For the negative $\alpha$ shown in Fig. 8-7(a), the bending moment components acting around both the $z$ and the $y$ axes are positive (see Fig. 2-2). The one around the $z$ axis is $M \cos \alpha$, and the one around the $y$ axis is $M \sin \alpha$. Figures 8-7(b) and (c) show alternative representations of these positive moment components.

The elastic flexure formula previously derived can be applied to each one of the moment components acting around a principal axis, and the combined stress follows by superposition. An example of superposition is in Fig. 8-8, where for simplicity a rectangular section is shown. Analogous results hold true in general and one has†

* In many books such bending is called *unsymmetrical*. However, as the problem considered is more general than something lacking symmetry, the word *skew* is used in this text. This corresponds to the use of the words *schiefe* in German and *kosoi* in Russian, which mean inclined or skew.

† It is possible to derive the flexure formula for arbitrarily directed $y$ and $z$ axes. Such a formula, equivalent to Eq. 8-5, is

$$\sigma_x = -\frac{M_{zz}I_{yy} - M_{yy}I_{yz}}{I_{yy}I_{zz} - I_{yz}^2}\,y + \frac{M_{yy}I_{zz} - M_{zz}I_{yz}}{I_{yy}I_{zz} - I_{yz}^2}\,z \qquad \textbf{(8-5a)}$$

where $I_{yy}$ and $I_{zz}$ are moments of inertia, and $I_{yz}$ is the product of inertia. For principal axes, $I_{yz} = 0$, and the above equation reverts to Eq. 8-5. For further details see, for example, D. J. Peery, *Aircraft Structures* (New York: McGraw-Hill Book Company, 1950).

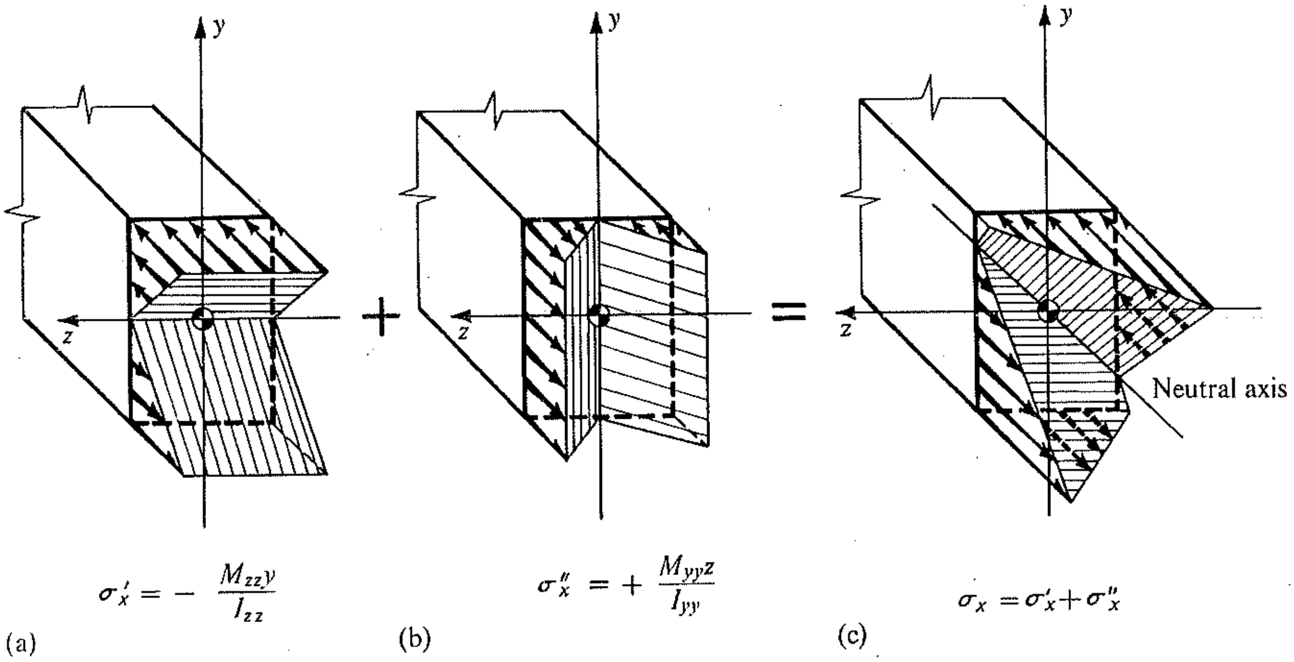


**Fig. 8-8.** Superposition of elastic bending stresses.

$$\sigma_x = -\frac{M_{zz}y}{I_{zz}} + \frac{M_{yy}z}{I_{yy}} \tag{8-5}$$

where the subscripts *yy* and *zz* on *M* and *I* refer to the respective principal axes of the cross-sectional area around which bending takes place. Note that the first term on the right side giving the stresses caused by bending around the *z* axis is negative just as Eq. 6-3 from which it comes. On the other hand, the second term, although analogous to Eq. 6-3, is taken positive to obtain the correspondence in sign between the normal stresses and the sense of the positive moment acting around the *y* axis. On this basis, in applying Eq. 8-5, if positive signs are associated with all quantities in conformity with the coordinate axes, positive results indicate tensile stresses; negative, compressive stresses. In most problems by thinking in terms of the physical action on the member, one can directly assign the sign of each term in Eq. 8-5, although the availability of the sign convention is desirable.

If, in general, the applied moment $M$ acts in a plane making a positive angle $\alpha$ with the $y$ axis, the bending-moment components are $M_{yy} = -M \sin \alpha$ and $M_{zz} = M \cos \alpha$, and Eq. 8-5 can be stated as

$$\sigma_x = -M\left(\frac{y}{I_{zz}} \cos \alpha + \frac{z}{I_{yy}} \sin \alpha\right) \tag{8-6}$$

From this relation an equation locating the neutral axis can be found by setting $\sigma_x = 0$. This yields

$$y = -z(I_{zz}/I_{yy}) \tan \alpha \tag{8-7}$$

A study of this equation using the procedures of analytic geometry shows that for skew bending, unless $I_{zz} = I_{yy}$, the neutral axis is not perpendicular to the plane of the applied moment. The neutral axis is, however, a straight line, and the "plane section" rotates around it. As in symmetrical bending, the largest stress occurs at the most remote point from the neutral axis. Note, however, that in skew bending the neutral axis does not coincide with either one of the principal axes and it is not located at right angles to the bending plane.

The analysis of inelastic beams for skew bending is very cumbersome and is beyond the scope of this text.*

**EXAMPLE 8-4**

A 4-in.-by-6-in. (actual size) wooden beam shown in Fig. 8-9(a) is used to support a uniformly distributed load of 1,000 lb (total) on a simple span of 10 ft. The applied load acts in a plane making an angle of 30°

* M. S. Aghbabian and E. P. Popov, "Unsymmetrical Bending of Rectangular Beams Beyond the Elastic Limit," *Proceedings, First U.S. National Congress of Applied Mechanics*, 1951, pp. 579–84 (published by ASME).

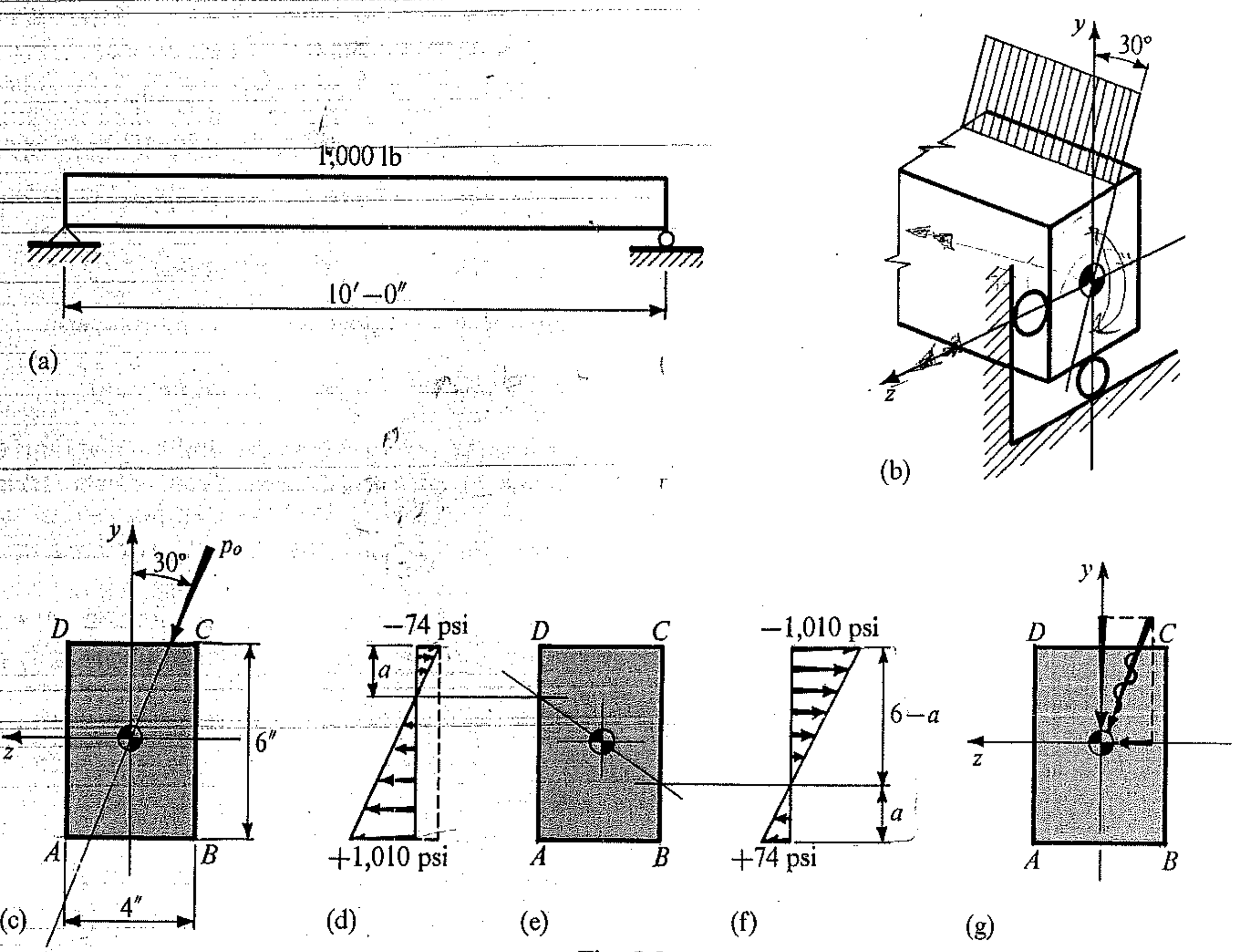


**Fig. 8-9**

with the vertical, as shown in Fig. 8-9(b) and again in Fig. 8-9(c). Calculate the maximum bending stress at midspan, and, for the same section, locate the neutral axis. Neglect the weight of the beam.

SOLUTION

The maximum bending in the plane of the applied load occurs at the midspan, and according to Example 2-6 it is equal to $p_o L^2/8$ or $WL/8$, where $W$ is the total load on the span $L$. Hence

$$M = WL/8 = 1{,}000(10)/8 = 1{,}250 \text{ ft-lb}$$

Here $\alpha = -30°$, and the moment components acting around their respective axes are

$$M_{zz} = M \cos \alpha = 1{,}250(\sqrt{3}/2)12 = 13{,}000 \text{ in-lb}$$

$$M_{yy} = -M \sin \alpha = -1{,}250(-0.5)12 = 7{,}500 \text{ in-lb}$$

By considering the nature of the flexural stress distribution about both principal axes of the cross section, one may conclude that the maximum tensile stress occurs at $A$. The value of this stress follows by applying Eq. 8-5 with $y = c_1 = -3$ in., and $z = c_2 = +2$ in. Stresses at the other corners of the cross section are similarly determined.

$$\sigma_A = -\frac{M_{zz}c_1}{I_{zz}} + \frac{M_{yy}c_2}{I_{yy}} = \frac{13{,}000(3)}{4(6)^3/12} + \frac{7{,}500(2)}{6(4)^3/12}$$

$$= +542 + 468 = +1{,}010 \text{ psi} \qquad \text{(tension)}$$

$$\sigma_B = +542 - 468 = +74 \text{ psi} \qquad \text{(tension)}$$

$$\sigma_C = -542 - 468 = -1{,}010 \text{ psi} \qquad \text{(compression)}$$

$$\sigma_D = -542 + 468 = -74 \text{ psi} \qquad \text{(compression)}$$

To locate the neutral axis the stress distribution diagrams along the sides in Fig. 8-9(d) or (f) can be used. From similar triangles, $a/(6 - a) = 74/1{,}010$, or $a = 0.41$ in. This locates the neutral axis in Fig. 8-9(e). Alternatively, Eq. 8-7 with $\alpha = -30°$ can be used.

When skew bending of a beam is caused by applied transverse forces, as in the above example, an equivalent procedure is usually more convenient. The applied forces are first resolved into components which act parallel to the principal axes of the cross-sectional area. Then the bending moments caused by these components around the respective axes are computed for use in the flexure formula. For the above example, such components of the applied load are shown in Fig. 8-9(g). To avoid torsional stresses the applied transverse forces must act through the shear center. For bilaterally symmetrical sections, e.g., a rectangle, a circle, an $I$ beam, etc., the shear center coincides with the centroid of the cross

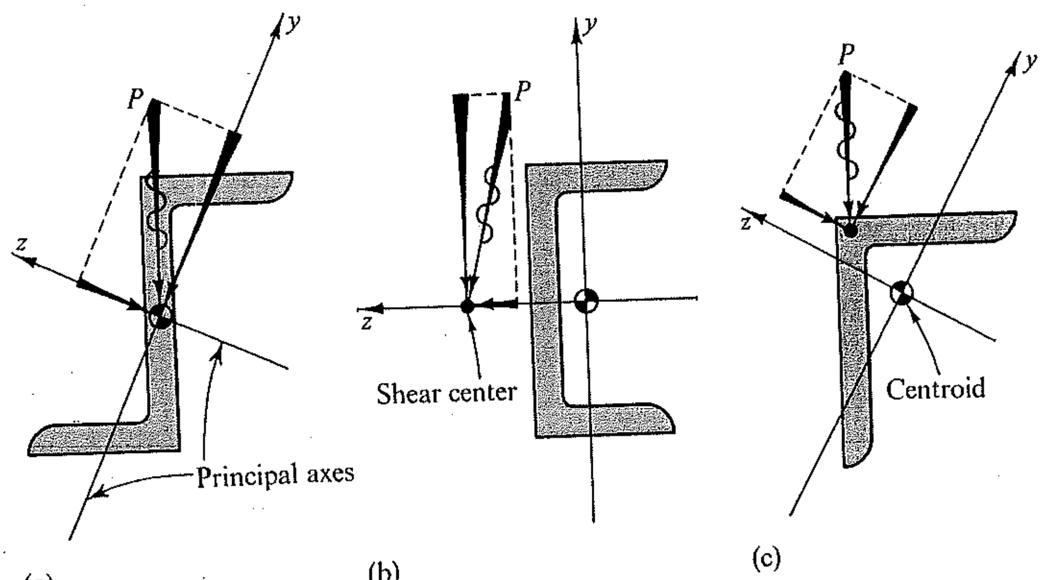


**Fig. 8-10.** Forces applied through shear center cause no torsion.

section. For other cross sections, such as channels, angles, $Z$ sections, etc., the shear center lies elsewhere (see Art. 7-7). In such problems the transverse force must be applied at the shear center to avoid torsional stresses. This approach is illustrated in Fig. 8-10. Otherwise, in addition to the bending stresses, the torsional stresses must be investigated. In such cases the applied torque equals the applied force multiplied by its moment arm measured from the shear center.

## 8-4. ECCENTRICALLY LOADED MEMBERS

Occasionally situations arise where a force $P$ acting parallel to the axis of the member is applied eccentrically with respect to the centroidal axis of the member, Fig. 8-11(a). By applying two equal and opposite forces $P$ at the centroid, as shown in Fig. 8-11(b), the problem is changed to that of an axially applied force $P$ and skew bending in the plane of the applied force $P$ and the axis of the member. This skew bending moment can be further resolved into the components $M_{yy} = Pz_o$ acting around the $y$ axis, and $M_{zz} = -Py_o$ acting around the $z$ axis, Figs. 8-11(d) and (e). Then, the compound normal stress at any point $(y, z)$ of the cross section, for an eccentrically loaded member, can be found by simply adding an axial stress term to Eq. 8-5. Hence

$$\sigma_x = \frac{P}{A} - \frac{M_{zz}y}{I_{zz}} + \frac{M_{yy}z}{I_{yy}} \tag{8-8}$$

where $P$ is taken positive for tensile forces. The remainder of the sign convention is the same as that for Eq. 8-5. Providing the $y$ and the $z$ axes

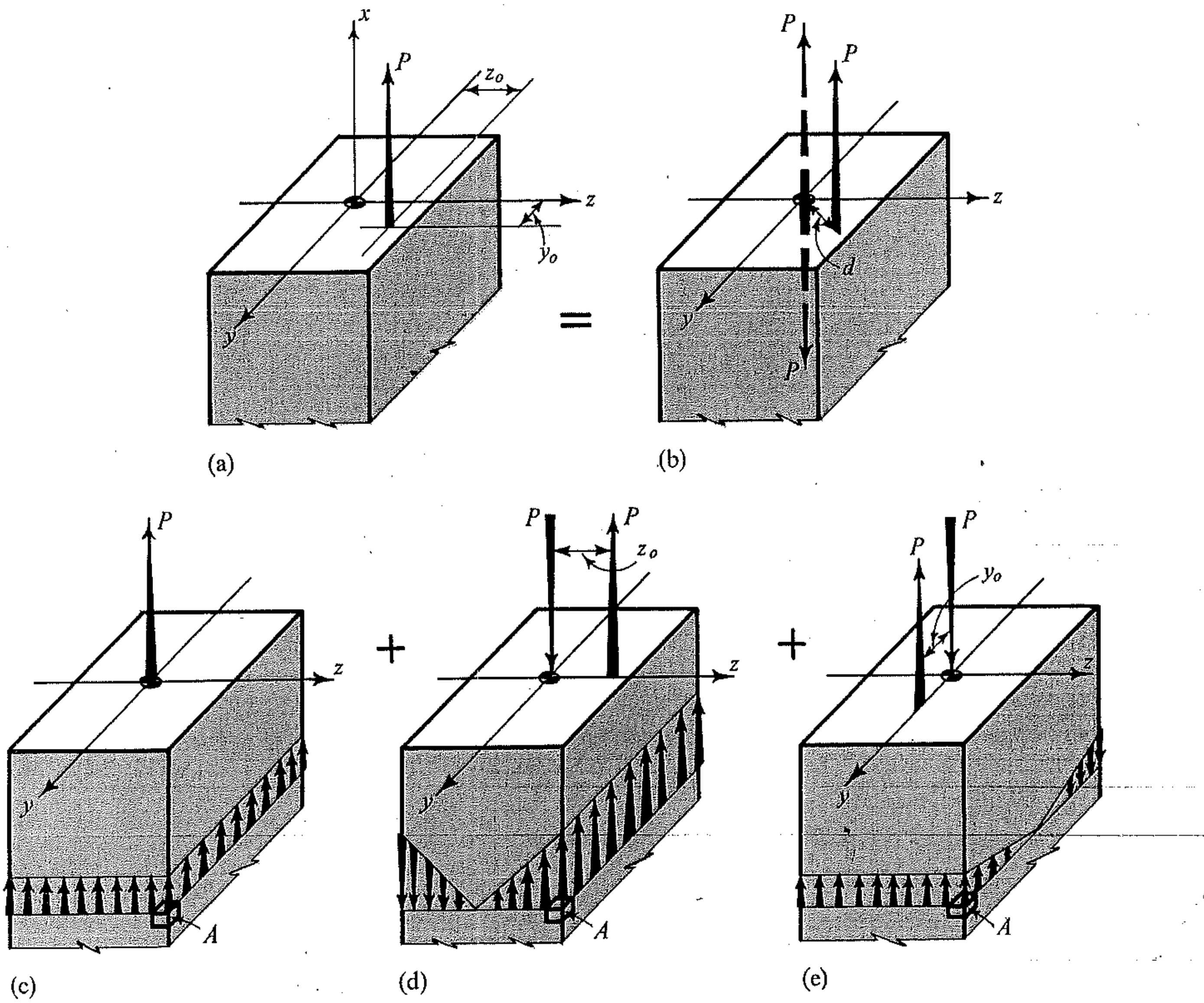


**Fig. 8-11.** Resolution of a problem into three problems, each one of which may be solved by the methods previously discussed.

are the principal axes, Eq. 8-8 is applicable to prismatic members of any cross-sectional shape.

For a given loading condition Eq. 8-8 can be rewritten as

$$\sigma_x = A + By + Cz \tag{8-9}$$

where $A$, $B$, and $C$ are constants. This is seen to be an equation of a plane; it clearly shows the nature of stress distribution. For the linearly elastic case under discussion, dividing through Eq. 8-9 by the elastic modulus $E$ recovers the basic kinematic assumption of the technical theory, i.e.,

$$\varepsilon_x = a + by + cz \tag{8-10}$$

where $a$, $b$, and $c$ are constants.

In some eccentrically loaded members it is possible to locate the line of zero stress within the cross-sectional area of a member by determining a line where $\sigma_x = 0$. This line is analogous to the neutral axis occurring in pure bending. Unlike the former case, however, with $P \neq 0$ this line does not pass through the centroid of a section. For large axial loads and small moments, it lies outside the cross section. Its significance lies in the fact that the normal stresses vary linearly from it.

This method is applicable for compression members providing their length is small in relation to their transverse dimensions. Slender bars in compression require special treatment (Chapter 14). Also, near the point of application of the force, the analysis developed here is incorrect. There the stress distribution is greatly disturbed and is similar to a local stress concentration (see Art. 4-18 and especially Fig. 4-30).

**EXAMPLE 8-5**

Find the stress distribution at the section $ABCD$ for the block shown in Fig. 8-12(a) if $P = 14.4$ kips. At the same section, locate the line of zero stress. Neglect the weight of the block.

**SOLUTION**

The forces acting on the section $ABCD$, Fig. 8-5(c), are $P = -14.4$ kips, $M_{yy} = -14.4(6) = -86.4$ kip-in., and $M_{zz} = -14.4(3 + 3) = -86.4$

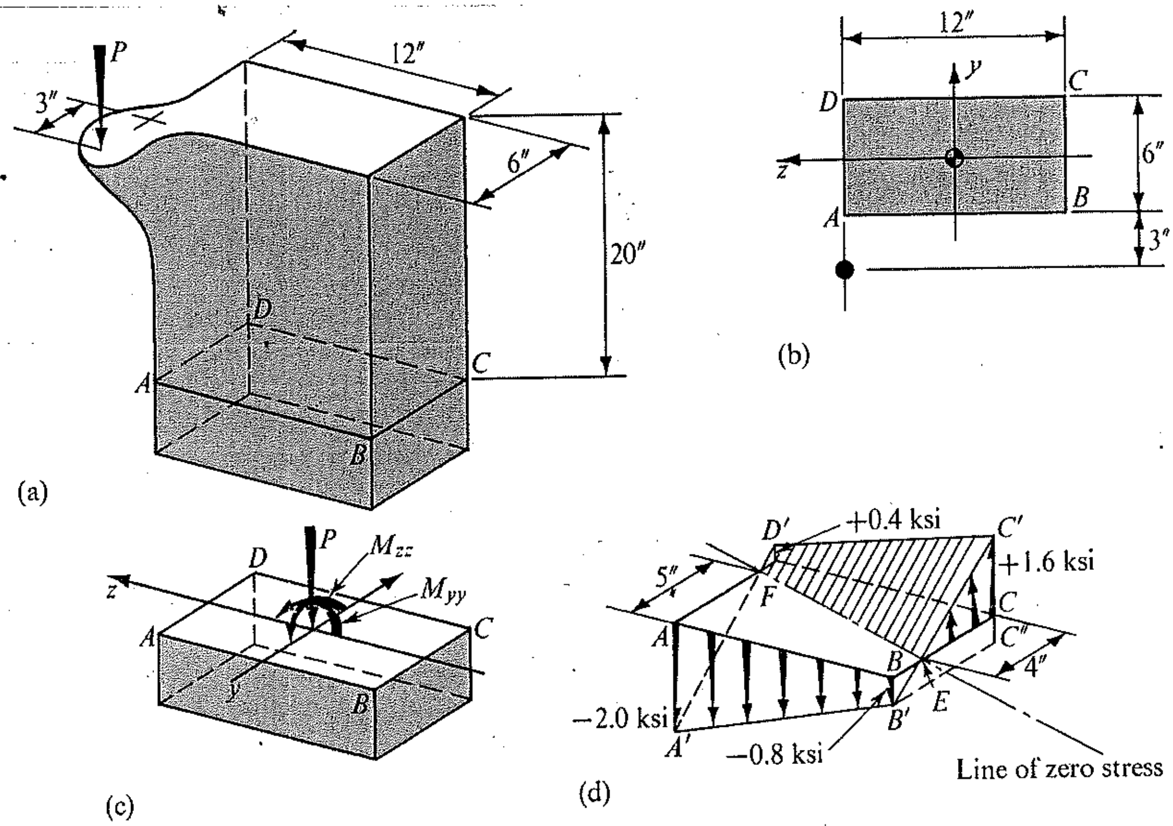


**Fig. 8-12**

kip-in. The cross section of the block $A = 6(12) = 72$ in.², and the respective section moduli are $S_{zz} = 12(6)^2/6 = 72$ in.³ and $S_{yy} = 6(12)^2/6 = 144$ in.³ Hence, using a relation equivalent to Eq. 8-8 gives the compound normal stresses for the corner elements:

$$\sigma = \frac{P}{A} \mp \frac{M_{zz}}{S_{zz}} \pm \frac{M_{yy}}{S_{yy}} = -\frac{14.4}{72} \pm \frac{86.4}{72} \mp \frac{86.4}{144} = -0.2 \pm 1.2 \mp 0.6$$

Here the units of stress are kips per square inch. The sense of the forces shown in Fig. 8-12(c) determines the signs of stresses. Therefore, if the subscript of the stress signifies its location, the corner normal stresses are:

$$\sigma_A = -0.2 - 1.2 - 0.6 = -2.0 \text{ ksi}$$

$$\sigma_B = -0.2 - 1.2 + 0.6 = -0.8 \text{ ksi}$$

$$\sigma_C = -0.2 + 1.2 + 0.6 = +1.6 \text{ ksi}$$

$$\sigma_D = -0.2 + 1.2 - 0.6 = +0.4 \text{ ksi}$$

These stresses are shown in Fig. 8-12(d). The ends of these four stress vectors at $A'$, $B'$, $C'$, and $D'$ lie in the plane $A'B'C'D'$. The vertical distance between the planes $ABCD$ and $A'B'C'D'$ defines the compound stress at any point on the cross section. The intersection of the plane $A'B'C'D'$ with the plane $ABCD$ locates the line of zero stress $FE$.

By drawing a line $B'C''$ parallel to $BC$, similar triangles $C'B'C''$ and $C'EC$ are obtained: thus the distance $CE = [1.6/(1.6 + 0.8)]6 = 4$ in. Similarly, the distance $AF$ is found to be 5 in. Points $E$ and $F$ locate the line of zero stress.

**EXAMPLE 8-6**

Find the zone over which the vertical downward force $P_o$ may be applied to the rectangular weightless block shown in Fig. 8-13(a) without causing any tensile stresses at the section $A$-$B$.

**SOLUTION**

The force $P = -P_o$ is placed at an arbitrary point in the first quadrant of the $y$-$z$ coordinate system shown. Then the same reasoning used in the preceding example shows that with this position of the force the greatest tendency for a tensile stress exists at $A$. With $P = -P_o$, $M_{zz} = +P_o y$ and $M_{yy} = -P_o z$, setting the stress at $A$ equal to zero fulfills the limiting condition of the problem. Using Eq. 8-8 allows the stress at $A$ to be expressed as:

$$\sigma_A = 0 = \frac{(-P_o)}{A} - \frac{(P_o y)(-b/2)}{I_{zz}} + \frac{(-P_o z)(-h/2)}{I_{yy}}$$

or

$$-\frac{P_o}{A} + \frac{P_o y}{b^2h/6} + \frac{P_o z}{bh^2/6} = 0$$

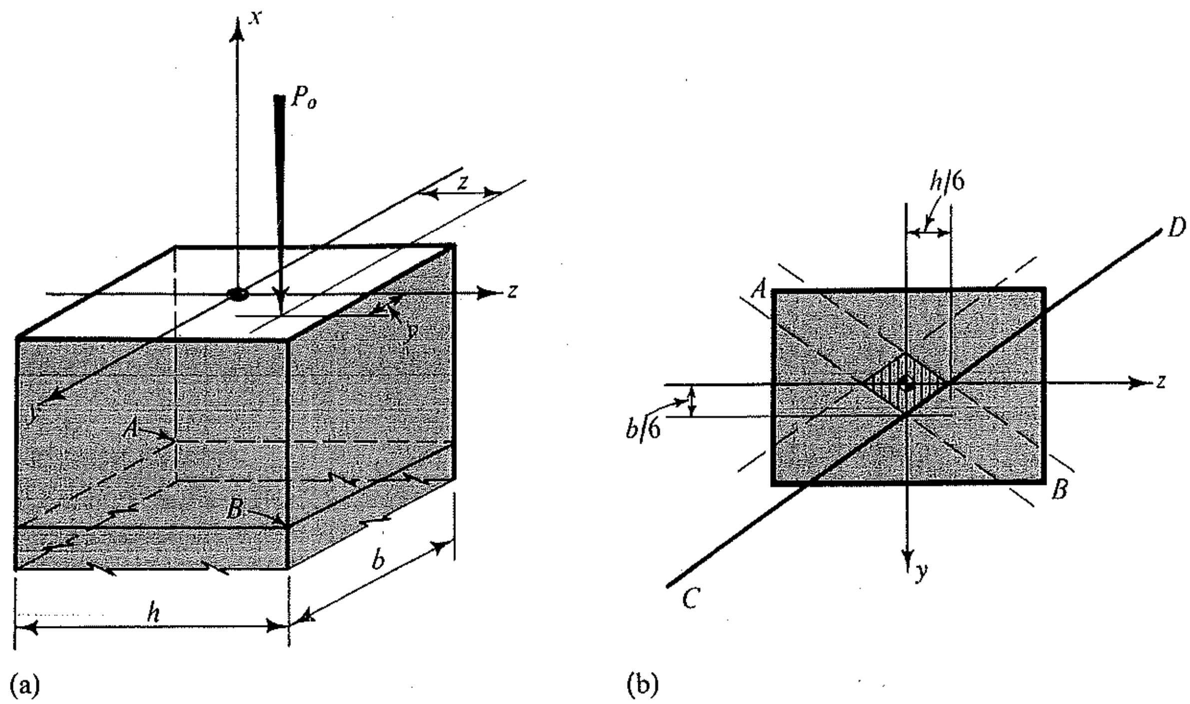

Fig. 8-13

Simplifying $[z/(h/6)] + [y/(b/6)] = 1$, which is an equation of a straight line. It shows that when $z = 0$, $y = b/6$; and when $y = 0$, $z = h/6$. Hence this line may be represented by the line $CD$ in Fig. 8-13(b). A vertical force may be applied to the block anywhere on this line and the stress at $A$ will be zero. Similar lines may be established for the other three corners of the section; these are shown in Fig. 8-13(b). If the force $P$ is applied on any one of these lines or on any line parallel to such a line toward the centroid of the section, there will be no tensile stress at the corresponding corner. Hence the force $P$ may be applied anywhere within the shaded area in Fig. 8-13(b) without causing tensile stress at any of the four corners or anywhere else. This zone of the cross-sectional area is called the *kern* of a section.

If for a rectangular block the location of the force $P$ is limited to one of the lines of symmetry, the maximum eccentricity $e = h/6$ to give zero stress along one of the edges, Figs. 8-14(a) and (b). This leads to a practical rule, much used in the past by designers of masonry structures: If the resultant of vertical forces acts within the middle third of a rectangular section, there is no tension in the material at that section. If, further, the applied load $P$ acts outside the middle third and the contact surfaces cannot transmit tensile forces, one has the case shown in Figs. 8-14(c) and (d). Here, assuming elastic action, the normal stress at $B$ may be expressed as

$$\sigma_B = -\frac{P}{xb} + P\left(\frac{x}{2} - a\right)\frac{6}{bx^2} = 0$$

where $(x/2) - a$ is the eccentricity of the applied force with respect to

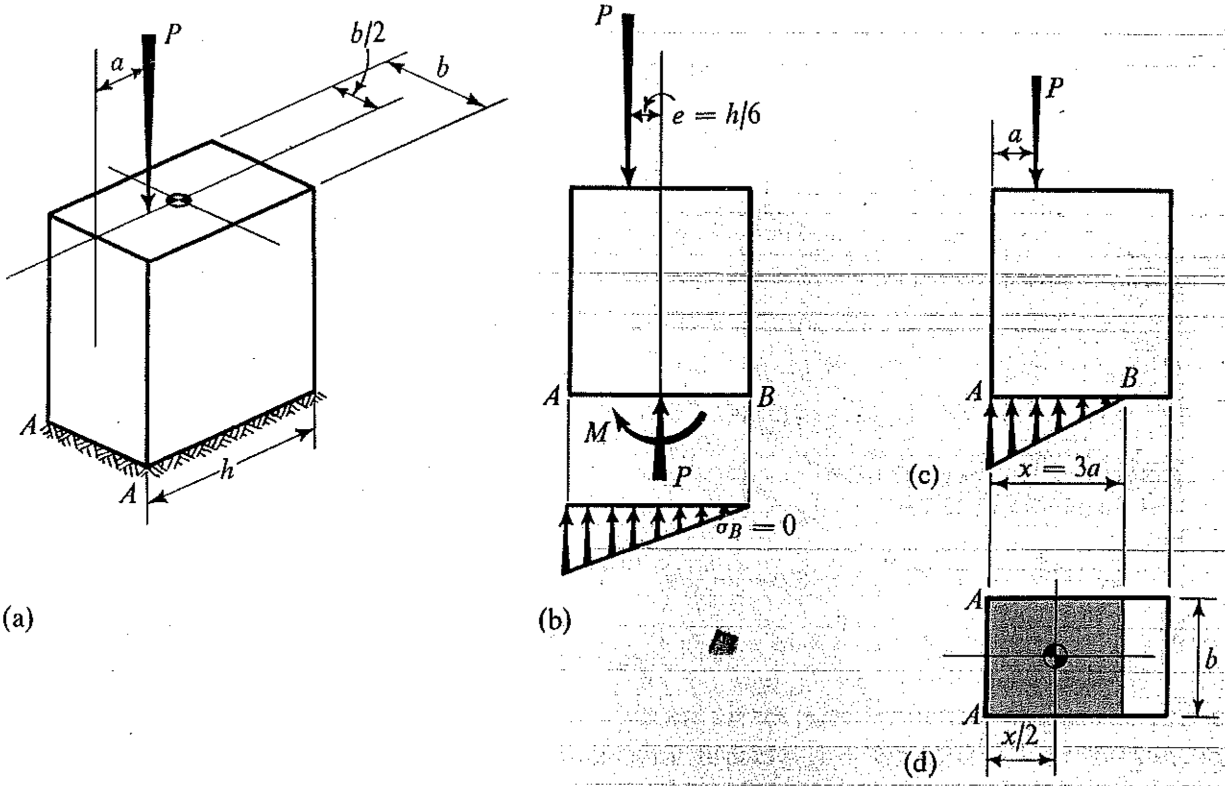


**Fig. 8-14.** (a) Eccentrically loaded block; (b) location of $P$ to cause zero stress at $B$; (c) elastic stress distribution between two surfaces which are unable to transmit tensile forces.

the centroidal axis of the shaded contact area, and $bx^2/6$ is its section modulus. Solving for $x$, one finds that $x = 3a$; the pressure distribution will be "triangular" as in Fig. 8-14(c) (why?). As $a$ decreases, the intensity of pressure on the line $A$-$A$ increases; when $a$ is zero, the block becomes unstable. Such problems are important in the design of foundations.

## 8-5. SUPERPOSITION OF SHEARING STRESSES

In the preceding part of the chapter superposition of the normal stresses $\sigma_x$ was the principal concern. In problems where both the elastic torsional and direct shearing stresses can be determined, the compound shearing stress also may be found by superposition. This corresponds to superposition of the off-diagonal stresses in Eq. 8-1. Here attention will be directed to instances where the shearing stresses being superposed not only act on the same element of area but also have the same line of action.* Only elastic stresses fall within the scope of this treatment.

* Noncolinear shearing stresses acting on the same element of area can be added vectorially.

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# 12

# Columns and Struts

**12·1.** *A* structural member whose lateral dimensions are small as compared to its length and subjected to compressive force is known as a strut. A vertical compression member is generally known as a column or stanchion.

A long column when subjected to direct load deflects in lateral direction which is known as buckling. The effect of lateral deflection is quite considerable in long columns. In contrast to long columns the effect of lateral deflections is negligible in short columns. In very long columns the effect of direct stresses is small as compared with bending stresses.

The main causes of bending in the columns are lack of straightness in member, *i.e.* initial curvature in the member, eccentricity of the load and non-homogeneity in the material of the column. Every column will have atleast small degree of eccentricity.

**Stable and Unstable Equilibrium**

From mechanics it is known that a body may be in three types of equilibrium—stable, neutral or unstable. A stable equilibrium is one in which body returns to its original position on being displaced from its position of equilibrium *e.g.* a ball resting on a concave surface as shown in Fig. 12·1 (*a*). A neutral equilibrium is one in

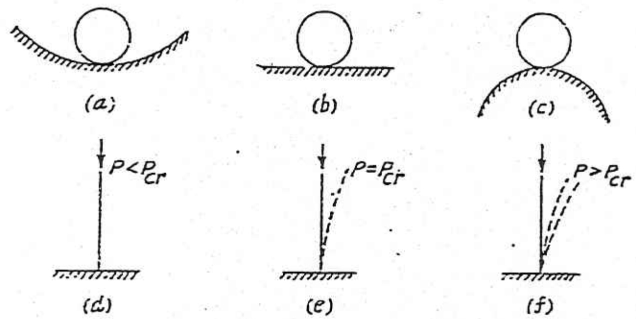


Fig. 12·1

which a body does not return to original position on being displaced but its motion stops *e.g.* a ball on a horizontal plane as shown in Fig. 12·1 (*b*). In unstable equilibrium body continues to move further away from its position of equilibrium on being displaced *e.g.* ball on a convex surface as shown in Fig. 12·1 (*c*).



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A long column subjected to small loads is in a state of stable equilibrium. If it is displaced slightly by lateral forces, it regains its original position on the removal of the force. When the axial load $P$ on column reaches certain critical value $P_{cr}$, the column will be in a state of neutral equilibrium. When it is displaced slightly from its straight position, it remains in deflected position. If the force $P$ exceeds the critical load $P_{cr}$, the column becomes in unstable equilibrium. The column either collapses or undergoes large deflections.

The critical load of column is defined as the load at which column is in neutral equilibrium.

**Slenderness Ratio**

In the long columns the effect of bending is to be considered while designing. The resistance of any member to bending is governed by its flexural rigidity $EI$, where $I=Ar^2$. Every structural member will have two principal moments of inertia of which one is maximum and the other is minimum. Resistance to bending is determined by least moment of inertia.

$I_{min}=Ar^2_{min}$ where $r_{min}$ is the least radius of gyration.

The ratio $\dfrac{\text{length of member}}{\text{least radius of gyration}}\left(\dfrac{l}{r_{min}}\right)$ is known as the slenderness ratio of the member. Whether a column is short or long is determined by the numerical value of the slenderness ratio.

The various end conditions encountered in columns are—fixed hinged and free. Thus considering end conditions, we have the following four categories of columns :

(*a*) One end free and the other fixed.
(*b*) Both ends hinged.
(*c*) Both ends fixed.
(*d*) One end fixed and the other end hinged.

Euler found out the failure load for various end conditions, considering stability of columns on assumption that column is initially straight, homogeneous, of uniform cross-section throughout and axially loaded.

**12·2 Column with one end free and the other fixed.**

B.M. at $x$ distance from fixed end

$$M_x=-P(a-y)$$

$$EI\frac{d^2y}{dx^2}=-M_x$$

$$=P(a-y)$$

$$EI\frac{d^2y}{dx^2}+Py=P.a$$

$$\frac{d^2y}{dx^2}+\frac{P}{EI}y=\frac{P}{EI}a.$$

Putting $$\frac{P}{EI}=\alpha^2$$

$$\frac{d^2y}{dx^2}+\alpha^2y=\alpha^2a$$

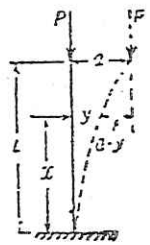



Fig. 12·2

The solution of this differential equation is

$$y = A\cos\alpha x + B\sin\alpha x + a.$$

Taking $x$ from the fixed end,

when $x=0,\quad y=0.$

$\therefore\quad 0=A+0+a$

$A=-a.$

$$\frac{dy}{dx} = -A\alpha\sin\alpha x + B\alpha\cos\alpha x.$$

At $x=0, \frac{dy}{dx}=0.$

$\therefore\quad 0=0+B\alpha$

$\therefore\quad B=0$

$\therefore\quad y=-a\cos\alpha x+a$

$=a(1-\cos\alpha x)$

At $x=L, y=a.$

$\therefore\quad a=a(1\cos-\alpha L)$

$\therefore\quad \cos\alpha L=0$

$$\alpha L=\frac{\pi}{2}, \frac{3\pi}{2}\ldots\ldots$$

Taking least value of $\alpha L$.

$\alpha L=\pi/2$

$$\therefore\quad \sqrt{\frac{P}{EI}}\times L=\frac{\pi}{2}$$

$$\therefore\quad \frac{P}{EI}\times L^2=\frac{\pi^2}{4}$$

$$\therefore\quad P=\frac{\pi^2 EI}{4L^2} \qquad \ldots(12\cdot1)$$

This is the least value which will cause the strut to buckle and is called Euler crippling load. In the above expression '$I$' is the least moment of inertia.

## 12·3. Column with both ends hinged

Let $y$ be the deflection at $x$ distance from the top hinge.

B.M. at $x$ distance from top.

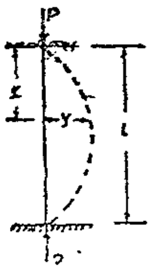


Fig. 12·3.

$$M_x=+Py$$

$$EI\frac{d^2y}{dx^2}=-M_x=-Py.$$

$$\therefore\quad EI\frac{d^2y}{dx^2}+\frac{P}{EI}y=0.$$

Putting $\frac{P}{EI}=\alpha^2$

$$\frac{d^2y}{dx^2}+\alpha^2 y=0.$$

The solution of this differential equation is

$$y=A\cos\alpha x+B\sin\alpha x.$$



At $x=0,\quad y=0.$

$\therefore\quad 0=A.$

At $x=L,\quad y=0.$

$\therefore\quad 0=0+B\sin\alpha L=0,$

As it is assumed that the column will buckle, $B$ cannot be zero.

$\therefore\quad \sin\alpha L=0.$

$\alpha L=0, \pi, 2\pi\ldots\ldots$

Taking the least possible value of $\alpha L$

$\alpha L=\pi$

$$\sqrt{\frac{P}{EI}}\times L=\pi$$

$$\frac{P}{EI}\times L^2=\pi^2$$

$$P=\frac{\pi^2 EI}{L^2} \qquad \ldots(12\cdot2)$$

## 12·4. Columns with both ends fixed

Let $M_0$ be the B.M. at fixed ends and $y$ be the deflection at $x$ distance from top end.

At a distance $x$ from top B.M.

$$M_x=Py-M_0$$

$$EI\frac{d^2y}{dx^2}=-M_x=-Py+M_0$$

$$\frac{d^2y}{dx^2}+\frac{P}{EI}y=\frac{M_0}{EI}$$

Putting $\frac{P}{EI}=\alpha^2,$

$$EI=\frac{P}{\alpha^2}$$

$$\frac{d^2y}{dx^2}+\alpha^2 y=\frac{M_0\alpha^2}{P}$$

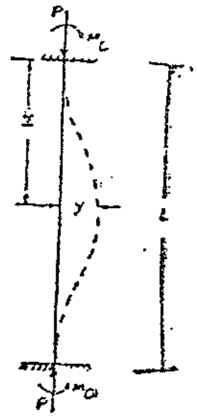


Fig. 12·4

The solution of this differential equation is

$$y=A\cos\alpha x+B\sin\alpha x+\frac{M_0}{P}.$$

$$\frac{dy}{dx}=-A\alpha\sin\alpha x+B\alpha\cos\alpha x.$$

At $x=0,\quad y=0$

$$\therefore\quad 0=A+\frac{M_0}{P}$$

or

$$A=-\frac{M_0}{P}.$$

At $x=0,\ \frac{dy}{dx}=0$

$0=0+B\alpha$

$\therefore\ B=0.$

$\therefore\ y=-\frac{M_0}{P}\cos\alpha x+\frac{M_0}{P}$

At $x=L,\ y=0.$

$0=-\frac{M_0}{P}\cos\alpha L+\frac{M_0}{P}$

$=\frac{M_0}{P}(1-\cos\alpha L)$

$\frac{M_0}{P}$ cannot be zero,

$\therefore\ (1-\cos\alpha L)=0$

$\therefore\ \cos\alpha L=1$

$\alpha L=0,\ 2\pi,\ 4\pi\ldots\ldots$

Taking the least possible value of $\alpha L$.

$\alpha L=2\pi$

$\sqrt{\frac{P}{EI}}\times L=2\pi$

$\frac{P}{EI}L^2=4\pi^2$

$P=\frac{4\pi^2 EI}{L^2}.$

## 12·5. Column with one end fixed and the other hinged.

Let $y$ be the deflection at '$x$' distance from the hinge.

B.M. at $x$ distance from the hinge.

$$M_x=Py+H_0x$$

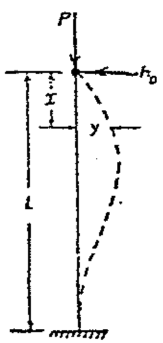


Fig. 12·5

$$EI\frac{d^2y}{dx^2}=-M_x=-Py\times H_0x$$

$\therefore\ \frac{d^2y}{dx^2}+\frac{P}{EI}y=-\frac{H_0}{EI}$

Putting $\frac{P}{EI}=\alpha^2,\quad EI=\frac{P}{\alpha^2}$

$\frac{d^2y}{dx^2}+\alpha^2y=-\frac{H_0\alpha^2}{P}x.$

The solution of this differential equation is

$$y=A\cos\alpha x+B\sin\alpha x-\frac{H_0}{P}x$$



$$\frac{dy}{dx}=-A\alpha\sin\alpha x+B\alpha\cos\alpha x-\frac{H_0}{P}$$

At $x0,\ y=0.$

$\therefore\ 0=A+0-0$

$\therefore\ A=0.$

At $x=L,\ y=0$

$\therefore\ 0=0+B\sin\alpha L-\frac{H_0L}{P}$

$\therefore\ B=\frac{H_0L}{P\sin\alpha L}$

At $x=L,\ \frac{dy}{dx}=0$

$\therefore\ 0=-0+B\alpha\cos\alpha L-\frac{H_0}{P}$

$\frac{H_0L}{P\sin\alpha L}\alpha\cos\alpha L-\frac{H_0}{P}=0$

$\therefore\ \tan\alpha L=\alpha L.$

The angle which satisfies the equation is given by

$\alpha L=4{\cdot}49$ radians

$\therefore\ \sqrt{\frac{P}{EI}}\times L=4{\cdot}49$ radians

$\frac{P}{EI}\ L^2=(4{\cdot}49)^2$

$P=\frac{20{\cdot}2\ EI}{L^2}$

$=\frac{2\pi^2 EI}{L^2}\left(\text{approximately}\right)\quad\ldots(12{\cdot}3)$

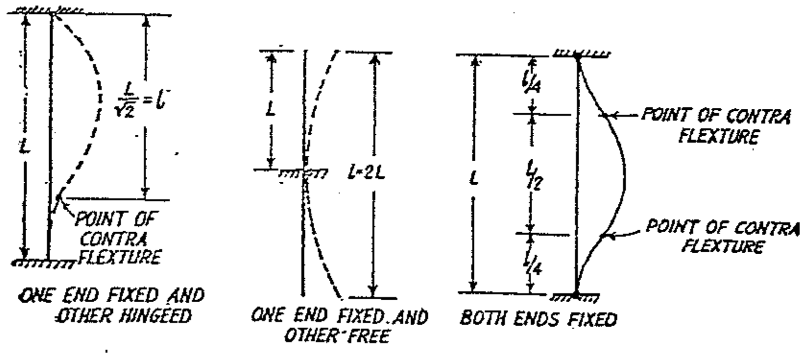
In general Euler's crippling load $P=\frac{\pi^2 EI}{l^2}$

where $l$ is the effective length, *i.e.*. equivalent length of both ends hinged column strength.

As $I=Ar^2$, where $r$ is the least radius of gyration.

$P=\frac{\pi^2 EA}{(l/r)^2}$

or $\frac{P}{A}$=Euler stress $\sigma_{b}=\frac{\pi^2 E}{\left(\frac{l}{r}\right)^2}$. This is also the average stress.




(a) (b) (c)

Fig. 12·6

The equivalent length for various end conditions are

(i) One end free and other fixed. $l=2L$

(ii) Both ends hinged $l=L$.

(iii) Both ends fixed $l=L/2$.

(iv) One end fixed and other hinged $l=L/\sqrt{2}$.

## 12·6. Limitation of Euler's Formula

In the Euler formula it was assumed that the member was absolutely straight and the load was axial. Furthermore the crippling load was derived from the differential equation of the elastic curve which is based on Hooke's law. Hooke's law is valid as long as stresses do not exceed proportional limit.

The critical stress $\sigma_c$ was found to be $\pi^2E/(l/r)^2$ where $l$ is the equivalent length of the column and $r$ least radius of gyration. $l/r$ is the slenderness ratio of the column.

Let $\sigma_p$ be the proportional limit of the material of the column. For Euler formula to be valid

$$\sigma_c \leqslant \sigma_p$$

$$\pi^2E/(l/r)^2 \leqslant \sigma_p$$

Consider for example low carbon steel with proportional limit of 2000 kg/cm².

$$l/r \geqslant \sqrt{\pi^2E/\sigma_p}$$

$$l/r \geqslant \pi\sqrt{2\times10^6/2000}=100 \text{ approximately}$$

∴ Euler formula is valid for $l/r$ ratio greater than 100. Similar limitations on $l/r$ ratio can be obtained for other materials.



## 12·7. Column with initial curvature

Let $ABC$ be the shape of the column before loading with central deflection $e$ and let $AC'B$ be the shape after loading with total central deflection $y_0$. At a point distance $x$ from top hinge, let $y_1$ be the deflection before loading and $y$ be the total deflection after loading.

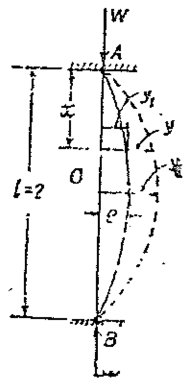



Fig. 12·7

As the curvature is very small. the curve may be assumed to be sine curve such that $y_1=e\sin\frac{\pi x}{l}$

It will be assumed that bending is uniplanar,

$$M=Wy$$

$$EI\frac{d^2(y-y_1)}{dx^2}=-M=-Wy.$$

$$\therefore \quad EI\frac{d^2y}{dx^2}=-Wy+EI\frac{d^2y_1}{dx^2}$$

$$y_1=e\sin\frac{\pi x}{l}$$

$$\frac{dy_1}{dx}=\frac{e\pi}{l}$$

$$\frac{d^2y_1}{dx^2}=-e\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}$$

$$EI\frac{d^2y}{dx^2}=-Wy-EIe\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}$$

$$\frac{d^2y}{dx^2}+\frac{W}{EI}y=-e\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}.$$

Solution of this differential equation is

$$y=A\cos\alpha x+B\sin\alpha x-e\frac{\frac{\pi^2}{l^2}\sin\frac{\pi x}{l}}{-\frac{\pi^2}{l^2}+\alpha^2}$$

$$=A\cos\alpha x+B\sin\alpha x-\frac{e\sin\pi x/l}{\frac{\alpha^2l^2}{\pi^2}-1}$$

At $x=0$, $y=0$,

∴ $0=A$,

$$\therefore \quad y = 0 + B\sin\alpha x - \frac{e\sin \pi x/l}{\frac{\alpha^2 l^2}{\pi^2} - 1}$$

$$\frac{dy}{dx} = B\alpha\cos\alpha x - \frac{e\frac{\pi}{l}\cos\frac{\pi x}{l}}{\frac{\alpha^2 l^2}{\pi^2} - 1}.$$

At $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$.

$$\therefore \quad 0 = B\alpha\cos\frac{\alpha l}{2} \quad \therefore \quad B = 0$$

$$\therefore \quad y = \frac{e\sin\frac{\pi x}{l}}{1 - \frac{\alpha^2 l^2}{\pi^2}} = \frac{e\sin\frac{\pi x}{l}}{1 - \frac{Wl^2}{EI\pi^2}}$$

But $P = \frac{\pi^2 EI}{l^2}$ = Crippling load.

$$\therefore \quad y = \frac{e\sin\frac{\pi x}{l}}{1 - \frac{W}{P}}.$$

At $x = l/2$, $y = y_0$

$$\therefore \quad y_0 = \frac{e\sin\pi/2}{1 - \frac{W}{P}} = \frac{e}{1 - \frac{W}{P}} = \frac{Pe}{P-W}.$$

Maximum B.M. $= Wy_0 = \frac{WPe}{P-W}$

Maximum compressive stress $= \frac{W}{A} + \frac{M}{Z}$

$$= \frac{W}{A} + \frac{W\frac{Pe}{P-W}}{Ar^2/y_c}$$

$$\text{as } Z = \frac{I}{y_c} = \frac{Ar^2}{y_c}$$

Maximum compressive stress $= \frac{W}{A}\left[1 + \frac{Pey_c}{(P-W)r^2}\right]$.



## 12·8. Column carrying eccentric load.

Let the eccentricity of load be $e$.

Let $ABC$ be the deflected form after loading. Consider a point at distance $x$ from the top hinge, let its distance from the direction of load be $y$ after loading.

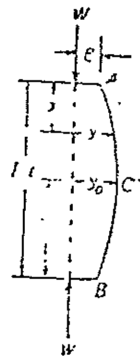



Fig. 12·8.

$$M_x = Wy$$

$$EI\frac{d^2y}{dx^2} = -M_x = -Wy$$

$$EI\frac{d^2y}{dx^2} + Wy = 0$$

$$\frac{d^2y}{dx^2} + \frac{W}{EI}y = 0.$$

Putting $\frac{W}{EI} = \alpha^2$

$$\frac{d^2y}{dx^2} + \alpha^2 y = 0.$$

The solution of this differential equation is

$$y = A\cos\alpha x + B\sin\alpha x.$$

At $x = 0$, $y = e$

$e = A$.

At $x = l$, $y = e$

$e = e\cos\alpha l + B\sin\alpha l$

$$\frac{e(1-\cos\alpha l)}{\sin\alpha l} = B$$

$$B = e\frac{2\sin^2\alpha l/2}{2\sin\frac{\alpha l}{2}\cos\frac{\alpha l}{2}} = e\tan\frac{\alpha l}{2}.$$

$$\therefore \quad y = e\cos\alpha x + e\tan\frac{\alpha l}{2}\sin\alpha x$$

$$= e\left[\cos\alpha x + \tan\frac{\alpha l}{2}\sin\alpha x\right].$$

At $x = l/2$, $y = y_0$

$$y_0 = e\cos\frac{\alpha l}{2} + e\tan\frac{\alpha l}{2}\sin\frac{\alpha l}{2}$$

$$= e\left[\cos\frac{\alpha l}{2} + \frac{\sin^2\frac{\alpha l}{2}}{\cos\frac{\alpha l}{2}}\right]$$

$$= \frac{e}{\cos\frac{\alpha l}{2}} = e\sec\frac{\alpha l}{2}$$

$$= e\sec\frac{l}{2}\sqrt{\frac{W}{EI}}.$$

Max. B.M. $= Wy_0$

$$= We \sec \frac{l}{2}\sqrt{\frac{W}{EI}}$$

Max. stress $= \frac{W}{A} + \frac{M}{I/y_c}$

$$= \frac{W}{A} + \frac{We \sec \frac{l}{2}\sqrt{\frac{W}{EI}}}{Ar^2/y_c}$$

$$= \frac{W}{A}\left[1 + \frac{e\,y_c}{r^2}\sec\frac{l}{2}\sqrt{\frac{W}{EI}}\right]$$

## 12·9. Laterally loaded columns

*Column with a concentrated load at mid-height.*
Let $F$ be the concentrated load applied.

$$M_x = \frac{F}{2}x + Wy$$

$$EI\frac{d^2y}{dx^2} = -M = -\frac{F}{2}x - Wy$$

$$\frac{d^2y}{dx^2} + \frac{W}{EI}y = \frac{-F}{2EI}x.$$

Let $\frac{W}{EI} = \alpha^2$.

$\therefore$ $\frac{d^2y}{dx^2} + \alpha^2 y = -\frac{F}{2EI}x$

$$(D^2+\alpha^2)y = -\frac{F}{2EI}x.$$

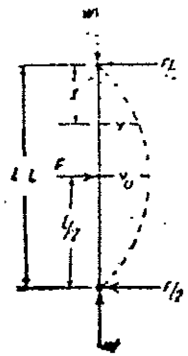


Fig. 12·9

The solution of this equation is

$y = A\cos\alpha x + B\sin\alpha x$ + Particular integral (P.I.)

$$\text{P.I.} = \frac{1}{D^2+\alpha^2}\left(-\frac{F}{2EI}x\right)$$

$$= \frac{1}{\alpha^2\left(1+\frac{D^2}{\alpha^2}\right)}\left(-\frac{F}{2EI}x\right)$$

$$= \frac{1}{\alpha^2}\left(1-\frac{D^2}{\alpha^2}\right)\left(-\frac{F}{2EI}x\right)$$

$$= -\frac{F}{2EI\alpha^2}x.$$

$\therefore$ Solution is, $y = A\cos\alpha x + B\sin\alpha x - \frac{F}{2EI\alpha^2}x.$

At $x=0,\ y+0.$

$\therefore$ $0 = A \quad \therefore A = 0$

$$\frac{dy}{dx} = \alpha B\cos\alpha x - \frac{P}{2EI\alpha^2}.$$



At $x = l/2,\ \frac{dy}{dx} = 0$

$\therefore$ $0 = \alpha B\cos\frac{\alpha l}{2} - \frac{F}{2EI\alpha^2}$

$\therefore$ $B = \frac{F}{2EI\,\alpha^3\cos\frac{\alpha l}{2}}.$

$\therefore$ $y = \frac{F}{2EI\,\alpha^3\cos\frac{\alpha l}{2}}\sin\alpha x - \frac{Fx}{2W}$

$$\left[\text{As } \frac{W}{EI} = \alpha^2\right]$$

At $x = l/2,\ y = y_0.$

$\therefore$ $y_0 = \frac{F\sin\frac{\alpha l}{2}}{2EI\,\alpha^3\cos\frac{\alpha l}{2}} - \frac{Fl}{4W}$

$$= \frac{F}{2\alpha W}\tan\frac{\alpha l}{2} - \frac{Fl}{4W}.$$

Max. B.M. $= \frac{Fl}{4} + Wy_0$

$$= \frac{Fl}{4} + \frac{WF}{2\alpha W}\tan\frac{\alpha l}{2} - \frac{WFl}{4W}$$

$$= \frac{F}{2\alpha}\tan\frac{\alpha l}{2}.$$

Max. stress $= \frac{W}{A} + \frac{M}{Ar^2/y_c}$

$$= \frac{W}{A} + \frac{F\tan\frac{\alpha l}{2}\times y_c}{2\alpha Ar^2}$$

$$= \frac{W}{A}\left[1 + \frac{Fy_c\tan\frac{\alpha l}{2}}{2W\alpha r^2}\right].$$

**Ex. 12·1.** *A straight circular bar of steel one cm in diameter and 120 cm long is mounted in testing machine and loaded axially in compression till it buckles. Assuming the Euler formula for pinned ends to apply, estimate the maximum central deflection before the material reaches its yield stress of 350 N/mm². $E = 0{\cdot}21\times10^5$ N/mm².*

**Sol.** $I = \frac{\pi}{64}\times d^4 = \frac{\pi}{64}\times 1^4 = \frac{\pi}{64}\text{cm}^4.$

Area $A = \frac{\pi}{4}\times 1^2 = \frac{\pi}{4}\text{cm}^2.$

Euler buckling load

$$P=\frac{\pi^2 EI}{l^2}=\frac{\pi^2\times 2\cdot 1\times 10^5}{1200\times 1200}\times\frac{\pi}{64}\times 10^4$$

$$P=\frac{70{,}000\times\pi^2}{64\times 48}=706\cdot 8\ \text{N}$$

Direct stress $\sigma_a=\frac{P}{A}=\frac{70{,}000\times\pi^2}{64\times 48}\times\frac{1}{\frac{\pi}{4}\times 100}$

$=9\ \text{N/mm}^2$.

Maximum stress $=\sigma_a+\sigma_b$.

When yield stress is reached

$\sigma_b=350-\sigma_a=350-9$

$=341\ \text{N/mm}^2$.

Corresponding bending moment

$$M=\frac{\sigma_{max}}{y_{max}}\times I=\frac{341}{50}\times\frac{\pi}{64}\times(10)^4$$

$=33480$ N-mm

Max. B.M. $=P\times$ deflection $\delta$

$33480=706\cdot 8\times\delta$

$$\delta=\frac{33480}{706\cdot 8}=47\cdot 37\ \text{mm}$$

**Ex. 12·2.** *Find the necessary diameter of a mild steel strut 2 metres long freely hinged at its ends if it has to carry a load of 100,000 N with possible deviation from the axis of 1/10th of the diameter. The greatest compressive stress is not to exceed 80 N/mm². $E=0\cdot 2\times 10^5$ N/mm².*

**Sol.** Let $d$ be the diameter of strut.

B.M. at $x$ distance from top,

$$M_x=+P(e+y)$$

$$EI\frac{d^2y}{dx^2}=-P(e+y).$$

$$\therefore\quad \frac{d^2y}{dx^2}+\frac{P}{EI}y=-\frac{Pe}{EI}.$$

Let $\frac{P}{EI}=\alpha^2$

$$\therefore\quad \frac{d^2y}{dx^2}+\alpha^2 y=\alpha^2 e$$

$$(D^2+\alpha^2)y=-\alpha^2 e.$$

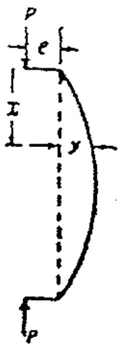



Fig. 12·10.

The solution of differential equation is

$$y=A\cos\alpha x+B\sin\alpha x+\frac{1}{D^2+\alpha^2}(-\alpha^2 e).$$

$$y=A\cos\alpha x+B\sin\alpha x-e.$$



At $x=0$, $y=0$.

$\therefore$ $0=A-e$

or $A=e$.

At $x=l$, $y=0$,

$0=e\cos\alpha l+B\sin\alpha l-e$

$$B=\frac{e(1-\cos\alpha l)}{\sin\alpha l}=\frac{e\,2\sin^2\frac{\alpha l}{2}}{2\sin\alpha l/2\cos\alpha l/2}$$

$=e\tan\alpha l/2$.

$$\therefore\quad y=e\cos\alpha x+e\tan\frac{\alpha l}{2}\times\sin\alpha x-e$$

At $x=l/2$,

$$y_0=e\cos\frac{\alpha l}{2}+e\tan\frac{\alpha l}{2}\sin\frac{\alpha l}{2}-e$$

$$=e\left[\cos\frac{\alpha l}{2}+\frac{\sin^2\frac{\alpha l}{2}}{\cos\frac{\alpha l}{2}}-1\right]$$

$$=e\left[\frac{\cos^2\frac{\alpha l}{2}+\sin^2\frac{\alpha l}{2}}{\cos\frac{\alpha l}{2}}-1\right]$$

$$=e\left(\sec\frac{\alpha l}{2}-1\right)$$

Max. B.M. $=P(e+y_0)$

$=Pe\sec\frac{\alpha l}{2}$.

Maximum stress $\sigma_{max}=\frac{P}{A}+\frac{M\times y_{max}}{I}$

$$=P\left[\frac{1}{A}+\frac{e\sec\frac{\alpha l}{2}\times y_{max}}{I}\right]$$

$A=\frac{\pi}{4}d^2$,

$I=Ar^2$.

$r=\frac{d}{4}$.

$y_{max}=d/2$, $e=d/10$.

$$\therefore\quad 80=\frac{100{,}000}{A}\left[1+\frac{d}{10}\times\frac{\sec\alpha l/2\times d/2}{(d/4)^2}\right]$$

$$\frac{80}{100{,}000}\times\frac{\pi}{4}\times d^2=1+0\cdot 8\sec\frac{\alpha l}{2}.$$

$$\frac{\pi d^2}{5000}=1+0\cdot 8\sec\frac{\alpha l}{2}.$$

$$\alpha^2=\frac{P}{EI}=\frac{100{,}000}{2\times10^5\times\frac{\pi}{64}\times d^4}=\frac{32}{d^4}$$

$$\alpha=\frac{5\cdot6569}{d^2}$$

$$\frac{\alpha l}{2}=\frac{5\cdot6569}{d^2}\times\frac{2000}{2}=\frac{5656\cdot9}{d^2}$$

$$\frac{\pi d^2}{5000}=1+0\cdot8\sec\frac{5656\cdot9}{d^2}$$

This equation can be solved by trial and error

$$d=69\cdot5 \text{ mm.}$$

$$\sec\frac{\alpha l}{2}=\frac{5656\cdot9}{(69\cdot5)^2}\times\frac{180^\circ}{\pi}=67^\circ$$

$$\therefore\quad 1+0\cdot8\sec\frac{\alpha l}{2}=1+\frac{0\cdot8\times1}{0\cdot389}=3\cdot035.$$

Left hand side $=\frac{\pi\times(69\cdot5^2)}{5000}=3\cdot056$

$d=69\cdot5$ cm satisfies the equation.

∴ Diameter of strus=70 mm.

**Ex. 12·3.** *Determine the maximum uniformly distributed lateral load (applied in the place of symmetry parallel to the longer side of the section) which can be carried by a 16 cm×8 cm timber strut 4 m long which is already subjected to an axial thrust of 20,000 N so that the maximum fibre stress does not exceed 14 N/mm².*

*Take E=10,000 N/mm² and assume pinned ends.*

**Sol.** Let $w$ kg/cm be the lateral load that can be carried by the strut.

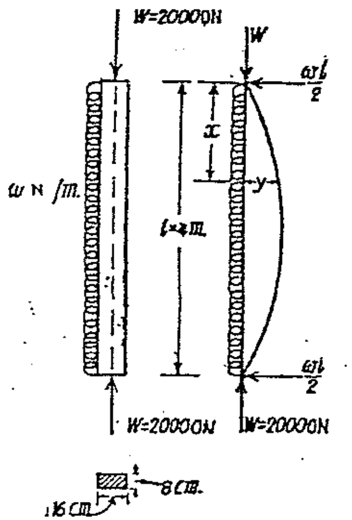



Fig. 12·11.

$$M_x=W\times y+\frac{wlx}{2}-\frac{wx^2}{2}$$

$$EI\frac{d^2y}{dx^2}=-M_x$$

$$=-Wy-\frac{wlx}{2}+\frac{wx^2}{2}$$

$$\frac{d^2y}{dx^2}+\frac{W}{EI}y=\frac{w}{2EI}(x^2-lx)$$

Let $\frac{W}{EI}=\alpha^2.$

$$EI=\frac{W}{\alpha^2}.$$

$$\therefore\quad\frac{d^2y}{dx^2}+\alpha^2y=\frac{w\alpha^2}{2W}(x^2-lx)$$

$$(D^2+\alpha^2)y=\frac{w\alpha^2}{2W}(x^2-lx).$$



Complementary function is

$$y=A\cos\alpha x+B\sin\alpha x$$

Particular integral $=\frac{1}{D^2+\alpha^2}\times\frac{w\alpha^2}{2W}(x^2-lx)$

$$=\frac{1}{\alpha^2\left(1-\frac{D^2}{\alpha^2}\right)}\times\frac{w\alpha^2}{2W}(x^2-lx)$$

$$=\frac{w}{2W}\left(1-\frac{D^2}{\alpha^2}\right)(x^2-lx)$$

$$=\frac{w}{2W}\left(x^2-lx-\frac{2}{\alpha^2}\right).$$

∴ Solution of differential equation is

$$y=A\cos\alpha x+B\sin\alpha x+\frac{w}{2W}\left(x^2-lx-\frac{x}{\alpha^2}\right).$$

At $x=0,\quad y=0$

$$\therefore\quad 0=A-\frac{w}{W\alpha^2}$$

$$\therefore\quad A=\frac{w}{W\alpha^2}$$

At $x=l,\quad y=0$

$$0=A\cos\alpha l+B\sin\alpha l+\frac{w}{2W}\left(l^2-l^2-\frac{2}{\alpha^2}\right)$$

$$=\frac{w}{W\alpha^2}\cos\alpha l+B\sin\alpha l-\frac{w}{W\alpha^2}$$

$$B\sin\alpha l=\frac{w}{W\alpha^2}(1-\cos\alpha l)$$

$$B=\frac{w}{W\alpha^2}\times\frac{2\sin^2\frac{\alpha l}{2}}{2\sin\alpha l/2\cos\frac{\alpha l}{2}}$$

$$=\frac{w}{W\alpha^2}\tan\frac{\alpha l}{2}$$

$$y=\frac{w}{W\alpha^2}\cos\alpha x+\frac{w}{W\alpha^2}\tan\frac{\alpha l}{2}\sin\alpha x+\frac{w}{2W}\left(x^2-lx-\frac{2}{\alpha^2}\right)$$

At $x=l/2,$

$$y_0=\frac{w}{W\alpha^2}\cos\frac{\alpha l}{2}+\frac{w}{W\alpha^2}\tan\frac{\alpha l}{2}\sin\frac{\alpha l}{2}+\frac{w}{2W}\left(\frac{l^2}{4}-\frac{l^2}{2}-\frac{2}{\alpha^2}\right)$$

$$y_0=\frac{w}{W\alpha^2}\left[\cos\frac{\alpha l}{2}+\frac{\sin\frac{\alpha l}{2}}{\cos\frac{\alpha l}{2}}\right]$$

$$+\frac{w}{2W}\left(-\frac{l^2}{4}-\frac{2}{\alpha^2}\right)$$

$$=\frac{w}{W\alpha^2}\times\sec\frac{\alpha l}{2}-\frac{wl^2}{8W}-\frac{w}{W\alpha^2}$$

$$=\frac{w}{W\alpha^2}\left(\sec\frac{\alpha l}{2}-1\right)-\frac{wl^2}{8W}$$

Max. B.M. $=W\times y_0+\dfrac{wl^2}{8}$

$$=\frac{w}{\alpha^2}\left(\sec\frac{\alpha l}{2}-1\right)-\frac{wl^2}{8}+\frac{wl^2}{8}$$

$$=\frac{w}{\alpha^2}\left(\sec\frac{\alpha l}{2}-1\right).$$

$$\alpha^2=\frac{W}{EI}=\frac{20000}{10{,}000\times\frac{1}{12}\times80\times160^3}$$

$$=\frac{3}{10\times(160)^2}$$

$$\alpha=\frac{\sqrt{3}}{6400}$$

$$\frac{\alpha l}{2}=\frac{4000}{2}\times\frac{\sqrt{3}}{6400}=\frac{\sqrt{3}}{3\cdot2}\text{ radians}$$

$$=\frac{\sqrt{3}}{3\cdot2}\times\frac{180}{\pi}=31^\circ$$

$$\sec\frac{\alpha l}{2}=\sec 31^\circ=1\cdot165.$$

Max. B.M. $=\dfrac{w}{\alpha^2}\left(\sec\dfrac{\alpha l}{2}-1\right)$

$$=\frac{w\times(160)^2\times10}{3}\times0\cdot165$$

$$=2{,}25{,}2800\ w\text{ N mm.}$$

Max. stress $=\dfrac{P}{A}+\dfrac{M\times y_{max}}{I}$

$$14=\frac{20{,}000}{160\times80}+\frac{2{,}252{,}800\ w\times80}{\frac{1}{12}\times80\times(160)^3}$$

$$14-\frac{200}{16\times8}=\frac{22{,}528\times w\times3}{256\times40}$$

$$14-1\cdot562=\frac{22{,}528\times w\times3}{256\times40}$$



$$w=\frac{12\cdot438\times256\times40}{22{,}528\times3}$$

$$=1\cdot885\text{ N/mm.}$$

Column can carry a lateral load of 1·885 N/mm.

**Ex. 12·4.** *A strut of length $l$ has its ends built into a material which exerts a constraining couple equal to K times the angular rotation in radius. Show that the buckling load P is given by the equation* $\tan\dfrac{\alpha l}{2}=-\dfrac{P}{\alpha K}$ *where* $\alpha^2=P/EI$.

**Sol.** Let $M_0$ be restraining couple at the end and $\theta$ be the slope.

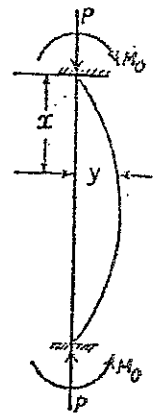


Fig. 12·12

$$M_0=K\theta$$
$$M_x=P\times y-M_0$$
$$EI\frac{d^2y}{dx^2}=-P\times y+M_0$$
$$\frac{d^2y}{dx^2}+\frac{P}{EI}\times y=+\frac{M_0}{EI}$$

Let $\dfrac{P}{EI}=\alpha^2.$

$\therefore$ $EI=\dfrac{P}{\alpha^2}$

Substituting

$$\frac{d^2y}{dx^2}+\alpha^2y=\alpha^2\frac{M_0}{P}$$

Solution of this differential equation is

$$y=A\cos\alpha x+B\sin\alpha x+\frac{M_0}{P}$$

At $x=0,\quad y=0.$

$\therefore$ $0=A+\dfrac{M_0}{P}$

$\therefore$ $A=-M_0/P.$

$$y=-\frac{M_0}{P}\cos\alpha x+B\sin\alpha x+\frac{M_0}{P}$$

$$\frac{dy}{dx}=+\frac{M_0}{P}\alpha\sin\alpha x+B\alpha\cos\alpha x.$$

At $x=\dfrac{l}{2};\ \dfrac{dy}{dx}=0.$

$$\therefore \quad 0=\frac{M_0}{P}\alpha \sin\frac{\alpha l}{2}+B\alpha \cos\frac{\alpha l}{2}$$

$$B=-\frac{M_0}{P}\tan\frac{\alpha l}{2}$$

$$\therefore \quad \frac{dy}{dx}=\frac{M_0}{P}\alpha \sin \alpha x-\frac{M_0}{P}\alpha \tan\frac{\alpha l}{2}\times\cos \alpha x.$$

At $x=0$, $\frac{dy}{dx}=\theta$

$$\therefore \quad \theta=\frac{M_0}{P}\alpha\times 0-\frac{M_0}{P}\alpha\tan\frac{\alpha l}{2}$$

$$=-\frac{M_0}{P}\alpha\tan\frac{\alpha l}{2}.$$

But $M_0=K\theta$

$$\therefore \quad \theta=-\frac{K\theta}{P}\alpha\tan\frac{\alpha l}{2}$$

$$\therefore \quad \tan\frac{\alpha l}{2}=-\frac{P}{K\alpha}.$$

Buckling load $P$ is given by the above equation.

## 12·10. Empirical formulae

As stated previously, for medium columns empirical formulae are used which take into account the practical considerations of initial curvature and eccentricity of loads. Such formulae are used in design and structural specifications. Following are some of the empirical formulae.

### 1. Straight line formula

In this formula it is assumed that the allowable stress depends on $l/r$ ratio and varies in a straight line fashion. This type of formula can be made sufficiently accurate over a given range. From the various test results, a mean straight line can be drawn to give allowable stresses.

$$\sigma_0=\frac{\sigma_y}{n}\left(1-\frac{al}{r}\right)$$

where $\sigma_0$ is allowable stress,

$\sigma_y$ = yield stress.

$a$ is a constant.

$n$ is factor of safety.

The values of $\sigma_y$ and $a$ can be obtained from Table 12·1.

### 2. Parabolic formula

For some materials the test results are such that the mean curve is best represented by a parabola. In this formula the allowable stress will vary as $(l/r)^2$.

$$\sigma_0=\frac{\sigma_y}{n}\left[1-a\left(\frac{l}{r}\right)^2\right]$$



where $\sigma_0$ is allowable stress.

$\sigma_y$ is yield stress.

$a$ is a constant.

$n$ is a factor of safety.

The value of $\sigma_y$ and $a$ can be obtained from Table 12·1.

TABLE 12·1

| *Material* | $\sigma_y$ *kg/cm²* | *Constant a in straight line formula* | *Constant a in parabolic formula* | *Constant a in Rankine formula* |
|---|---|---|---|---|
| Mild steel | 3200 | 0·0053 | 0·000057 | 1/7500 |
| Wrought iron | 2500 | 0 0053 | 0·000039 | 1/9000 |
| Cast iron | 5500 | 0·C08 | 0·00016 | 1/1600 |

### 3. Rankine's formula

Rankine suggested the following empirical formula :

$$\frac{1}{P_R}=\frac{1}{P_c}+\frac{1}{P}$$

where $P_R$ is the Rankine's crippling load, $P_c$ is crippling load for short columns in which there is no buckling. $P$ is the Euler's crippling load.

$$P_c=\sigma_y\times A, \text{ where } \sigma_y \text{ is the yield stress}$$

$$P=\frac{\pi^2 EI}{l^2}$$

$$\therefore \quad \frac{1}{P_R}=\frac{1}{\sigma_y A}+\frac{1}{\pi^2 EI/l^2}=\frac{1}{\sigma_y A}+\frac{l^2}{\pi^2 EI}$$

$$=\frac{1}{\sigma_y A}+\frac{l^2}{\pi^2 EAr^2}=\frac{1}{\sigma_y A}\left[1+\frac{\sigma_y (l/r)^2}{\pi^2 E}\right]$$

$$\therefore \quad P_R=\frac{\sigma_y A}{1+\frac{\sigma_y}{\pi^2 E}(l/r)^2}$$

$$=\frac{\sigma_y A}{1+a\left(\frac{l}{r}\right)^2} \text{ where } a=\frac{\sigma_y}{\pi^2 E}.$$

The formula is absolutely empirical, the values of $\sigma_y$ and $a$ are found experimentally. The values of $\sigma_y$ and $a$ can be obtained from Table 12·1.

### 4. Secant formula or Perry's formula

For a column with eccentric load.

$$\text{Max. stress}=\frac{W}{A}\left[1+\frac{ey_c}{r^2}\sec\frac{l}{2}\sqrt{\frac{W}{EI}}\right]$$
...(Art. 12·8)

$$\sigma_0=\text{Working stress}=\text{Max. stress}$$
$$=\frac{W}{A}\left[1+\frac{ey_c}{r^2}\sec\frac{l}{2}\sqrt{\frac{W}{EI}}\right]$$
$$=\frac{W}{A}\left[1+\frac{ey_c}{r^2}\cdot\sec\frac{l}{2}\sqrt{\frac{W}{\frac{\pi^2 EI}{l^2}}\cdot\frac{l^2}{\pi^2}}\right]$$
$$=\frac{W}{A}\left[1+\frac{ey_c}{r^2}\sec\frac{\pi}{2}\sqrt{\frac{W}{P}}\right]$$

where $P=\frac{\pi^2 EI}{l^2}$ Euler's crippling load.

$\sec\frac{\pi}{2}\sqrt{P}$ was found to be nearly equal to $\frac{1\cdot2P}{P-W}$ by **Perry**.

$$\therefore\quad \sigma_0=\frac{W}{A}\left[1+\frac{ey_c}{r^2}\times\frac{1\cdot2P}{P-W}\right]$$
$$=\frac{W}{A}\left[1+\frac{1\cdot2\,ey_c}{r^2\left(1-\frac{W}{P}\right)}\right]$$
$$\therefore\quad \frac{W}{A}=\frac{\sigma_0}{1+\frac{1\cdot2\,ey_c}{r^2\left(1-\frac{W}{P}\right)}}$$
$$=\frac{\sigma_y/n}{1+\frac{1\cdot2\,ey_c}{r^2\left(1-\frac{W}{P}\right)}}$$

where $\sigma_y$ = yield stress
$n$ = factor of safety
$W$ = safe working load.

**5. Indian Standards Institution code formula**

As per I.S.I. code formula, working stress $\sigma_0$ is given by

$$\sigma_0=\sigma_0'=\frac{\sigma_y/n}{1+0\cdot20\sec\left(\frac{l}{r}\sqrt{\frac{n\sigma_0'}{4E}}\right)}$$
for $l/r=0$ to 160,

where $\sigma_0$ = average allowable axial stress
$\sigma_0'$ = value obtained from above secant formula.



$\sigma_y$ = yield stress
$n$ = factor of safety = 1·68.

For values of $l/r \geqslant 160$

$$\sigma_0=\sigma_0'\left(1\cdot2-\frac{l}{800r}\right)$$

Some values for various $l/r$ ratios for steel conforming to I.S. 226—1962 are given in Table 12·2.

Table 12·2 Allowable Compressive Stress Per I.S. Code Formula

| $l/r$ | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 130 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma_0(N/mm^2)$ | 125·0 | 124·6 | 123·9 | 122·4 | 120·3 | 117·2 | 113·0 | 107·5 | 100·7 | 92·8 | 84·0 | 75·3 | 67·1 | 59·7 |
| $l/r$ | 140 | 150 | 160 | 170 | 180 | 190 | 200 | 210 | 220 | 230 | 240 | 250 | 300 | 350 |
| $\sigma_0(N/mm^2)$ | 53·1 | 47·4 | 42·3 | 37·7 | 33·6 | 30·0 | 27·0 | 24·3 | 21·0 | 19·9 | 18·1 | 16·6 | 1[illegible] | 7·6 |

**Note.** Intermediate values may be obtained by linear interpolation.

**Ex. 12·5.** *A hollow cast iron column, hinged at both ends is 4 metres long. Its external diameter is 20 cm and internal diameter 15 cm. Find the maximum load it can carry if factor of safety is 4. Use Rankine's formula*

$$P_R=\frac{\sigma_y A}{1+a\left(\frac{l}{r}\right)^2}.$$

**Sol.** For cast iron

$$\sigma_y=550\ \text{N/mm}^2$$
$$a=\frac{1}{1600}$$
$$l=400\ \text{cm}$$
$$A=\frac{\pi}{4}[20^2-15^2]=\frac{\pi}{4}\times35\times5$$
$$=145\cdot3\ \text{cm}^2.$$
$$=14{,}530\ \text{mm}^2$$
$$r=\sqrt{\frac{I}{A}}=\sqrt{\frac{\frac{\pi}{64}(20^4-15^4)}{\frac{\pi}{4}(20^2-15^2)}}$$
$$=\sqrt{\frac{20^2+15^2}{16}}=\sqrt{\frac{625}{16}}$$
$$=6\cdot25\ \text{cm}$$
$$\frac{l}{r}=\frac{400}{6\cdot25}=64.$$

walls) have sufficient stiffness to restrain the tops and bottoms of columns so they can be classified as braced elements. If a designer is uncertain as to the effectiveness of bracing elements, ACI Code § 10.11.4 provides two quantitative criteria, only one of which must be satisfied:

*Method 1.* Columns in a given story may be considered *braced* or *nonsway* elements if the column end moments produced by a *second-order* structural analysis are not more that 5 percent larger than the moments predicted by a *first-order* analysis. A first-order analysis is based on the initial geometry of the structure and assumes behavior is elastic. A second-order analysis, which is more complicated, includes the influence of joint displacements and changes in geometry on the forces in structures. Today more and more computer programs have the capability to carry out both a first- and second-order analysis. If lateral displacements are small, both types of analysis produce about the same results.

*Method 2.* A story may be considered braced if:

$$\text{Stability Index, } Q = \frac{\Sigma P_u \Delta_0}{V_u l_c} \leq 0.05 \tag{7.7}$$

where $\Sigma P_u$ and $V_u$ = the total vertical load and story shear, respectively, in the story being investigated

$l_c$ = the length of column, measured from center to center of joints

$\Delta_0$ = the relative lateral deflection between the top and bottom floors of the story due to $V_u$ computed using a first-order elastic analysis. In this analysis, ACI Code § 10.11.1 specifies that the influence of flexural cracking, creep, and other factors on member stiffness be accounted for by using reduced values of moment of inertia based on the *gross* area of the cross section, i.e., $0.70I_g$ for columns and $0.35I_g$ for beams (see Table 7.1 for additional details).

The use of Eq. (7.7) to classify a frame as braced or unbraced is illustrated in Example 7.1.

If a structural frame is not attached to an effective bracing element but depends on the bending stiffness of its columns and girders to provide lateral resistance, it is termed an *unbraced* or *sway frame*. Examples of braced and unbraced frames are shown in Fig. 7.10.

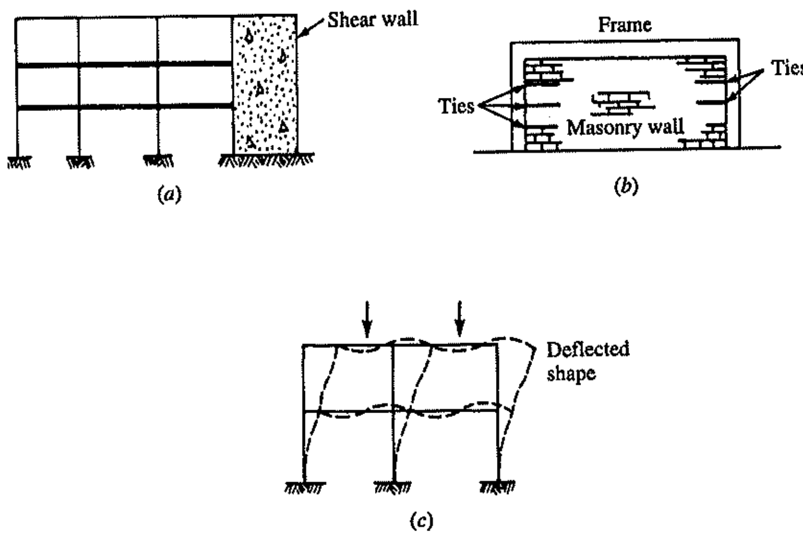



**FIGURE 7.10** Examples of braced and unbraced frames: (*a*) frame braced by a shear wall, (*b*) rigid frame braced by connection to masonry wall, (*c*) unbraced frame.

Given two identical frames, one braced and the other unbraced, the effective length of the columns will always be greater in the unbraced frame than in the braced frame. Since the strength of a column, like the stiffness of a structure, decreases as the effective length increases, the designer should ensure that bracing elements are incorporated into a structure.

**EXAMPLE 7.1.** Under factored gravity and wind loads, a first-order structural analysis determines that the third floor of a reinforced concrete building frame displaces laterally, with respect to the second floor, a distance $\Delta_0 = 0.48$ inches. The analysis produces the forces shown in Fig. 7.11. Verify if the columns in the story are considered members of a *braced* or *unbraced* frame using Eq. (7.7) to check the magnitude of the Stability Index, $Q$.

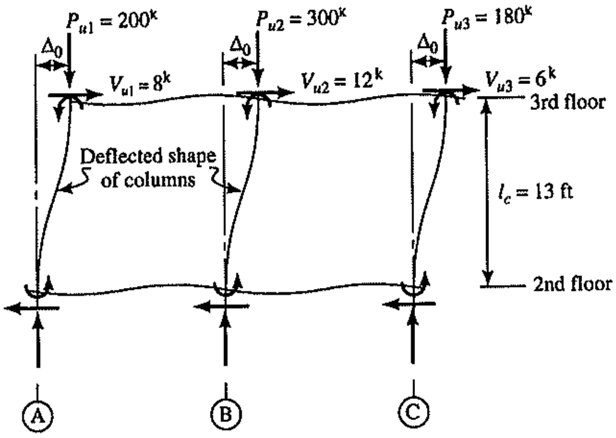



**FIGURE 7.11** Section of a reinforced concrete frame showing both the column forces and the relative lateral displacement between floors ($\Delta_0 = 0.48$ in) created by factored wind and gravity loads.

**Solution.** To be classified as a *braced* frame, $Q = \Sigma P_u \Delta_0/(V_u l_c)$ must not exceed 0.05.

$$Q = \frac{(200 + 300 + 180)0.48 \text{ in}}{(8 + 12 + 6)13 \times 12} = 0.08 > 0.05 \qquad \text{Frame classified as } \textit{unbraced}$$

## 7.4 EFFECTIVE-LENGTH FACTORS FOR COLUMNS OF RIGID FRAMES

In a reinforced concrete frame, columns are rigidly attached to girders and adjacent columns. The effective length of a particular column between stories will depend on how the frame is braced and on the bending stiffness of the girders. As a column bends in response to applied loads, the ends of the attached girders must rotate with the column because of the rigid joint. If the girders are stiff and do not bend significantly, they will provide full rotational restraint to the column, like a fixed support (Fig. 7.12*a*). If the girders are flexible and bend easily, as in Fig. 7.12*b*, they provide only a small degree of rotational restraint, and the end conditions for the column approach those of a pin support that allows unrestrained rotation.

The Jackson and Moreland alignment charts[2] (Fig. 7.13) can be used to evaluate the influence of girder bending stiffness on the effective-length factor of a column that is part of a rigid frame. The charts are entered with values of $\psi$ for the joints at each end of a column. For a rigid frame whose members are prismatic, $\psi$, the ratio of the sum of the relative bending stiffnesses of the columns to that of the girders, is defined as

$$\psi = \frac{\Sigma(E_c I_c/L_c)}{\Sigma(E_g I_g/L_g)} \tag{7.8}$$

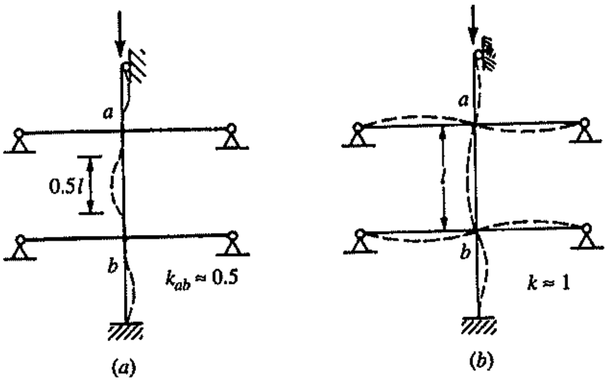



**FIGURE 7.12** Influence of girder stiffness on the effective length of a column in a braced frame: (*a*) rigid girders, (*b*) flexible girders.

where $I_c$ = effective moment of inertia of column = $0.7I_{\text{gross}}$
$I_g$ = effective moment of inertia of girder = $0.35I_{\text{gross}}$
$L_c$ = length of column, center to center of joints
$L_g$ = length of girder, center to center of joints
$E_g, E_c$ = modulus of elasticity of girders and columns, respectively

The intersection of the straight line connecting the two $\psi$ values with the vertical line labeled $k$ gives the value of the column's effective-length factor. Since the strength of a column is influenced by the presence or absence of lateral support, charts are given for both *braced* and *unbraced* frames.

The value of the $k$ factor is based on the assumption that all columns in a *braced* frame buckle simultaneously and that the girders bend into single curvature with equal but opposite rotations at each end (Fig. 7.14*a*). For the *unbraced* frame, the $k$ factors are based on the assump-

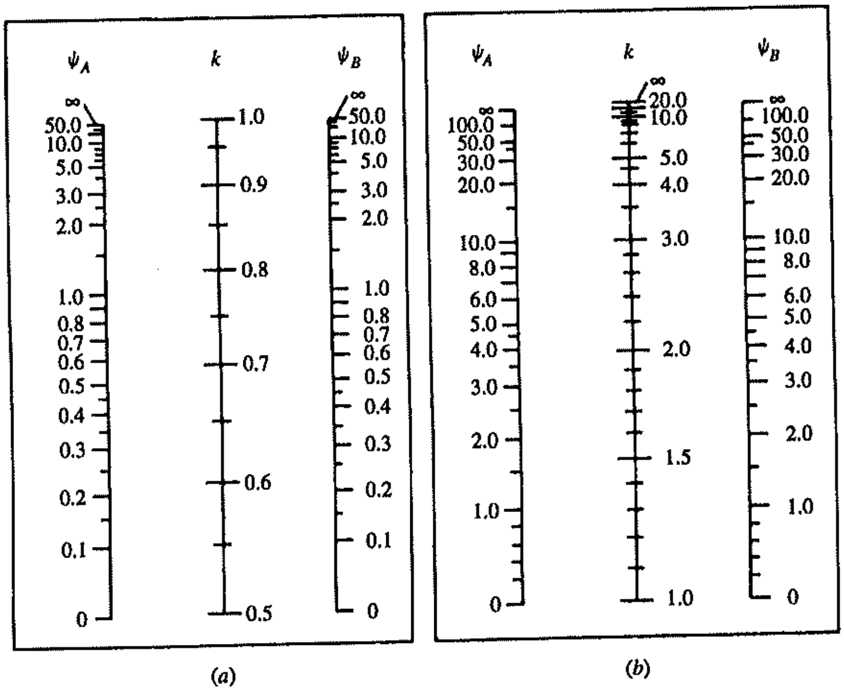



**FIGURE 7.13** Jackson-Moorland alignment charts for the effective-length factor $k$: (*a*) braced frames, (*b*) unbraced frames. $\psi$ = ratio of $\Sigma(EI/L_c)$ of compression members to $\Sigma(EI/L_g)$ of flexural members in a plane at one end of a compression member, $k$ = effective-length factor.[1,2]

## CODE

**6.2.4** Additional analysis methods that are permitted include 6.2.4.1 through 6.2.4.4.

**6.2.4.1** Two-way slabs shall be permitted to be analyzed for gravity loads in accordance with (a) or (b):

(a) Direct design method in 8.10
(b) Equivalent frame method in 8.11

**6.2.4.2** Slender walls shall be permitted to be analyzed in accordance with 11.8 for out-of-plane effects.

**6.2.4.3** Diaphragms shall be permitted to be analyzed in accordance with 12.4.2.

**6.2.4.4** A member or region shall be permitted to be analyzed and designed using the strut-and-tie method in accordance with Chapter 23.

**6.2.5** Slenderness effects shall be permitted to be neglected if (a) or (b) is satisfied:

(a) For columns not braced against sidesway

$$\frac{k\ell_u}{r} \le 22 \qquad (6.2.5a)$$

(b) For columns braced against sidesway

$$\frac{k\ell_u}{r} \le 34 + 12(M_1/M_2) \qquad (6.2.5b)$$

and

$$\frac{k\ell_u}{r} \le 40 \qquad (6.2.5c)$$

where $M_1/M_2$ is negative if the column is bent in single curvature, and positive for double curvature.

If bracing elements resisting lateral movement of a story have a total stiffness of at least 12 times the gross lateral stiffness of the columns in the direction considered, it shall be permitted to consider columns within the story to be braced against sidesway.

**6.2.5.1** The radius of gyration, $r$, shall be permitted to be calculated by (a), (b), or (c):

(a) $$r = \sqrt{\frac{I_g}{A_g}} \qquad (6.2.5.1)$$

(b) 0.30 times the dimension in the direction stability is being considered for rectangular columns

## COMMENTARY

Finite element analysis was introduced in the 2014 Code to explicitly recognize a widely used analysis method.

**R6.2.5** Second-order effects in many structures are negligible. In these cases, it is unnecessary to consider slenderness effects, and compression members, such as columns, walls, or braces, can be designed based on forces determined from first-order analyses. Slenderness effects can be neglected in both braced and unbraced systems, depending on the slenderness ratio ($k\ell_u/r$) of the member.

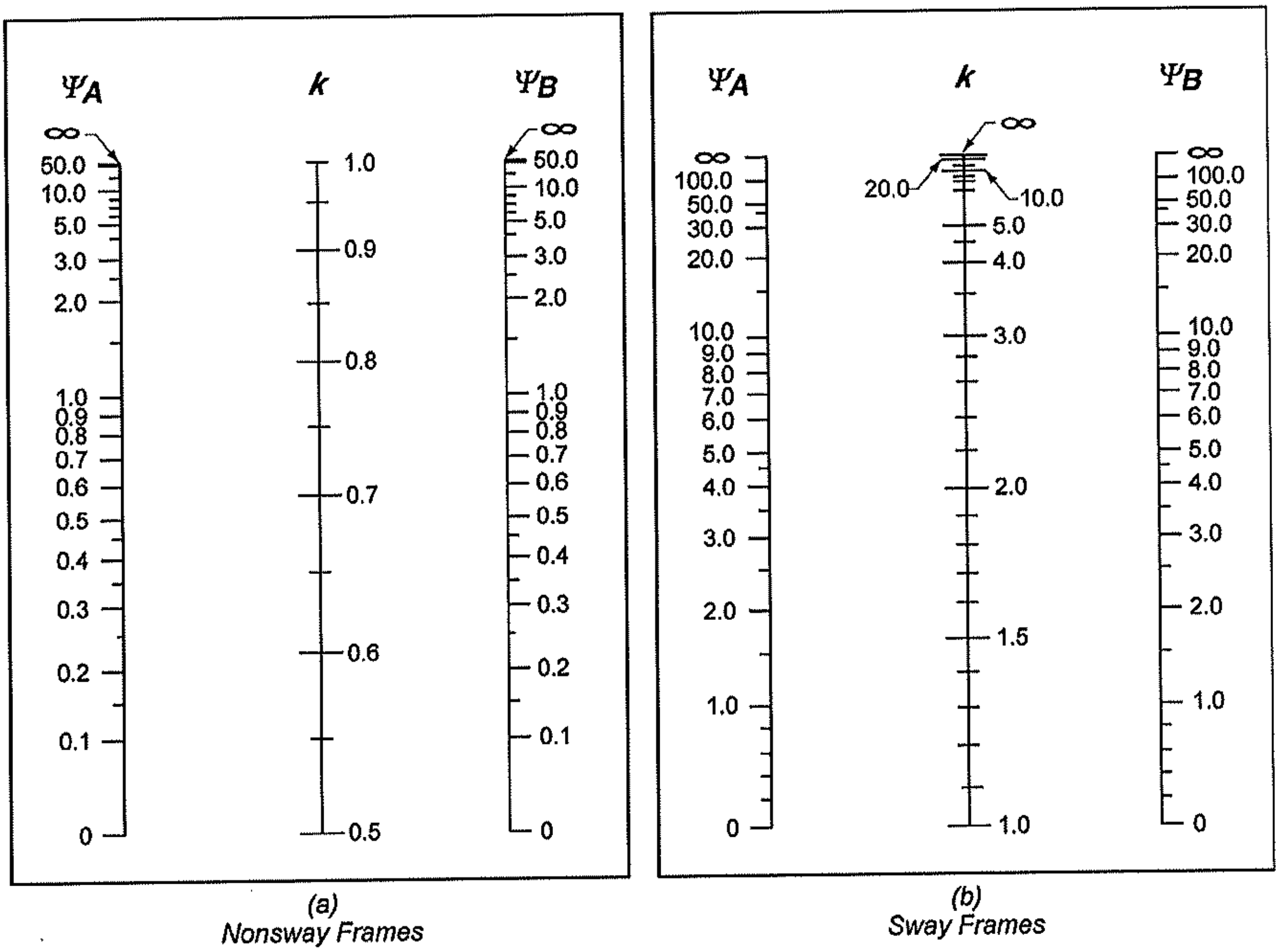
The sign convention for $M_1/M_2$ has been updated so that $M_1/M_2$ is negative if bent in single curvature and positive if bent in double curvature. This reflects a sign convention change from the 2011 Code.

The primary design aid to estimate the effective length factor $k$ is the Jackson and Moreland Alignment Charts (Fig. R6.2.5), which provide a graphical determination of $k$ for a column of constant cross section in a multi-bay frame (ACI SP-17(09); Column Research Council 1966).

Equations (6.2.5b) and (6.2.5c) are based on Eq. (6.6.4.5.1) assuming that a 5 percent increase in moments due to slenderness is acceptable (MacGregor et al. 1970). As a first approximation, $k$ may be taken equal to 1.0 in Eq. (6.2.5b) and (6.2.5c).

The stiffness of the lateral bracing is considered based on the principal directions of the framing system. Bracing elements in typical building structures consist of shear walls or lateral braces. Torsional response of the lateral-force-resisting system due to eccentricity of the structural system can increase second-order effects and should be considered.

**CODE** | **COMMENTARY**




Ψ = ratio of Σ(*EI*/ℓ*c*) of columns to Σ(*EI*/ℓ) of beams in a plane at one end of a column

ℓ = span length of beam measured center to center of joints

*Fig. R6.2.5—Effective length factor* k.

(c) 0.25 times the diameter of circular columns

**6.2.5.2** For composite columns, the radius of gyration, $r$, shall not be taken greater than:

$$r = \sqrt{\frac{(E_c I_g/5) + E_s I_{sx}}{(E_c A_g/5) + E_s A_{sx}}} \quad (6.2.5.2)$$

Longitudinal bars located within a concrete core encased by structural steel or within transverse reinforcement surrounding a structural steel core shall be permitted to be used in calculating $A_{sx}$ and $I_{sx}$.

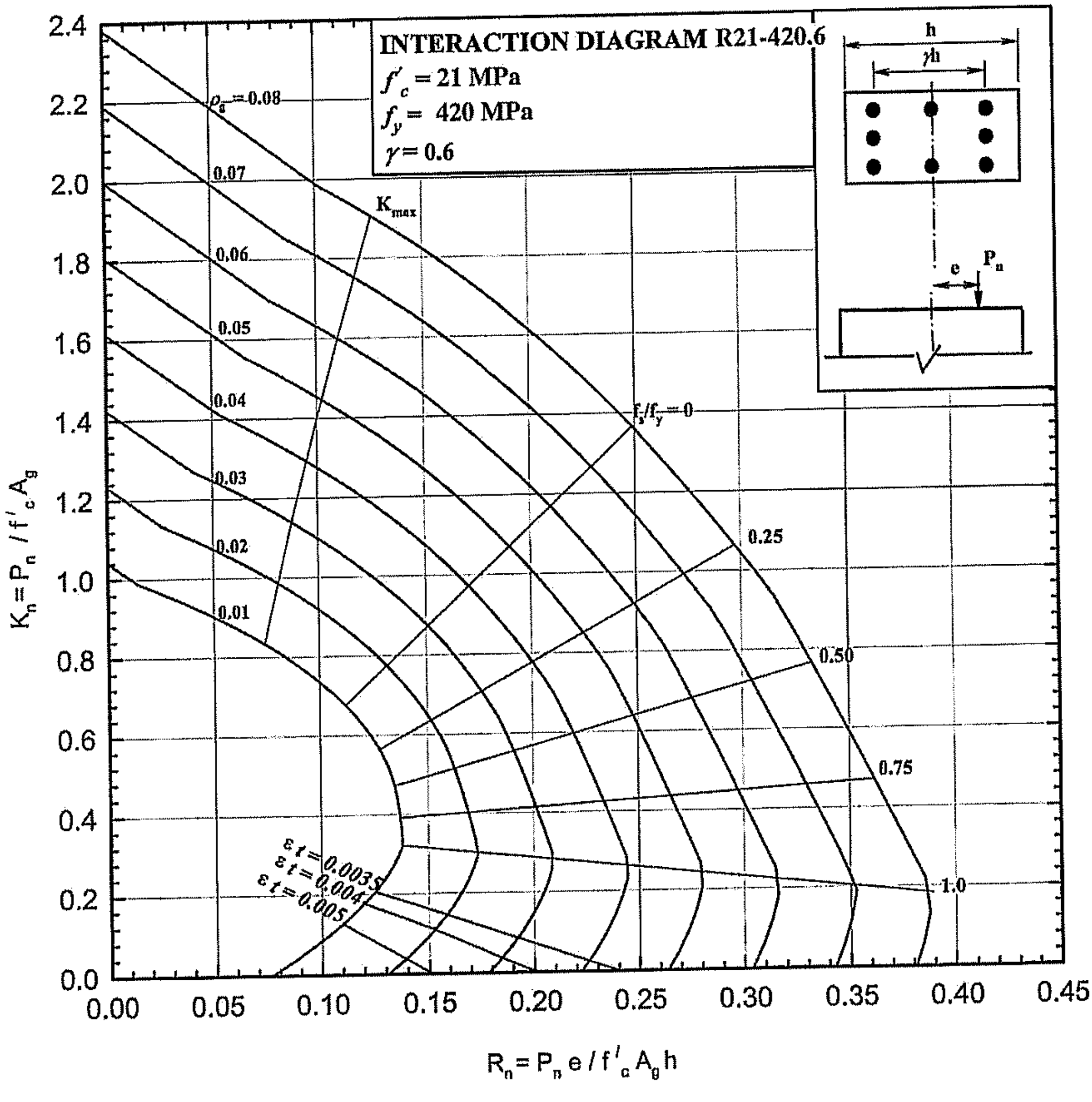
**R6.2.5.2** Equation (6.2.5.2) is provided because the provisions in 6.2.5.1 for estimating the radius of gyration are overly conservative for concrete-filled tubing and are not applicable for members with enclosed structural shapes.

**6.2.6** Unless slenderness effects are neglected as permitted by 6.2.5, the design of columns, restraining beams, and other supporting members shall be based on the factored forces and moments considering second-order effects in accordance with 6.6.4, 6.7, or 6.8. $M_u$ including second-order effects shall not exceed $1.4M_u$ due to first-order effects.

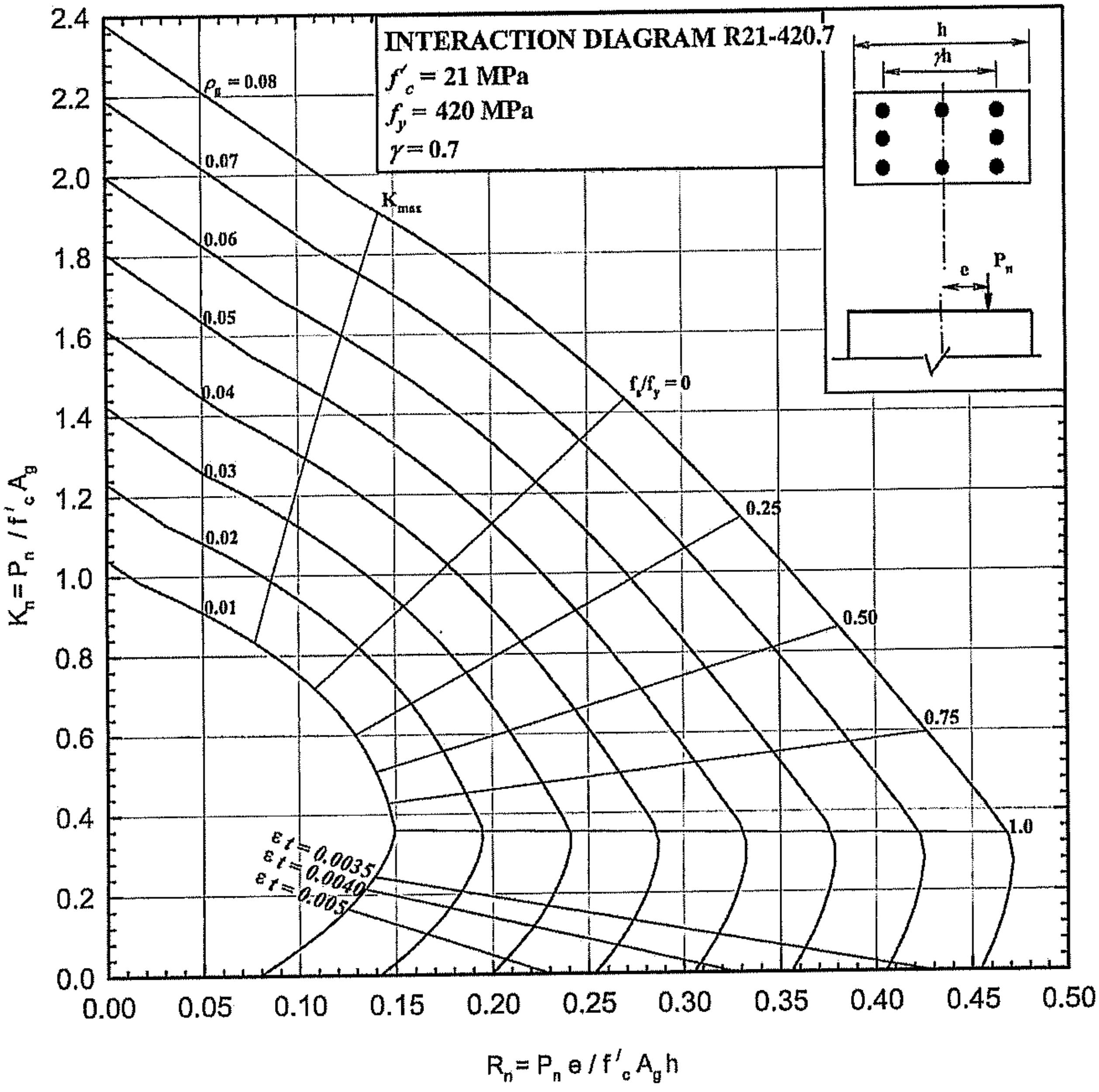
**R6.2.6** Design considering second-order effects may be based on the moment magnifier approach (MacGregor et al. 1970; MacGregor 1993; Ford et al. 1981), an elastic second-order analysis, or a nonlinear second-order analysis. Figure R6.2.6 is intended to assist designers with application of the slenderness provisions of the Code.

3.5—Columns design aids

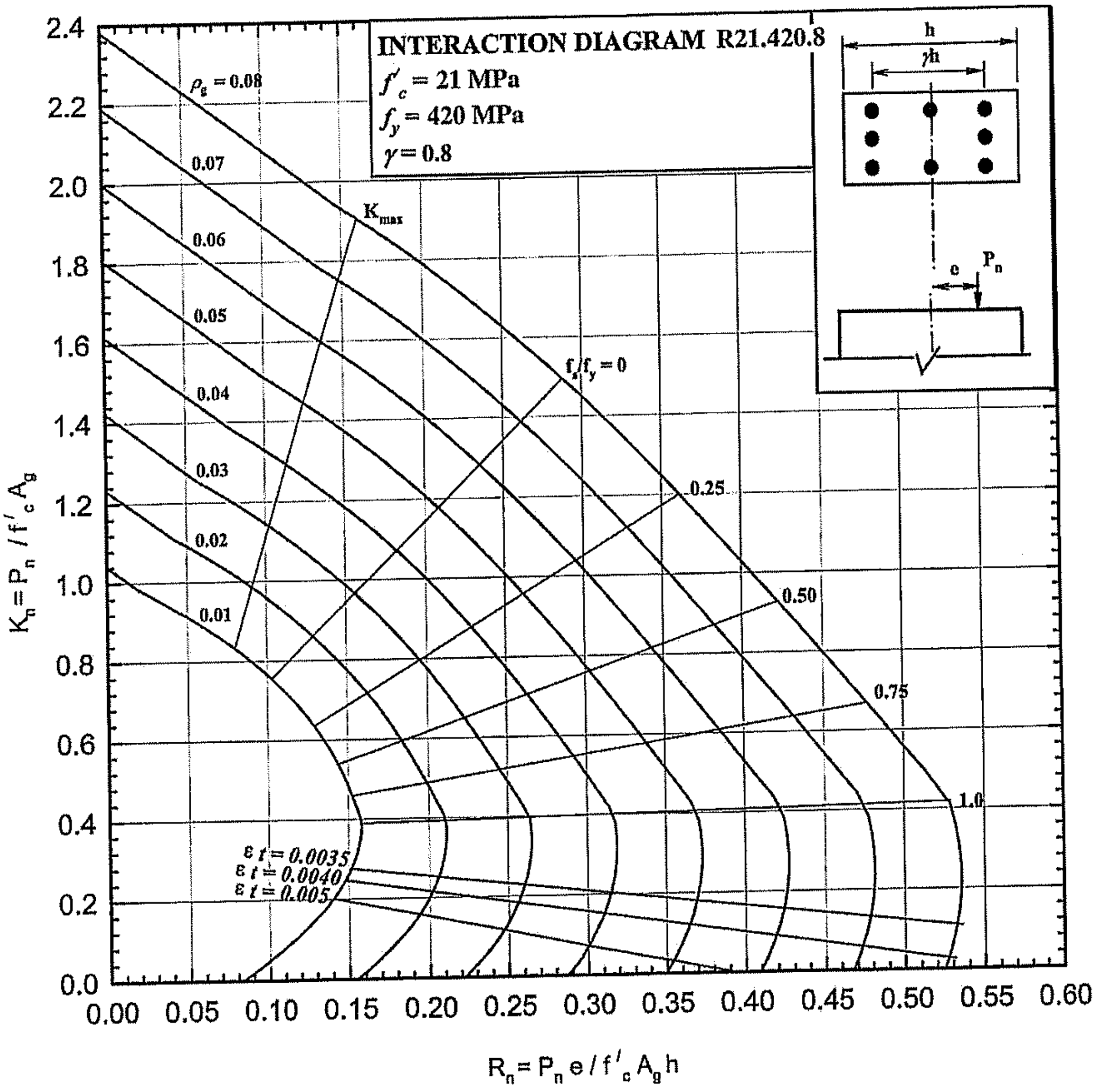
## COLUMNS 3.1.1 - Nominal load-moment strength interaction diagram, R21-420.6



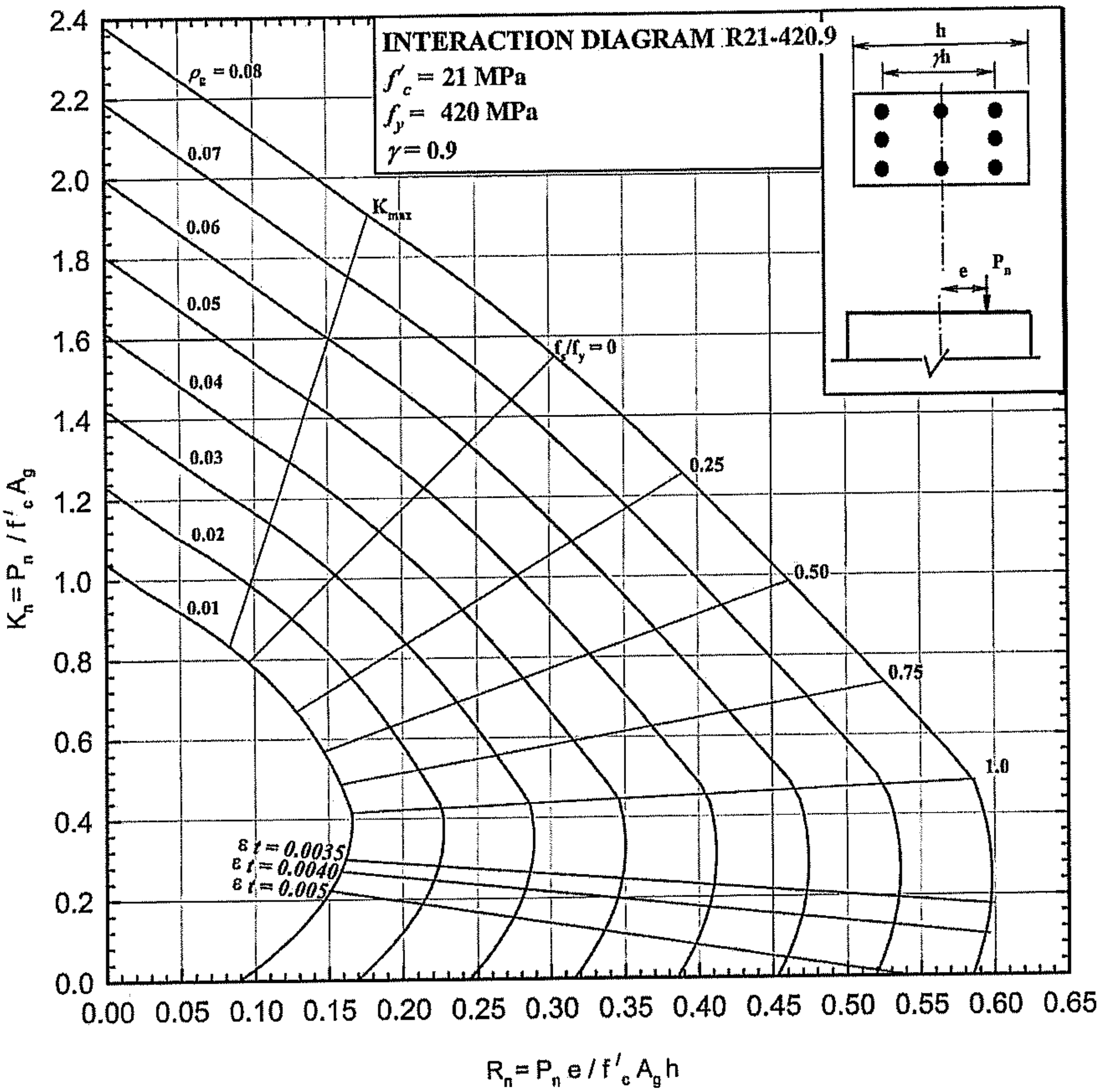
## COLUMNS 3.1.2 - Nominal load-moment strength interaction diagram, R21-420.7



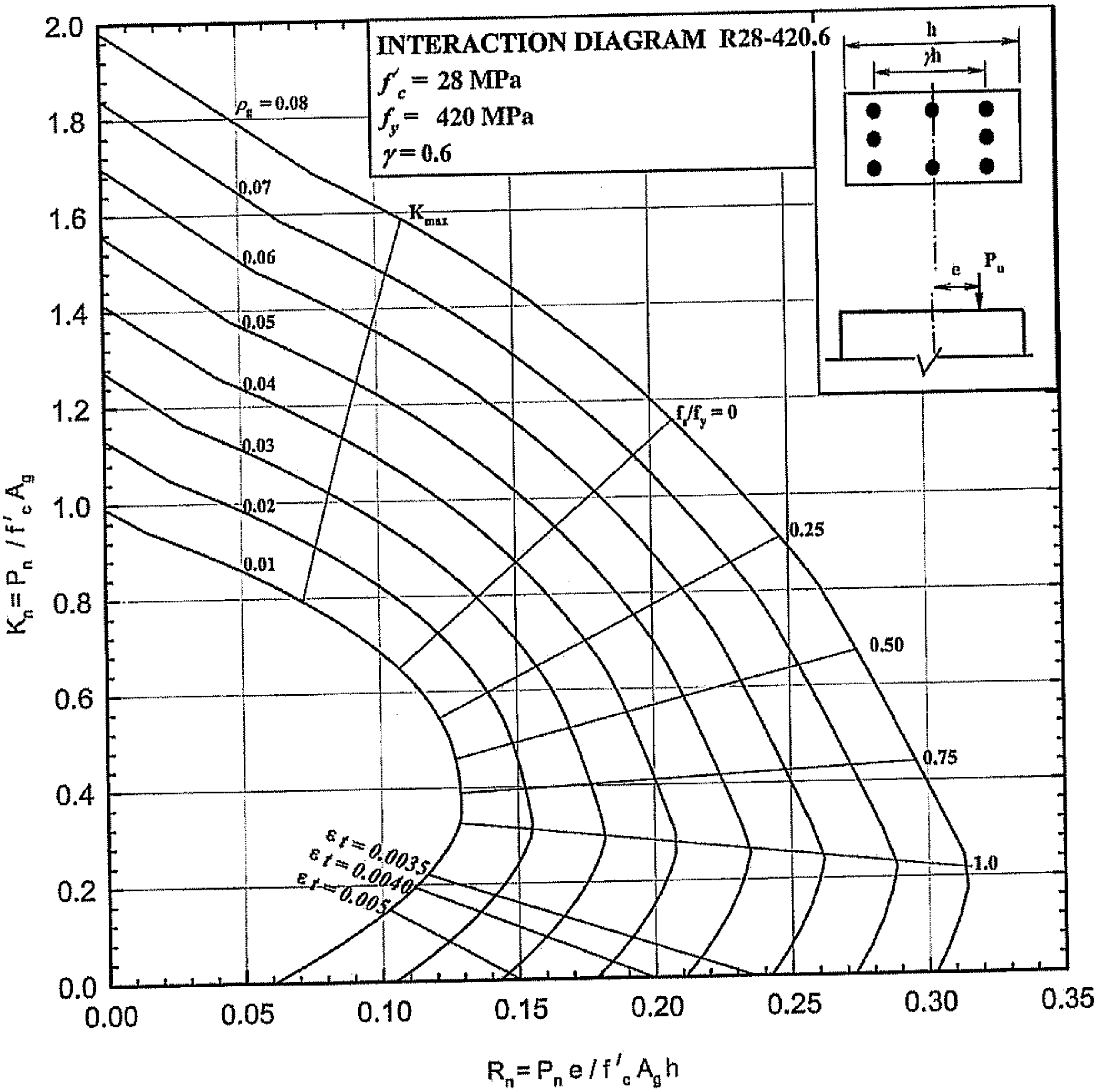
## COLUMNS 3.1.3 - Nominal load-moment strength interaction diagram, R21.420.8



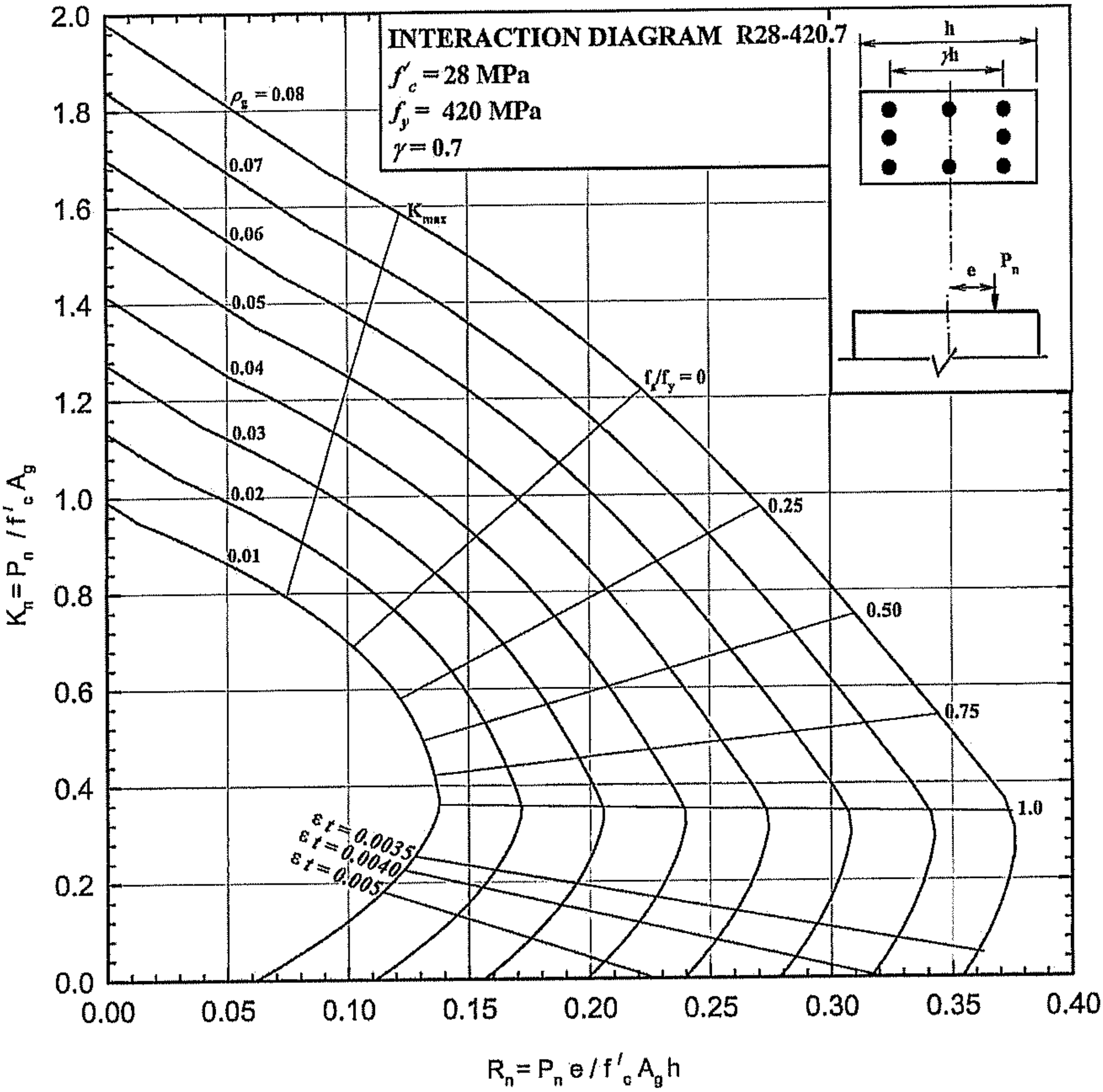
## COLUMNS 3.1.4 - Nominal load-moment strength interaction diagram, R21-420.9



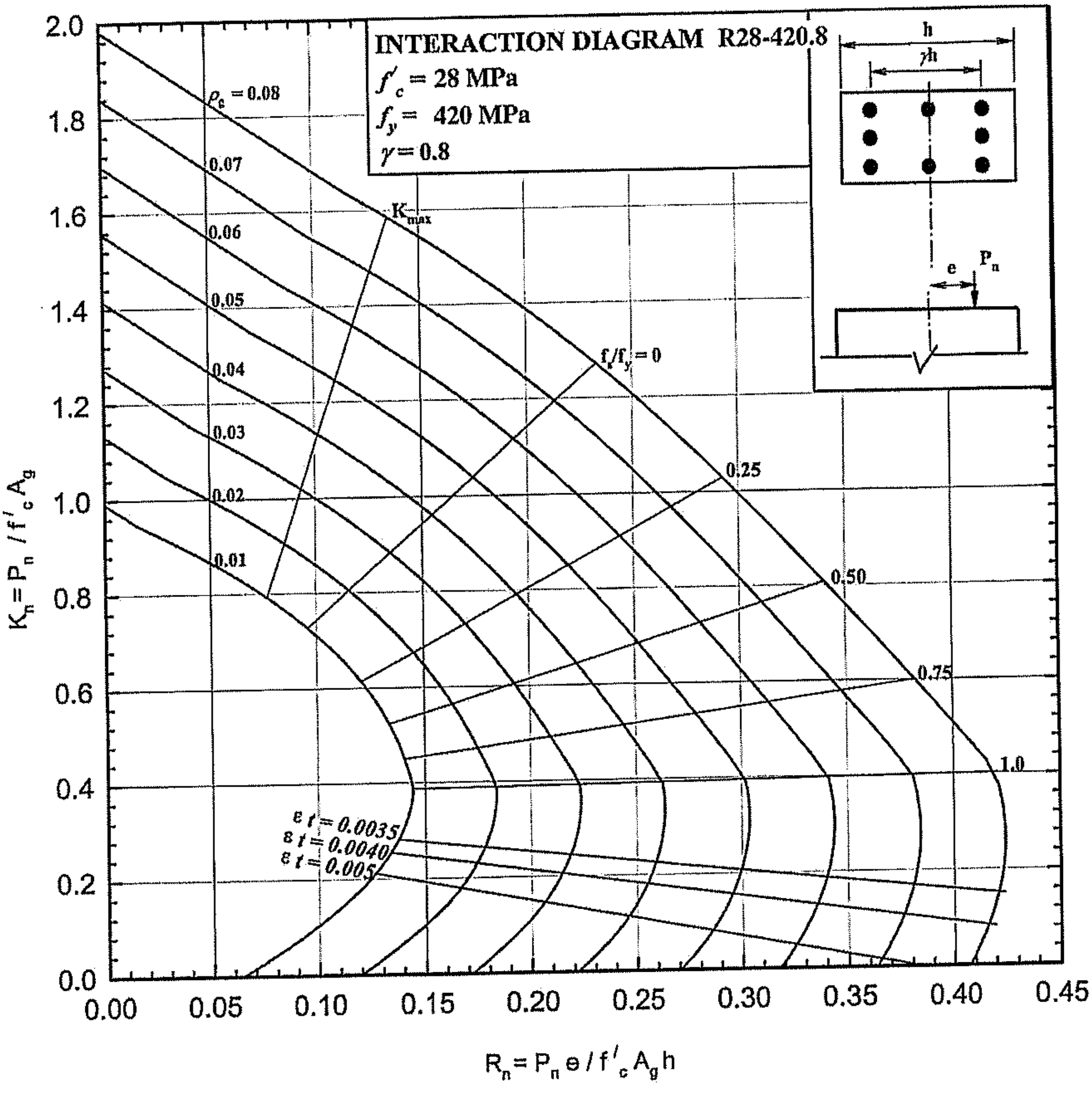
## COLUMNS 3.2.1 - Nominal load-moment strength interaction diagram, R28-420.6



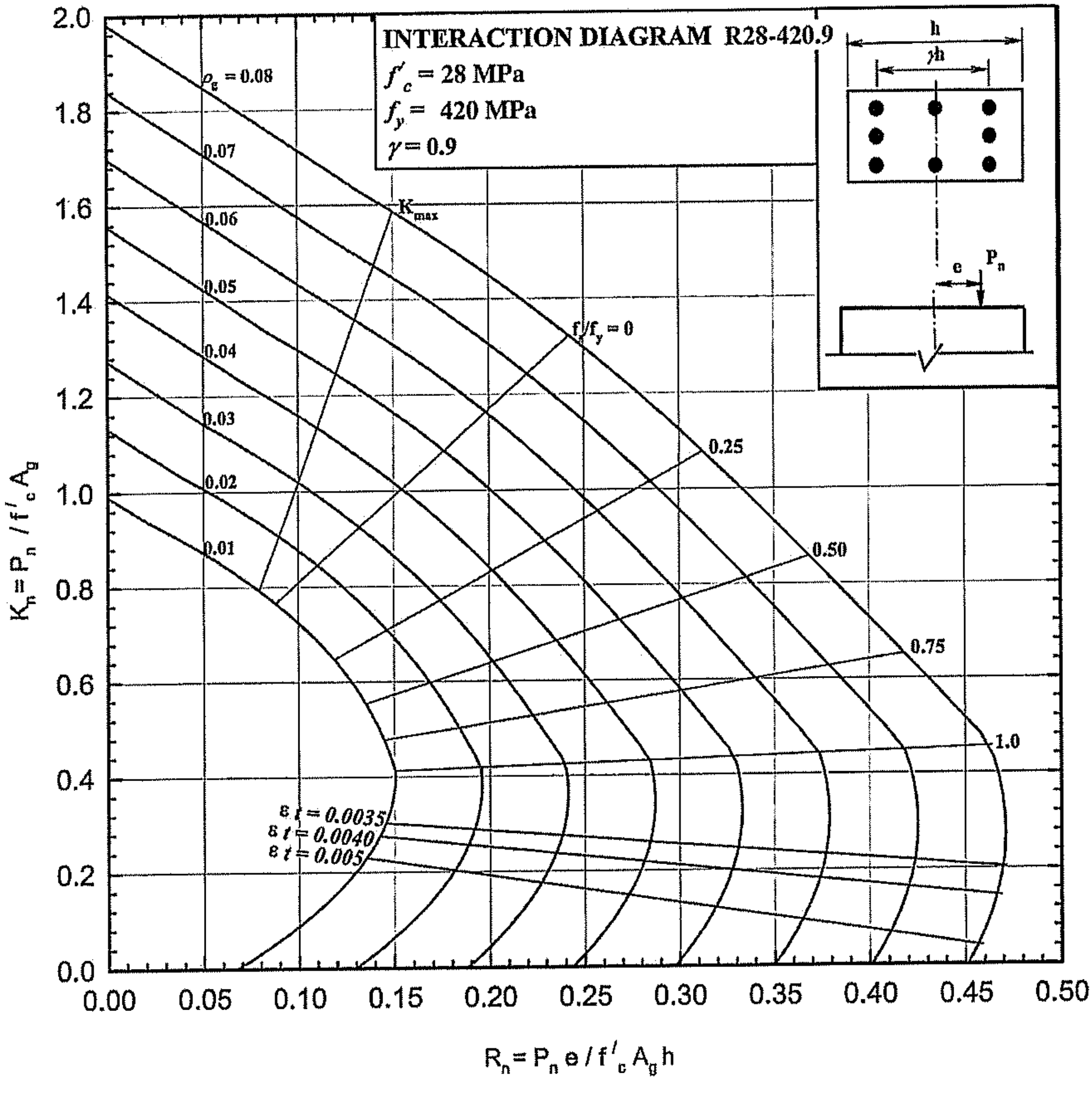
**COLUMNS 3.2.2 - Nominal load-moment strength interaction diagram, R28-420.7**



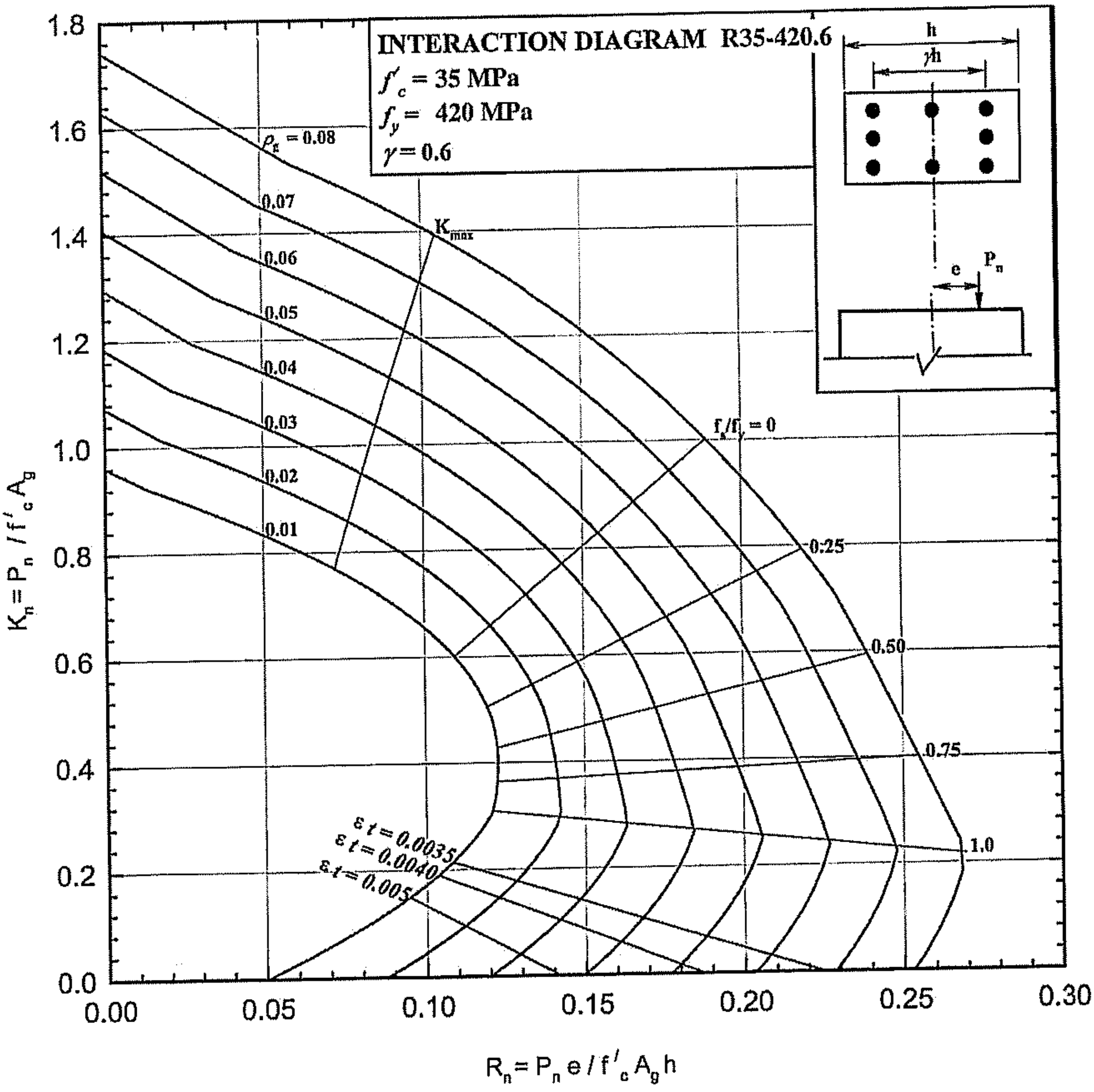
## COLUMNS 3.2.3 - Nominal load-moment strength interaction diagram, R28-420.8



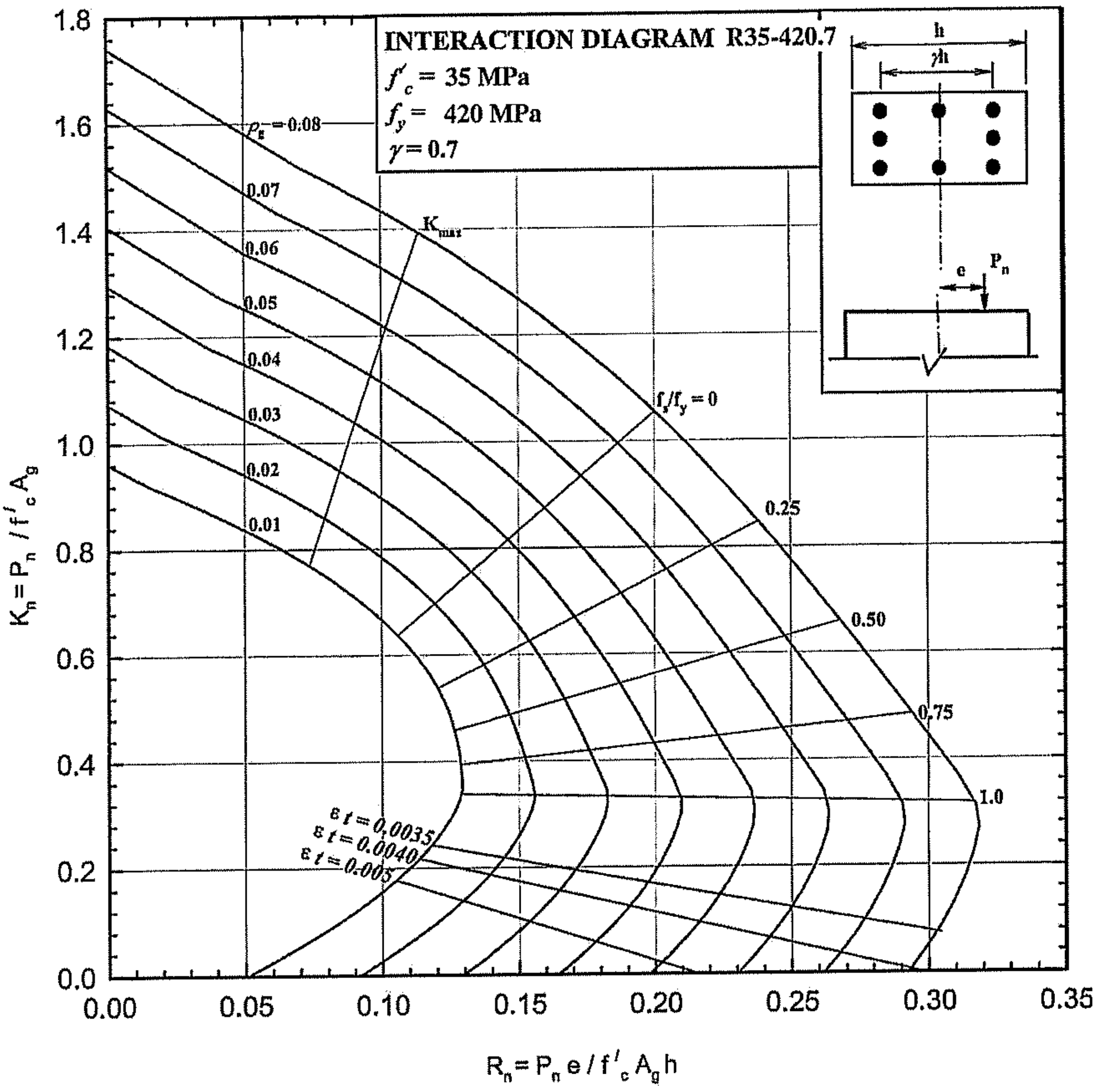
## COLUMNS 3.2.4 - Nominal load-moment strength interaction diagram, R28-420.9



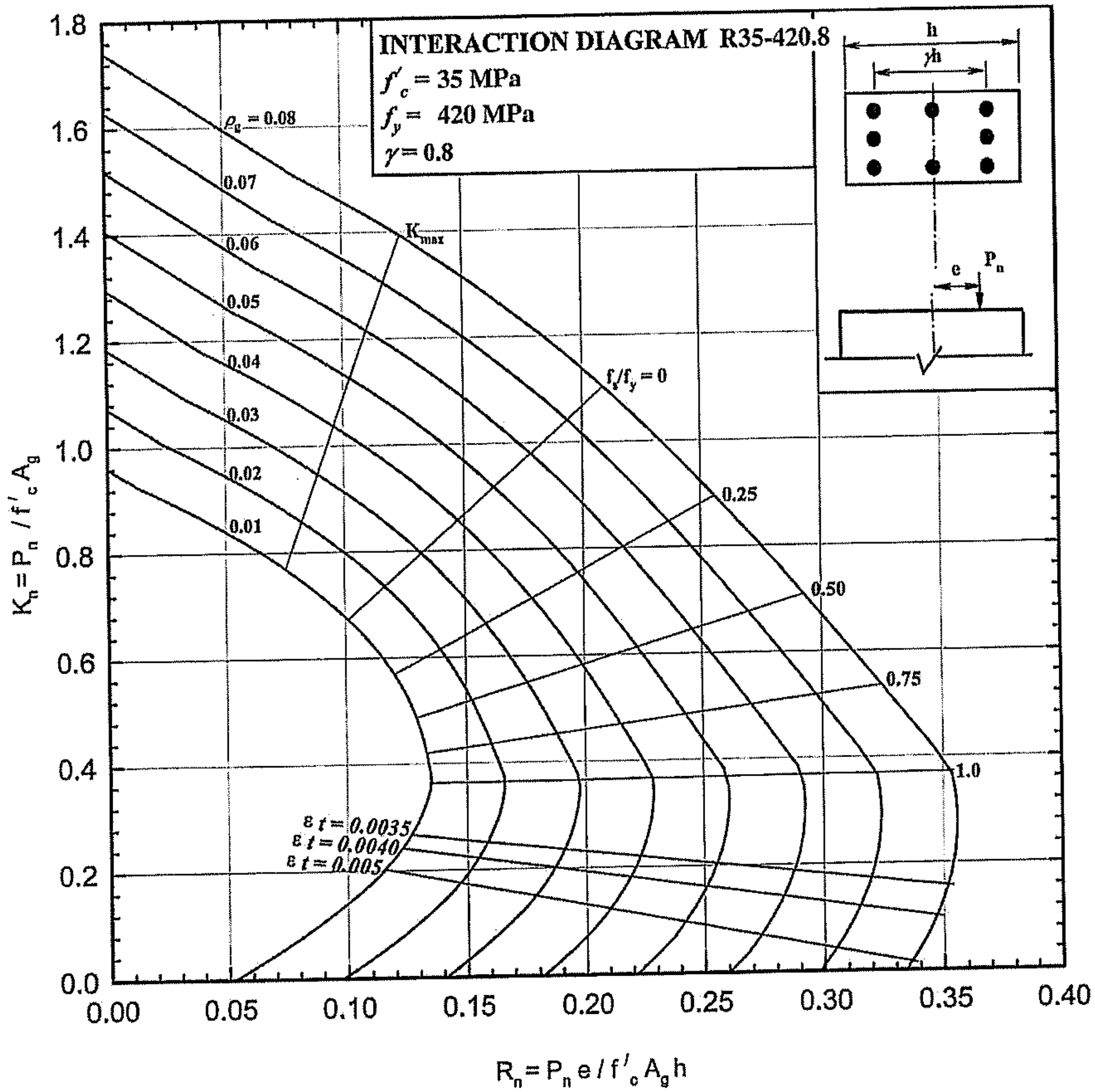
## COLUMNS 3.3.1 - Nominal load-moment strength interaction diagram, R35-420.6



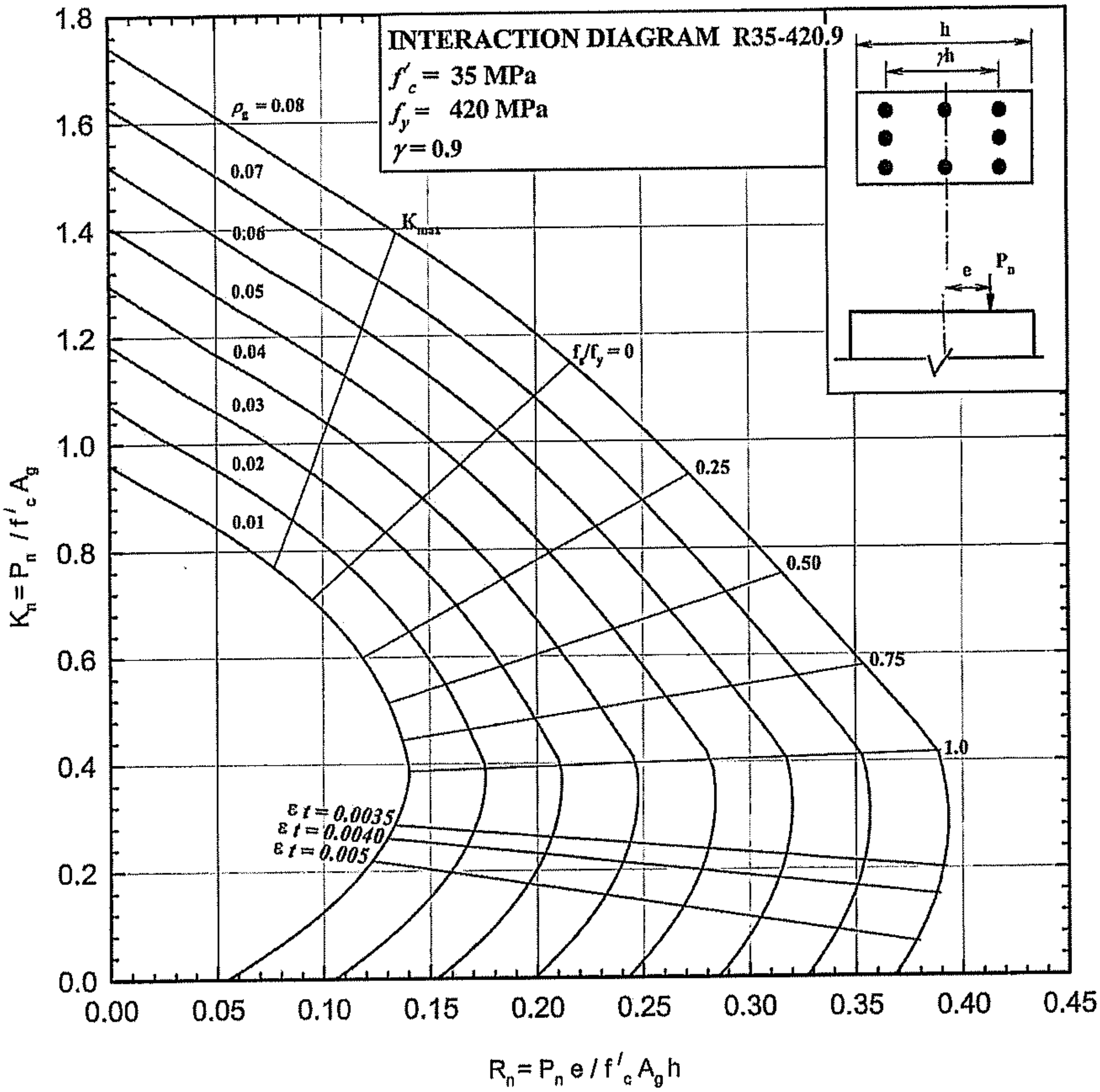
## COLUMNS 3.3.2 - Nominal load-moment strength interaction diagram, R35-420.7



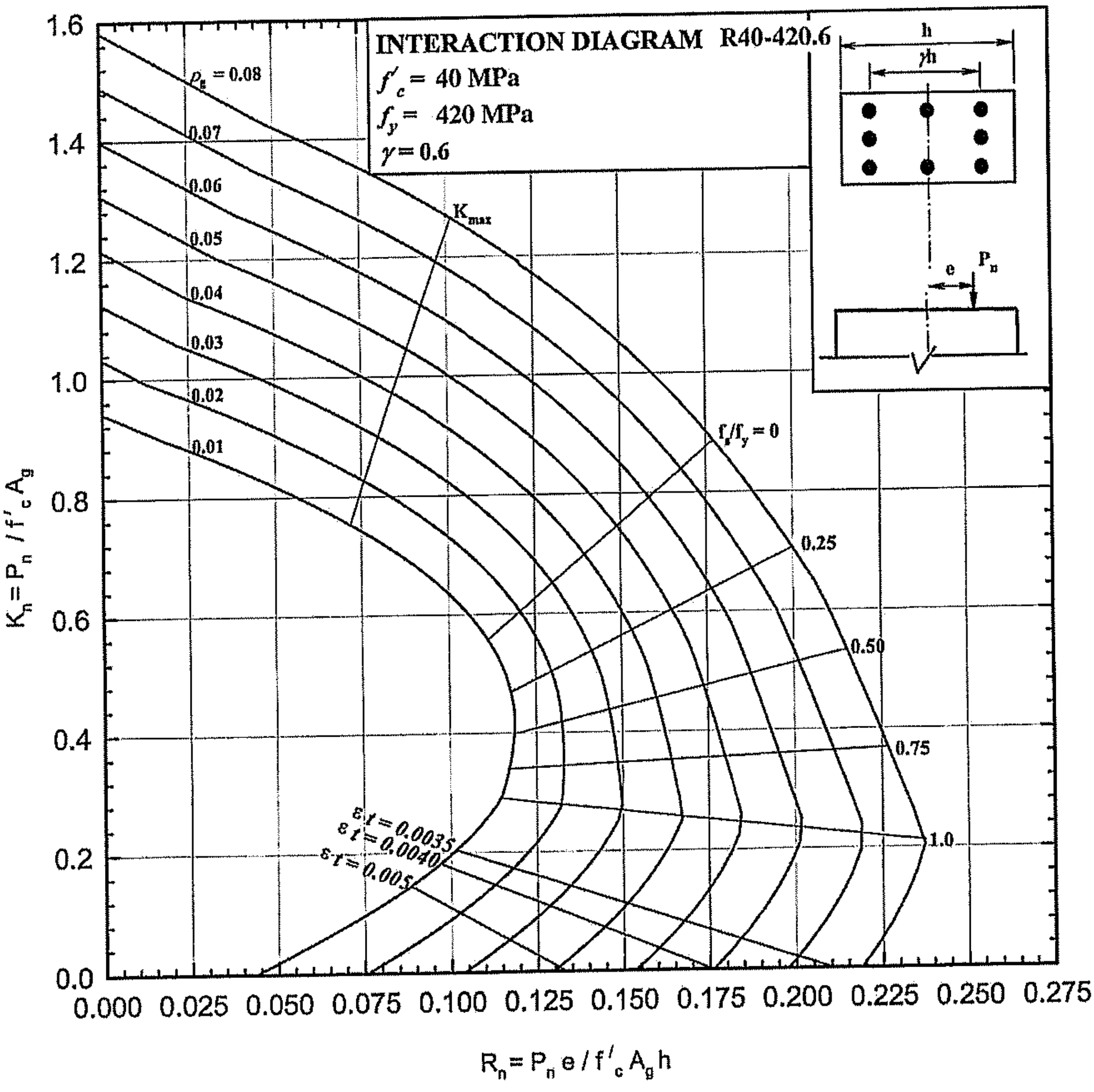
## COLUMNS 3.3.3 - Nominal load-moment strength interaction diagram, R35-420.8



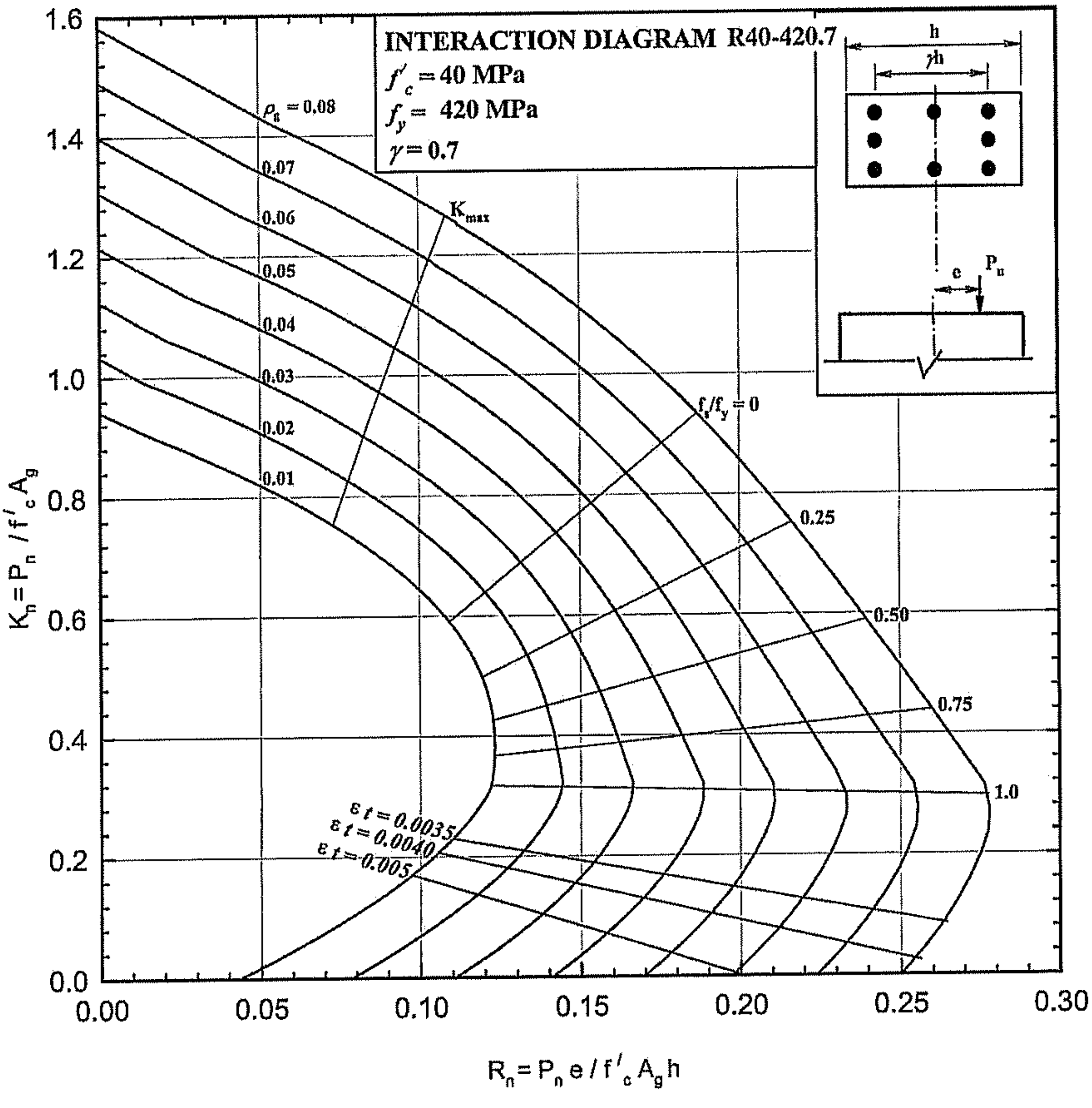
**COLUMNS 3.3.4 - Nominal load-moment strength interaction diagram, R35-420.9**



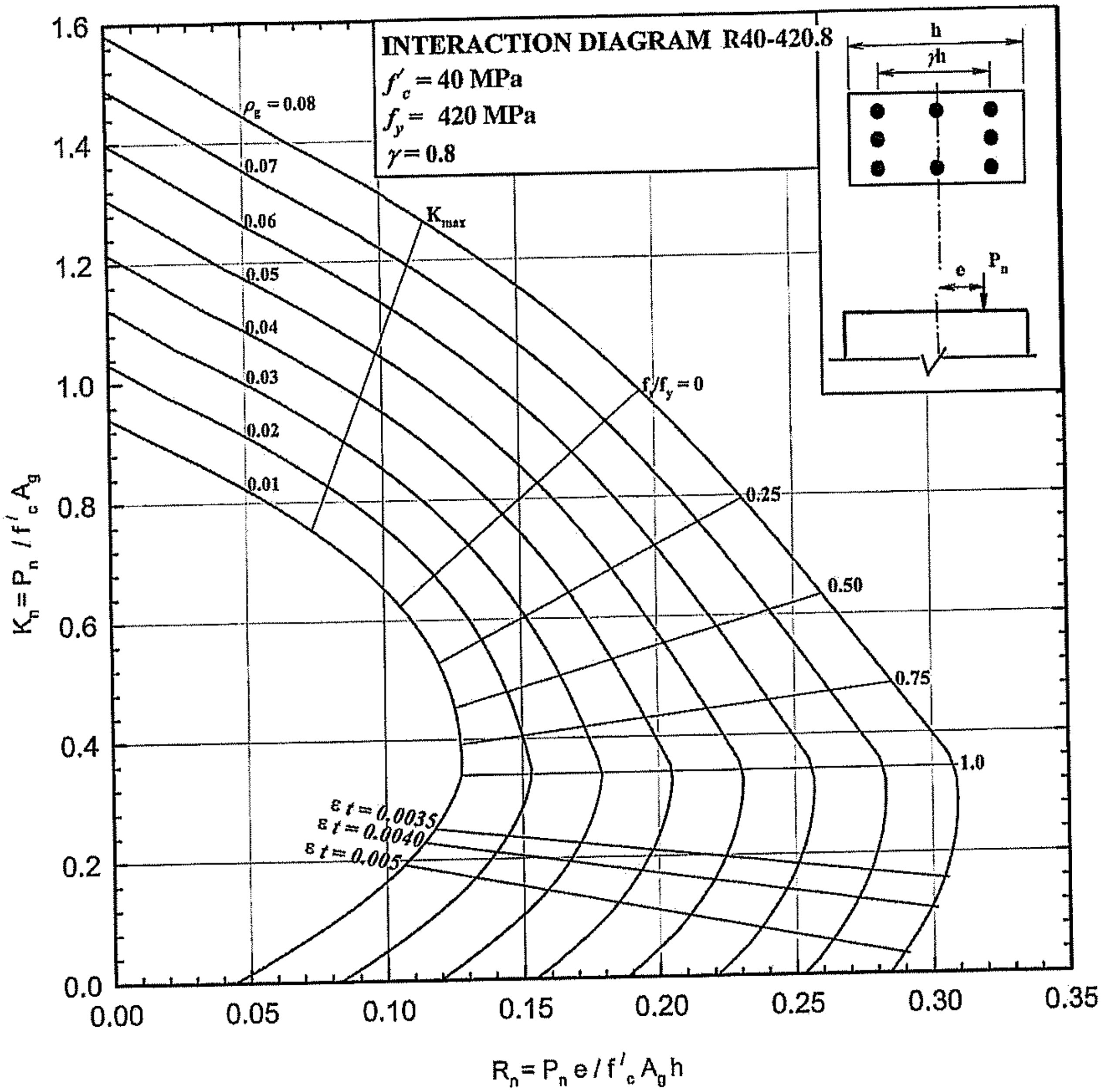
## COLUMNS 3.4.1 - Nominal load-moment strength interaction diagram, R40-420.6



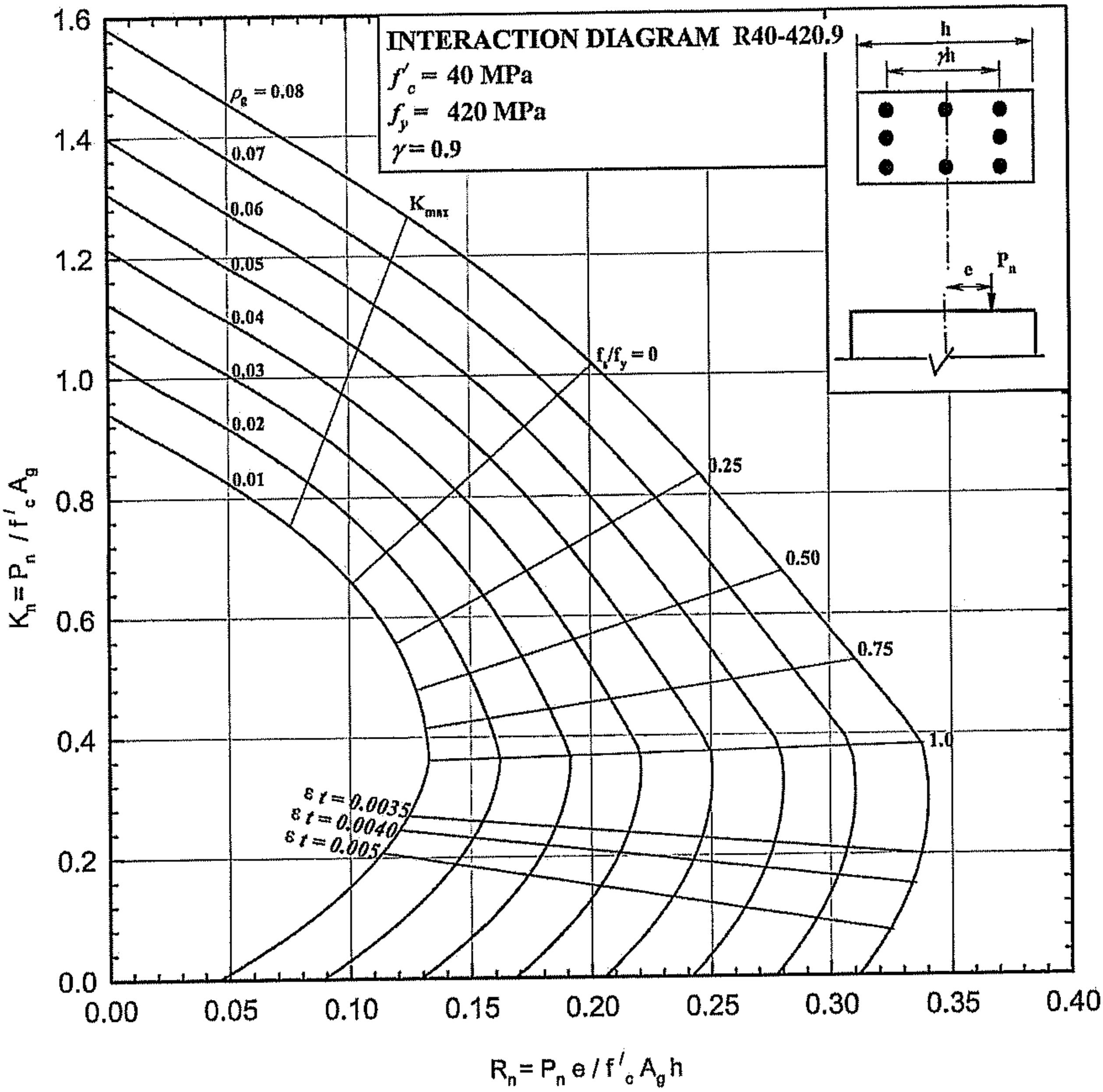
# COLUMNS 3.4.2 - Nominal load-moment strength interaction diagram, R40-420.7



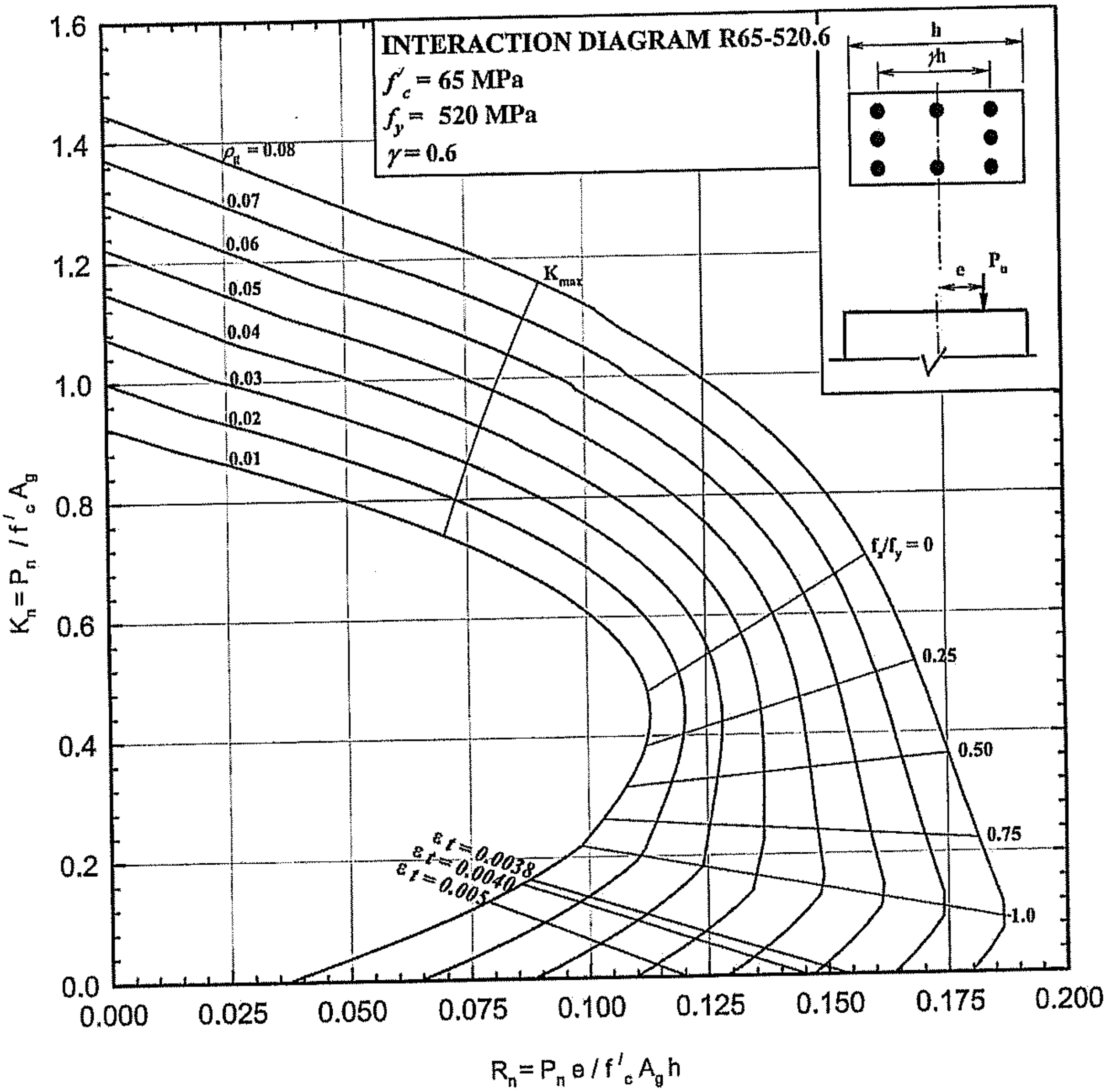
# COLUMNS 3.4.3 - Nominal load-moment strength interaction diagram, R40-420.8



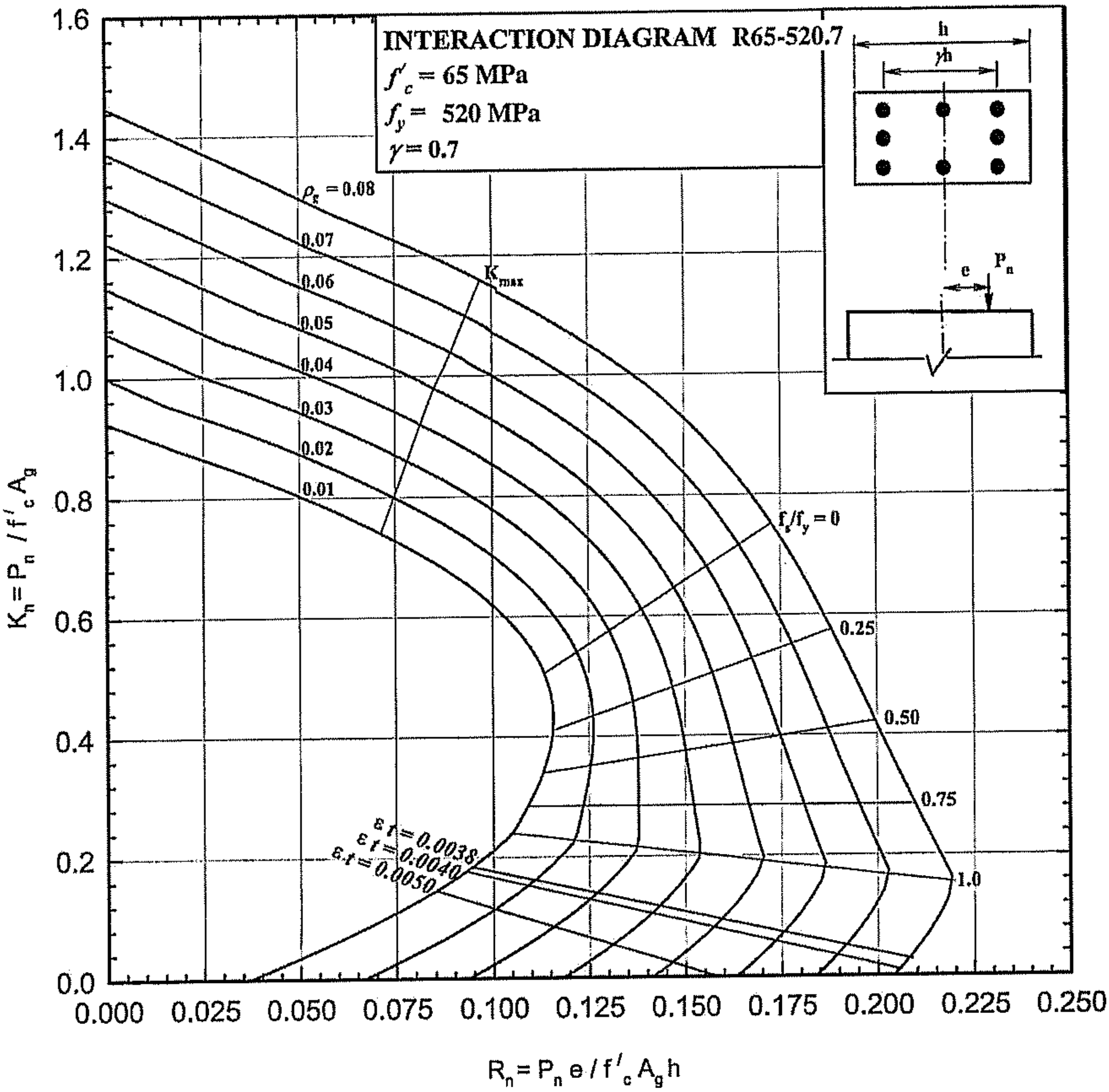
## COLUMNS 3.4.4 - Nominal load-moment strength interaction diagram, R40-420.9



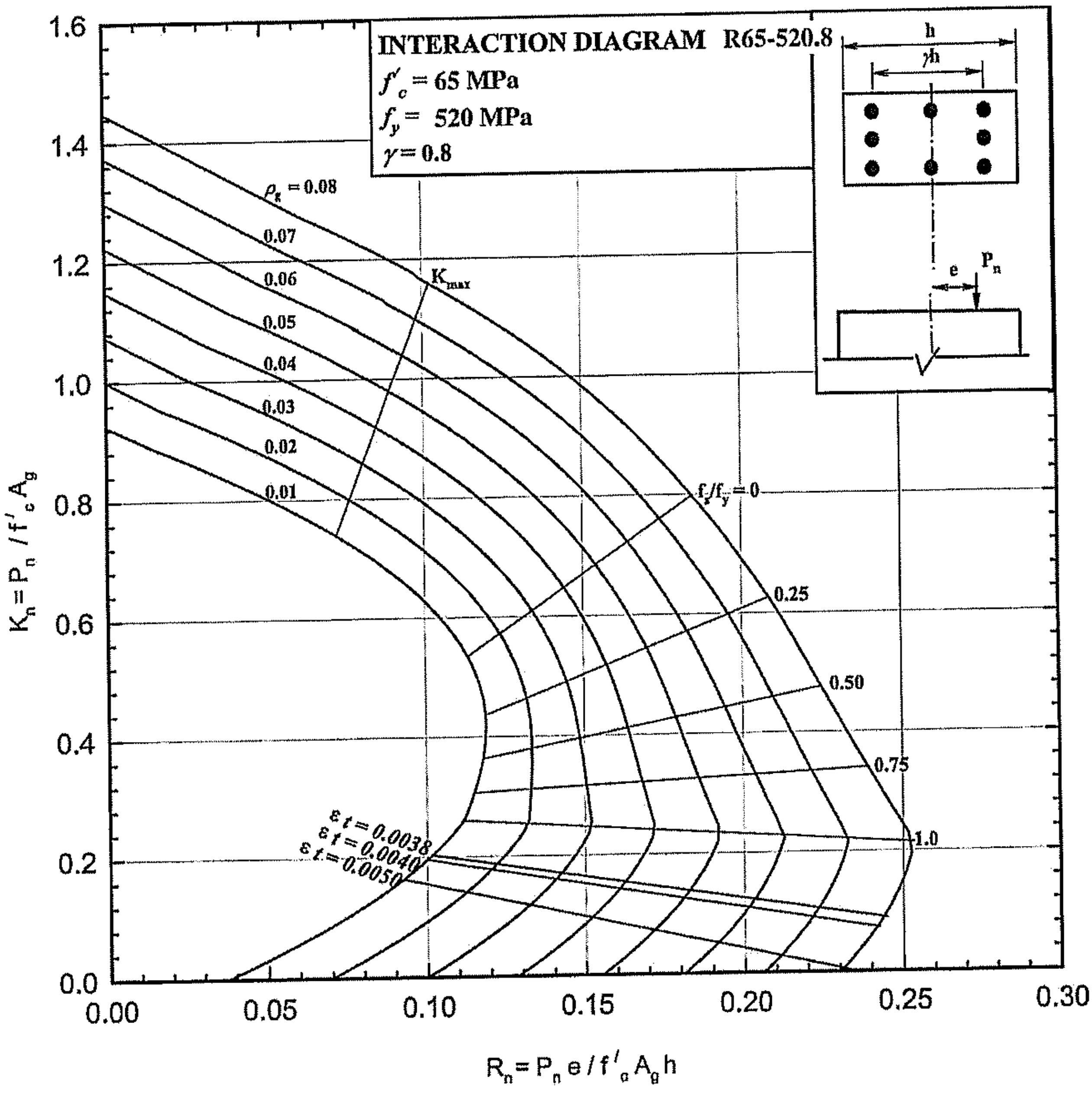
## COLUMNS 3.5.1 - Nominal load-moment strength interaction diagram, R65-520.6



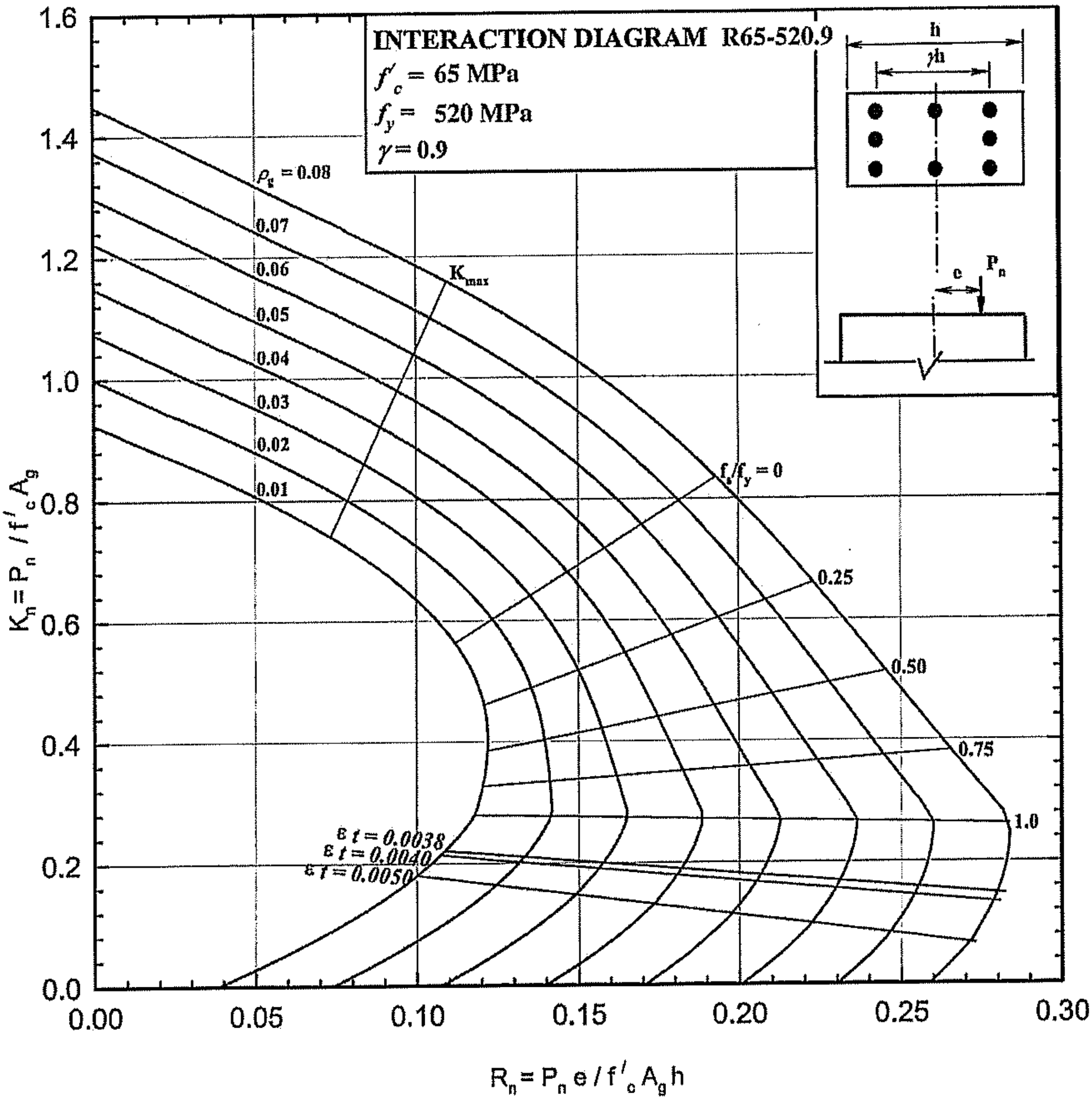
**COLUMNS 3.5.2 - Nominal load-moment strength interaction diagram, R65-520.7**



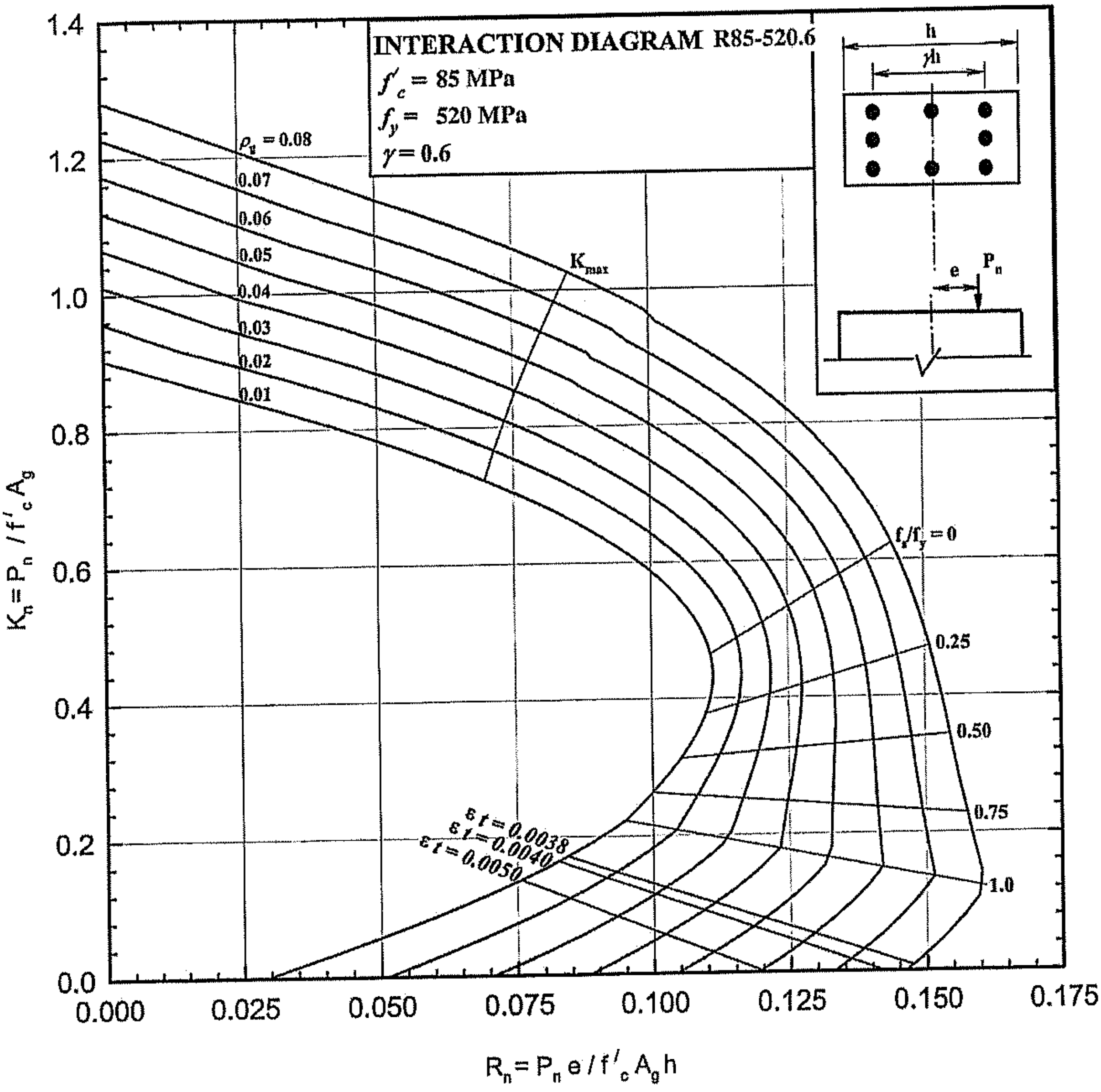
## COLUMNS 3.5.3 - Nominal load-moment strength interaction diagram, R65-520.8



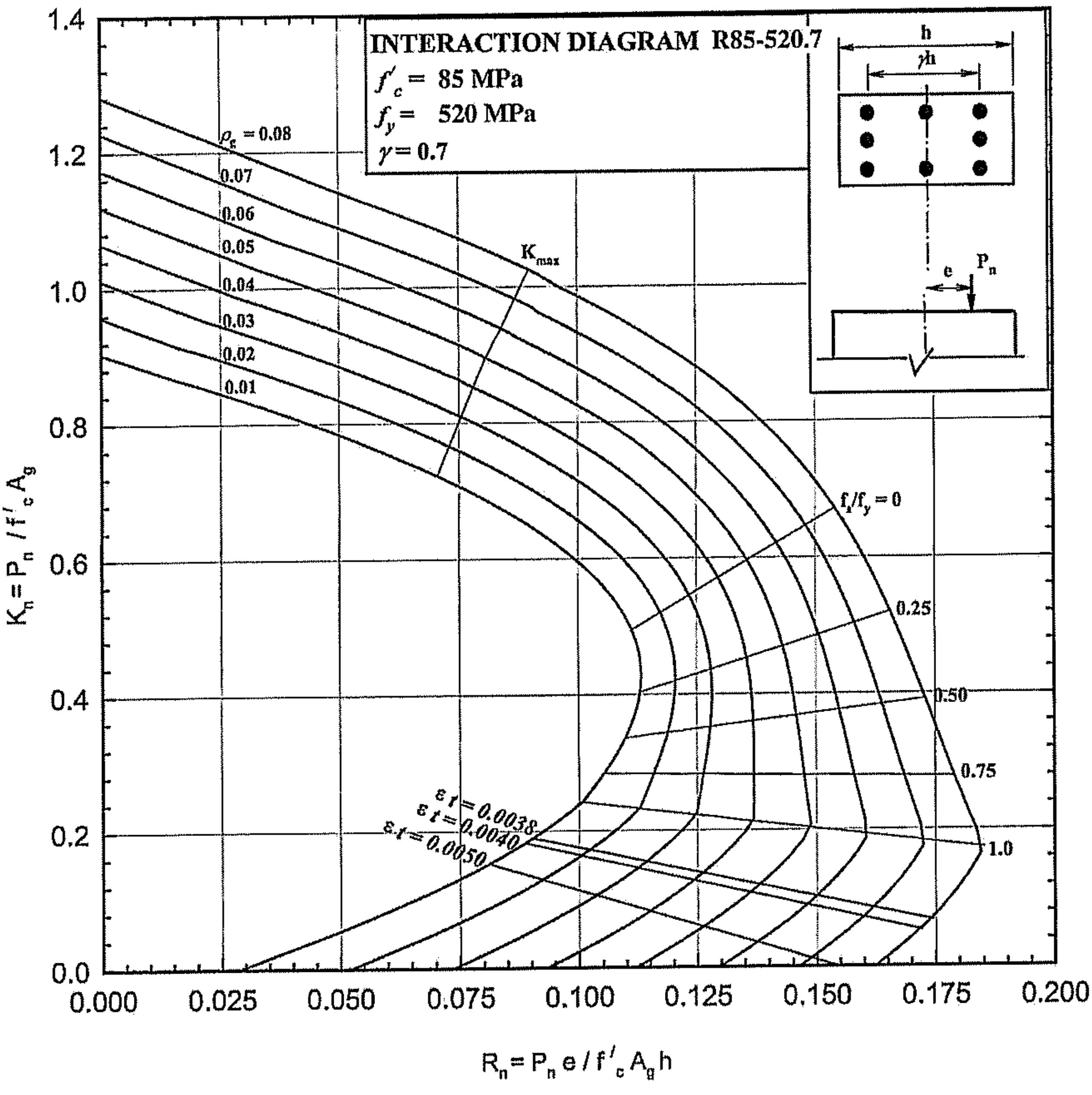
# COLUMNS 3.5.4 - Nominal load-moment strength interaction diagram, R65-520.9



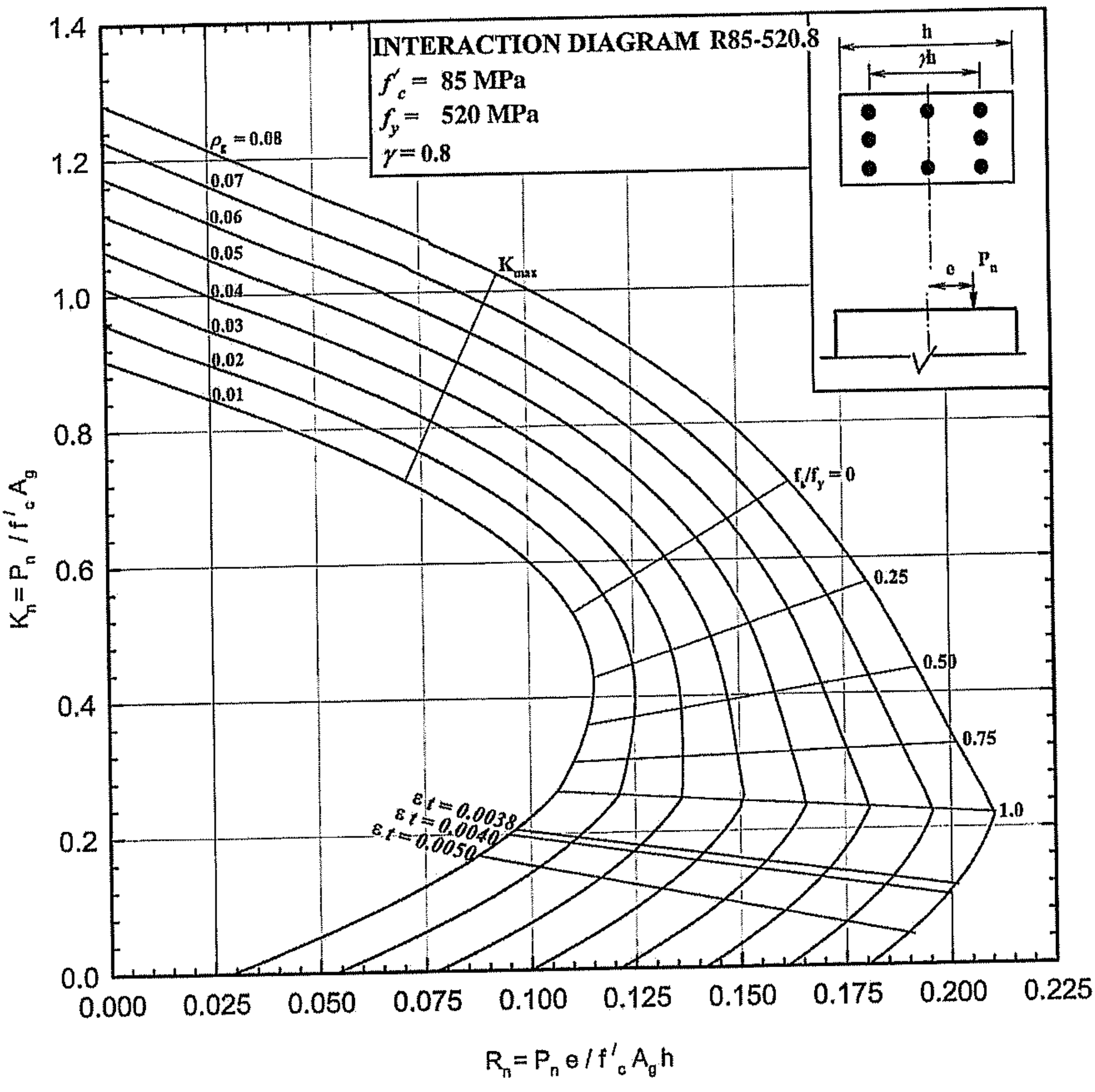
## COLUMNS 3.6.1 - Nominal load-moment strength interaction diagram, R85-520.6



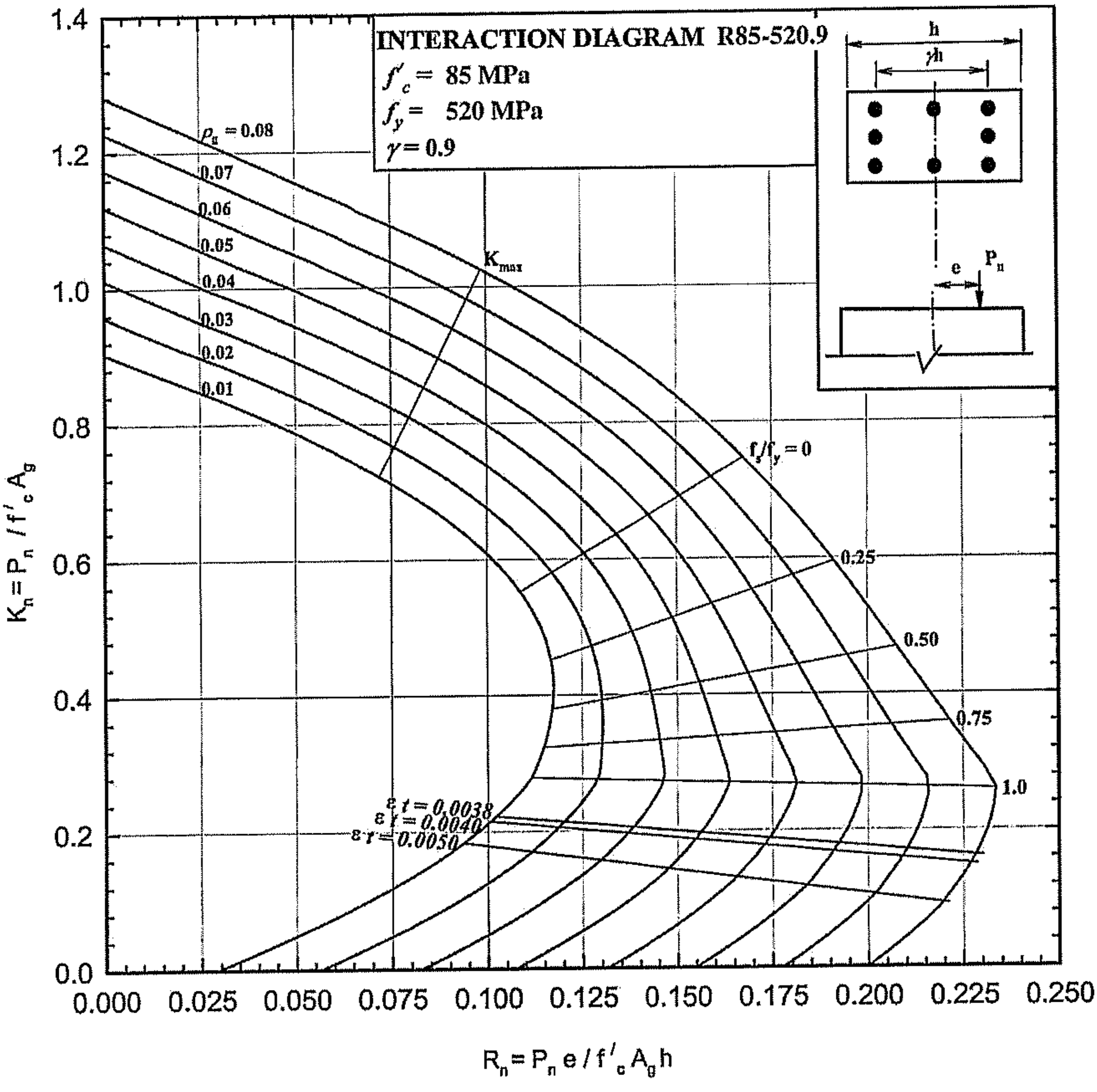
**COLUMNS 3.6.2 - Nominal load-moment strength interaction diagram, R85-520.7**



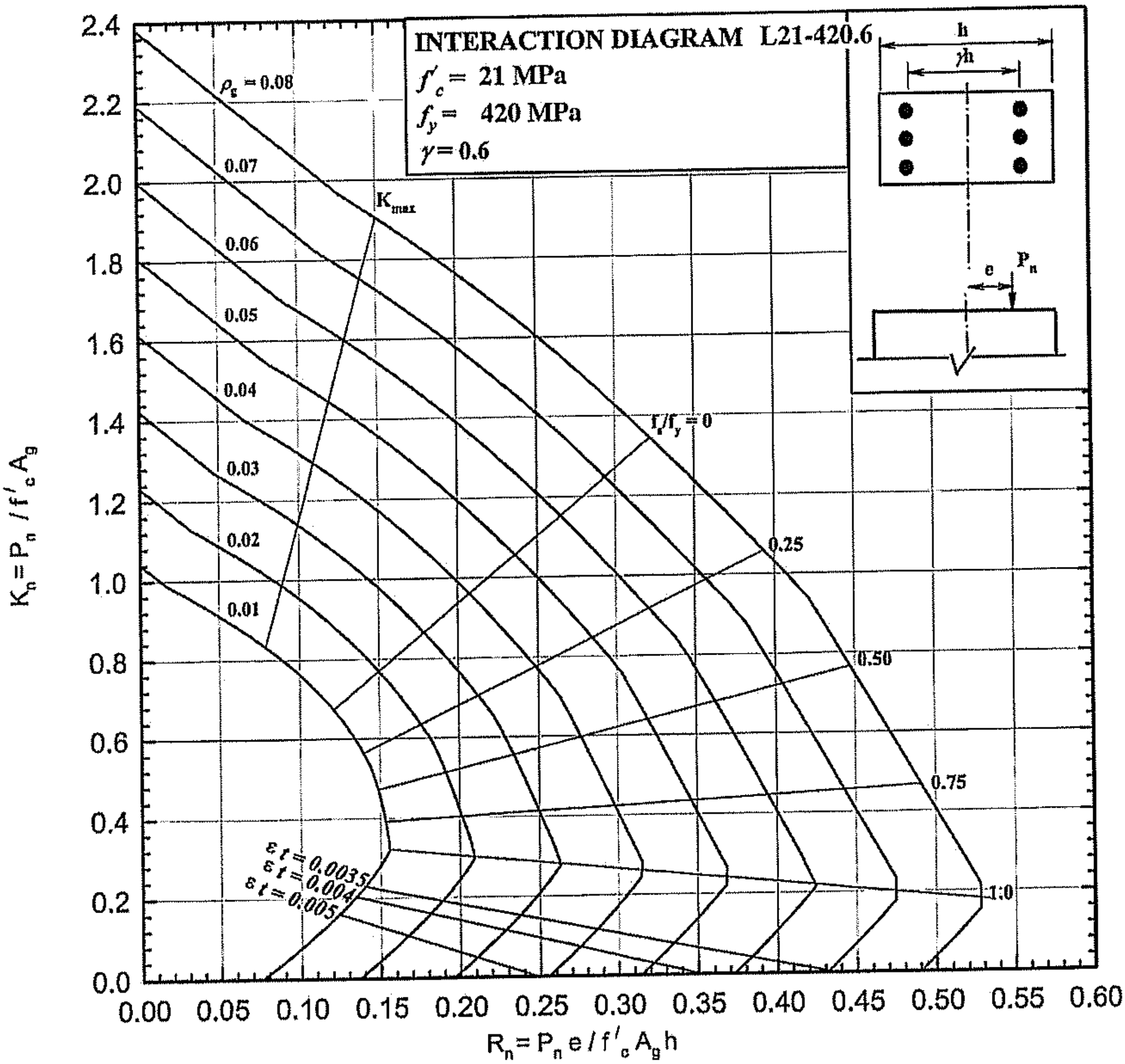
# COLUMNS 3.6.3 - Nominal load-moment strength interaction diagram, R85-520.8



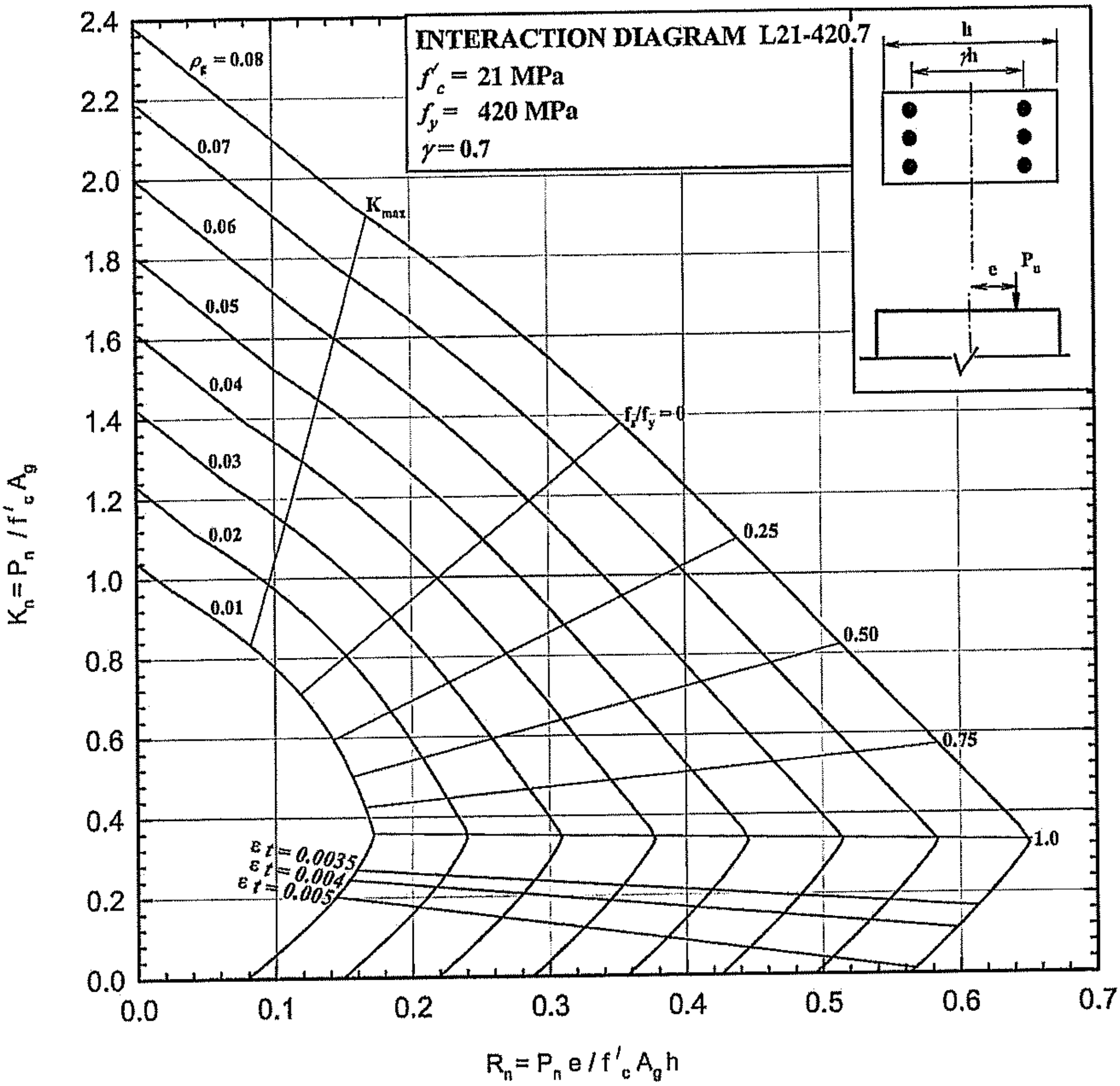
**COLUMNS 3.6.4 - Nominal load-moment strength interaction diagram, R85-520.9**



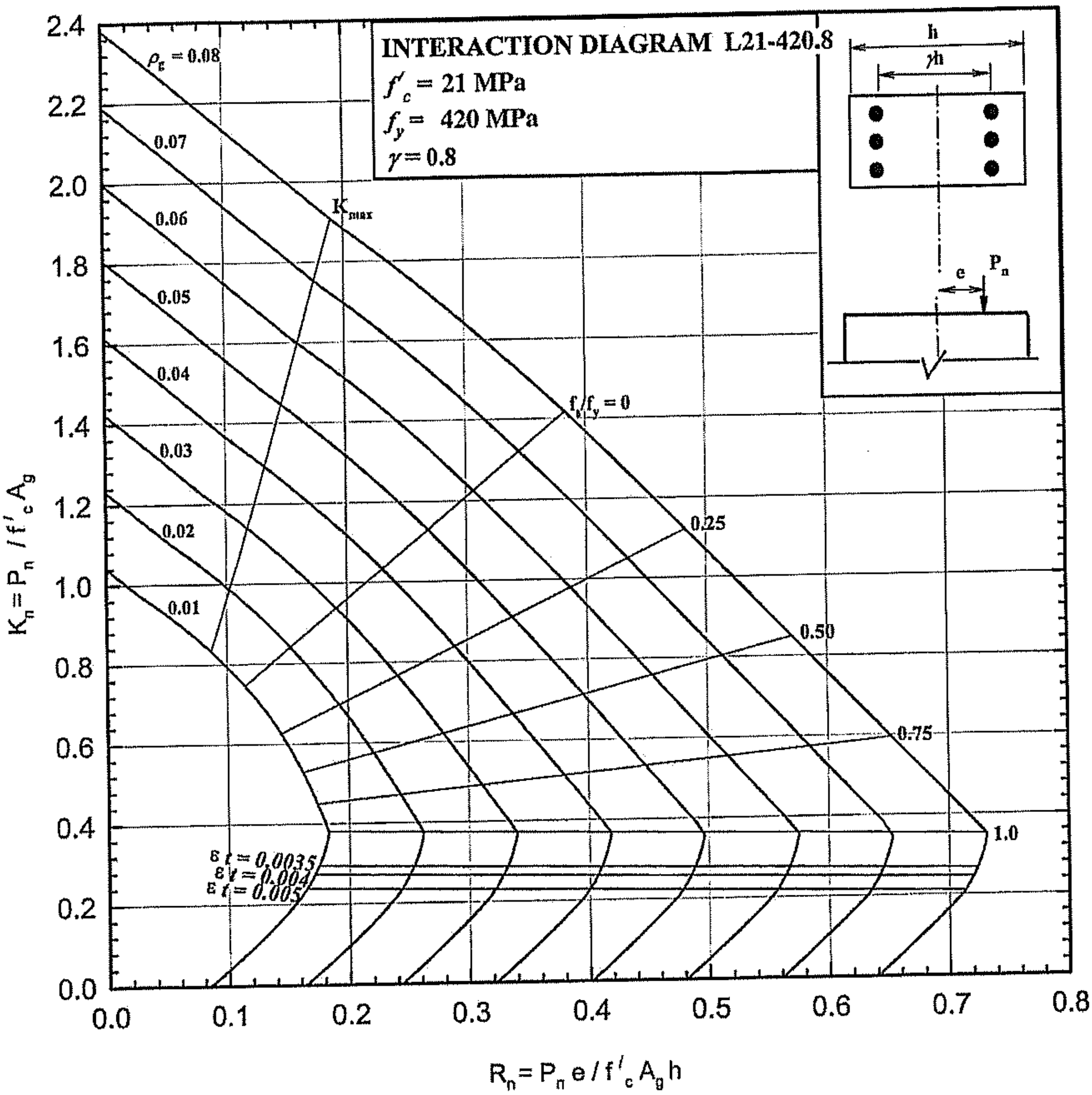
## COLUMNS 3.7.1 - Nominal load-moment strength interaction diagram, L21-420.6



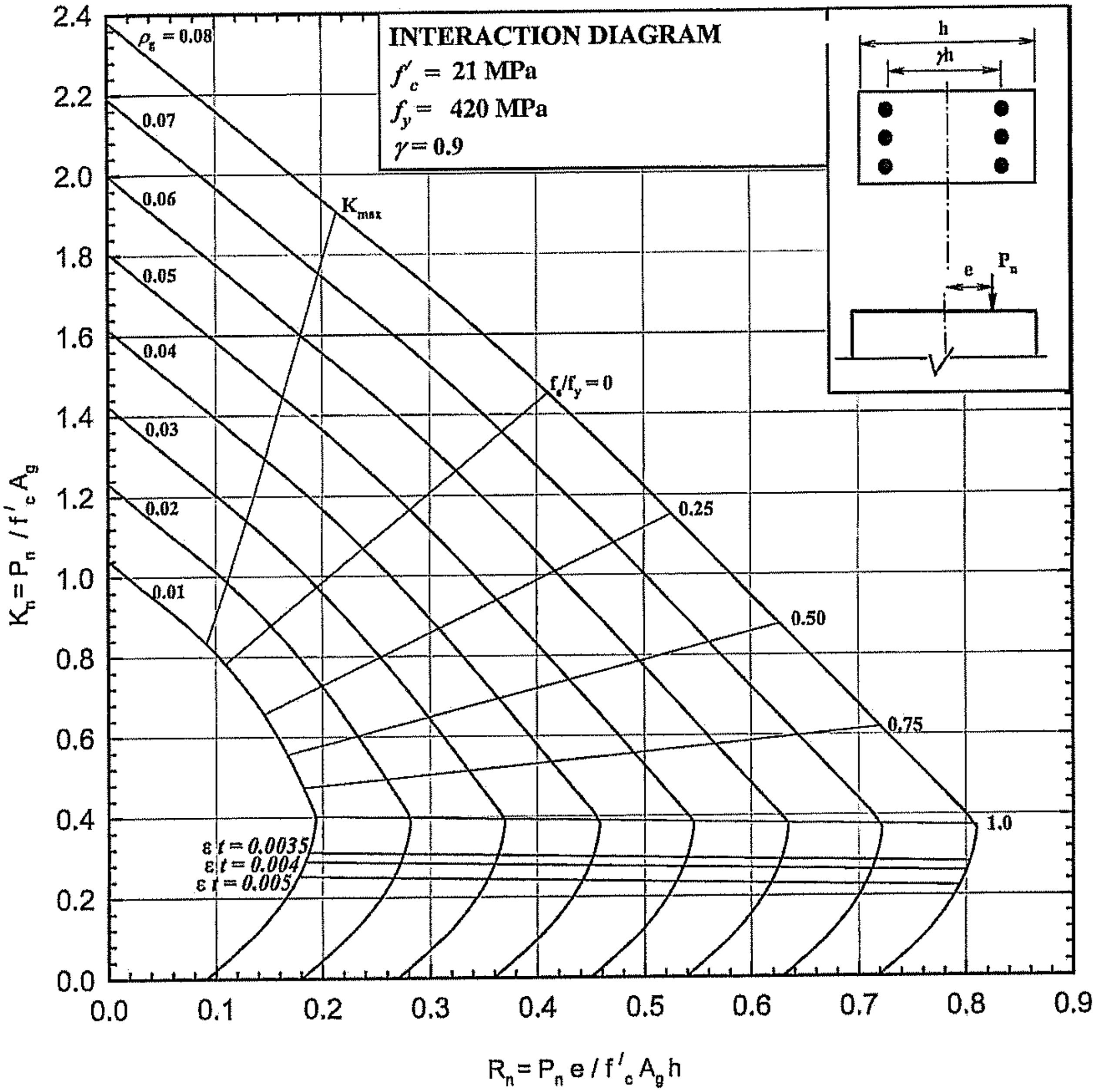
## COLUMNS 3.7.2 - Nominal load-moment strength interaction diagram, L21-420.7



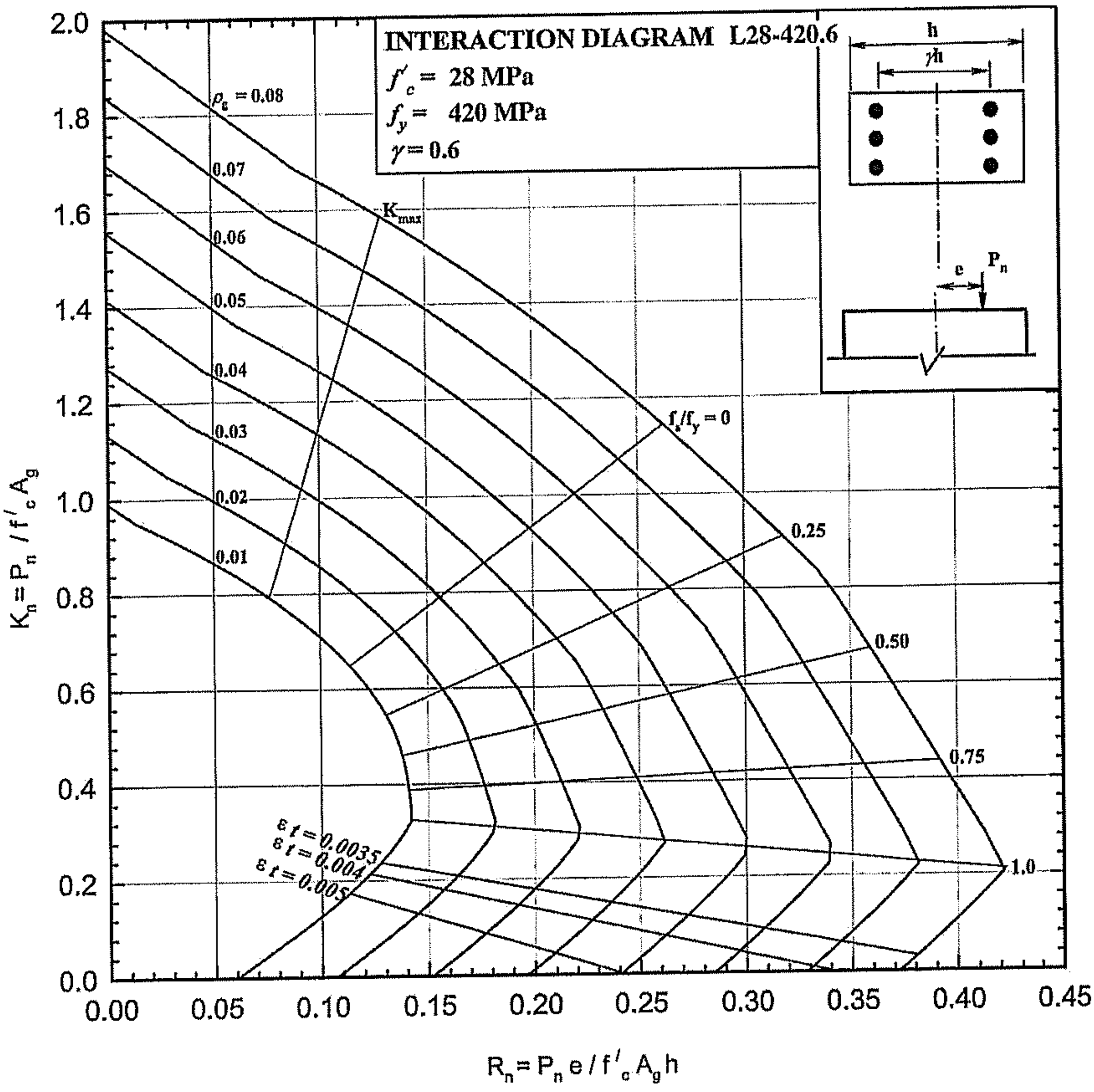
## COLUMNS 3.7.3 - Nominal load-moment strength interaction diagram, L21-420.8



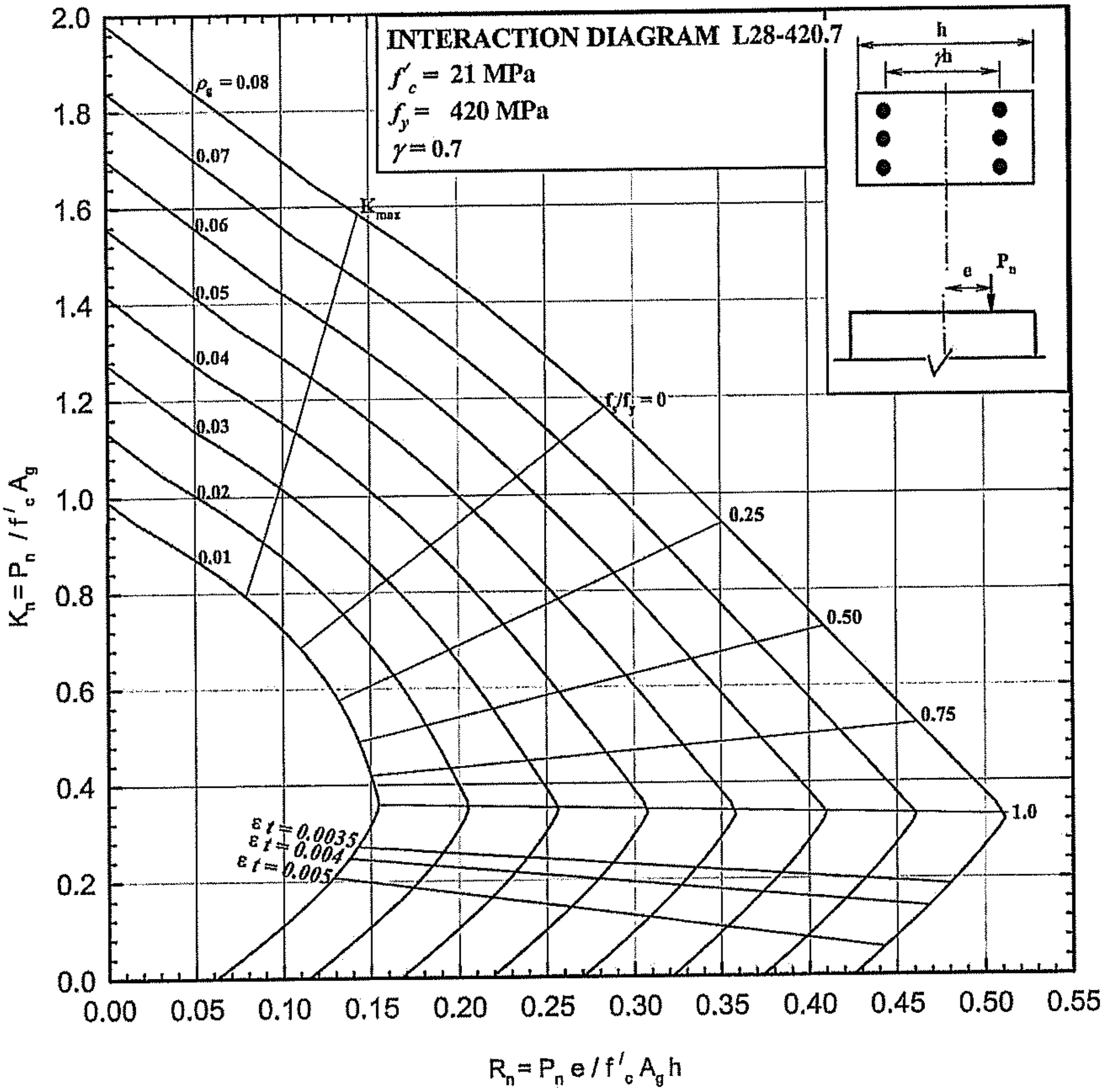
**COLUMNS 3.7.4 - Nominal load-moment strength interaction diagram, L21-420.9**



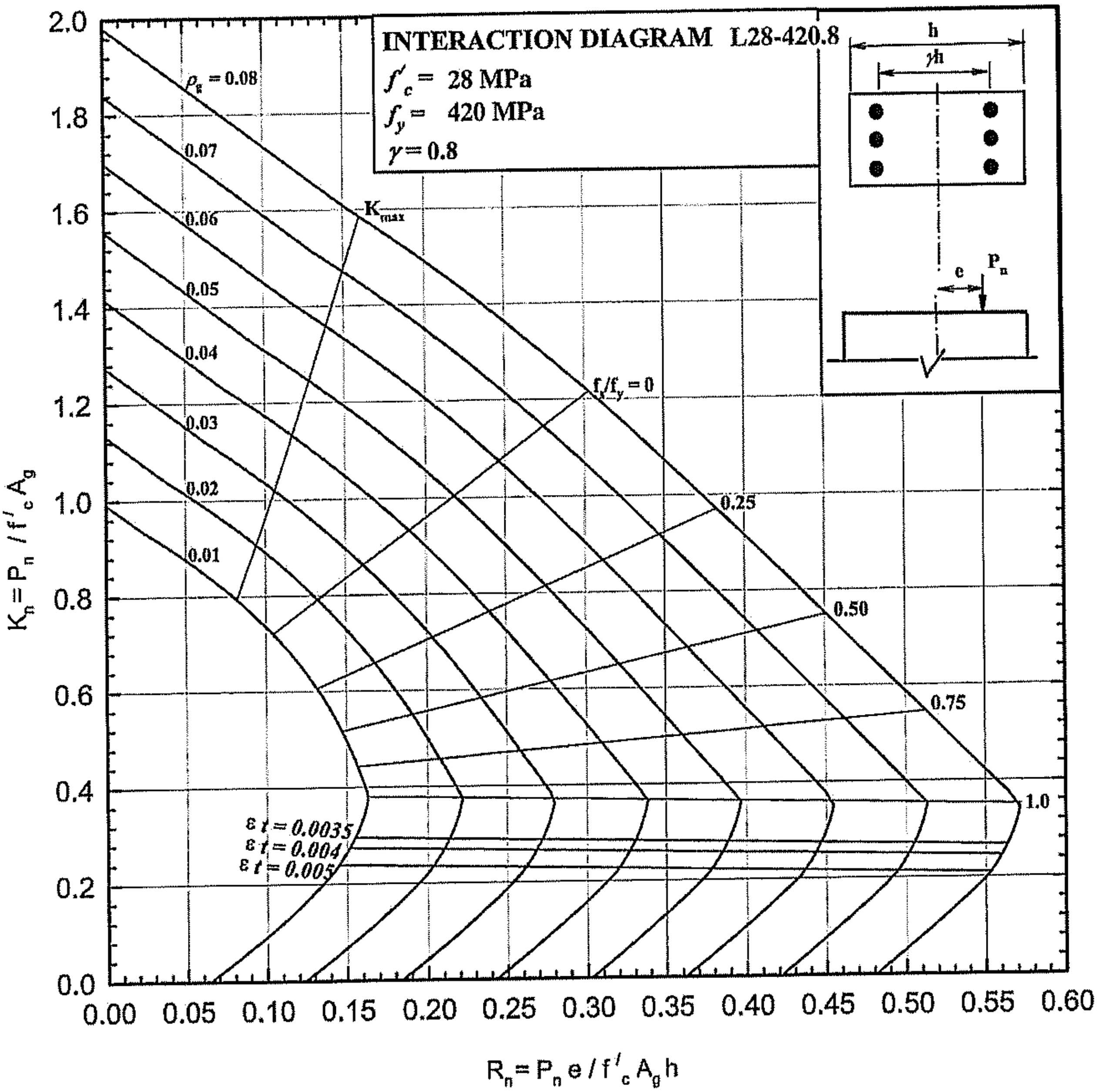
## COLUMNS 3.8.1 - Nominal load-moment strength interaction diagram, L28-420.6



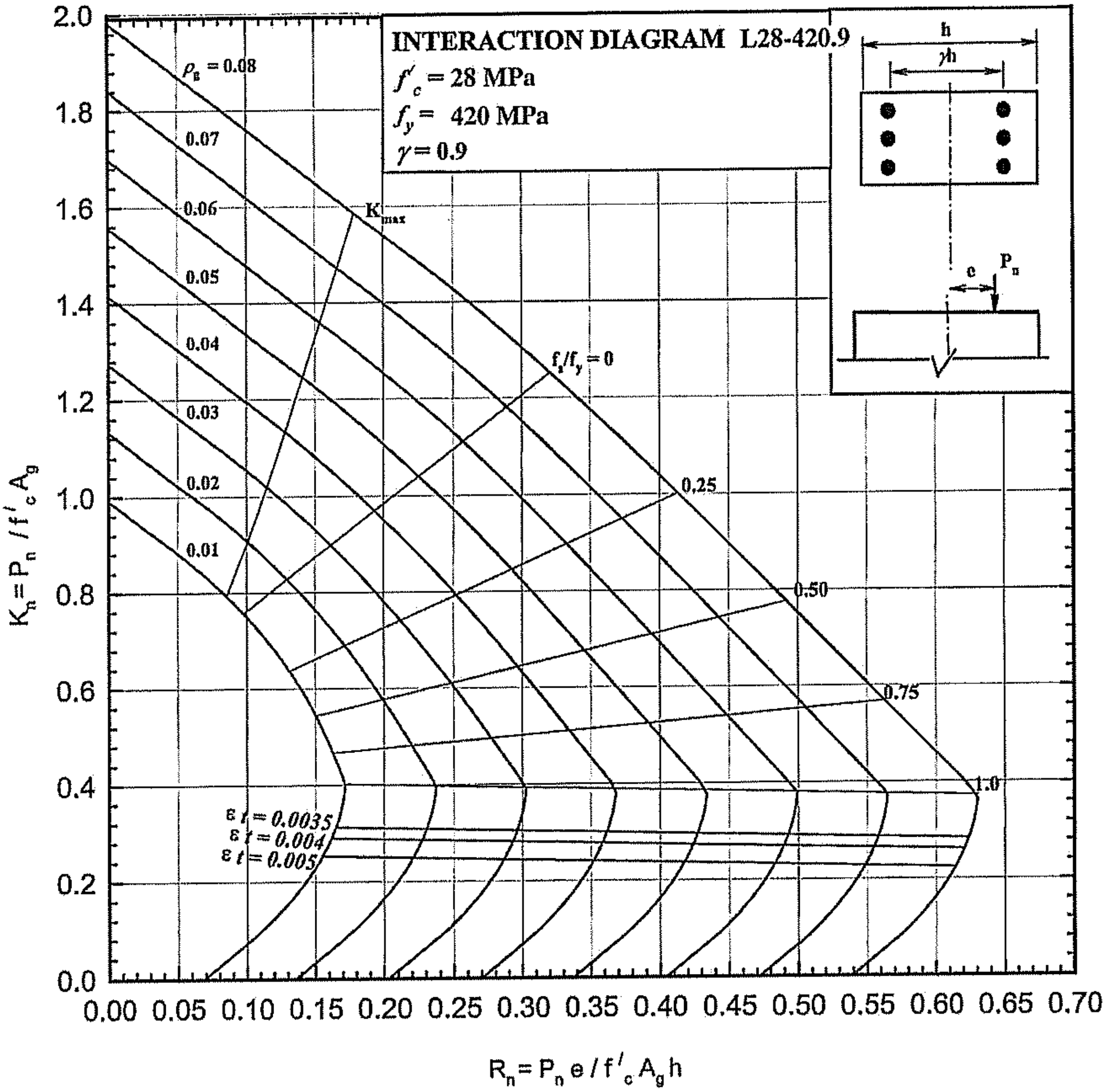
## COLUMNS 3.8.2 - Nominal load-moment strength interaction diagram, L28-420.7



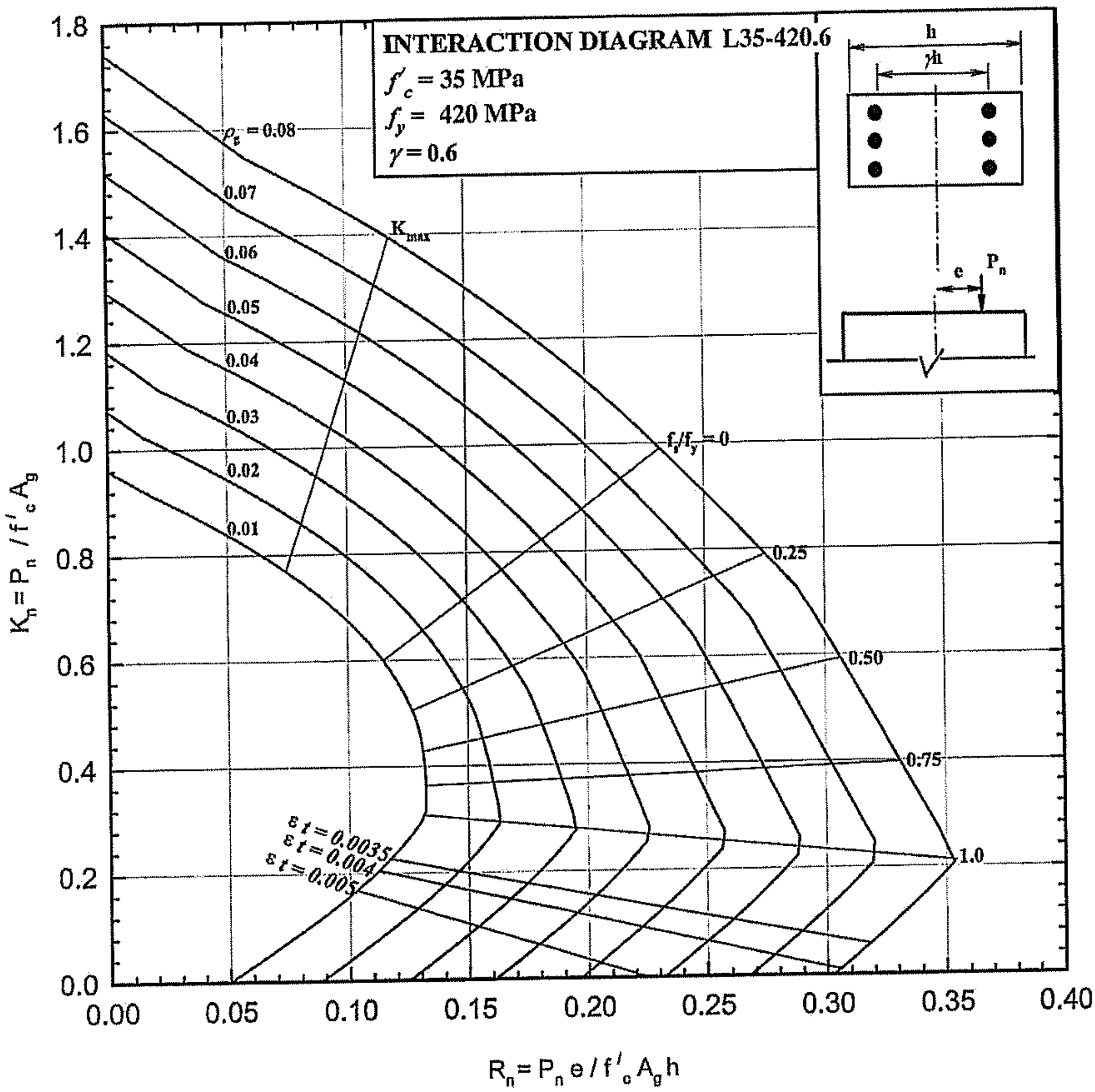
## COLUMNS 3.8.3 - Nominal load-moment strength interaction diagram, L28-420.8



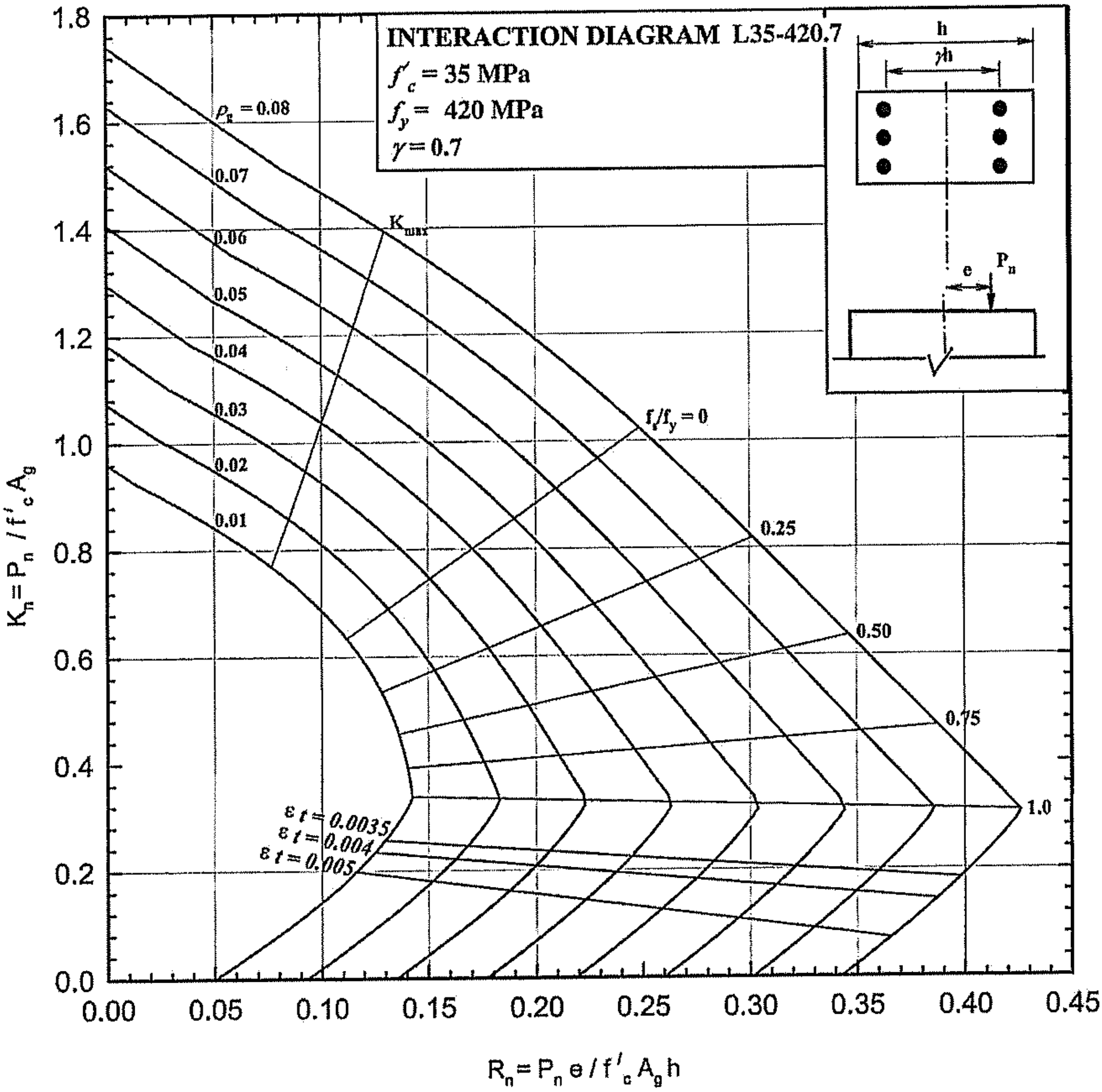
## COLUMNS 3.8.4 - Nominal load-moment strength interaction diagram, L28-420.9



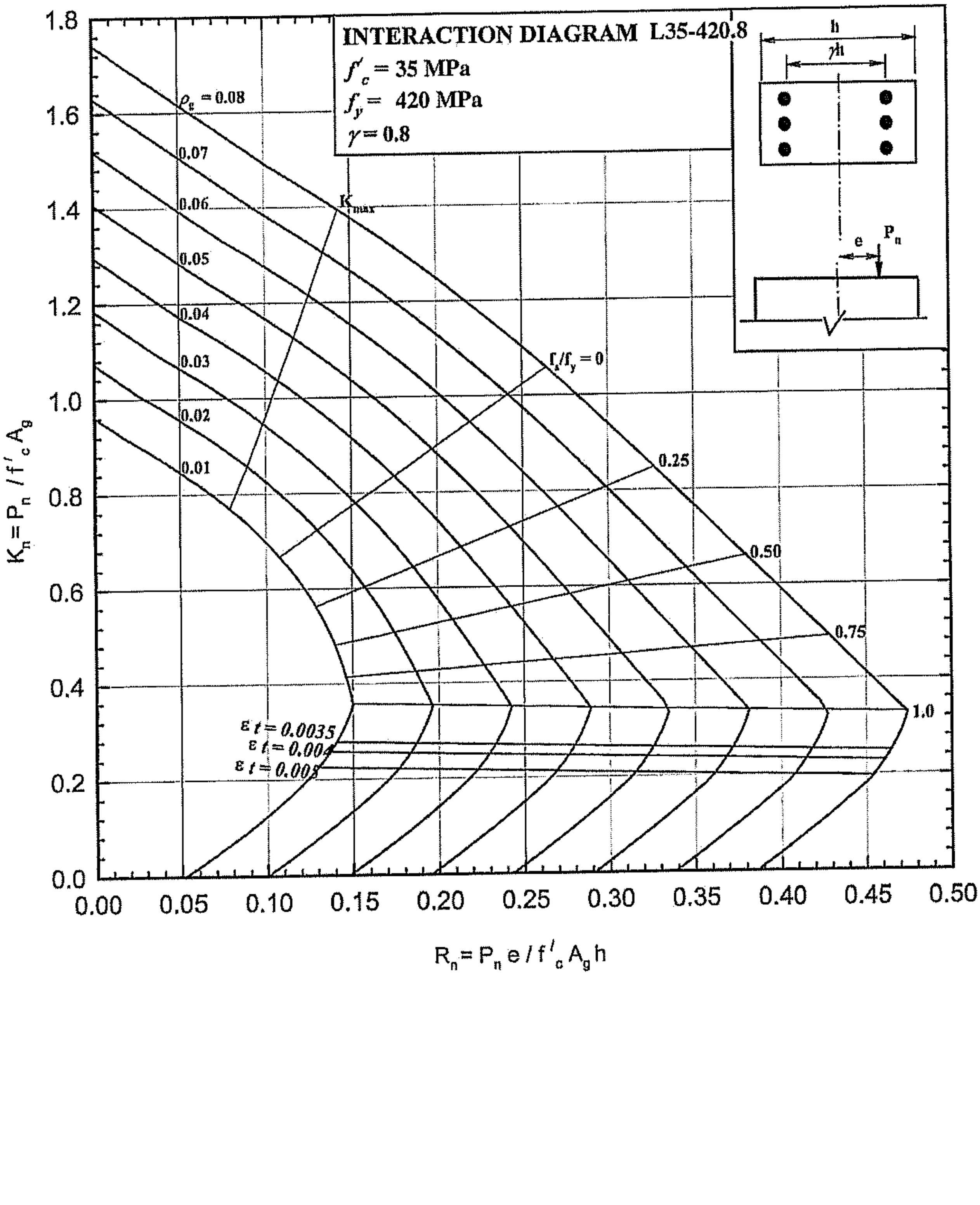
## COLUMNS 3.9.1 - Nominal load-moment strength interaction diagram, L35-420.6



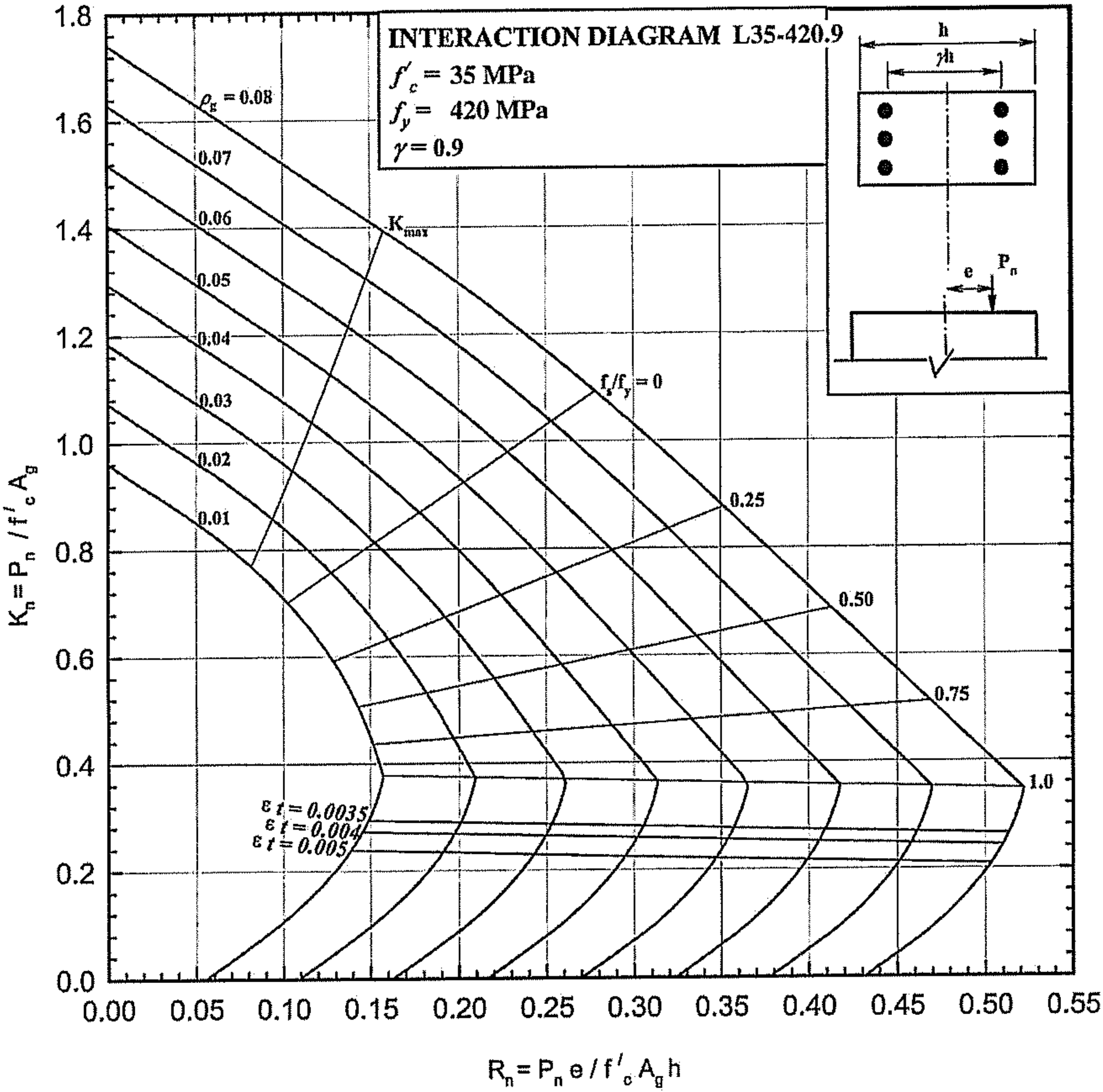
## COLUMNS 3.9.2 - Nominal load-moment strength interaction diagram, L35-420.7



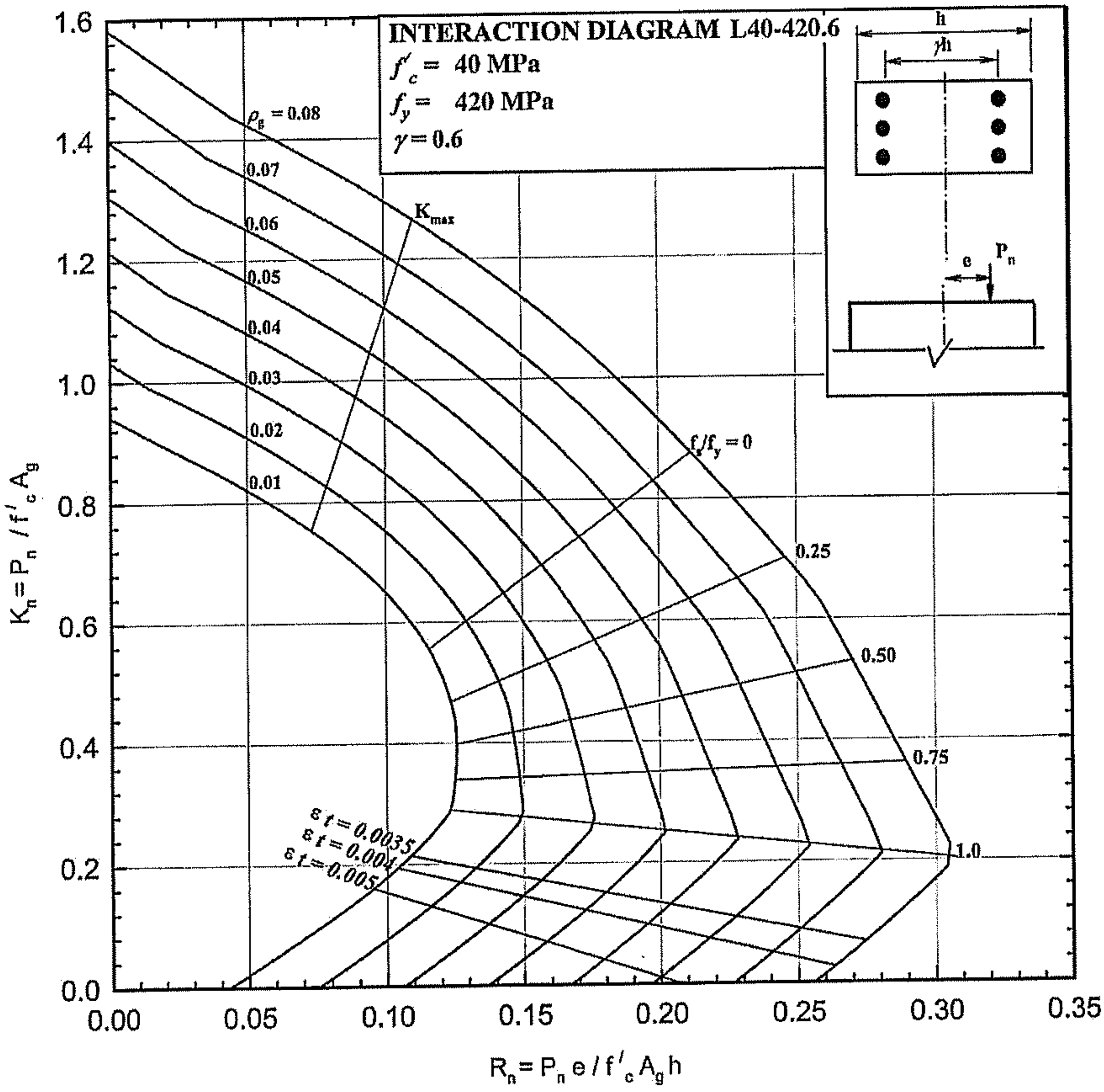
## COLUMNS 3.9.3 - Nominal load-moment strength interaction diagram, L35-420.8



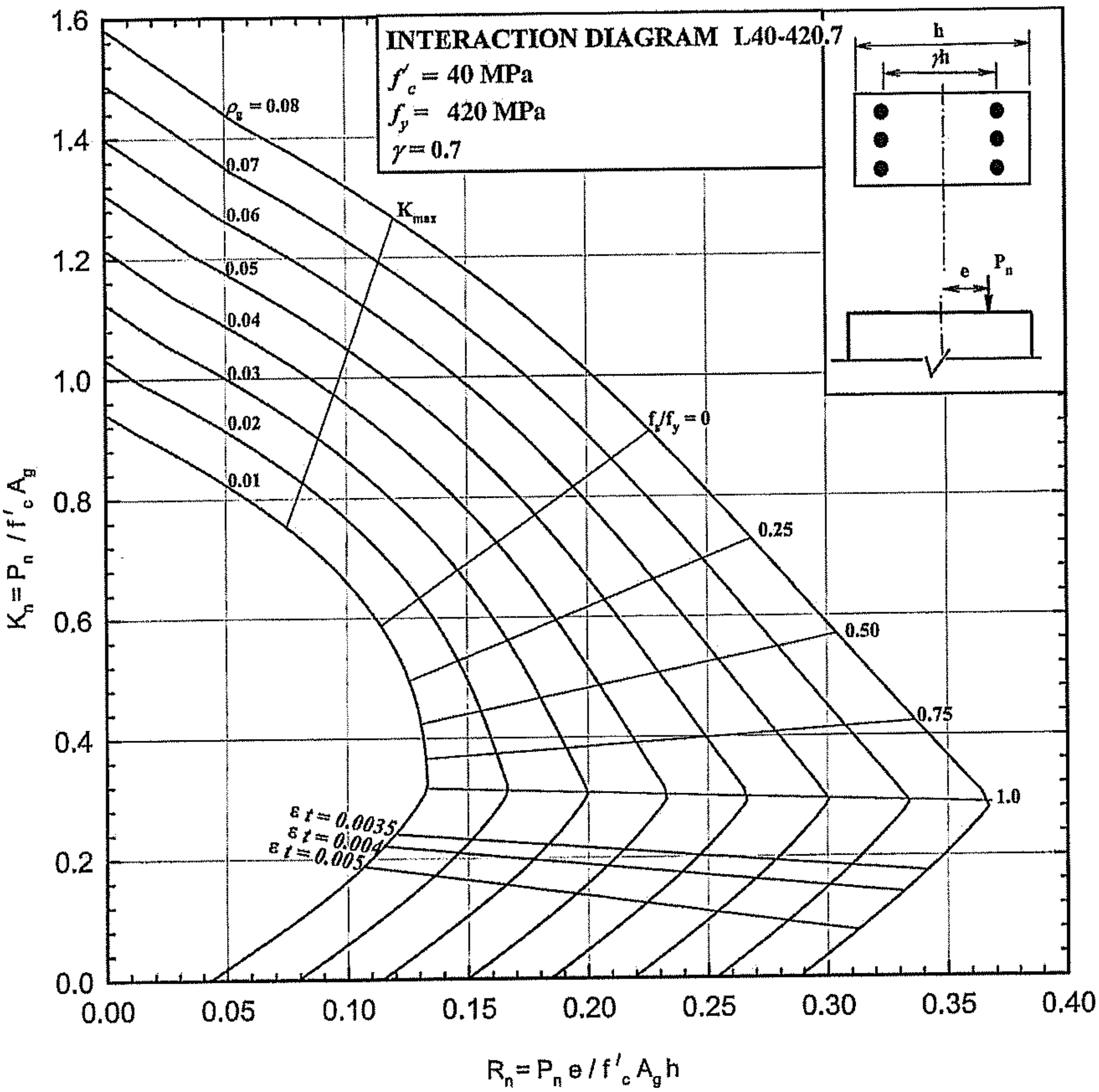
**COLUMNS 3.9.4 - Nominal load-moment strength interaction diagram, L35-420.9**



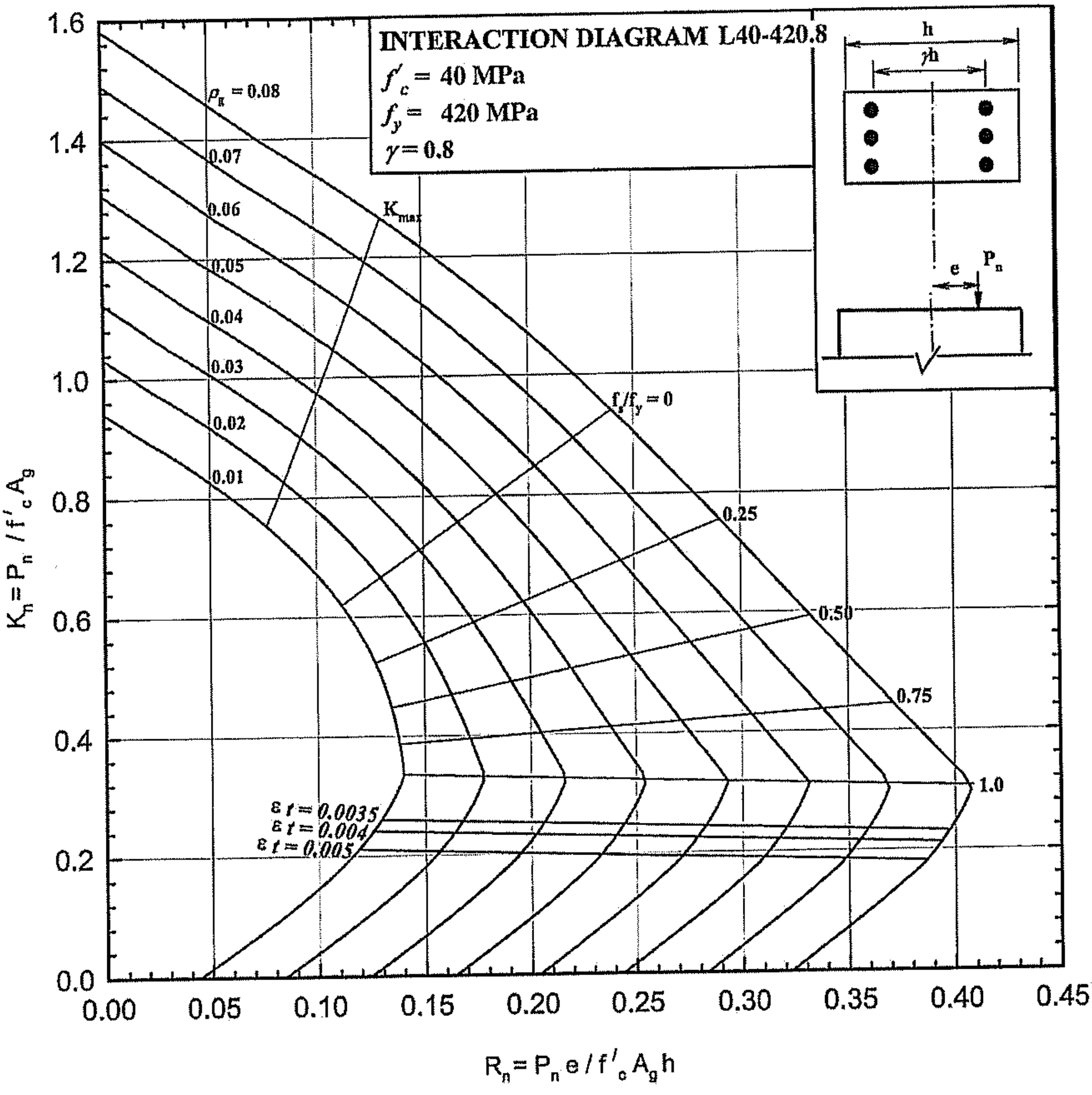
## COLUMNS 3.10.1 - Nominal load-moment strength interaction diagram, L40-420.6



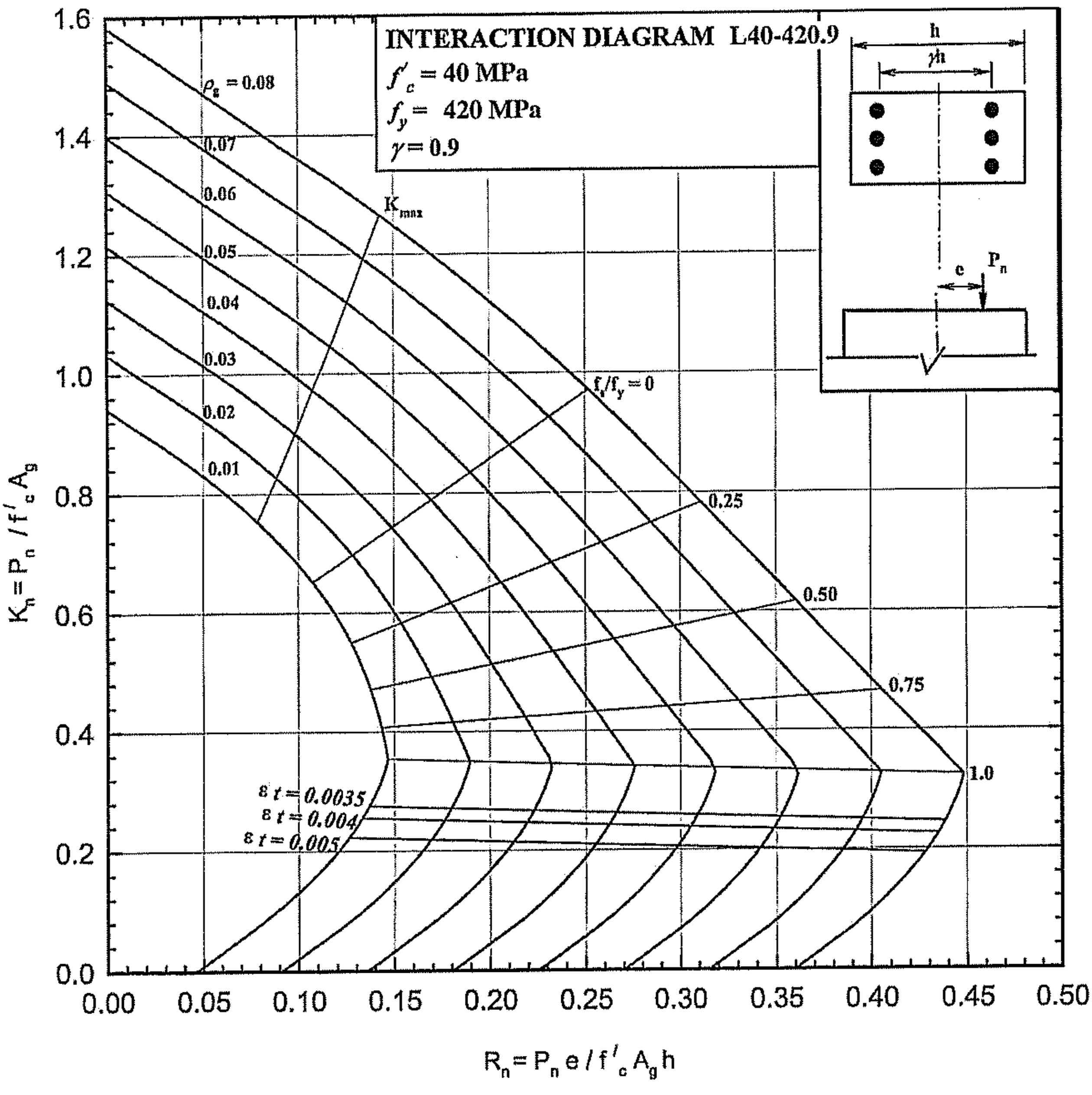
## COLUMNS 3.10.2 - Nominal load-moment strength interaction diagram, L40-420.7



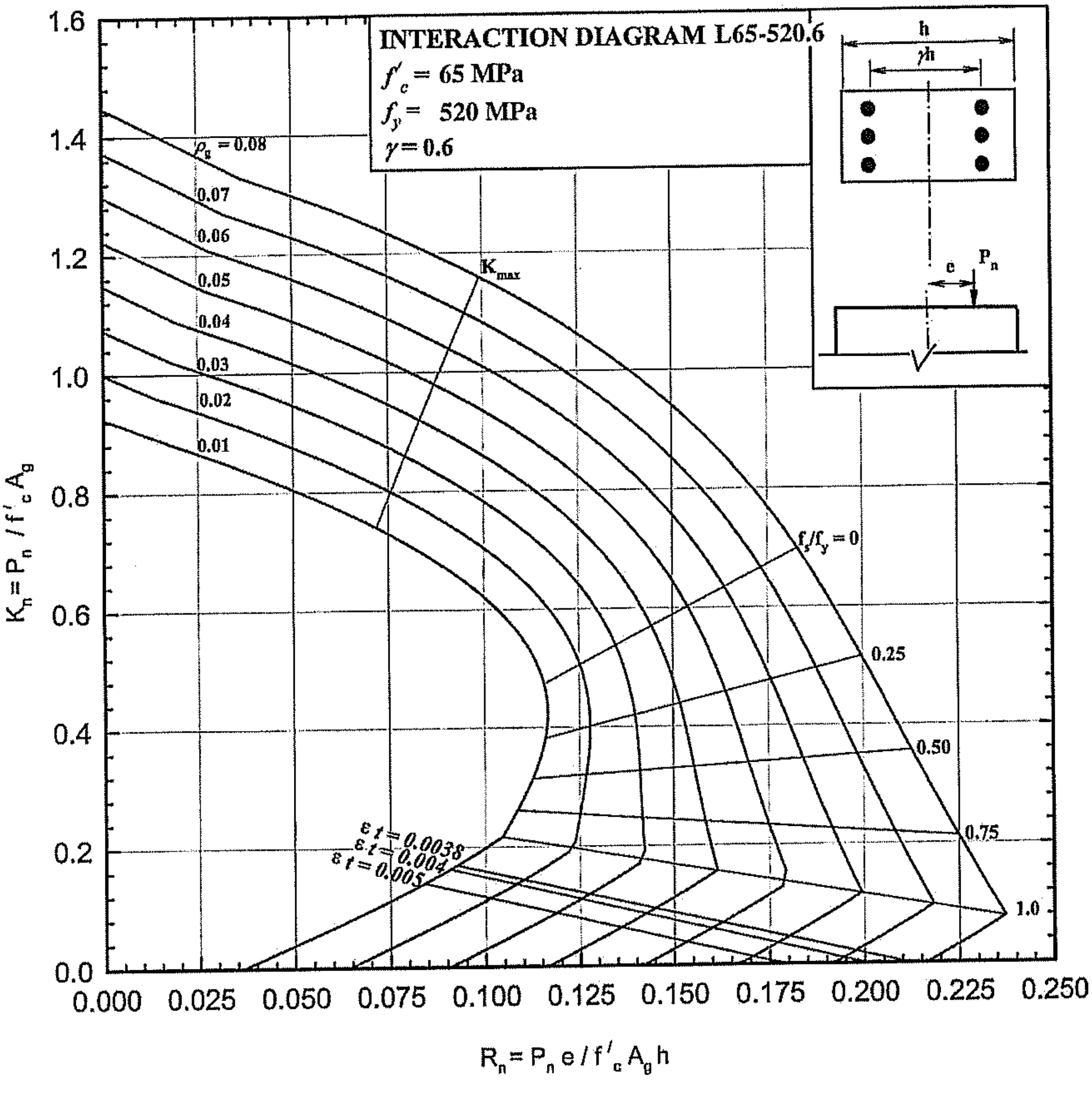
## COLUMNS 3.10.3 - Nominal load-moment strength interaction diagram, L40-420.8



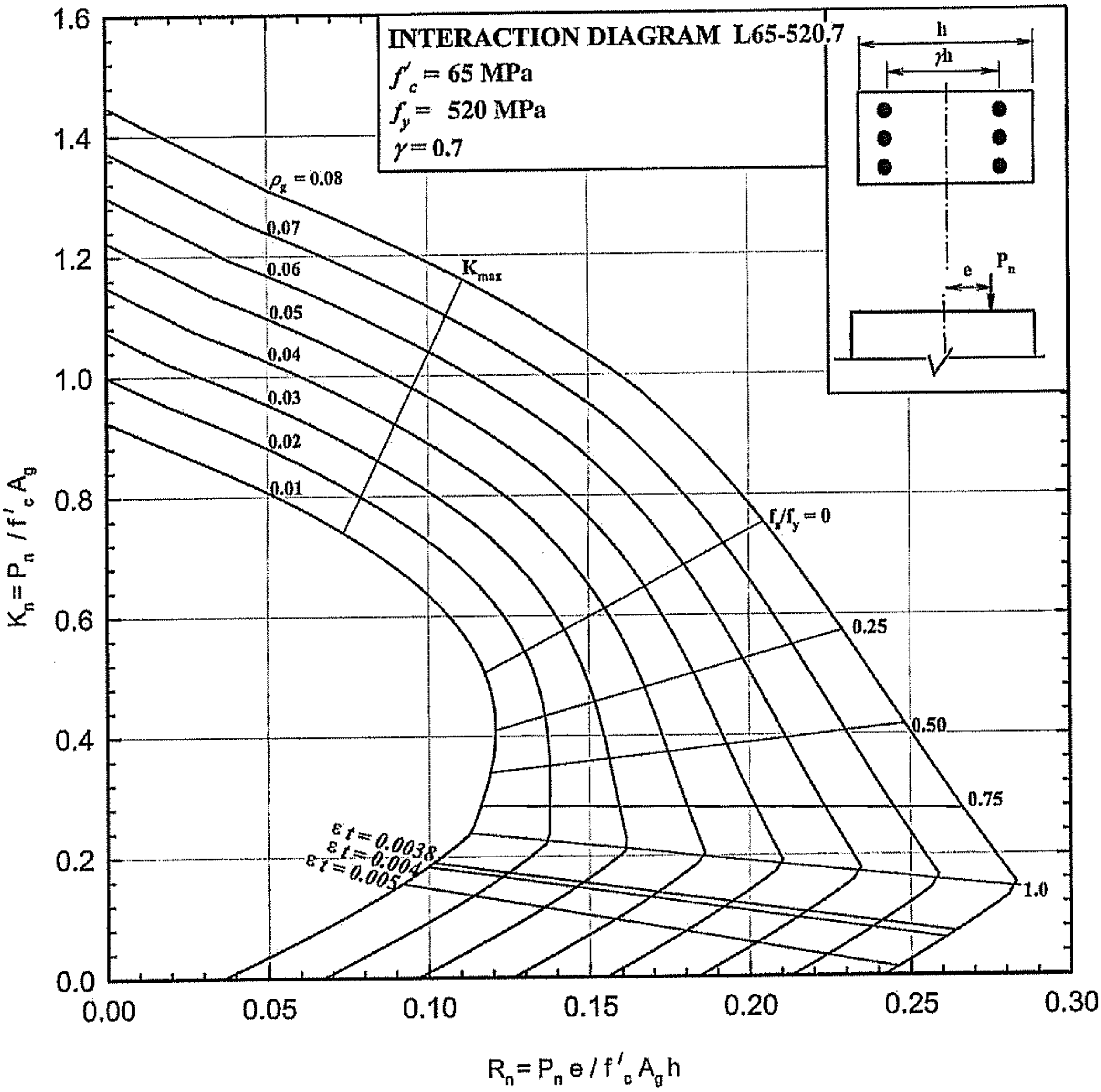
# COLUMNS 3.10.4 - Nominal load-moment strength interaction diagram, L40-420.9



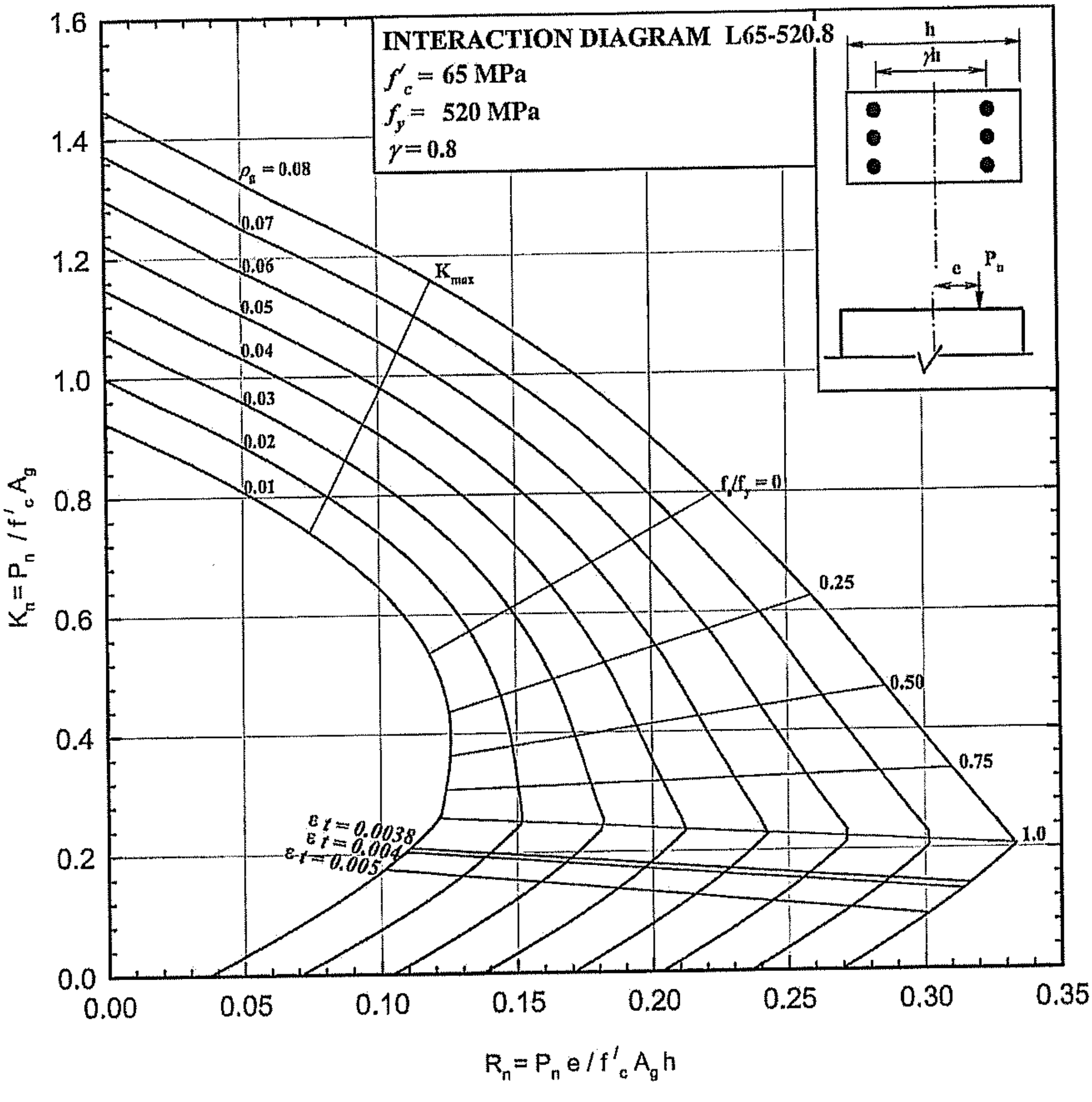
## COLUMNS 3.11.1 - Nominal load-moment strength interaction diagram, L65-520.6



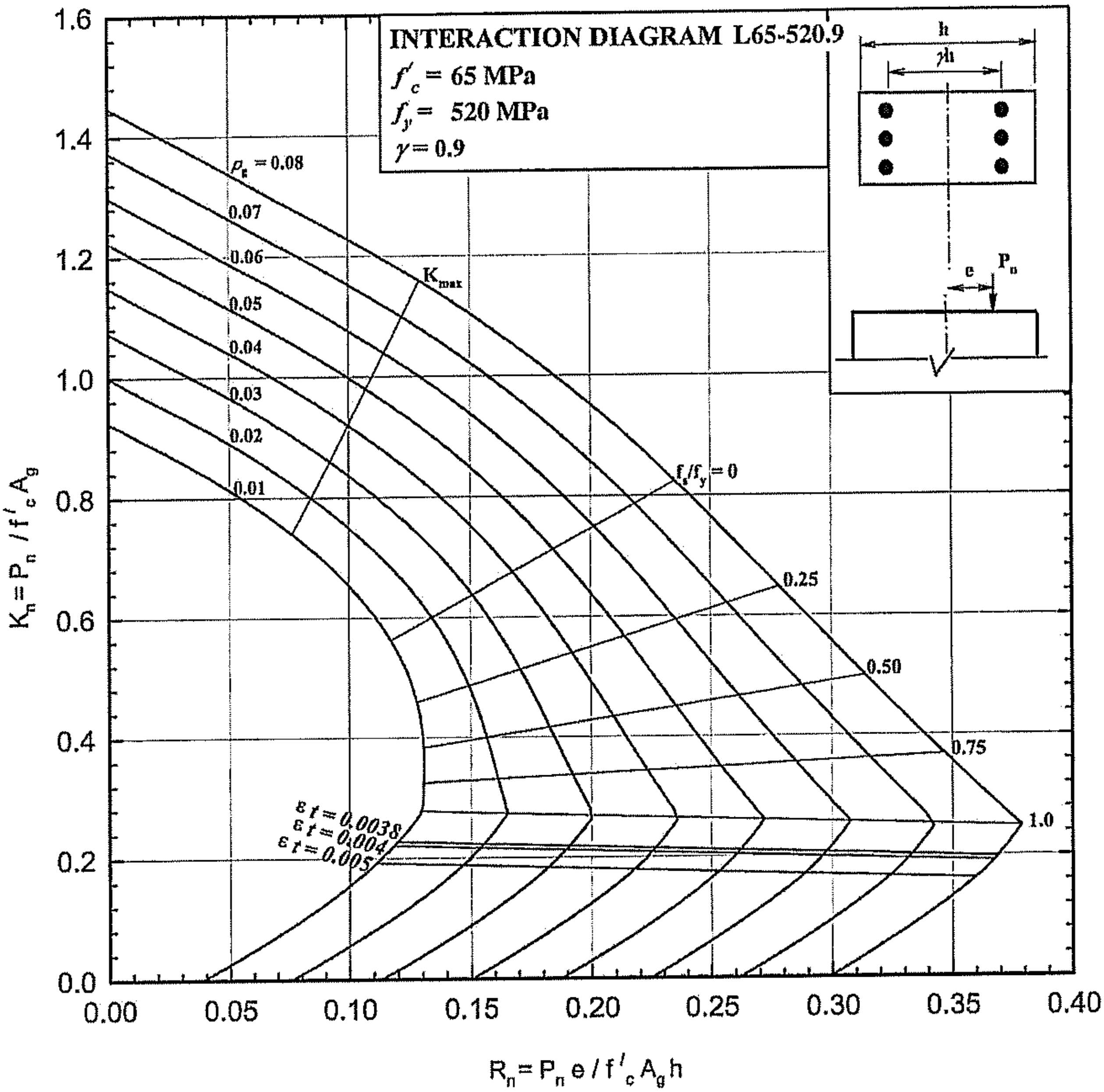
**COLUMNS 3.11.2 - Nominal load-moment strength interaction diagram, L65-520.7**



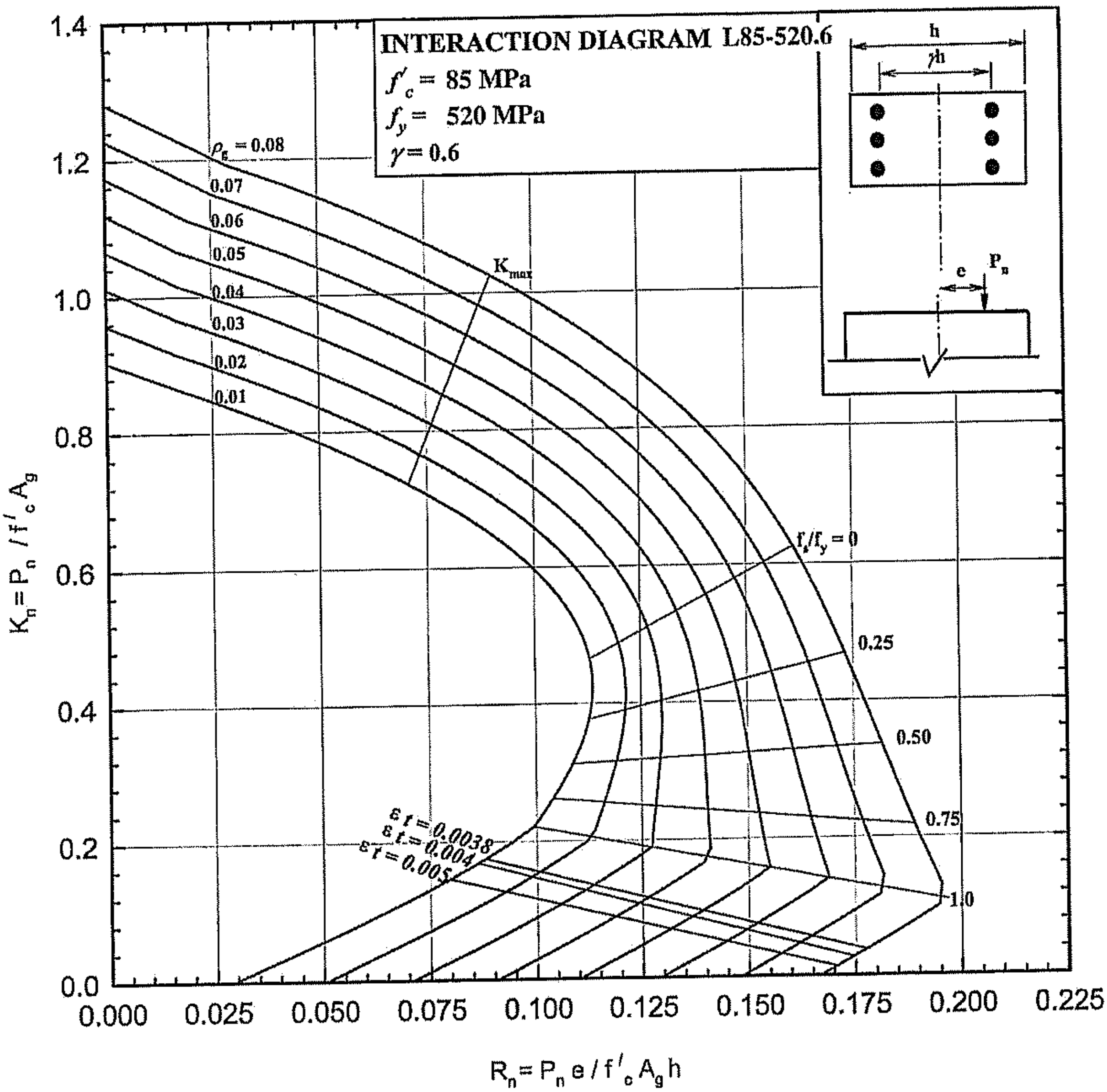
## COLUMNS 3.11.3 - Nominal load-moment strength interaction diagram, L65-520.8



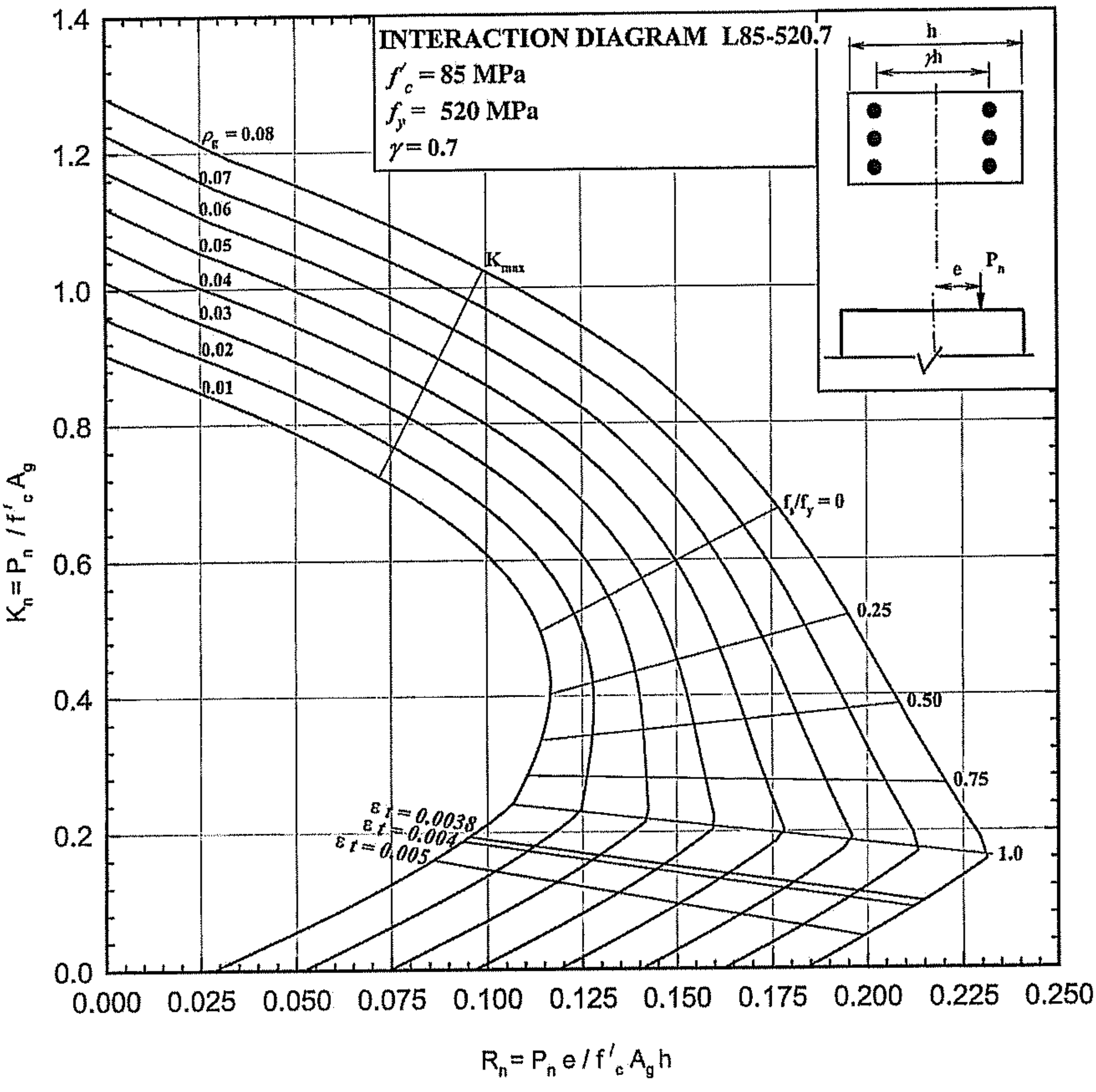
# COLUMNS 3.11.4 - Nominal load-moment strength interaction diagram, L65-520.9



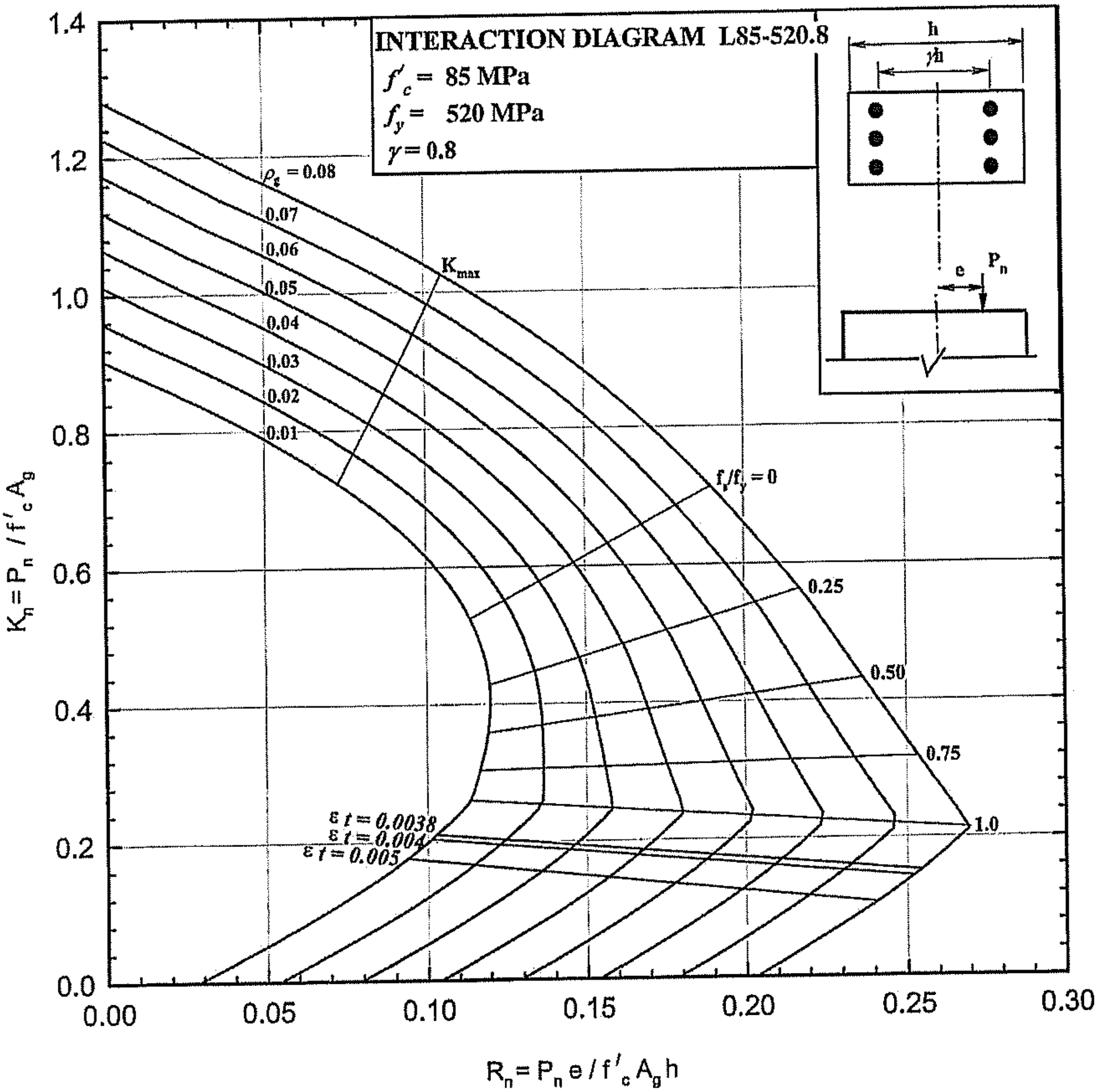
## COLUMNS 3.12.1 - Nominal load-moment strength interaction diagram, L85-520.6



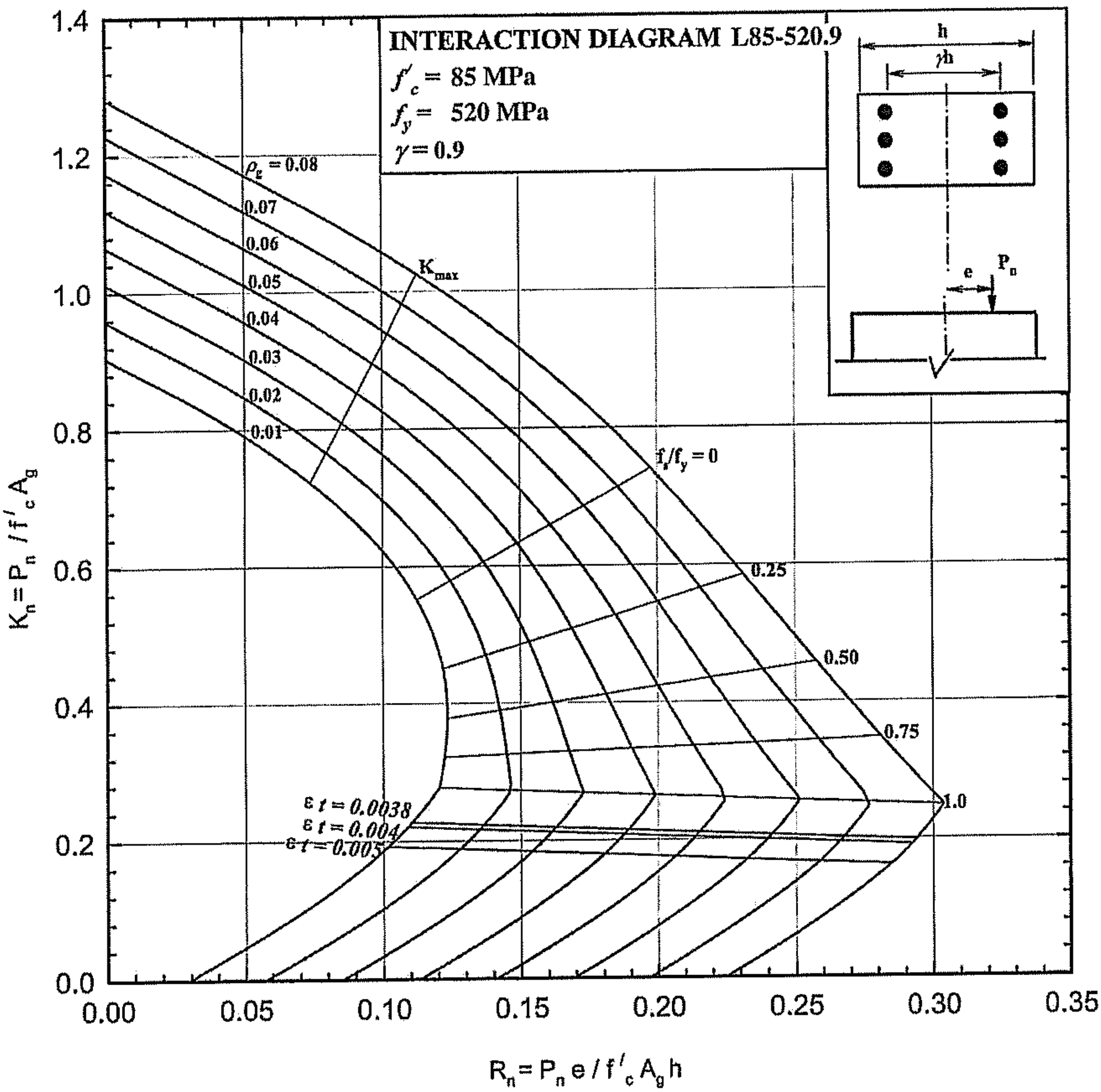
**COLUMNS 3.12.2 - Nominal load-moment strength interaction diagram, L85-520.7**



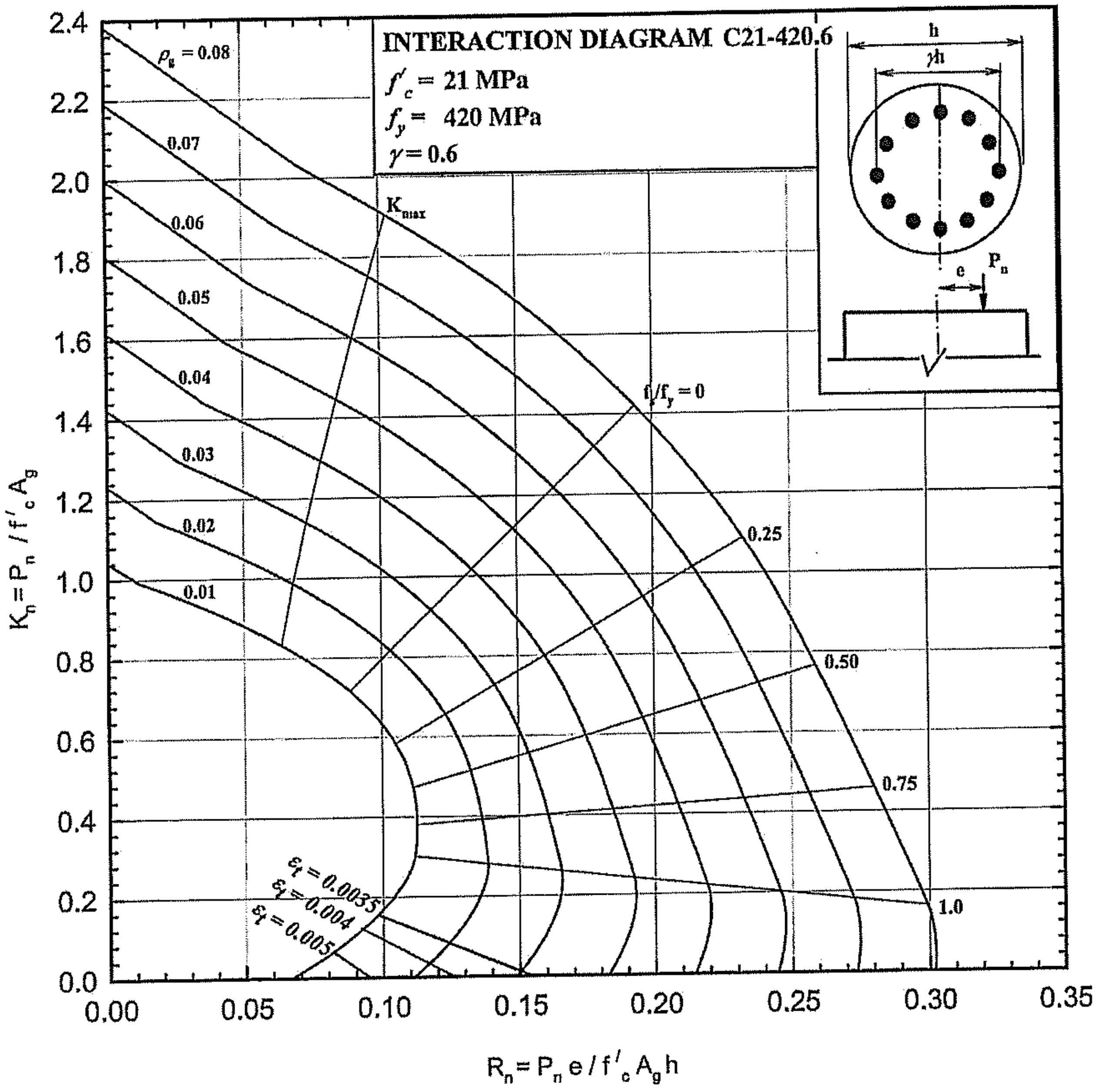
**COLUMNS 3.12.3 - Nominal load-moment strength interaction diagram, L85-520.8**



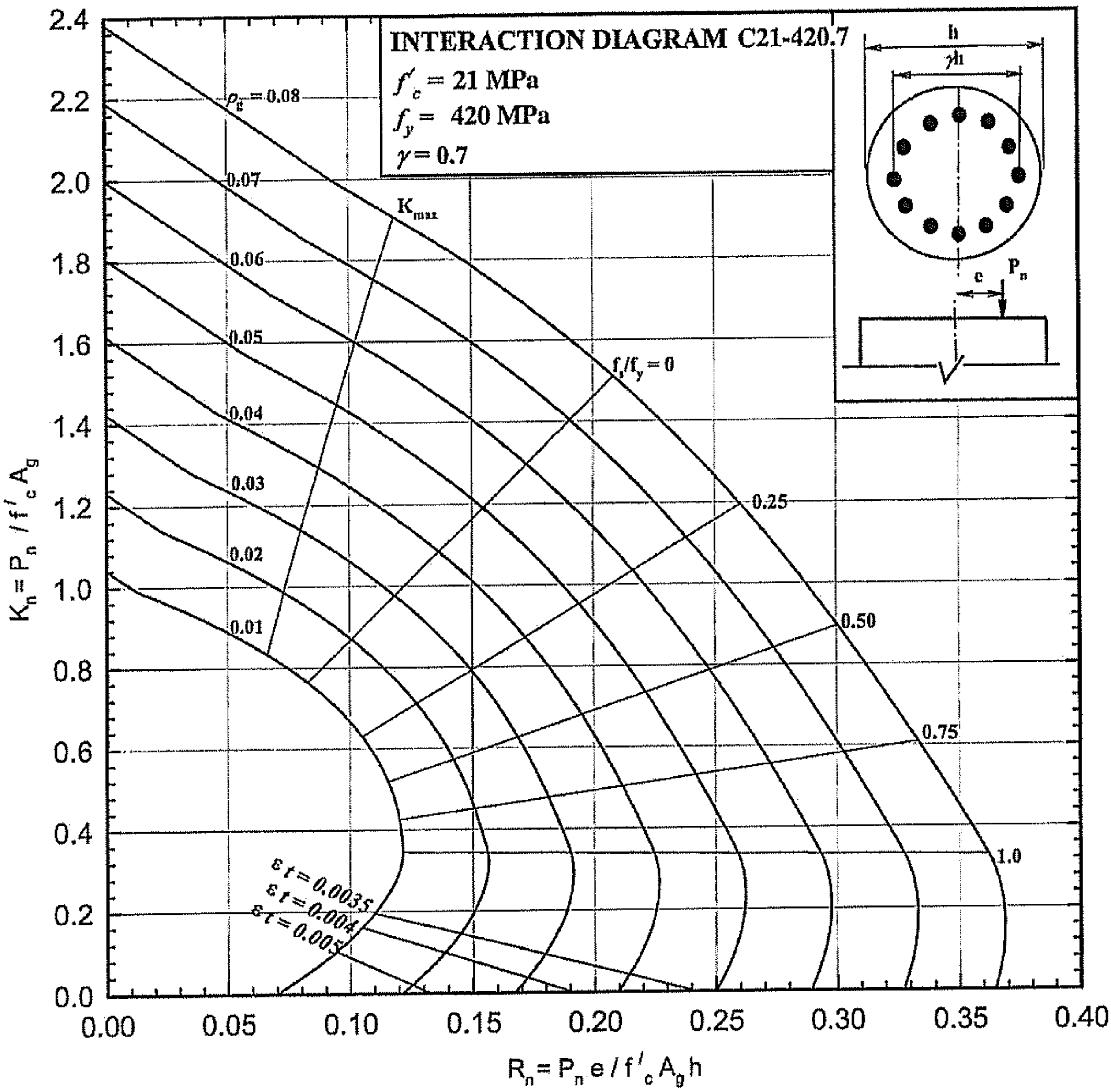
## COLUMNS 3.12.4 - Nominal load-moment strength interaction diagram, L85-520.9



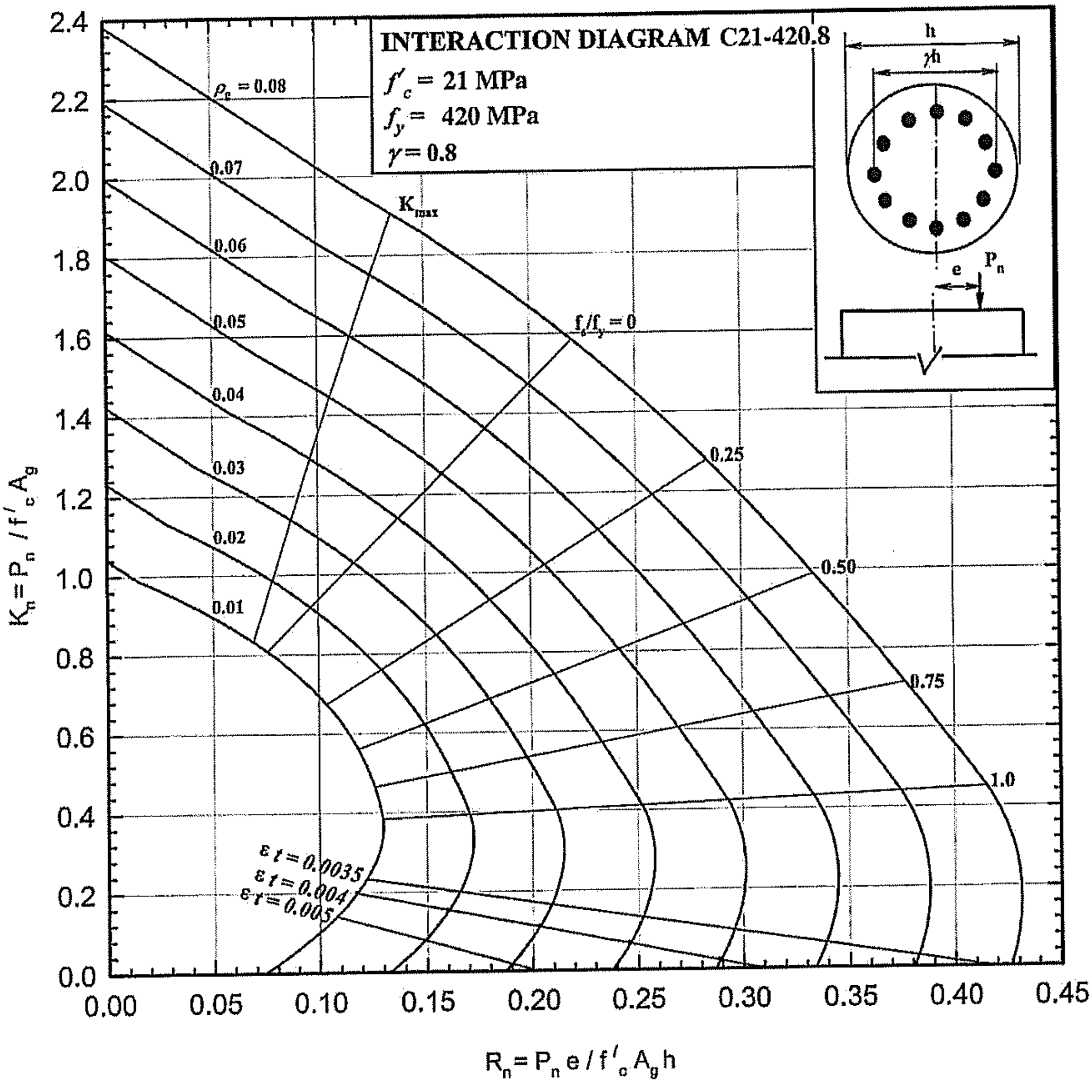
## COLUMNS 3.13.1 - Nominal load-moment strength interaction diagram, C21-420.6



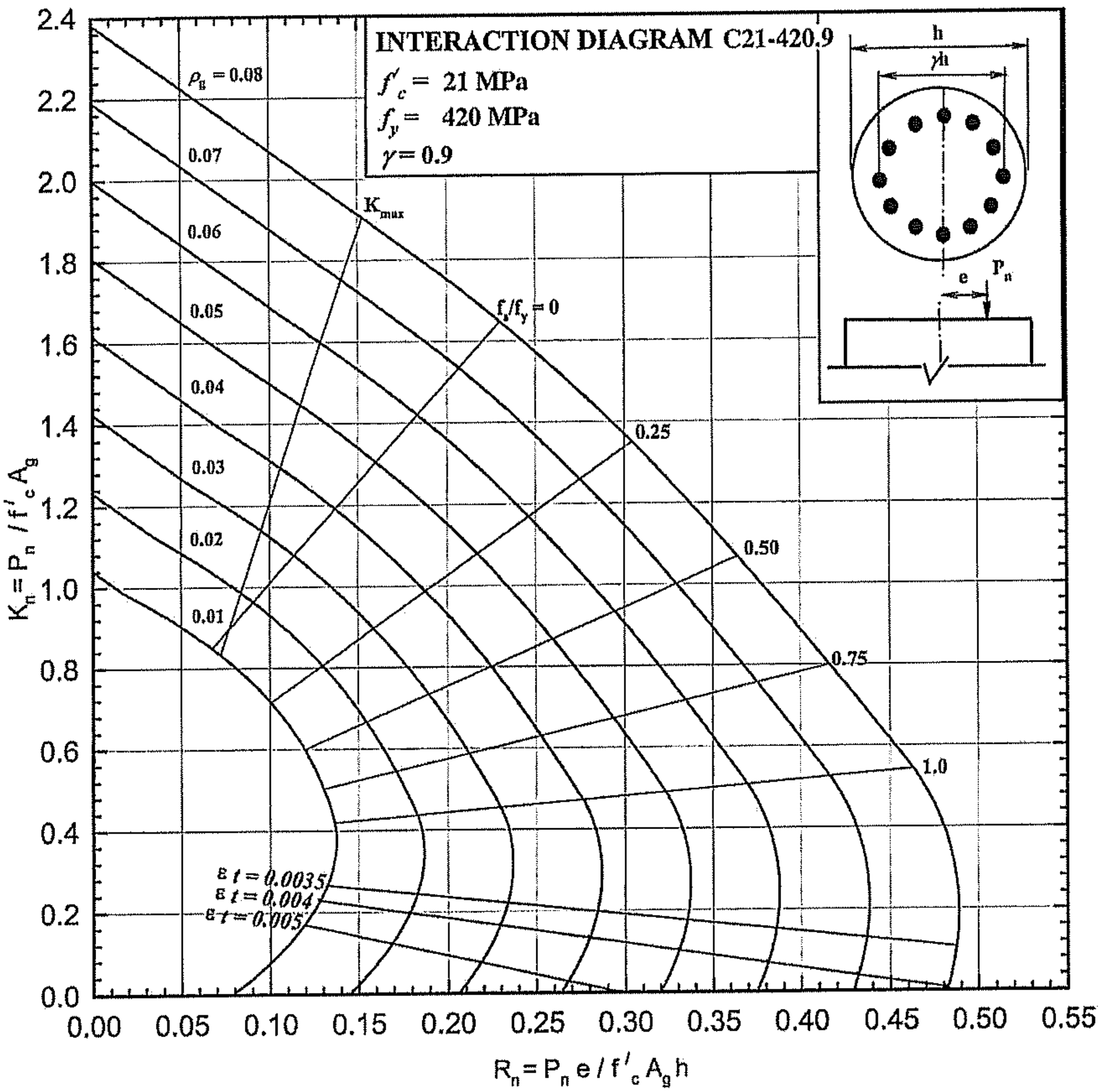
## COLUMNS 3.13.2 - Nominal load-moment strength interaction diagram, C21-420.7



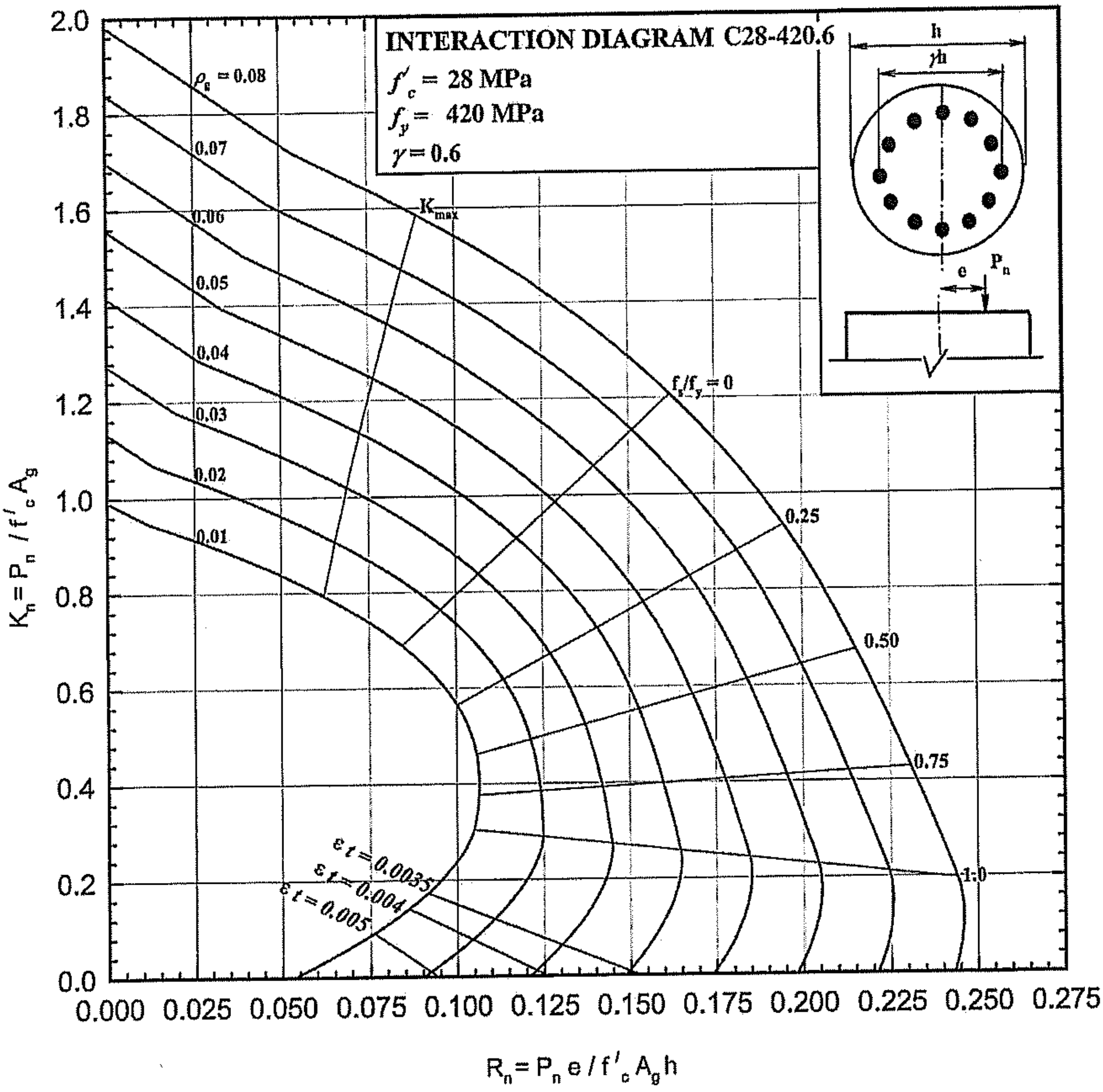
## COLUMNS 3.13.3 - Nominal load-moment strength interaction diagram, C21-420.8



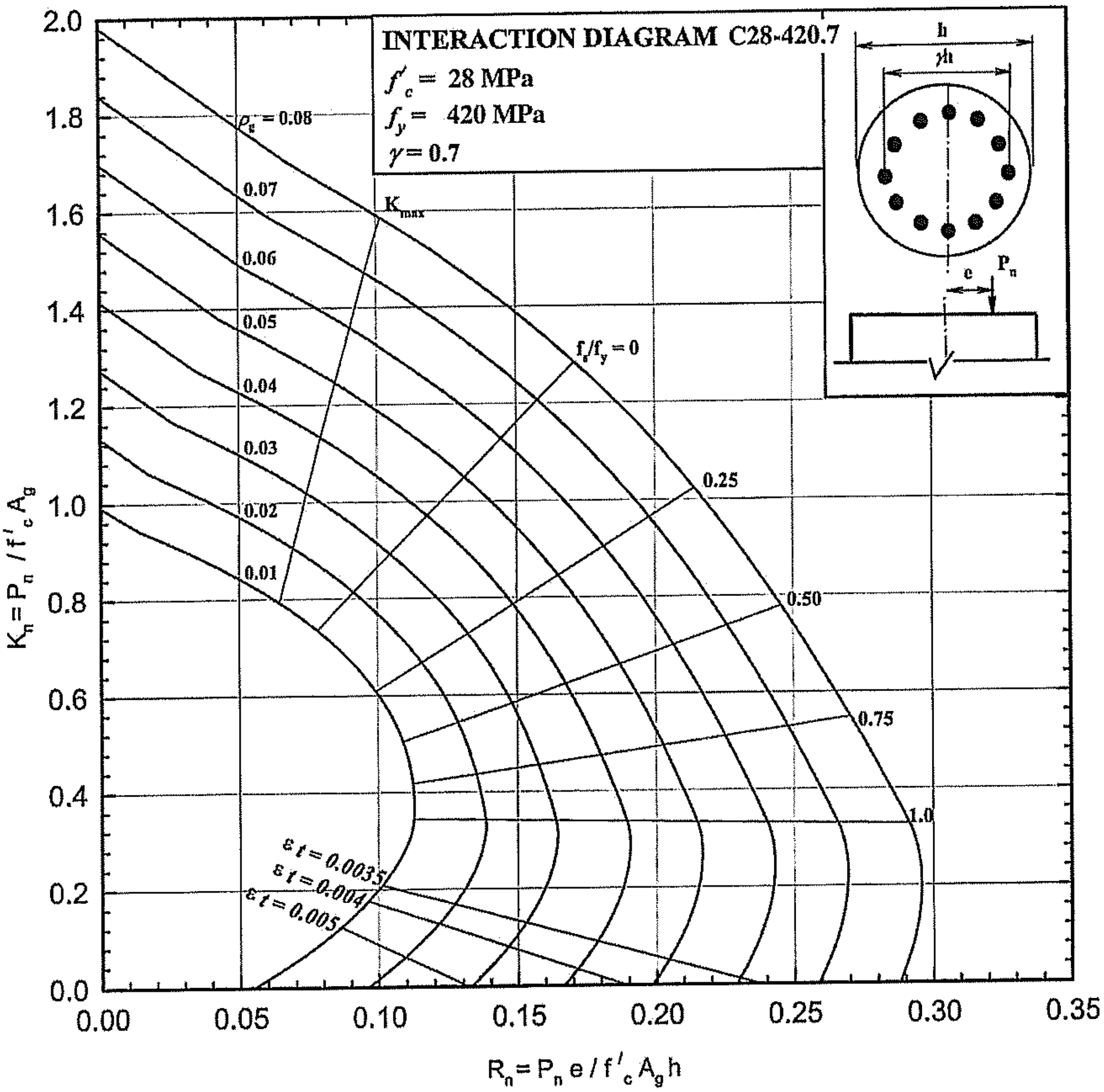
## COLUMNS 3.13.4 - Nominal load-moment strength interaction diagram, C21-420.9



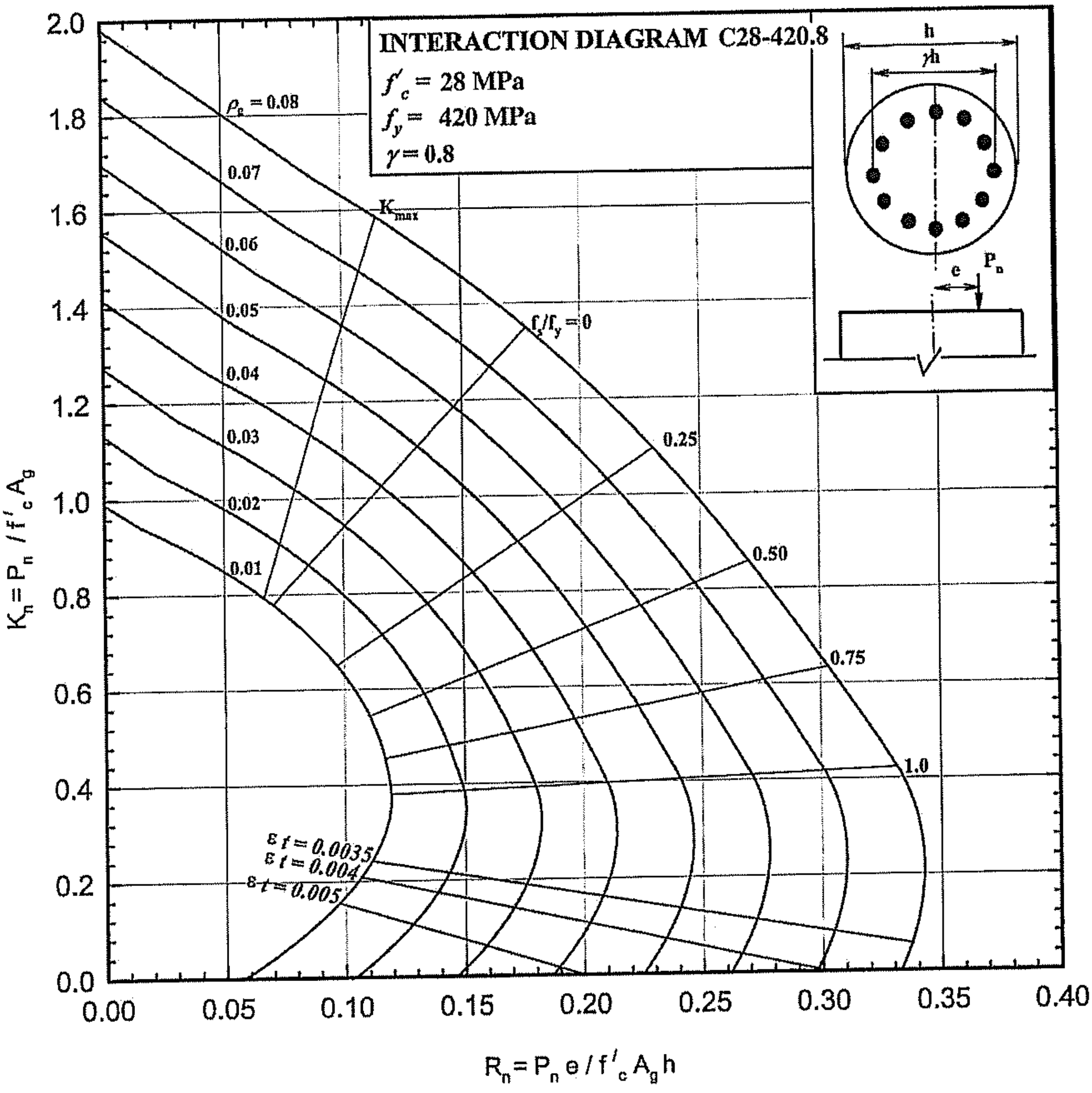
## COLUMNS 3.14.1 - Nominal load-moment strength interaction diagram, C28-420.6



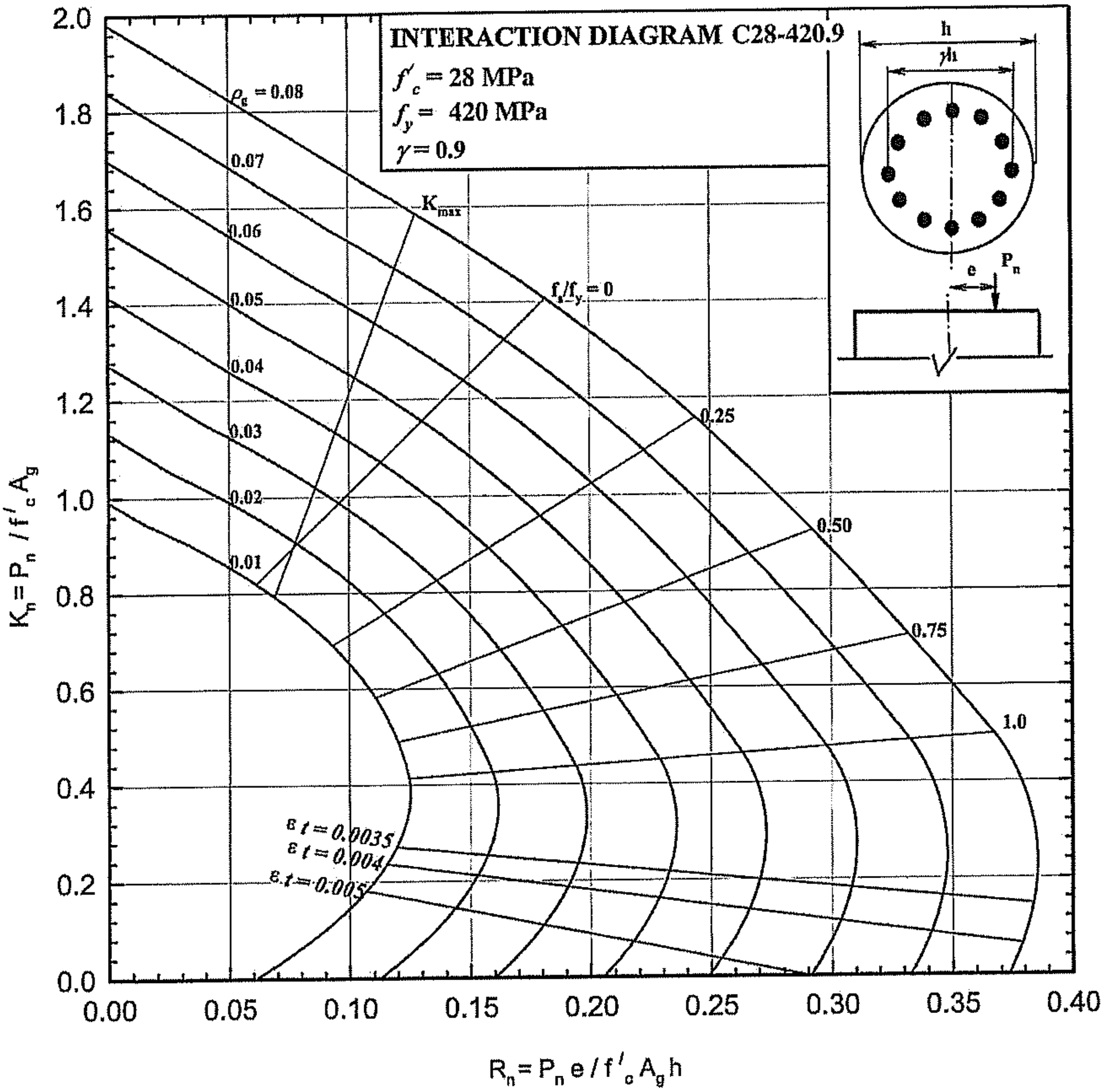
## COLUMNS 3.14.2 - Nominal load-moment strength interaction diagram, C28-420.7



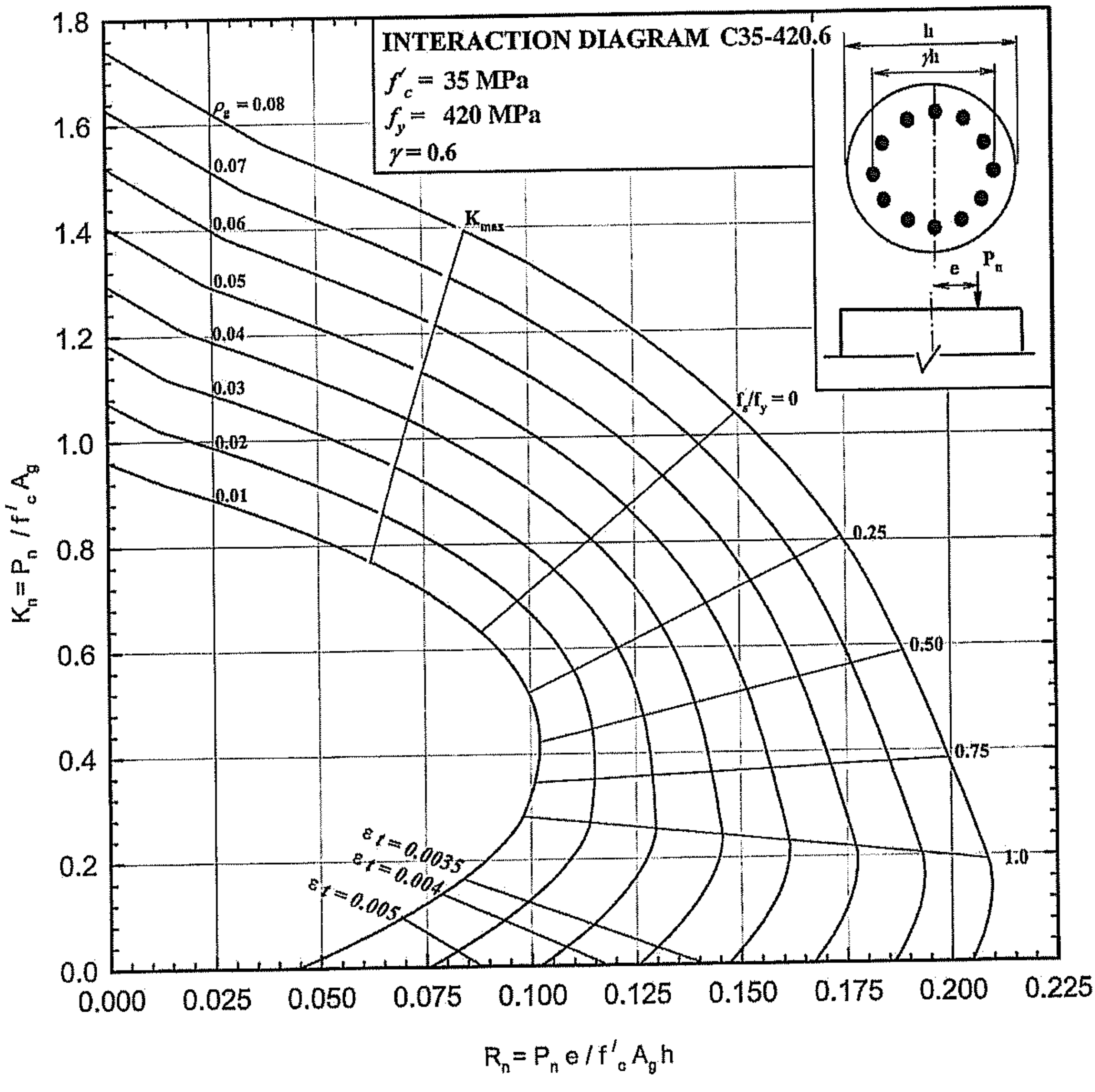
**COLUMNS 3.14.3 - Nominal load-moment strength interaction diagram, C28-420.8**



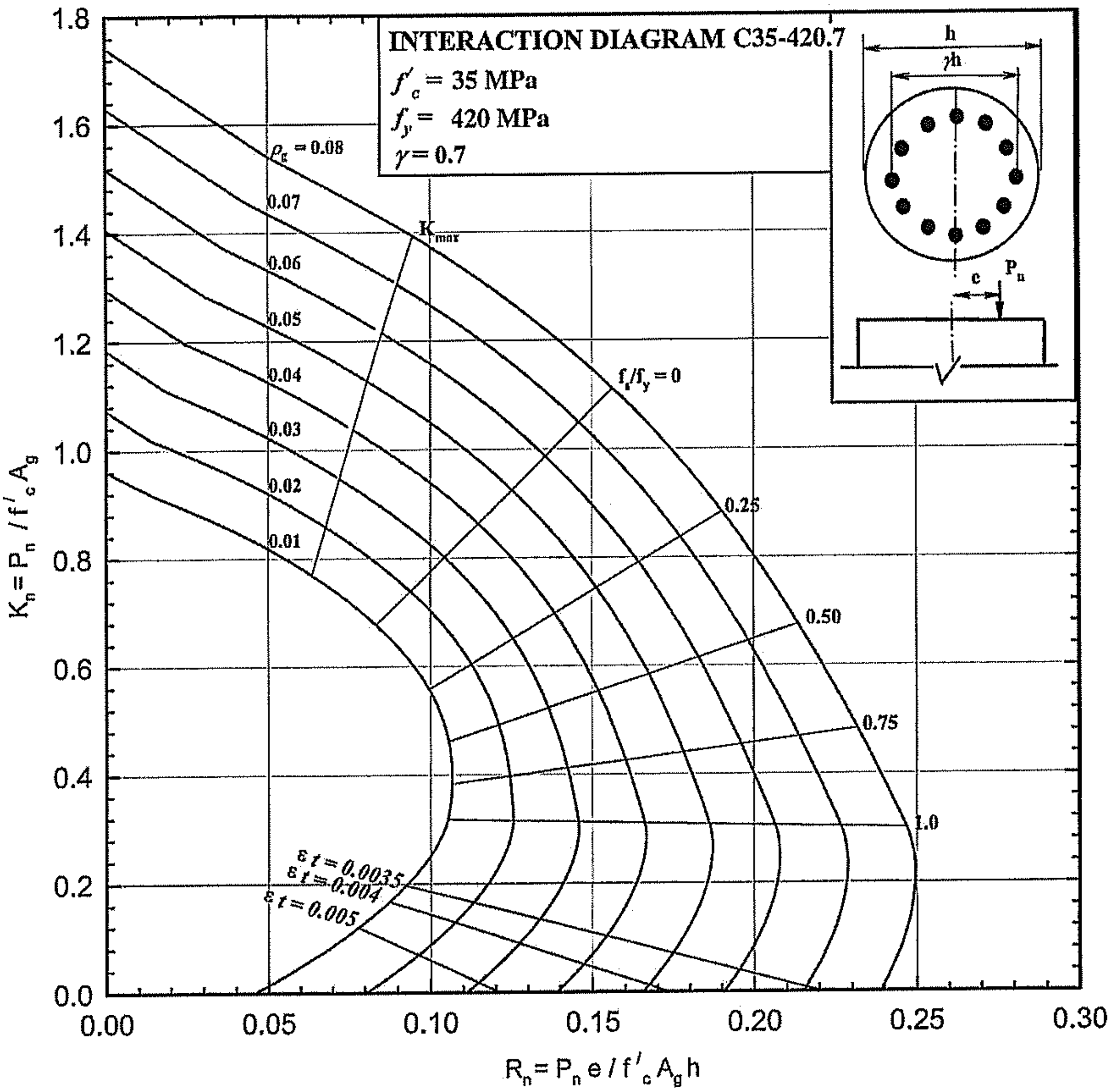
## COLUMNS 3.14.4 - Nominal load-moment strength interaction diagram, C28-420.9



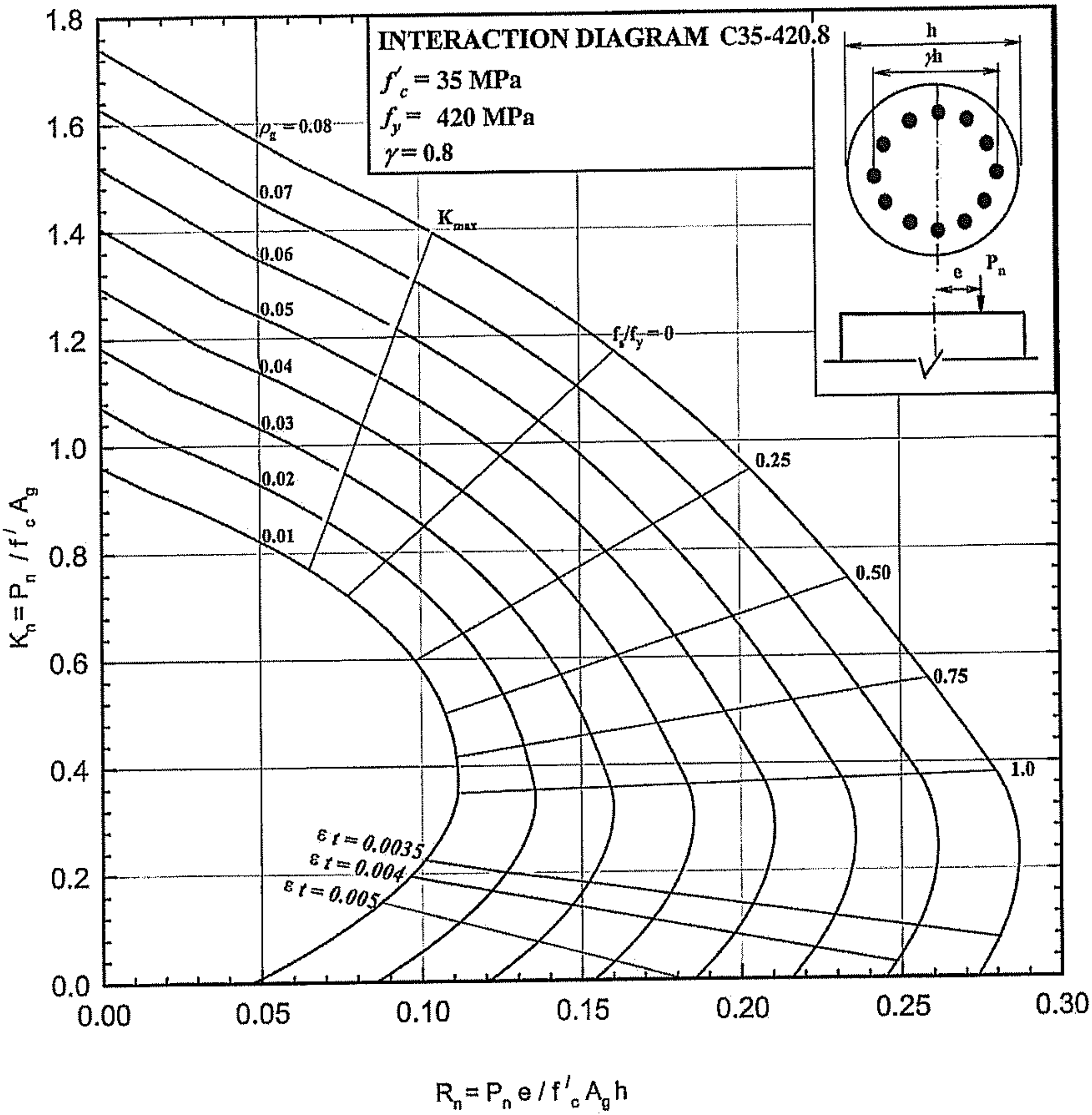
## COLUMNS 3.15.1 - Nominal load-moment strength interaction diagram, C35-420.6



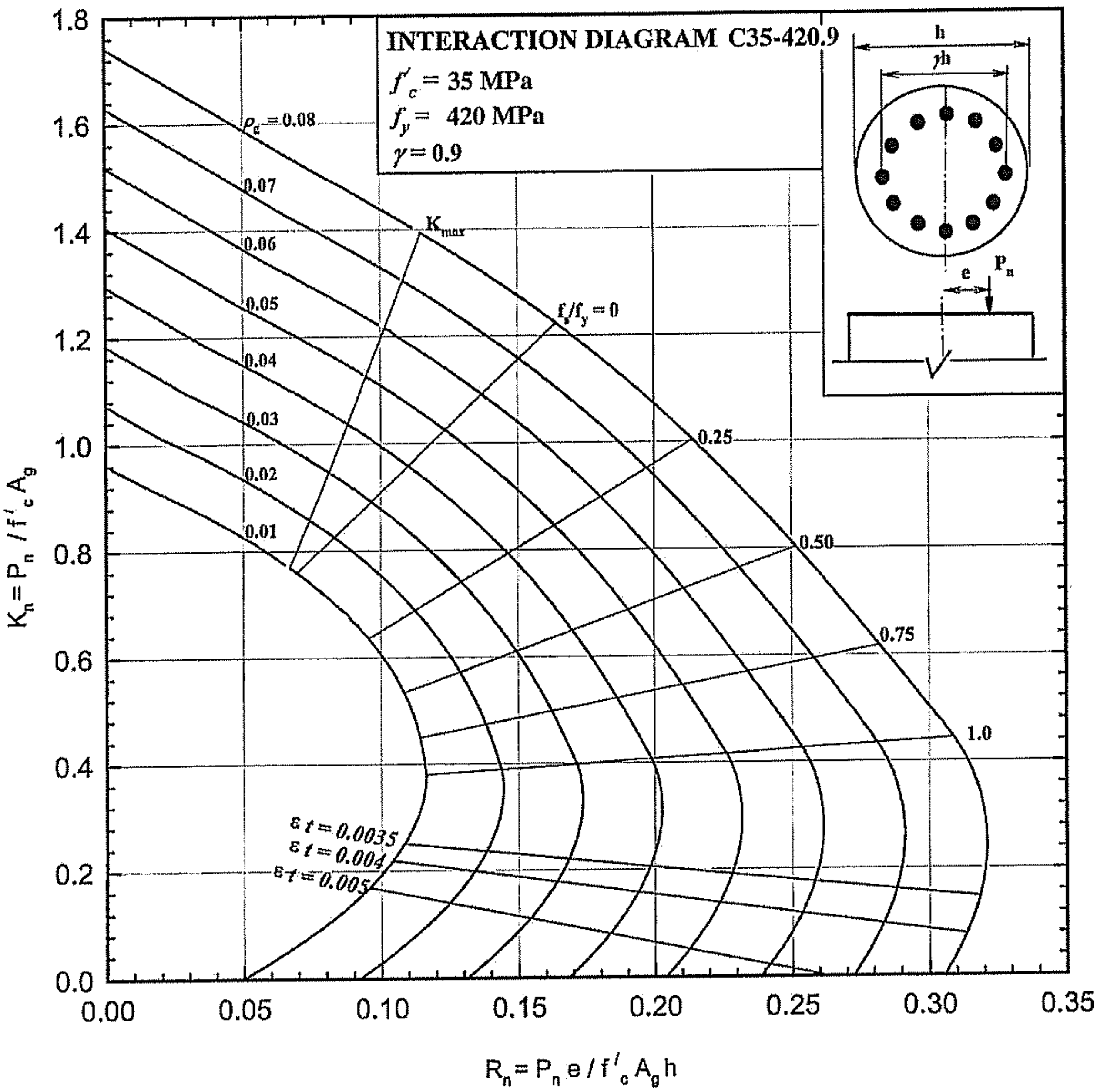
**COLUMNS 3.15.2 - Nominal load-moment strength interaction diagram, C35-420.7**



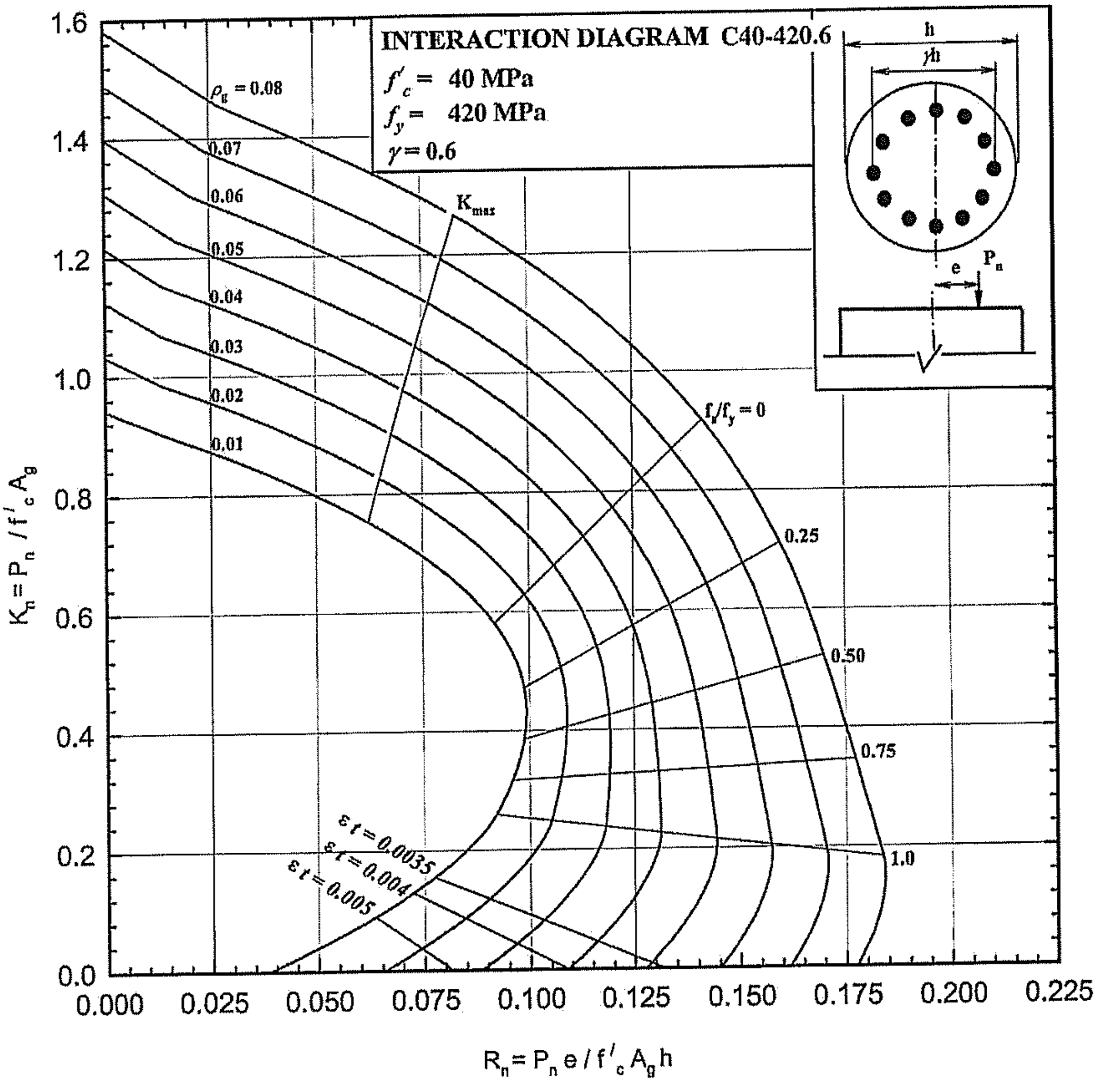
# COLUMNS 3.15.3 - Nominal load-moment strength interaction diagram, C35-420.8



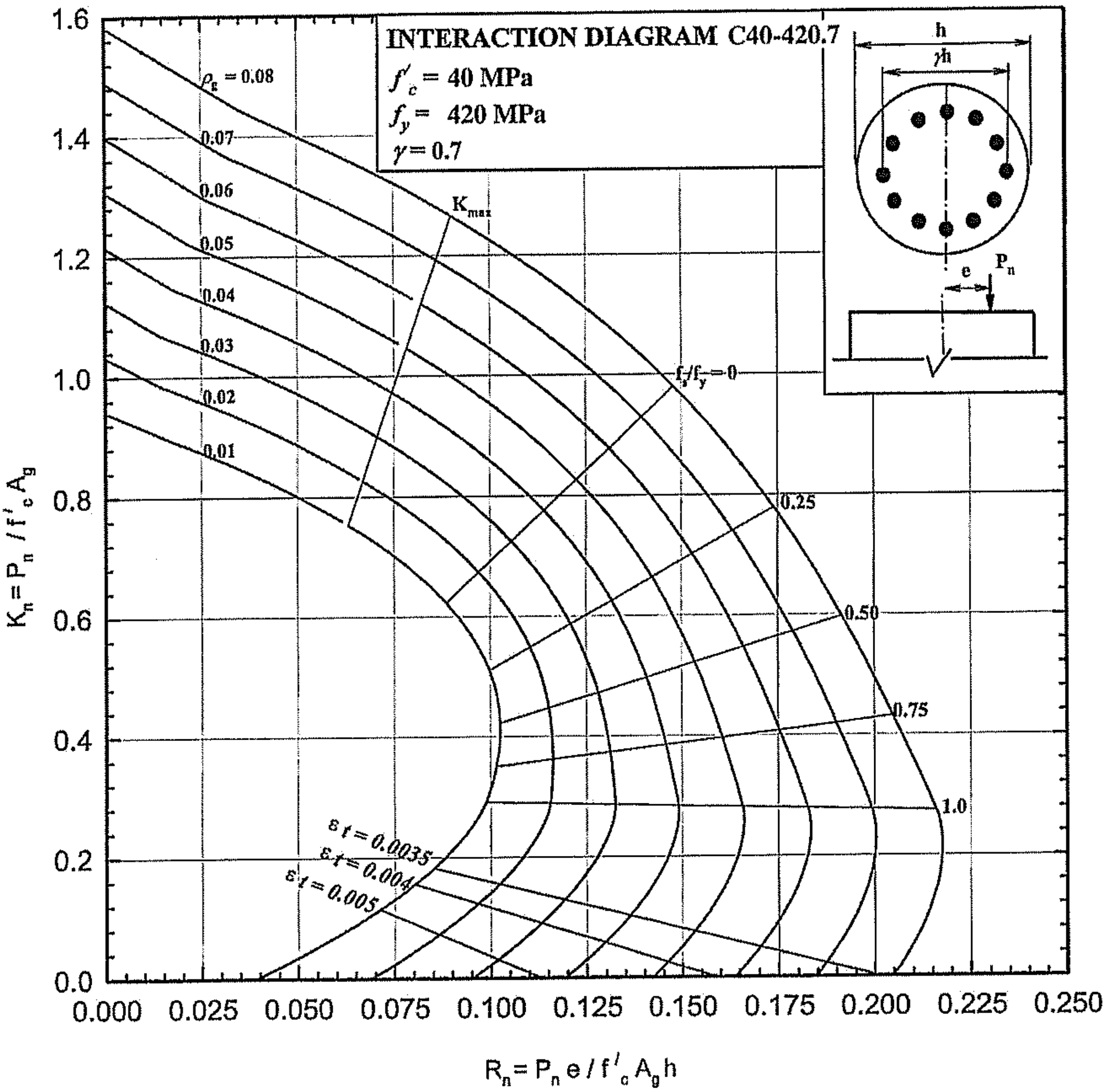
# COLUMNS 3.15.4 - Nominal load-moment strength interaction diagram, C35-420.9



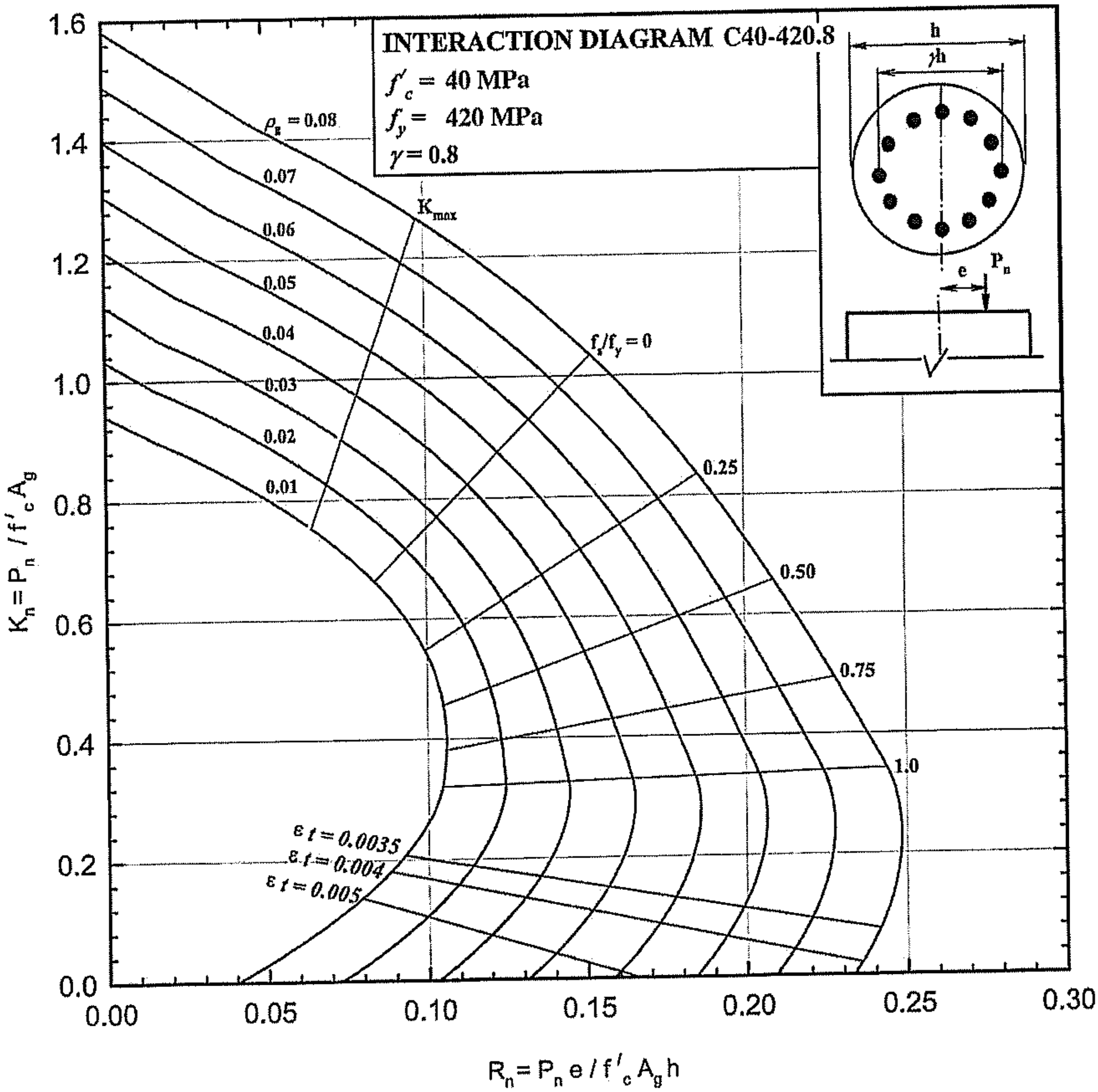
## COLUMNS 3.16.1 - Nominal load-moment strength interaction diagram, C40-420.6



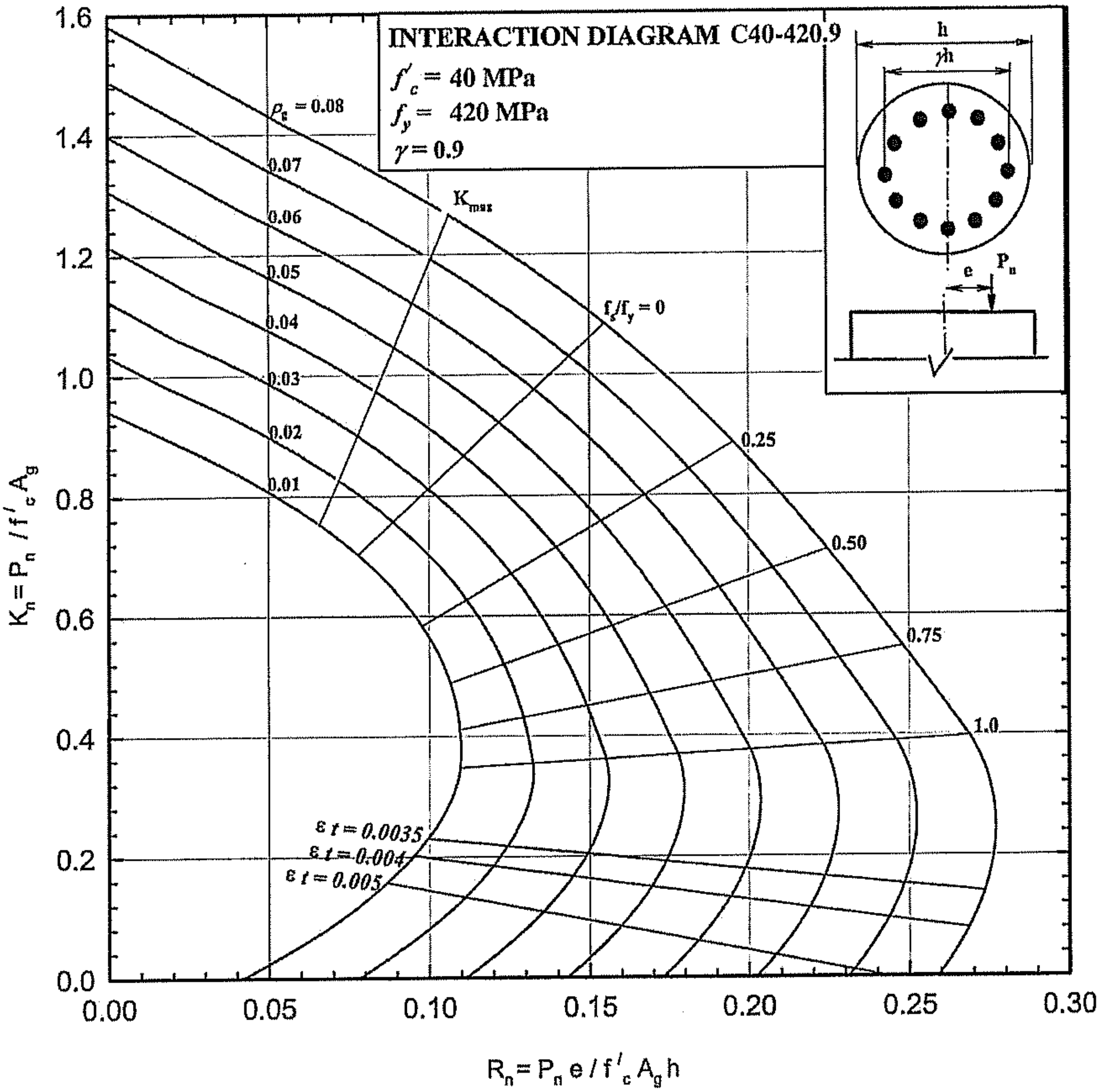
**COLUMNS 3.16.2 - Nominal load-moment strength interaction diagram, C40-420.7**



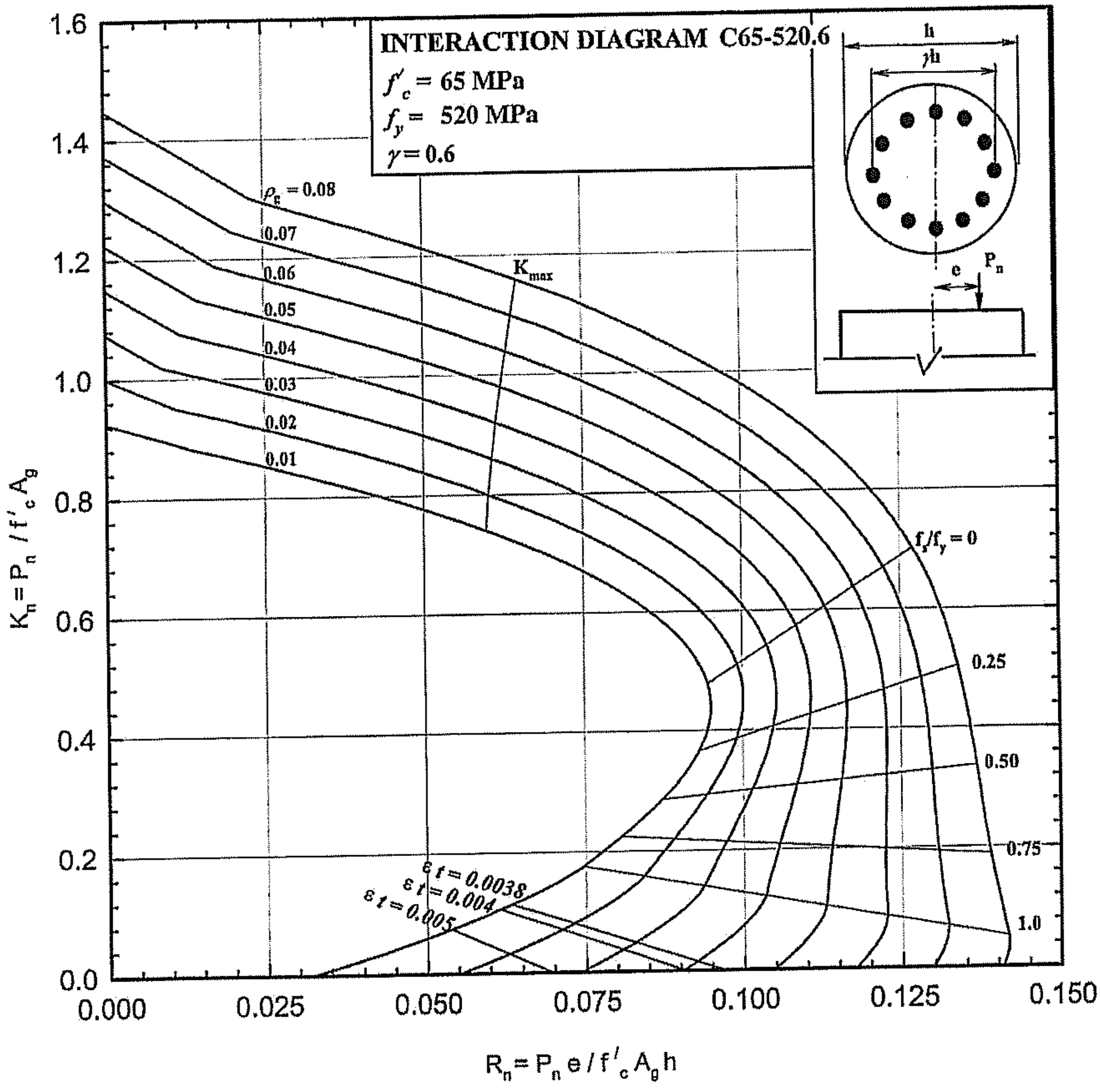
**COLUMNS 3.16.3 - Nominal load-moment strength interaction diagram, C40-420.8**



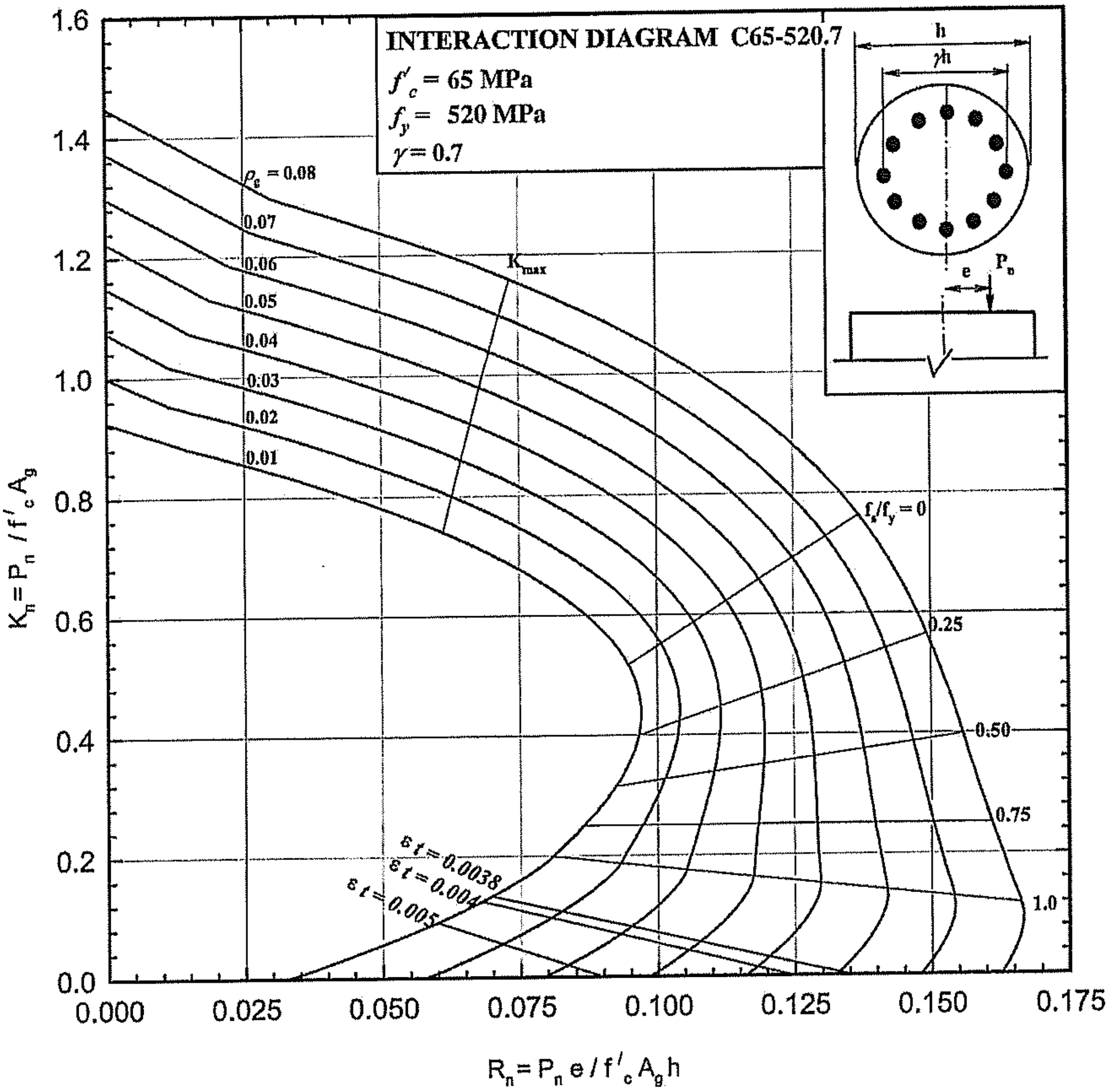
## COLUMNS 3.16.4 - Nominal load-moment strength interaction diagram, C40-420.9



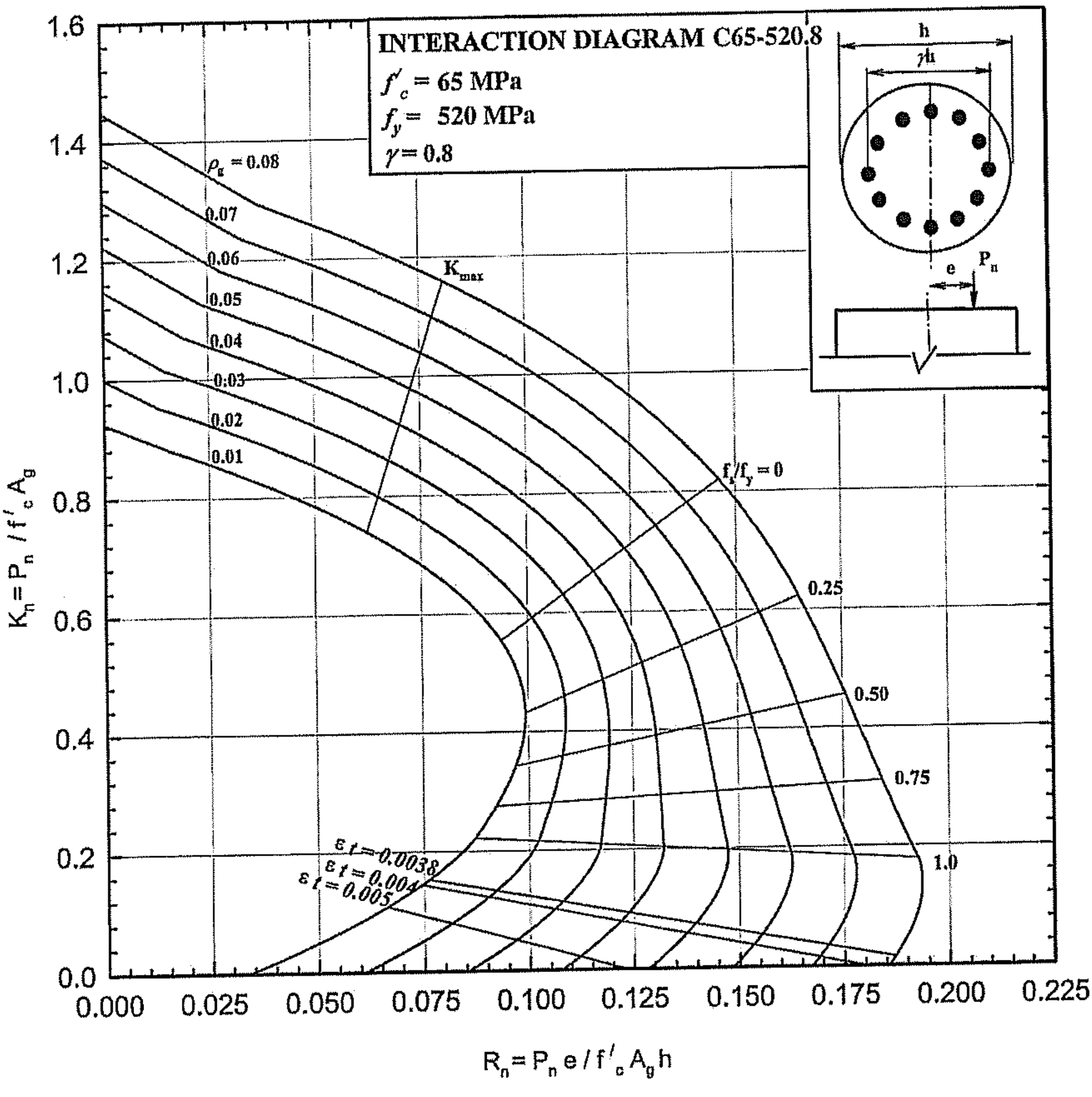
# COLUMNS 3.17.1 - Nominal load-moment strength interaction diagram, C65-520.6



## COLUMNS 3.17.2 - Nominal load-moment strength interaction diagram, C65-520.7



# COLUMNS 3.17.3 - Nominal load-moment strength interaction diagram, C65-520.8



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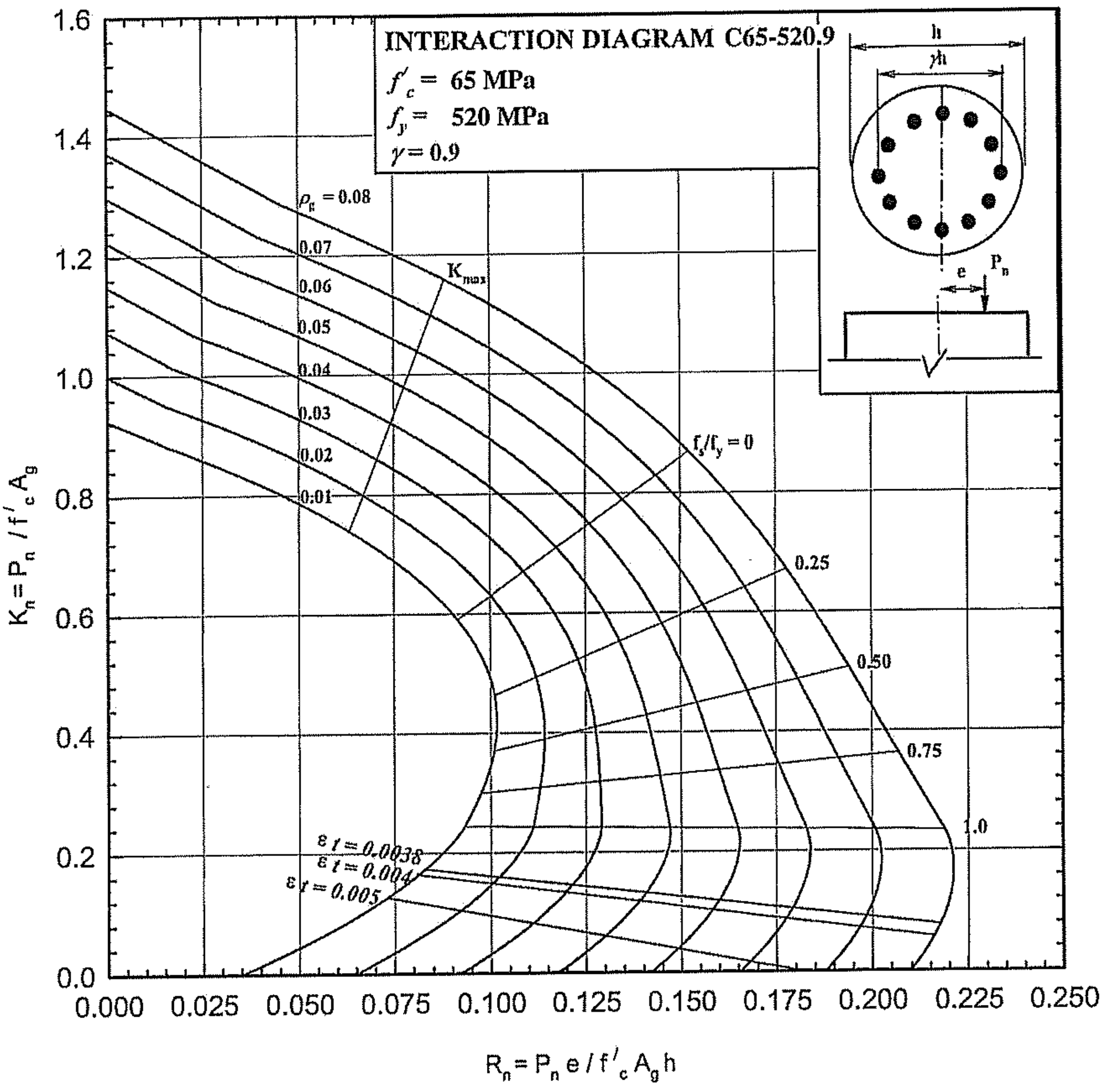

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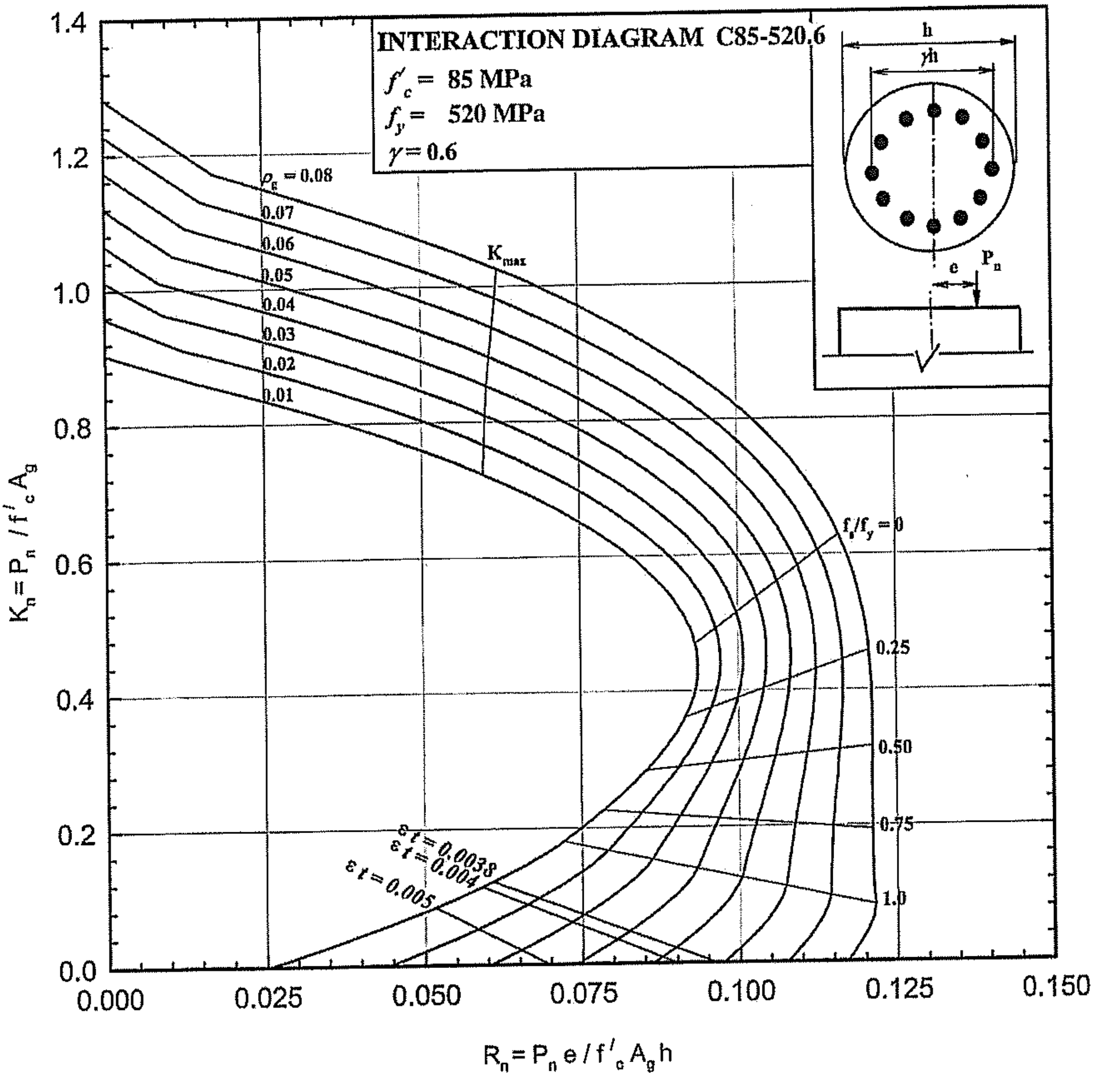

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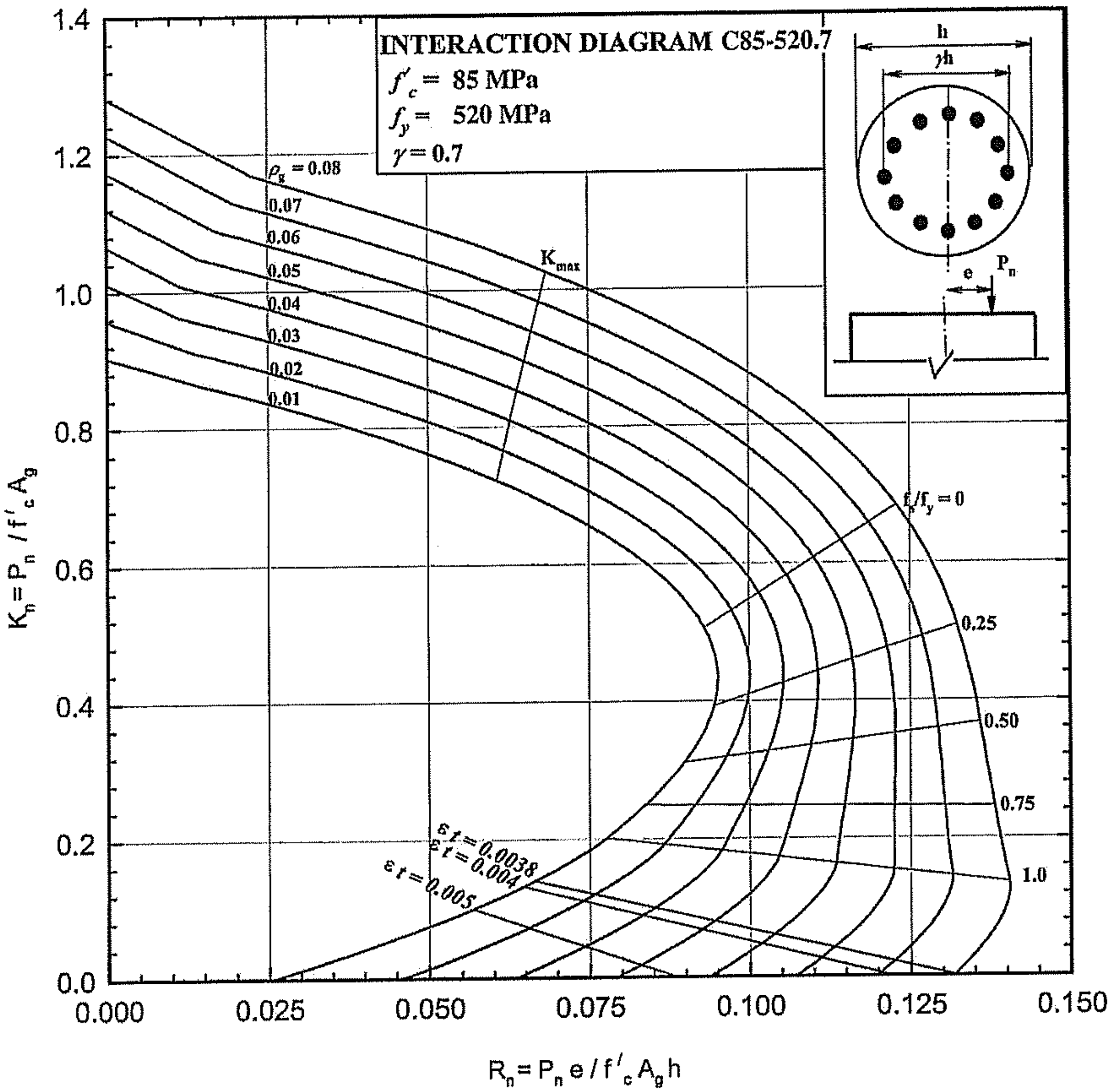

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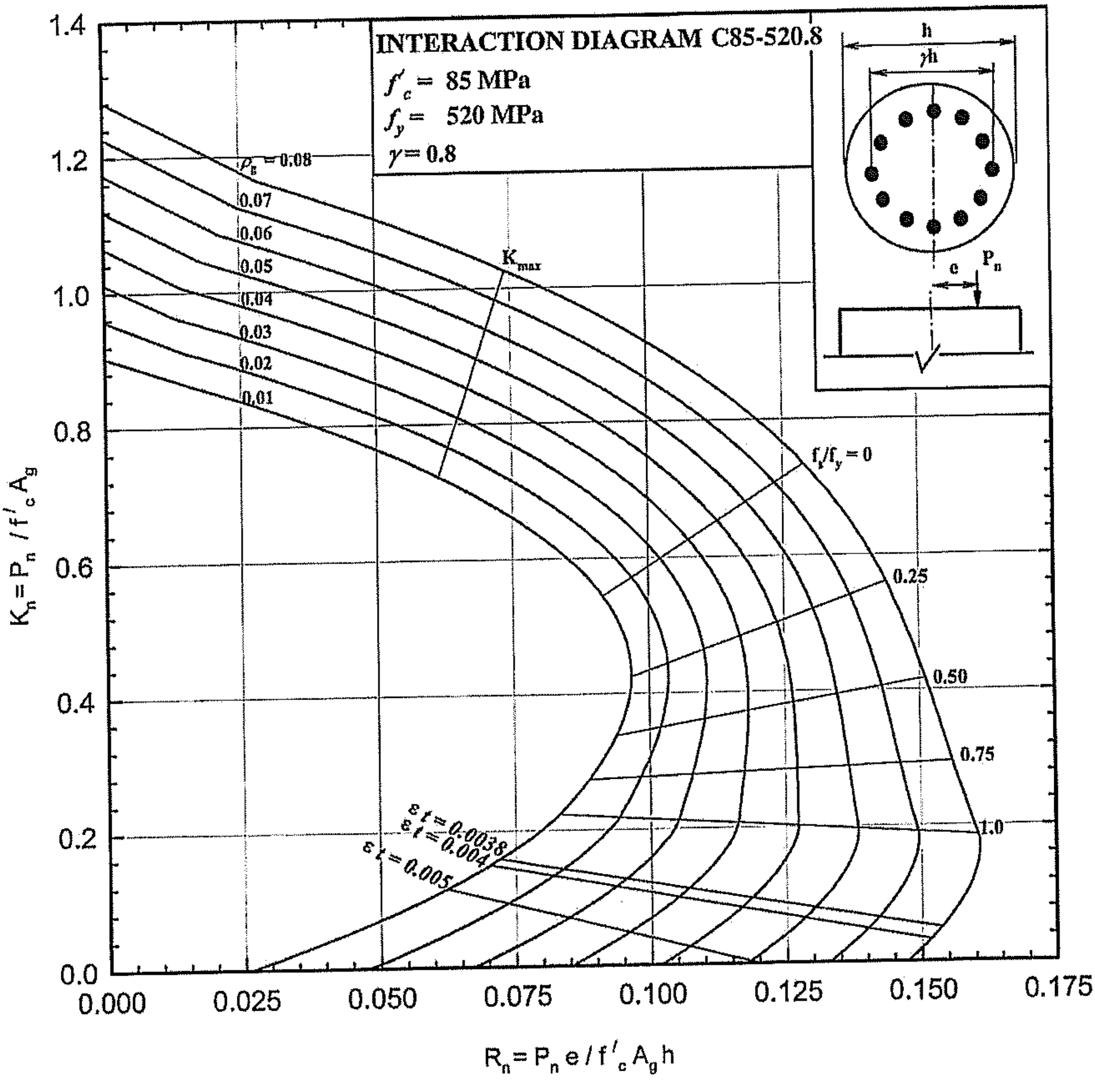

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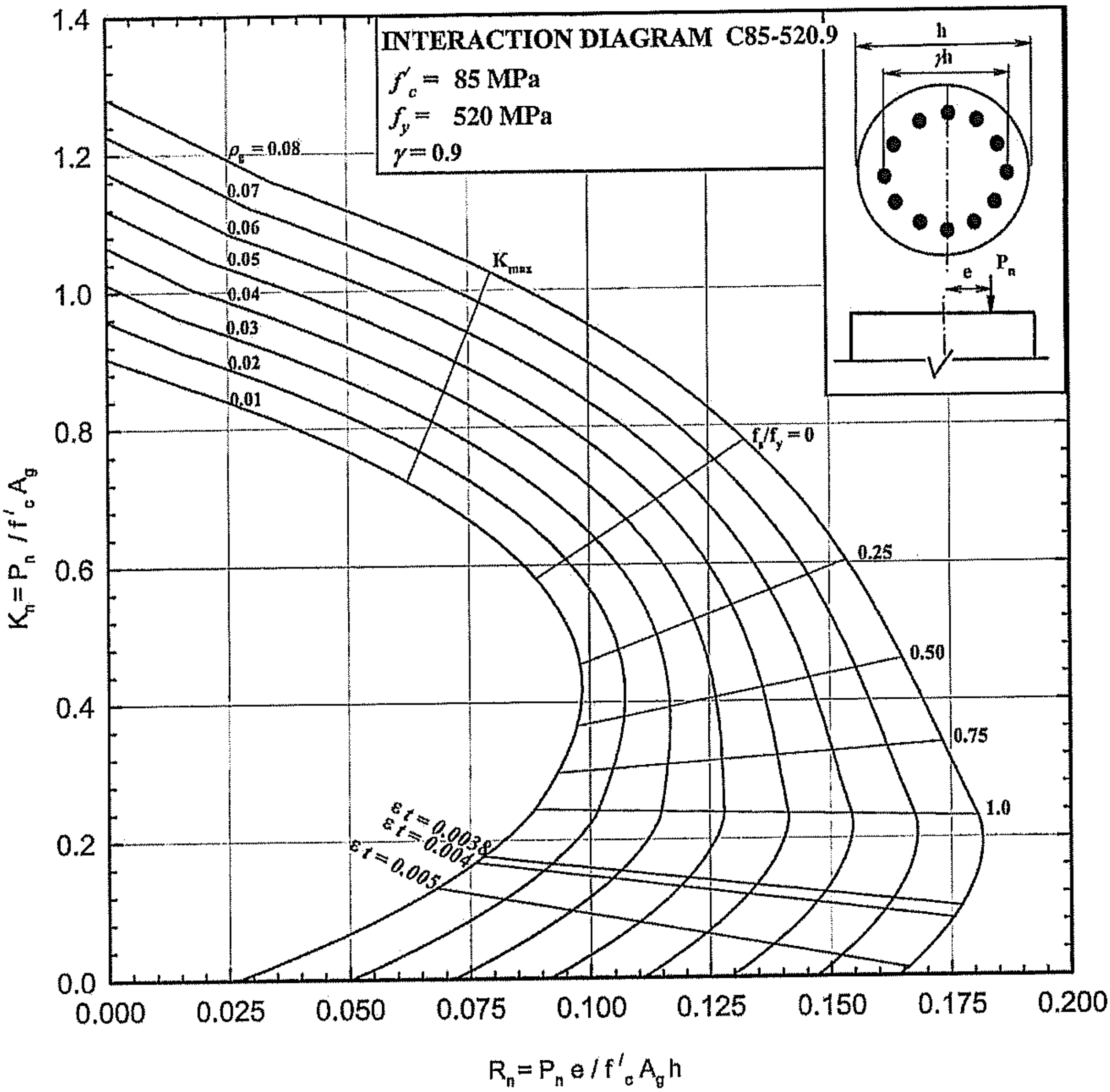

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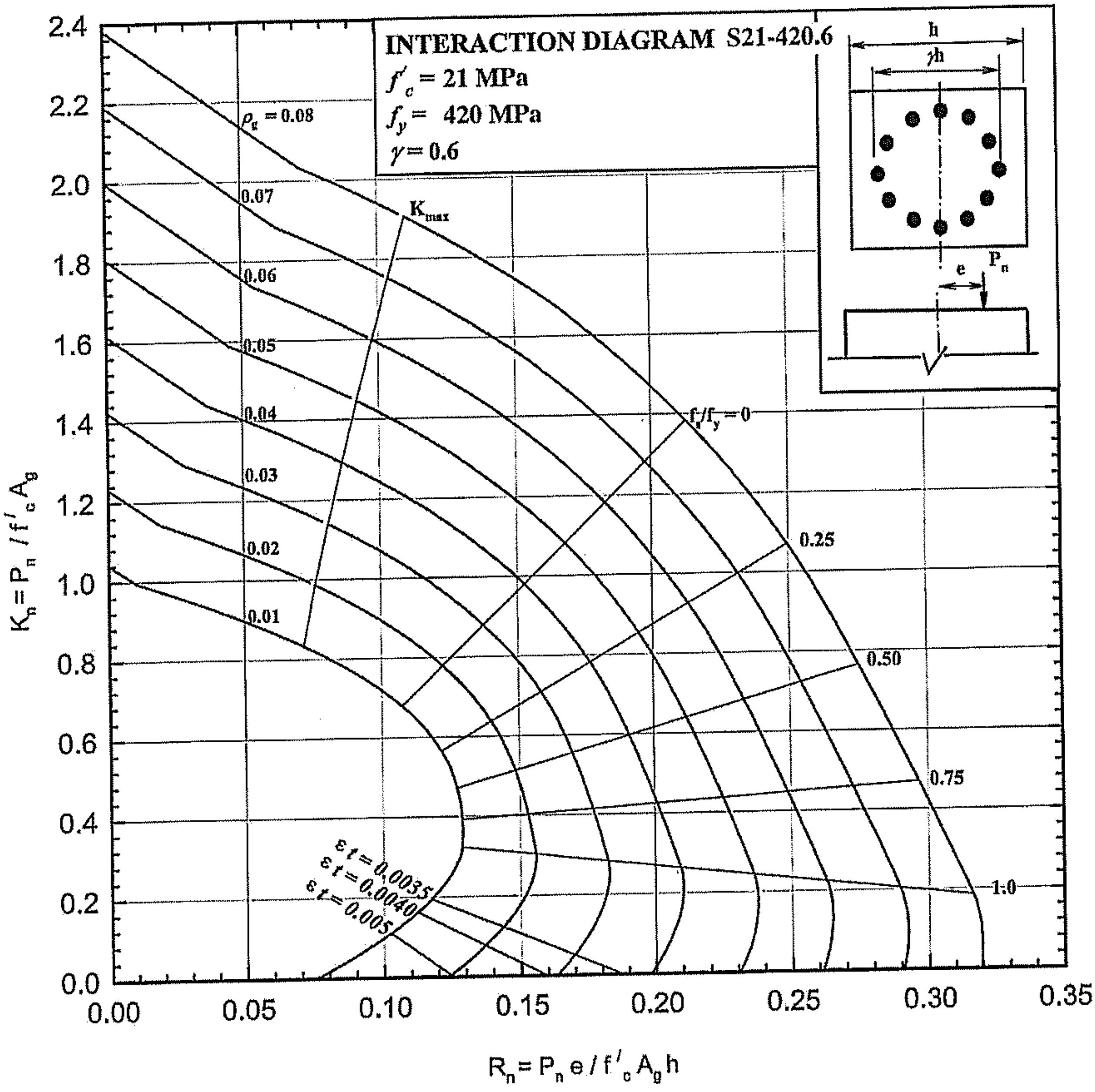

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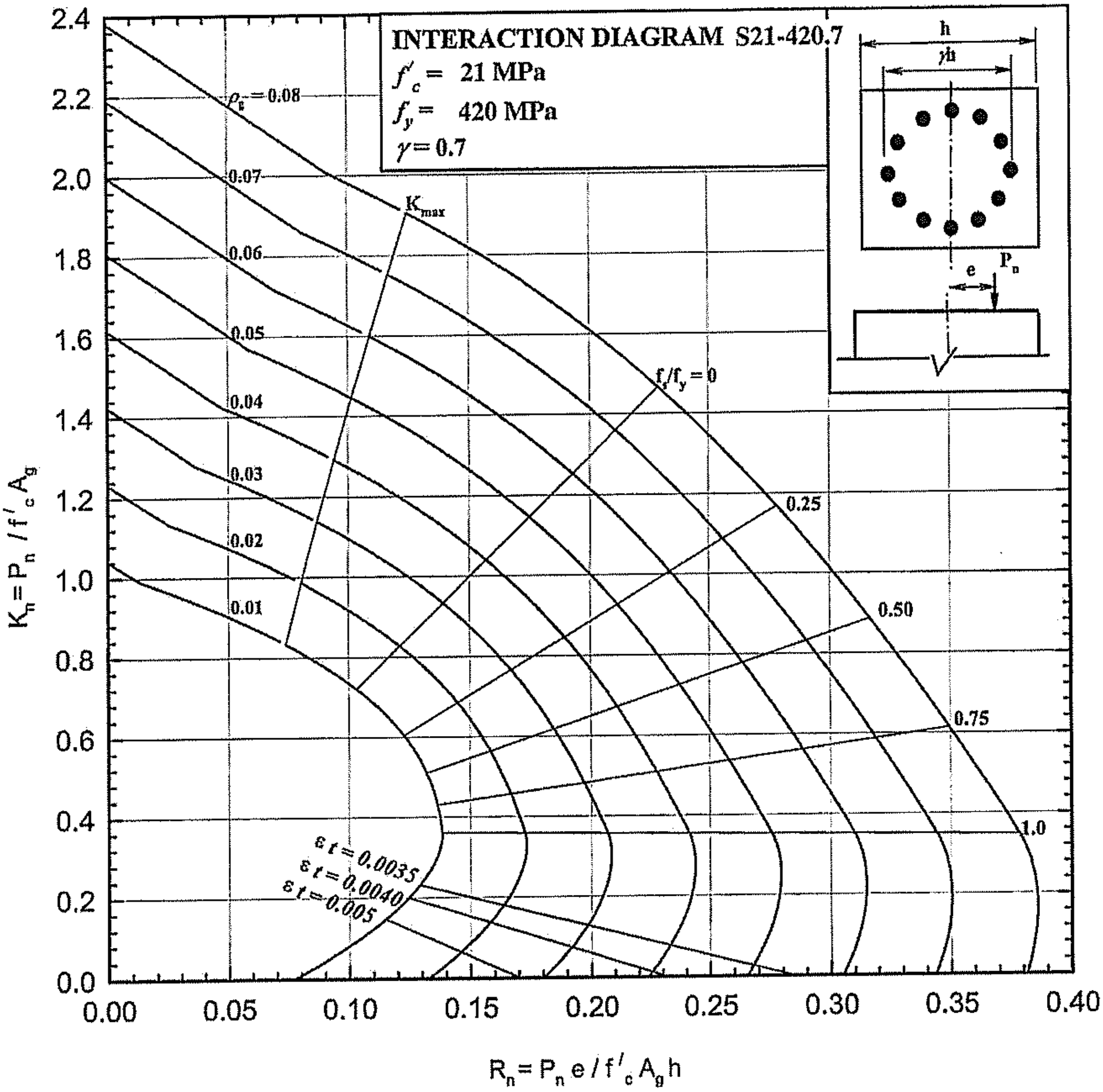

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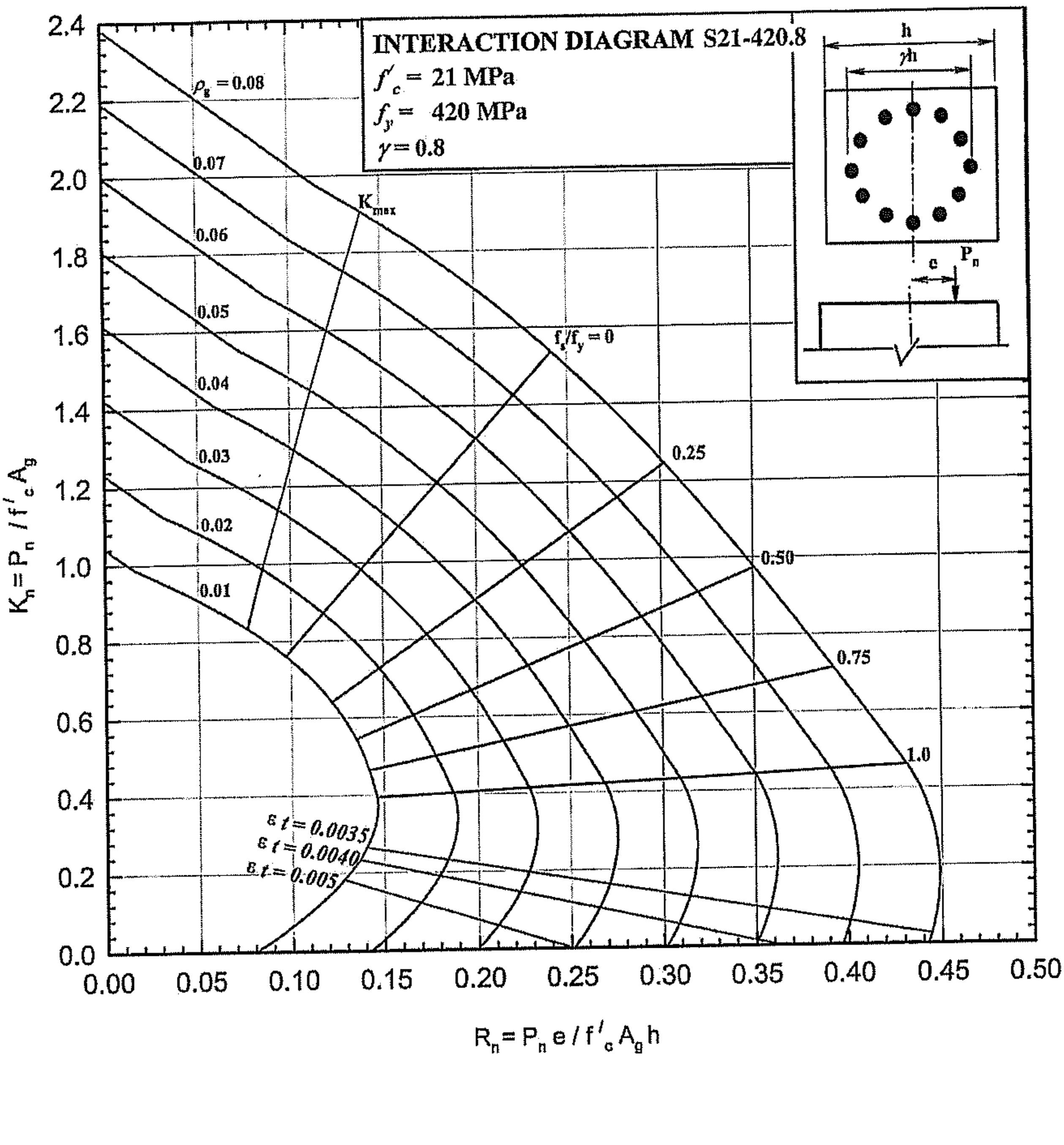

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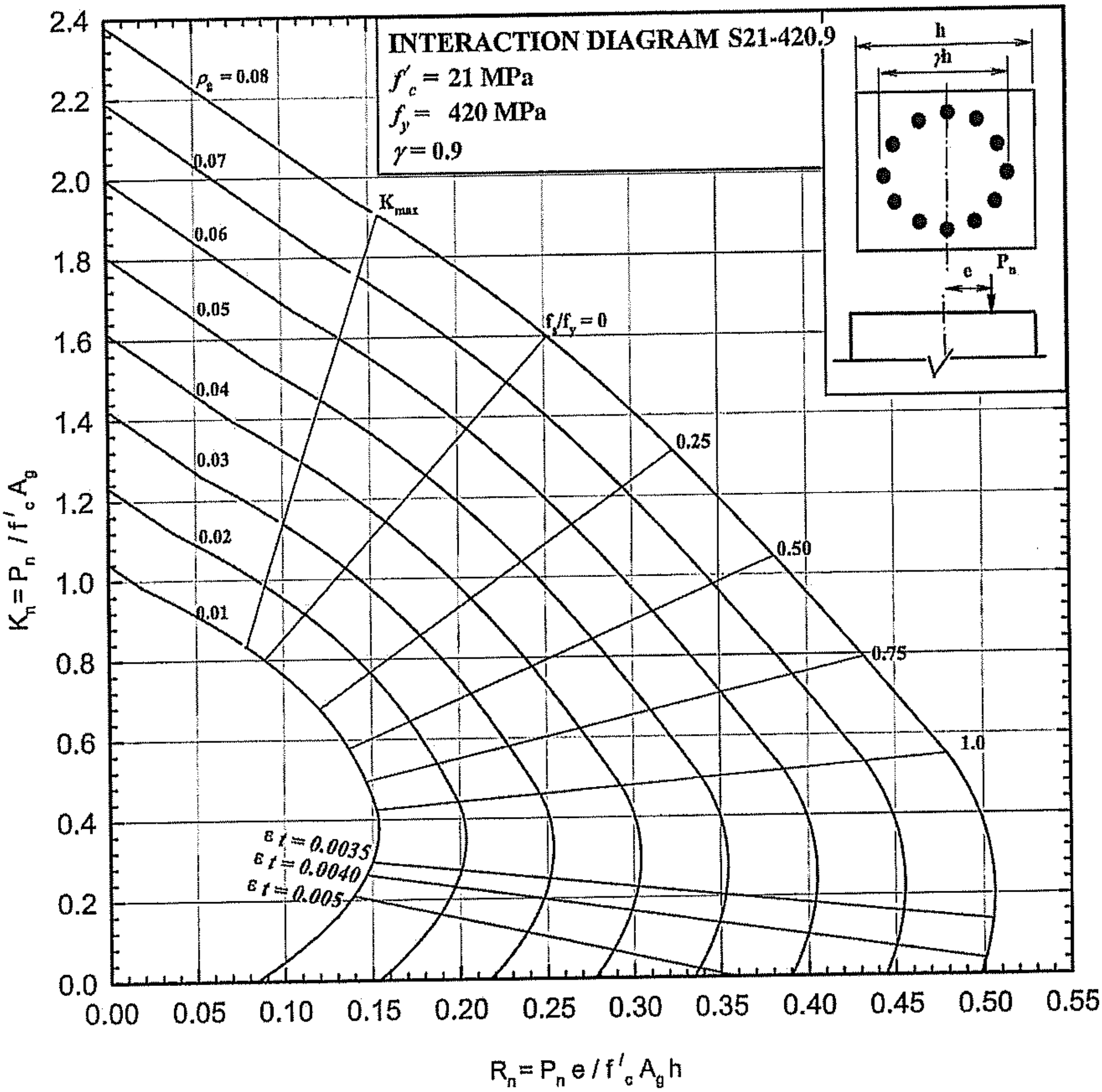
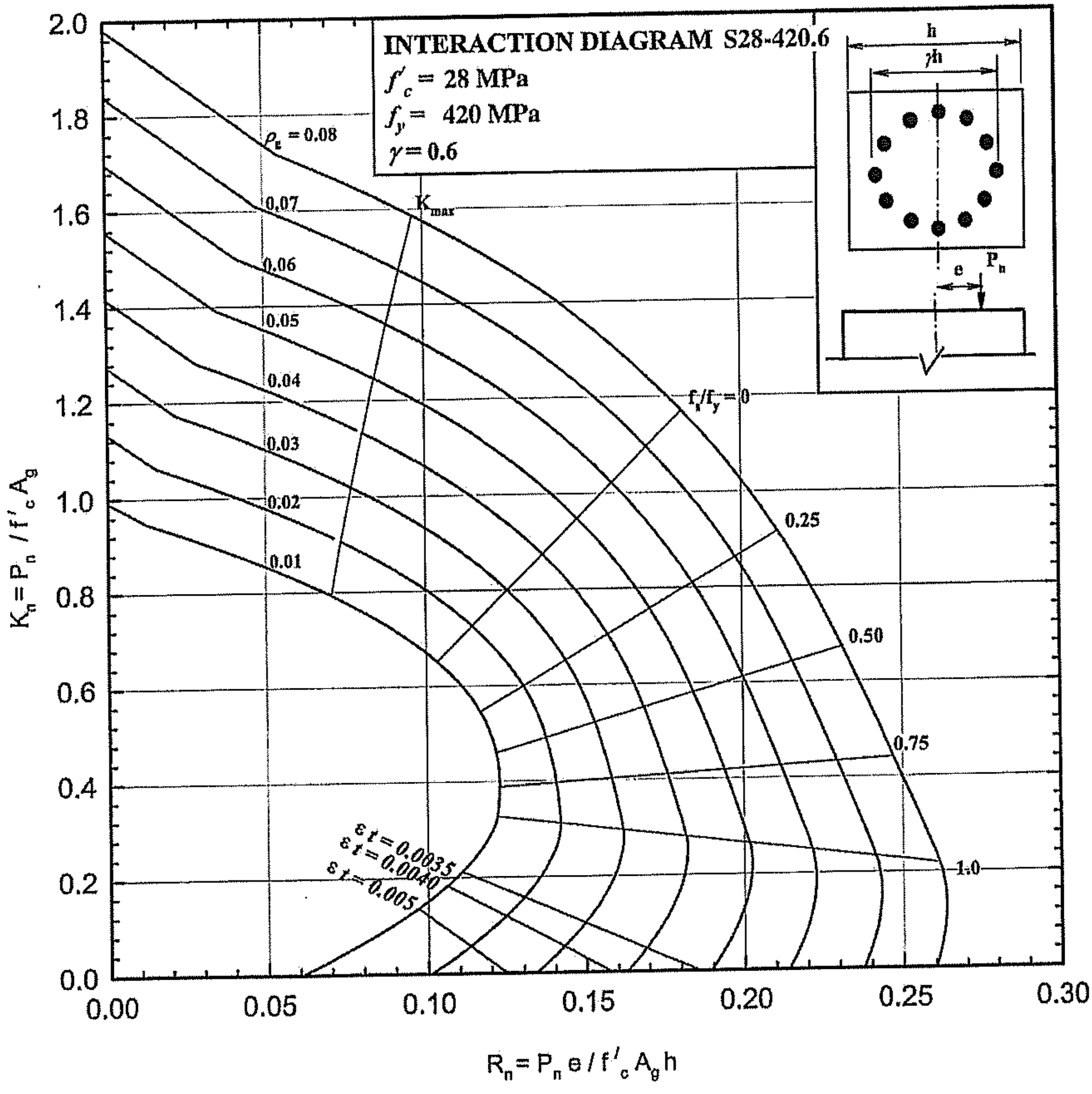


## COLUMNS 3.20.1 - Nominal load-moment strength interaction diagram, S28-420.6



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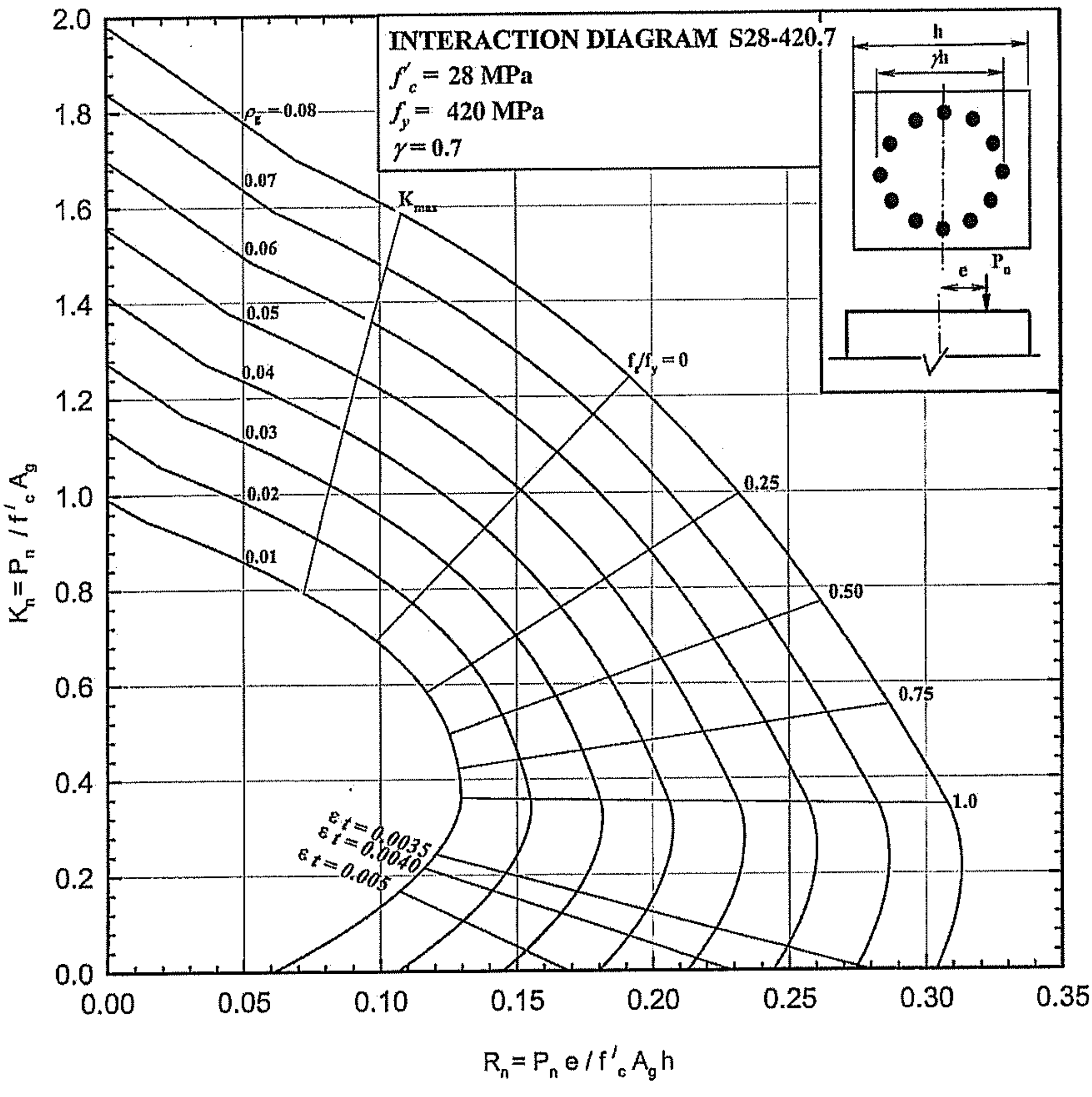

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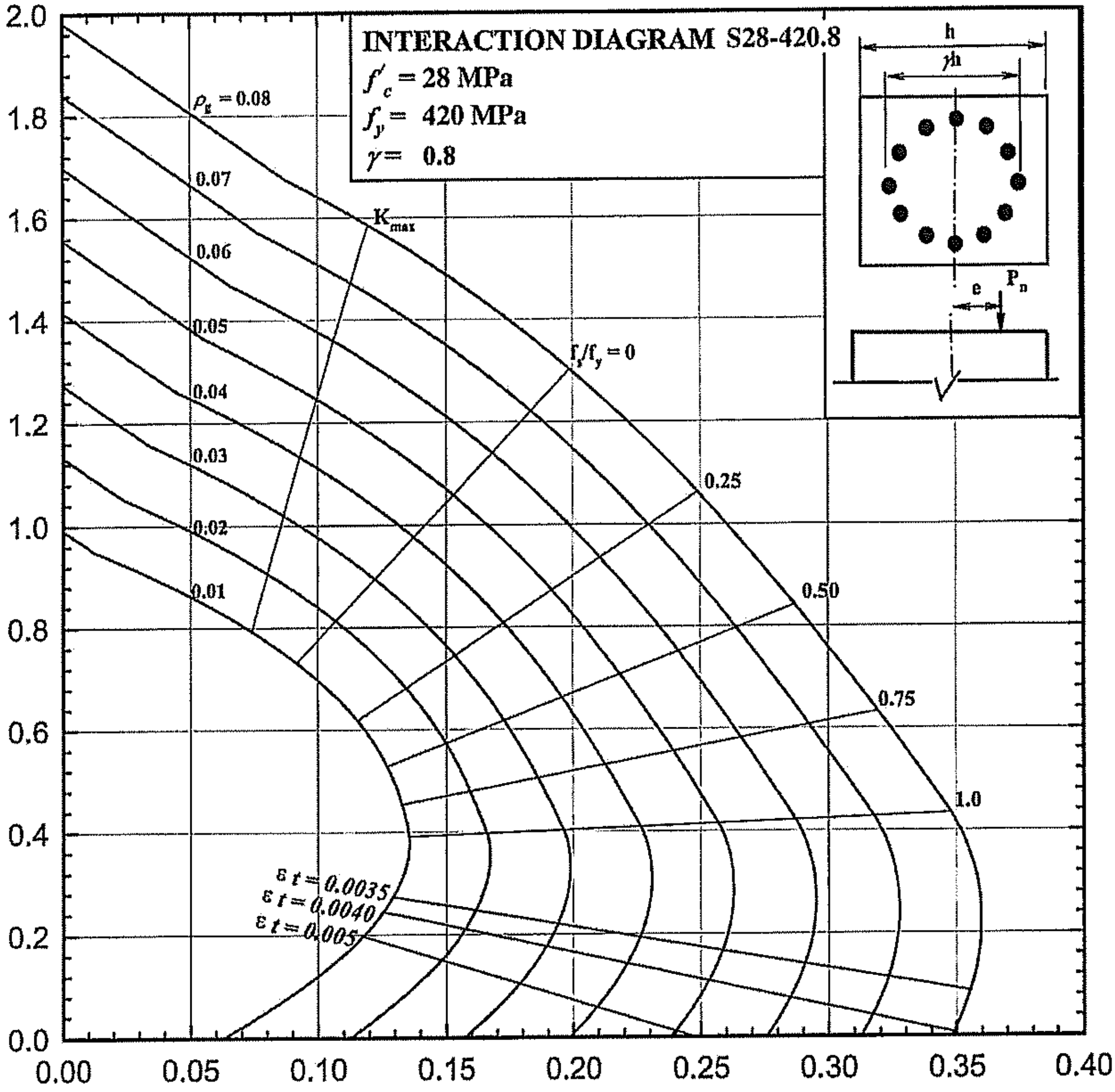

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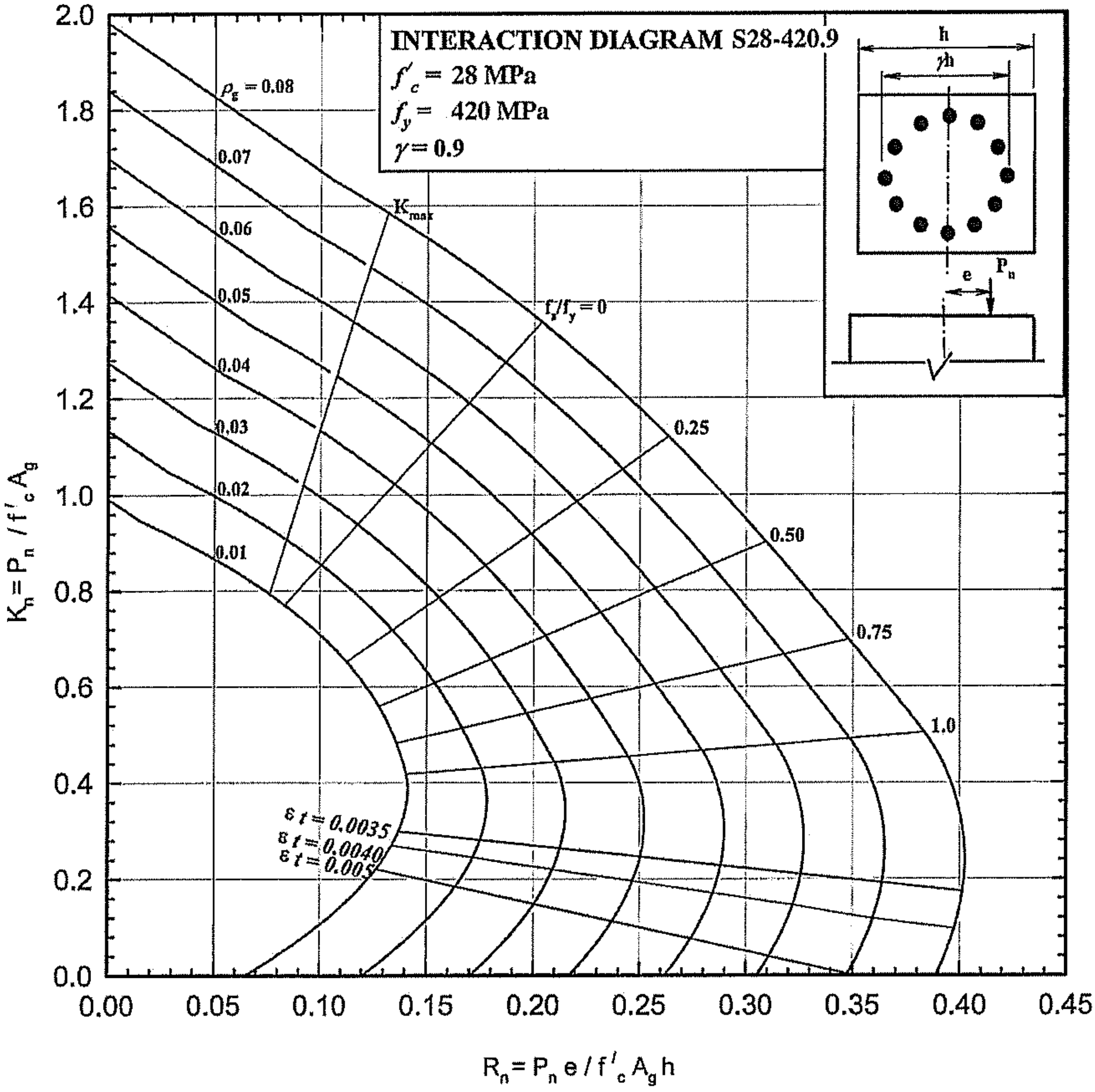

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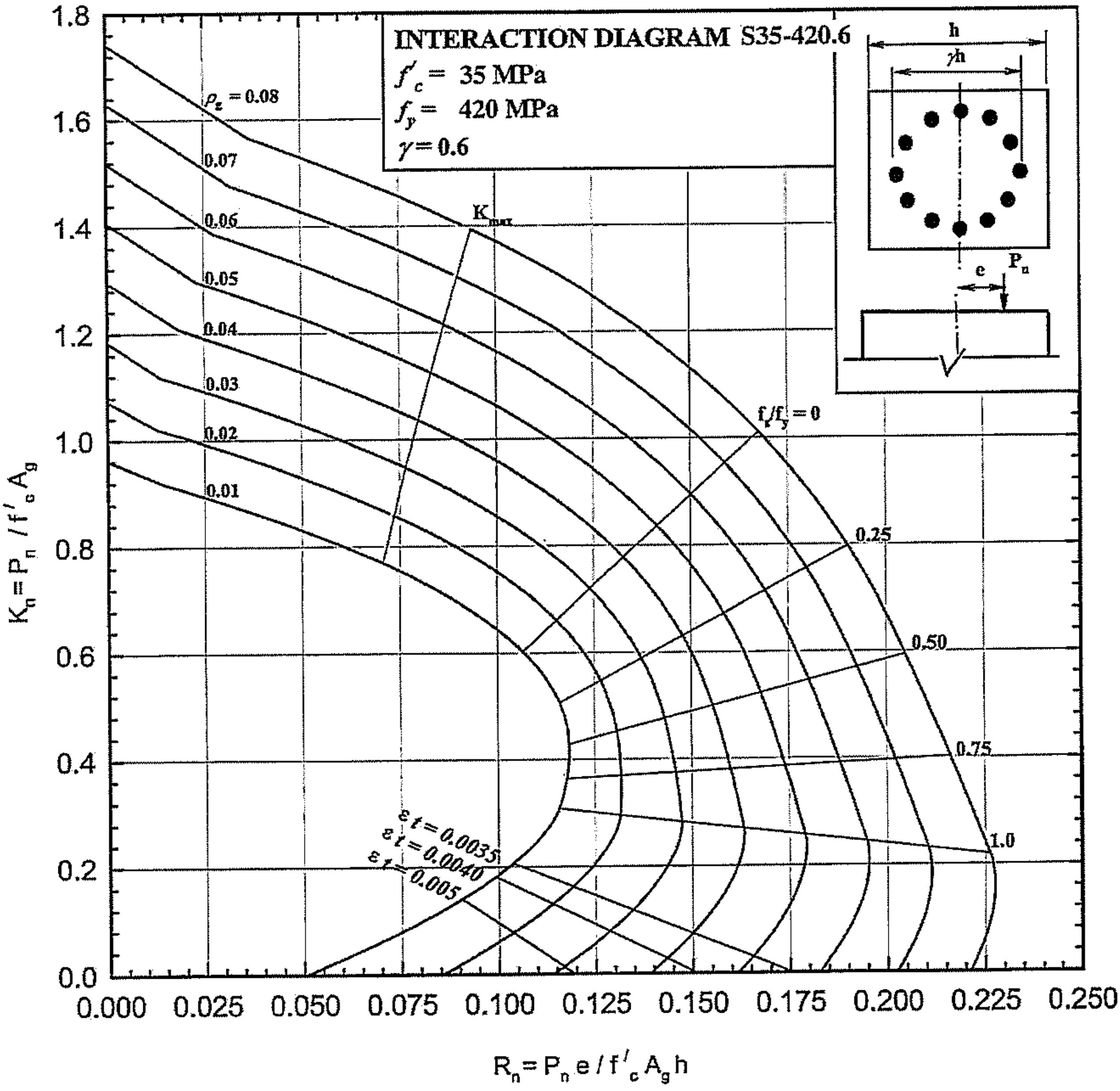

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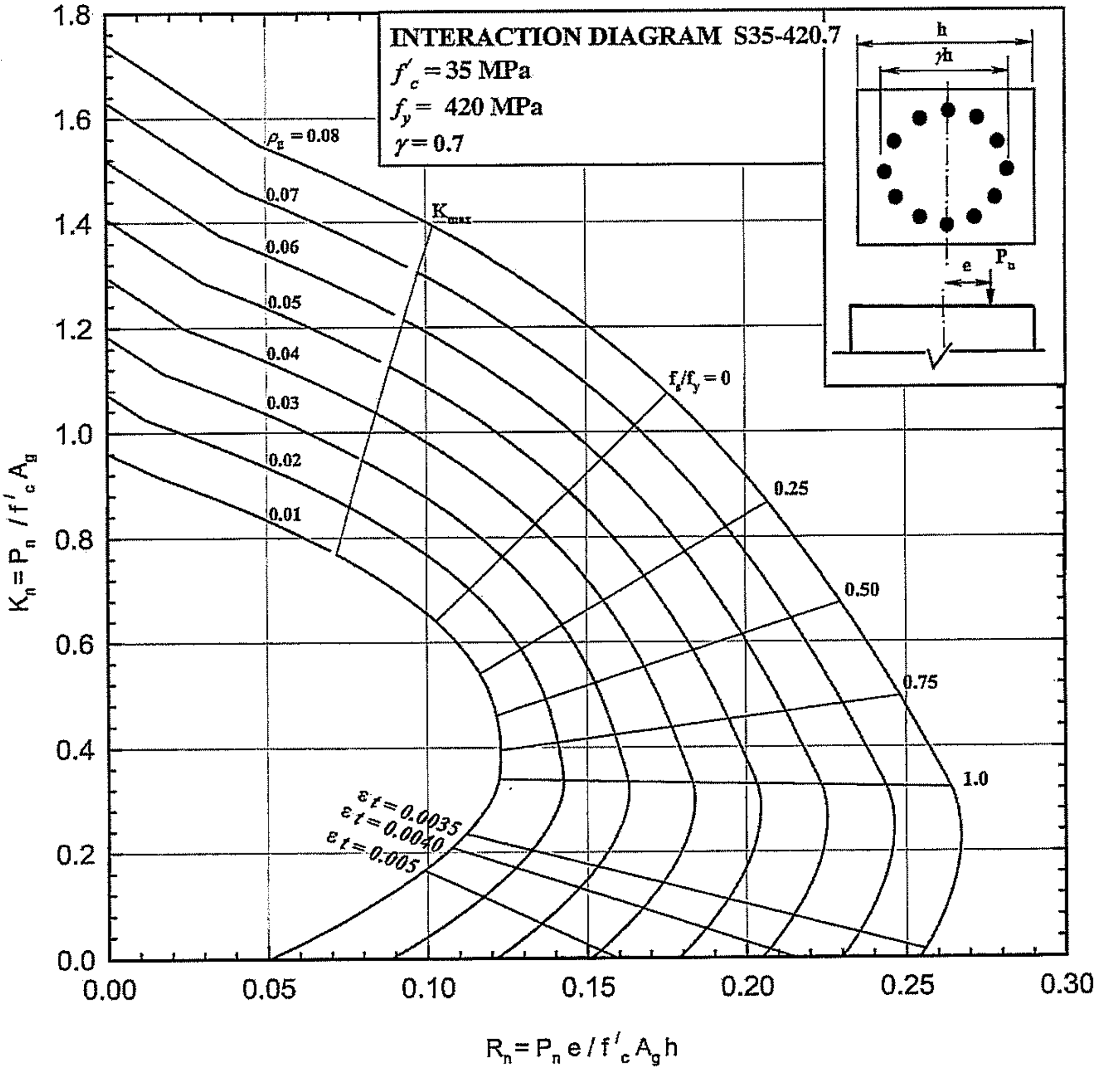

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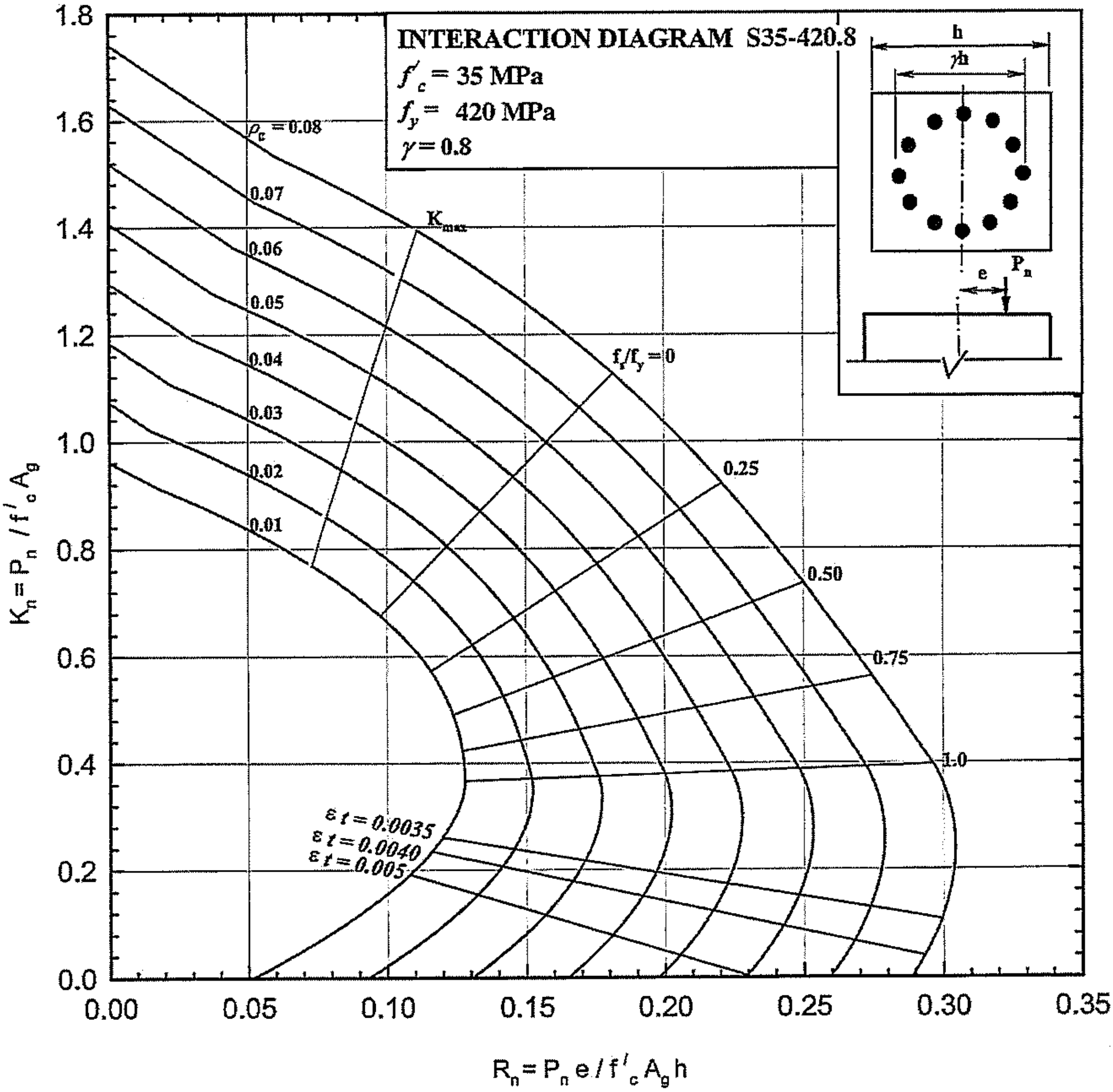

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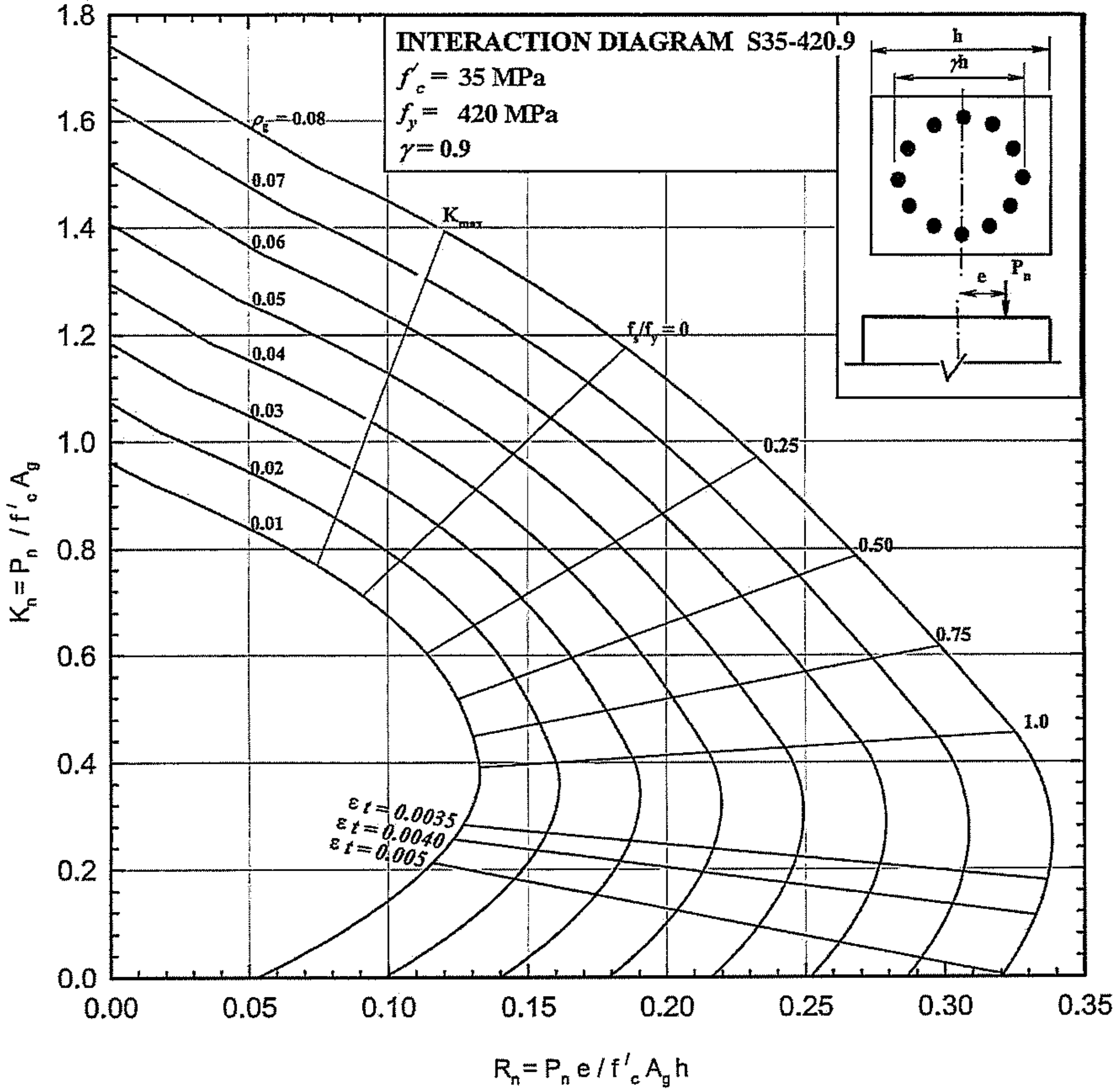

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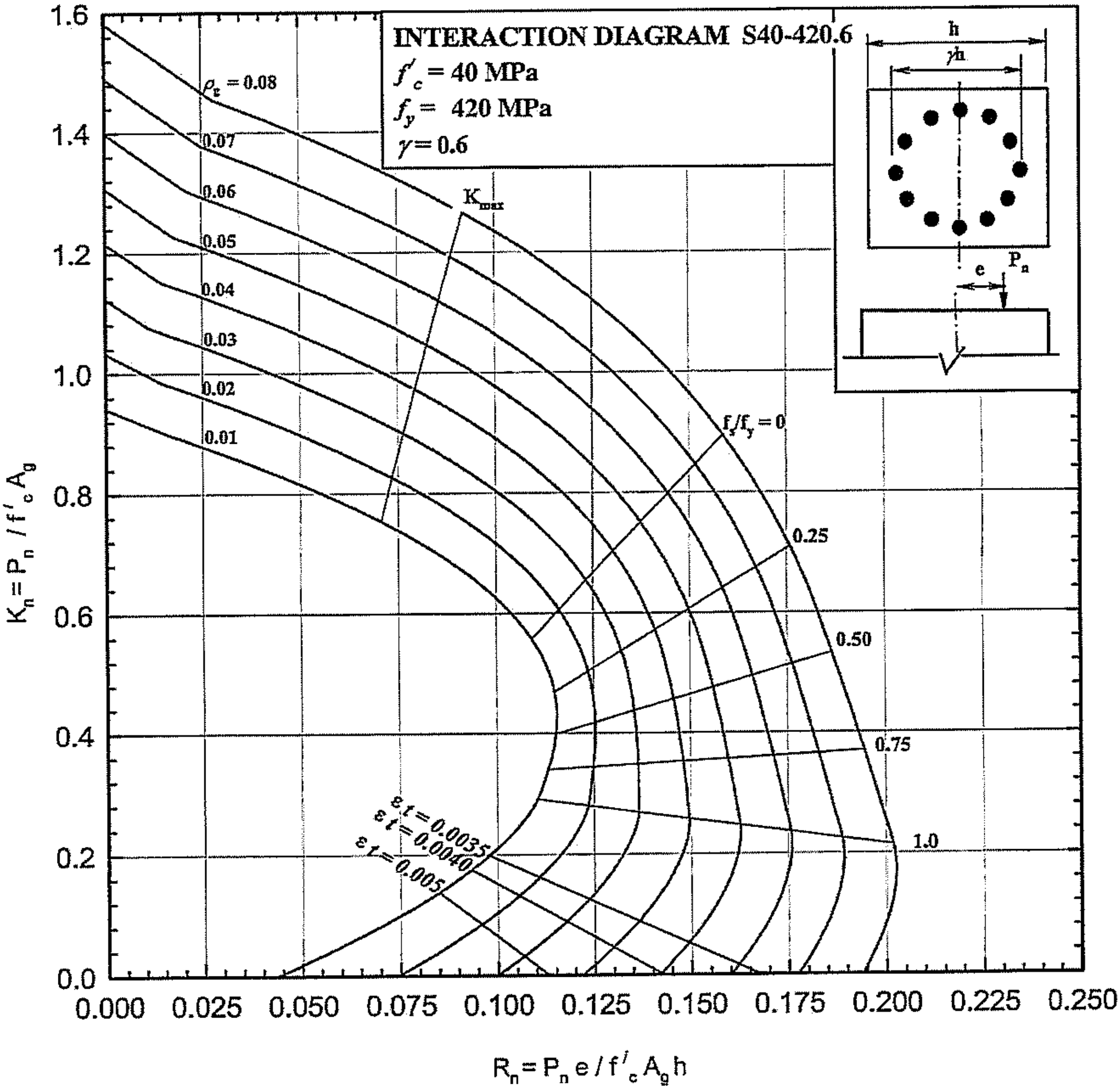
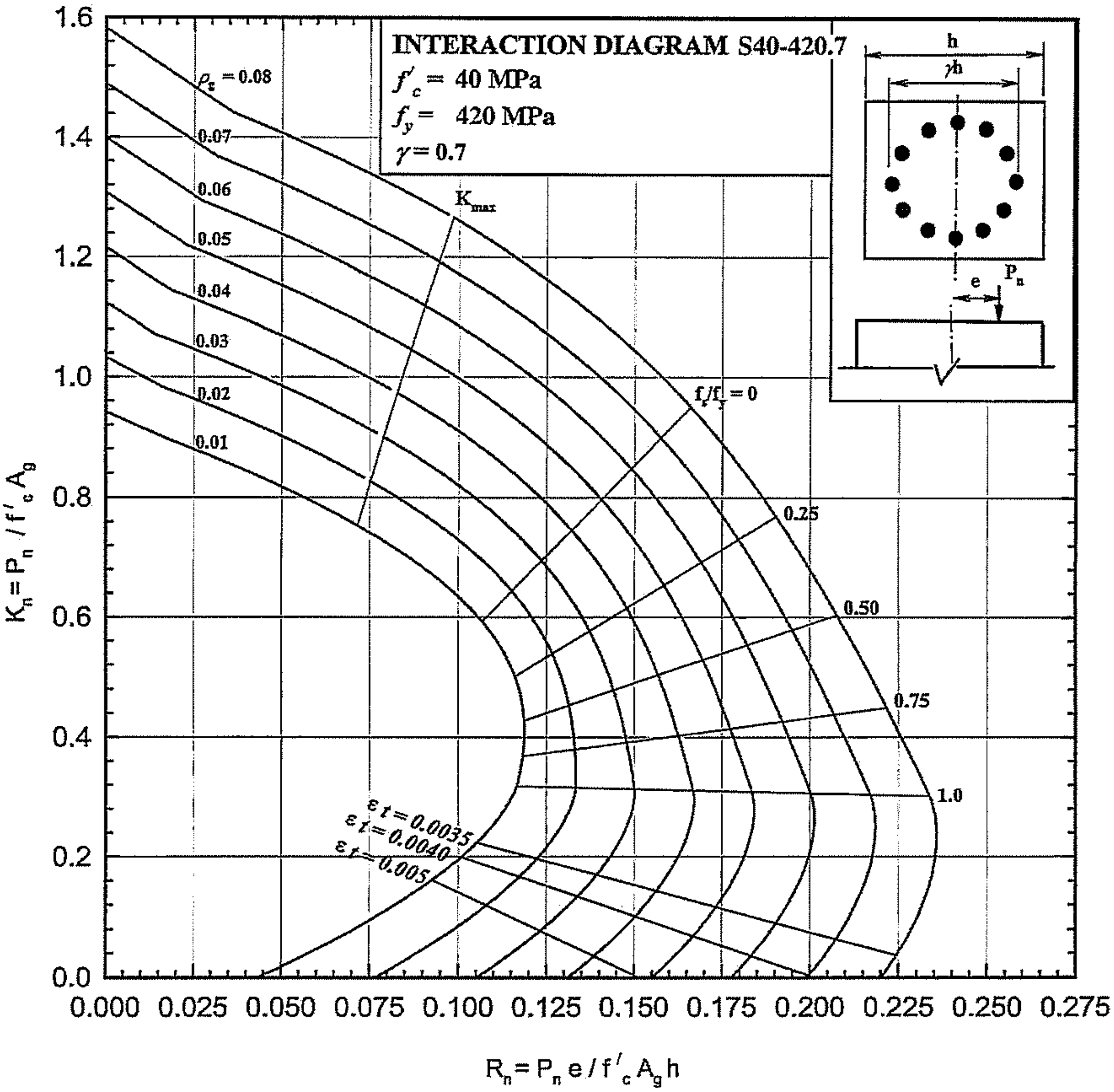


**COLUMNS 3.22.2 - Nominal load-moment strength interaction diagram, S40-420.7**



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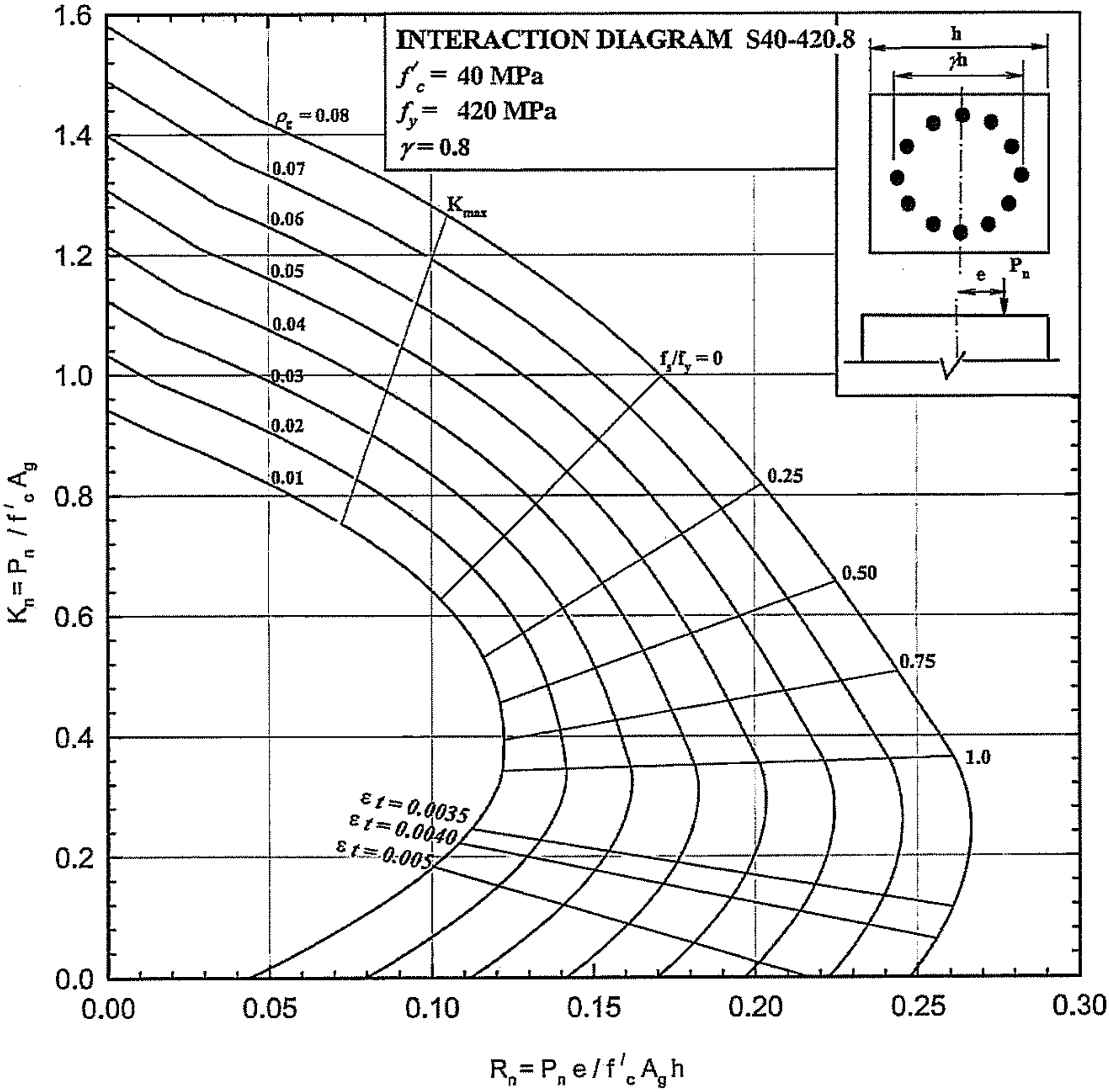

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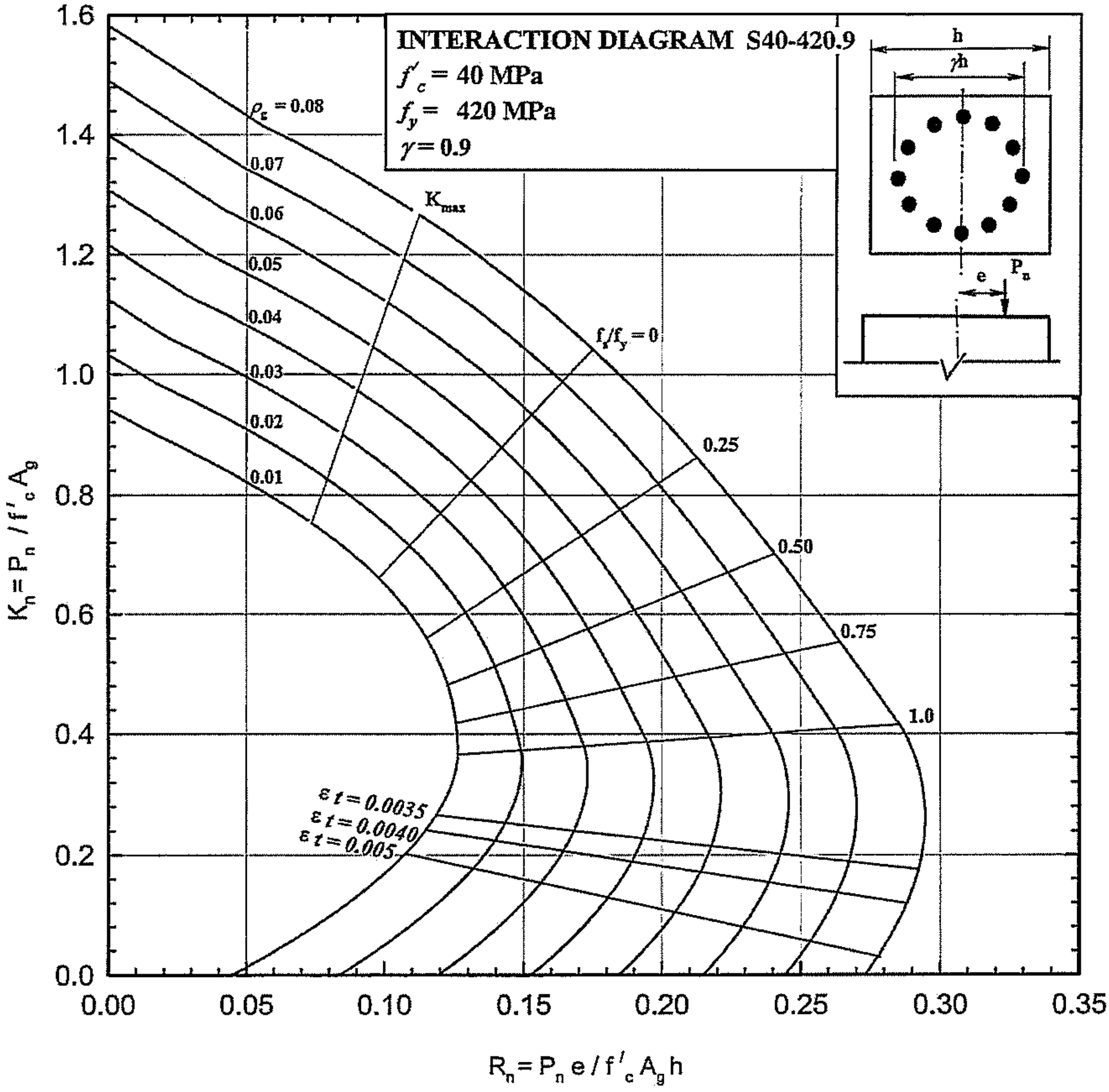

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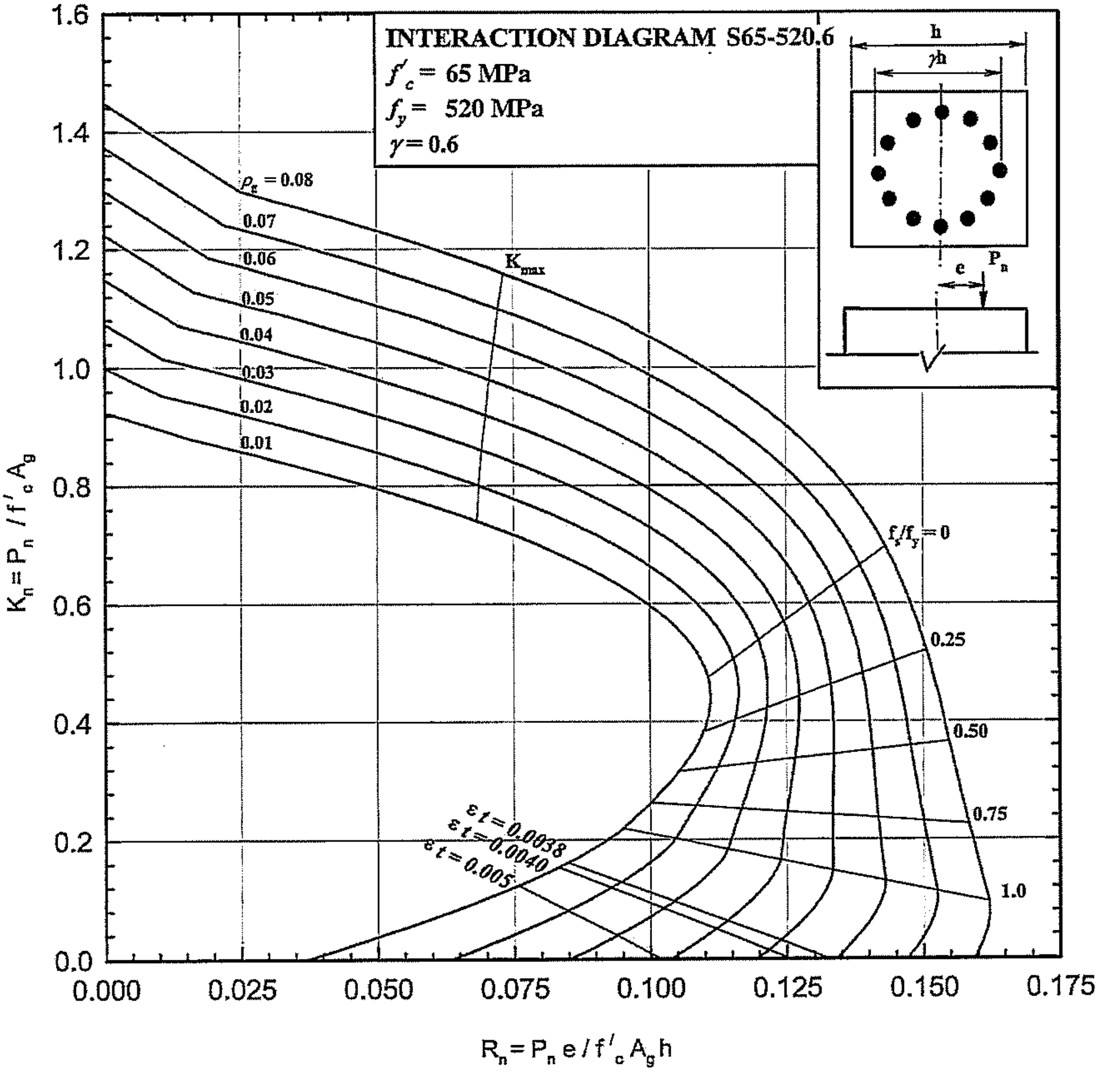

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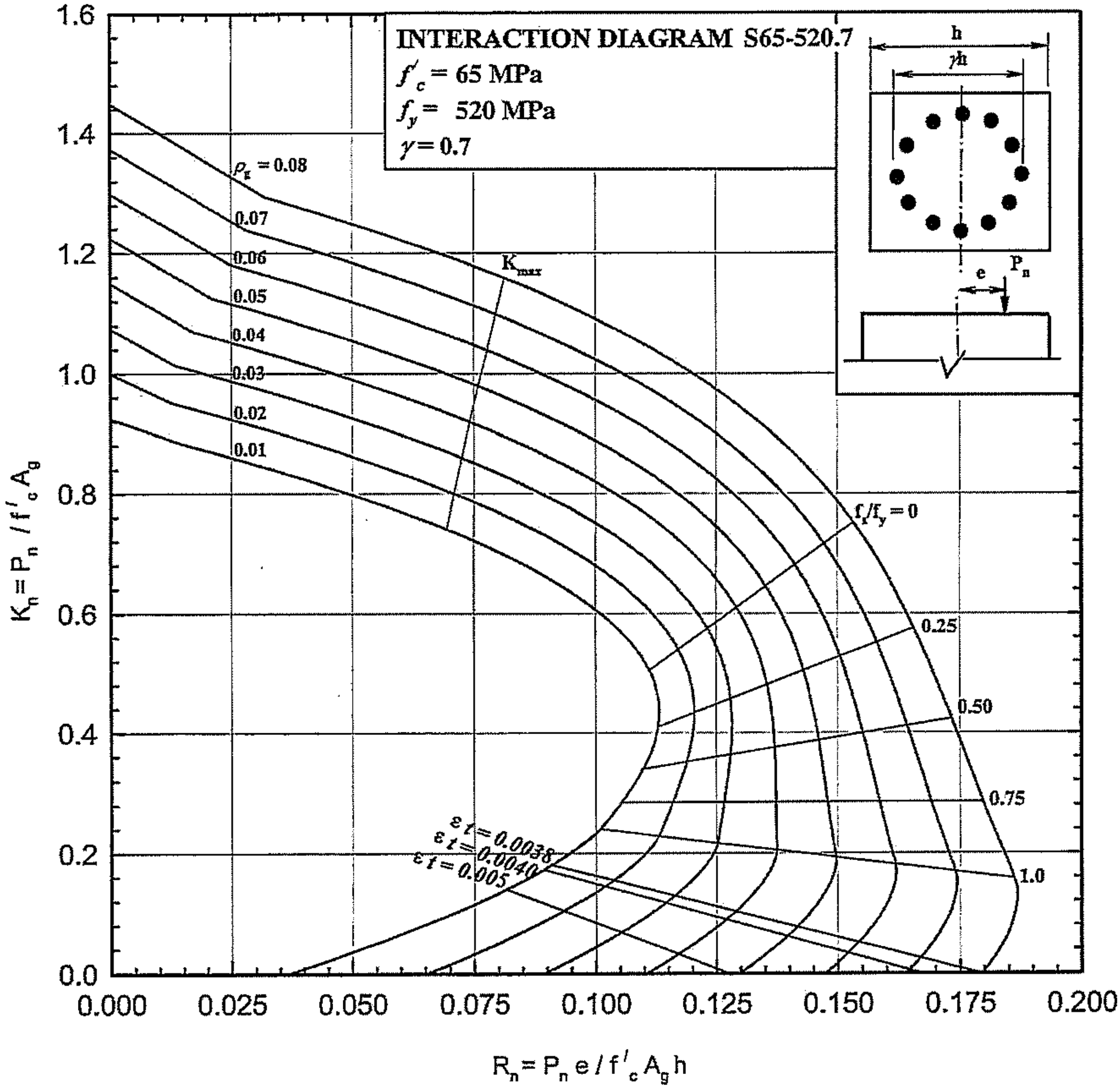

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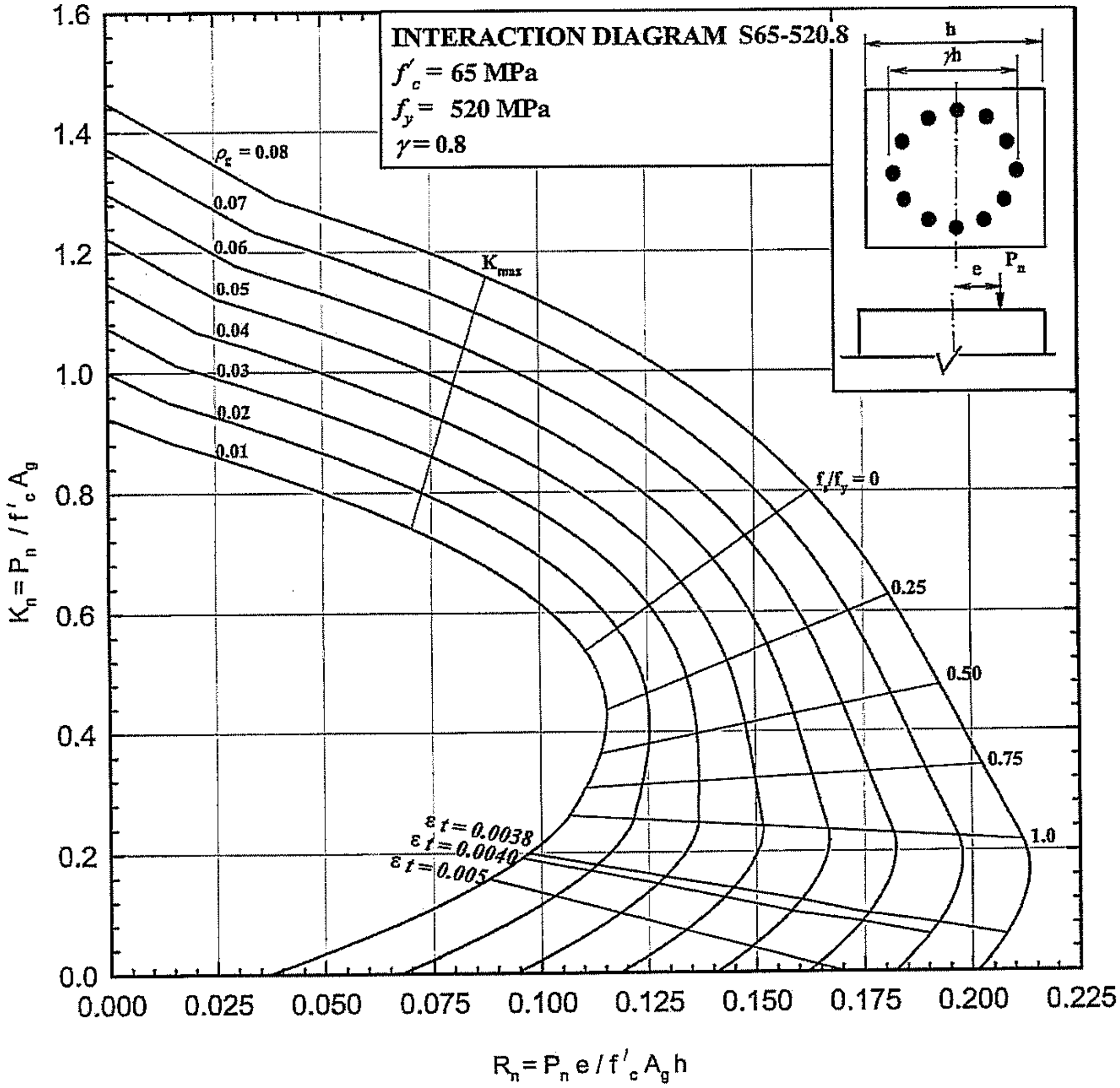

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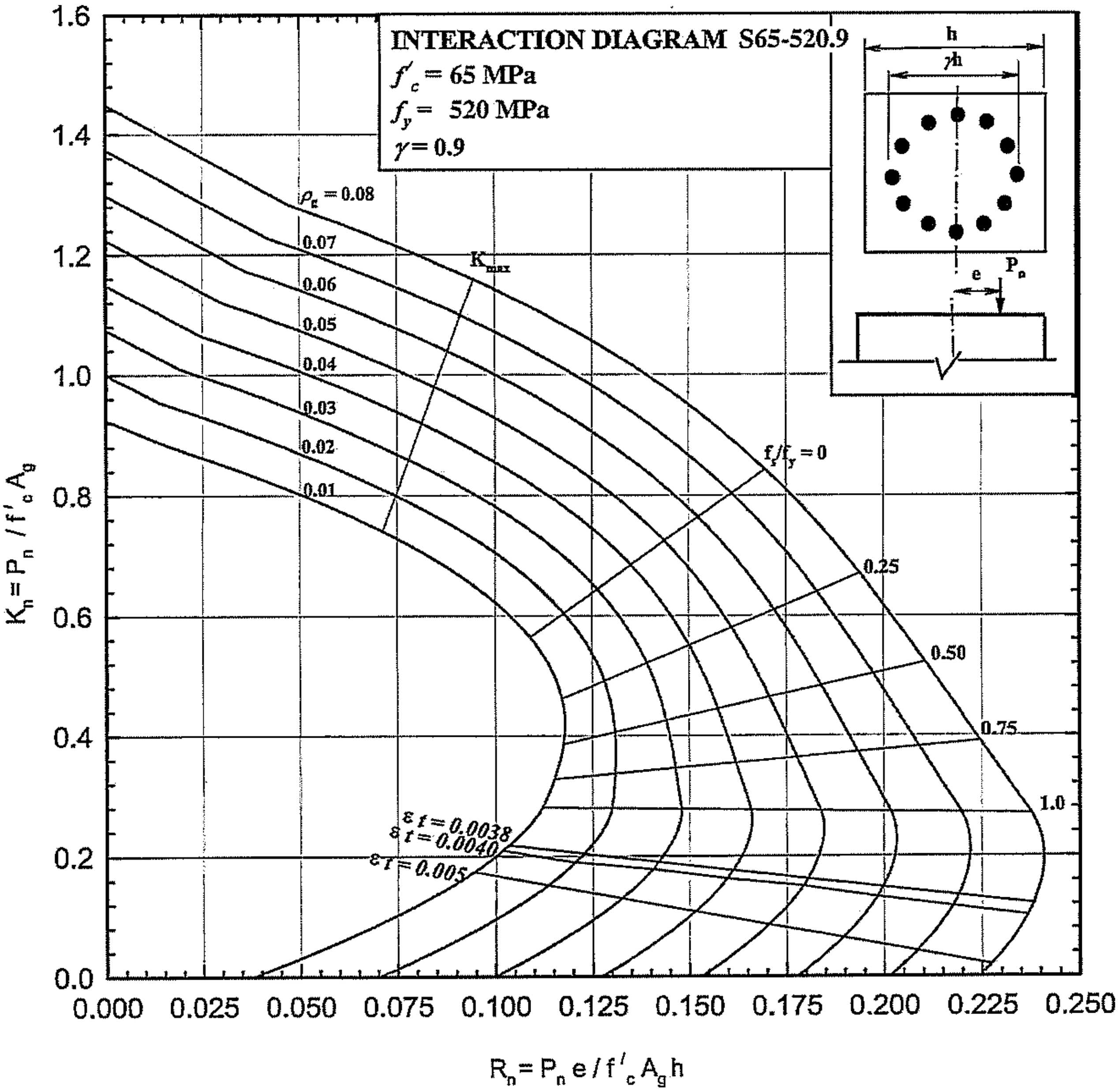
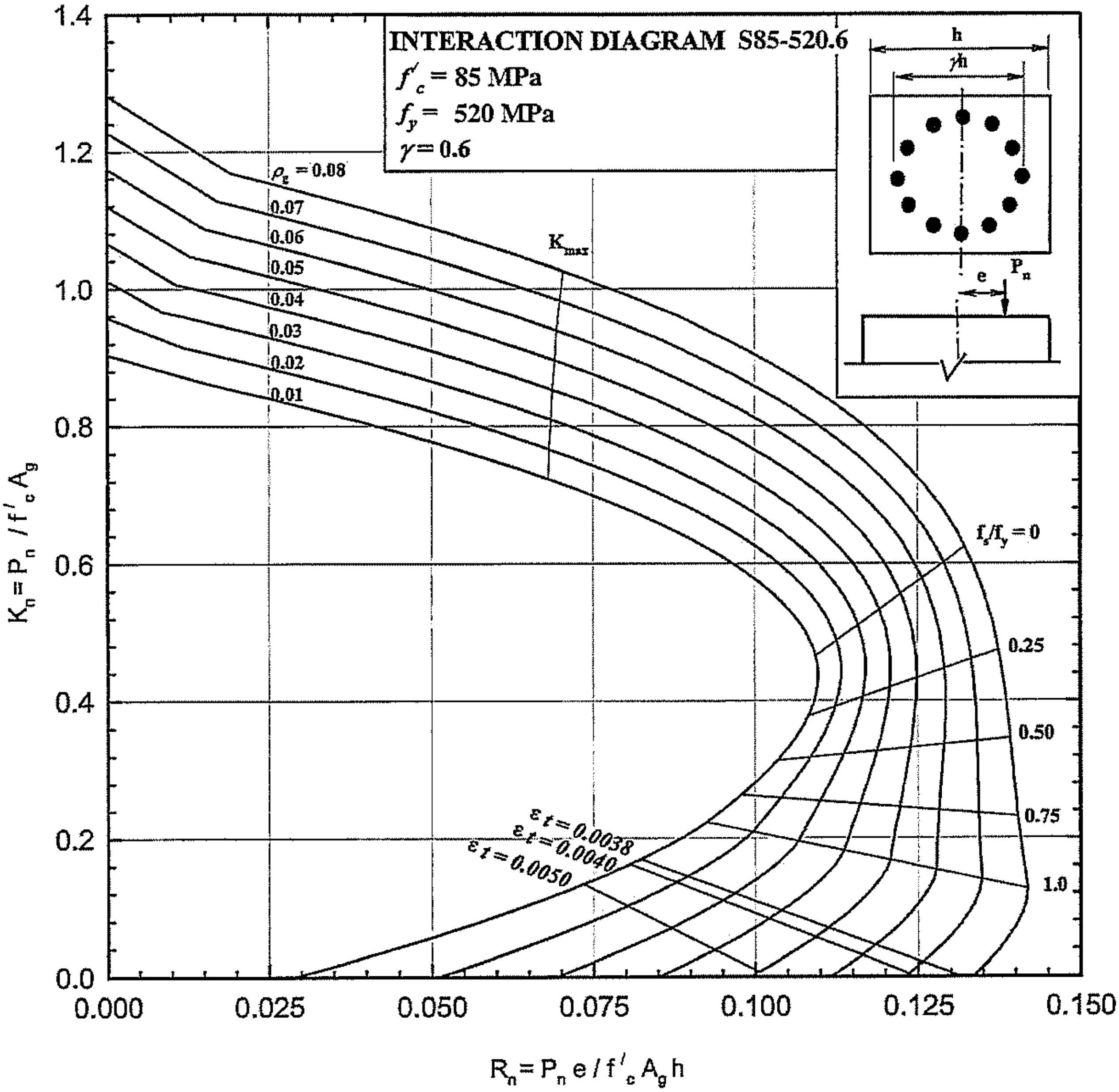
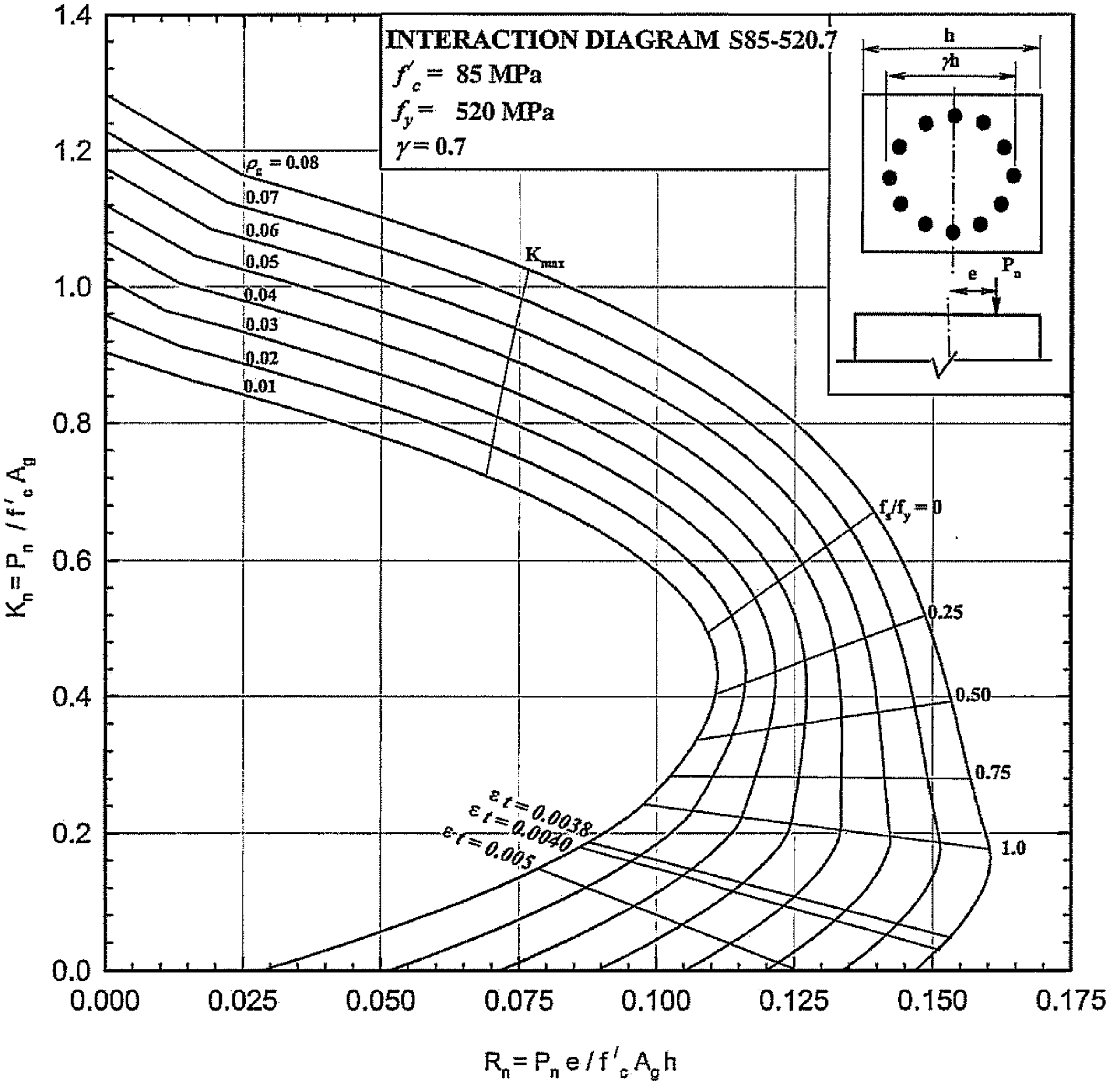


## COLUMNS 3.24.1 - Nominal load-moment strength interaction diagram, S85-520.6



**COLUMNS 3.24.2 - Nominal load-moment strength interaction diagram, S85-520.7**



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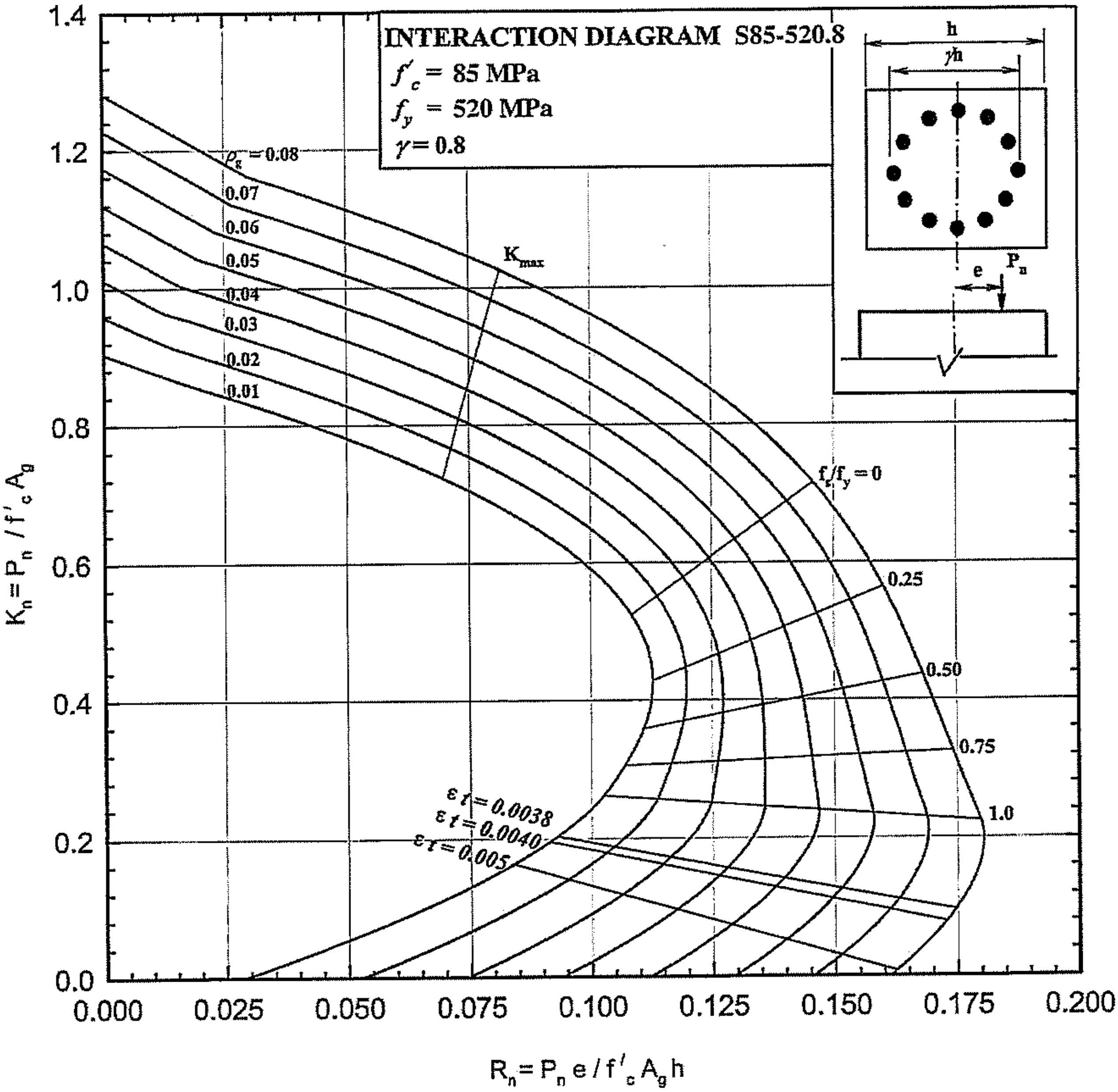

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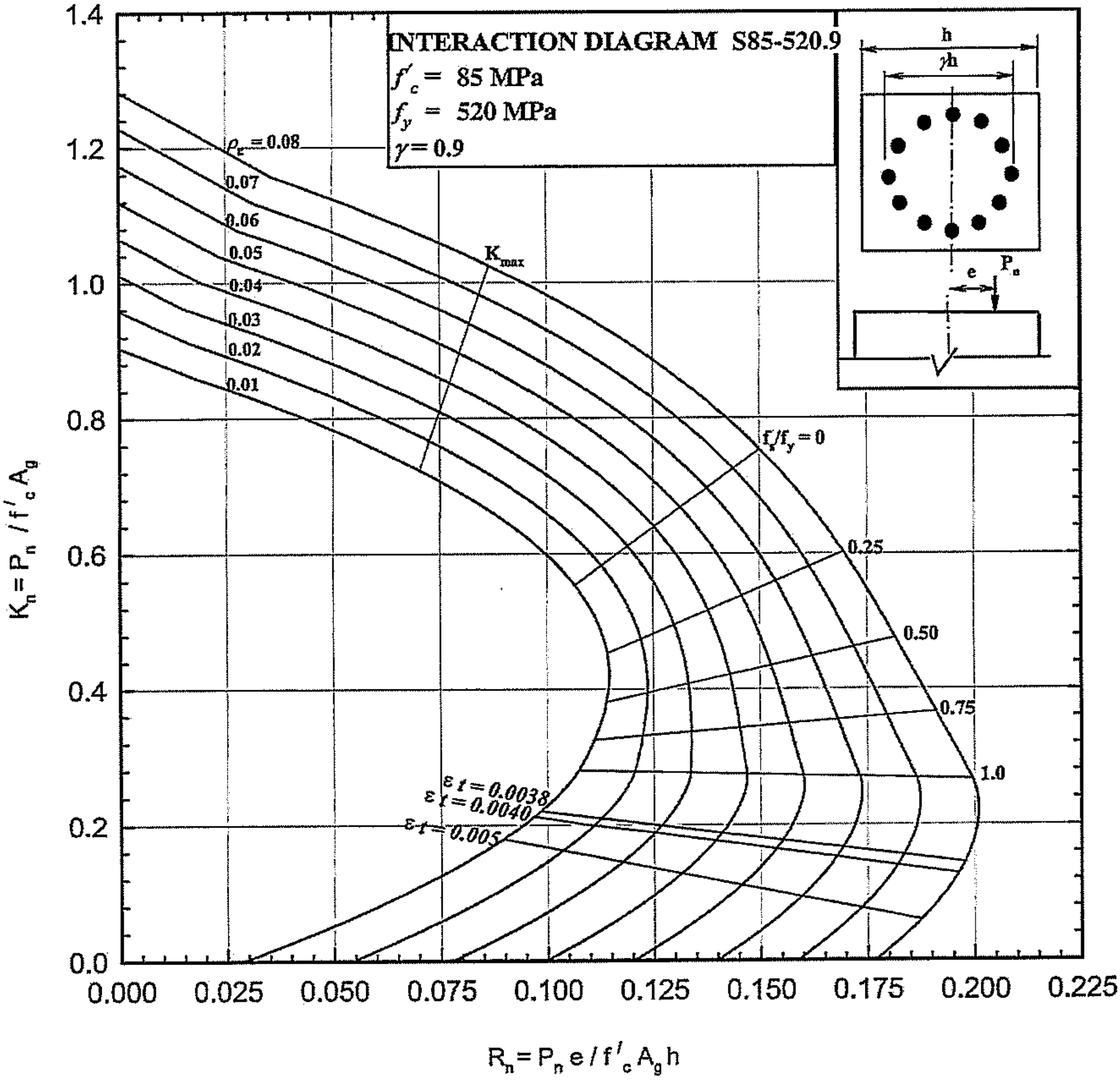


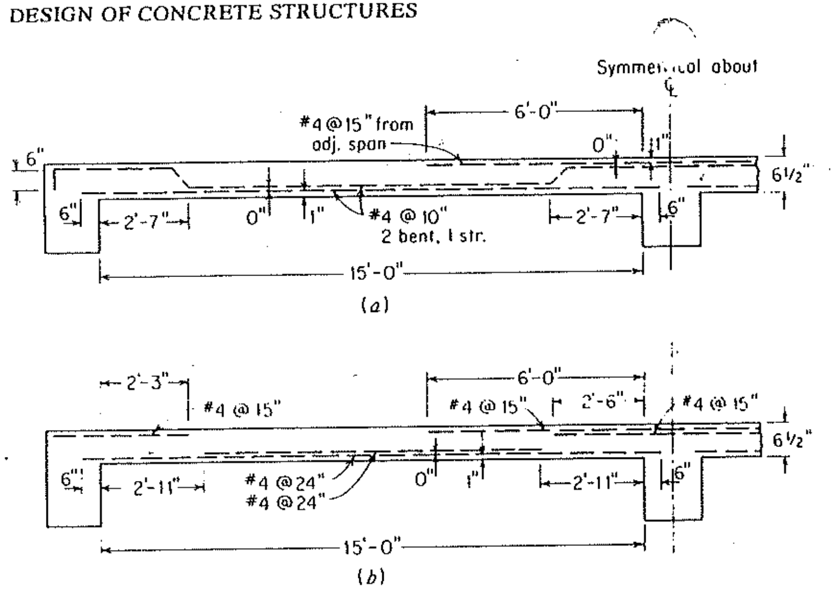



Figure 5.3

at the ends of those bars, where there will be concentrations of stress in the concrete. The design would be improved if the negative bars were cut off at 3 ft from the face of supports, rather than 2 ft 3 in and 2 ft 6 in as shown, and if the positive steel were cut off at 2 ft 2 in, rather than at 2 ft 11 in. This would result in an overlap of approximately $2d$ of the cut positive and negative bars.

The required area of steel to be placed normal to the main reinforcement for purposes of temperature and shrinkage crack control is $0.14\ in^2$. This will be provided by No. 4 bars at 16-in spacing, placed directly on top of the main reinforcement in the positive-moment region and below the main steel in the negative-moment zone.

## PROBLEMS

**5.1** A small bridge consisting of a concrete slab supported by steel stringers is to carry a uniformly distributed service load of 300 psf in addition to its own weight. The four stringers of the bridge, spanning in the long direction, are spaced 8 ft on centers. The concrete slab spans in the transverse direction and is continuous over the two interior stringers. Find the required thickness of the slab, and design and detail the bar reinforcement, using $f_y = 50{,}000$ psi and $f'_c = 3000$ psi. Bent bars will be used in preference to all-straight-bar reinforcement. The ACI moment coefficients do not apply. Use overload factors of 1.4 and 1.7 applied to dead and live loads, respectively. Use a maximum steel ratio of $0.50\rho_b$.

**5.2** Redesign the bridge slab of Prob. 5.1 using all straight bars rather than bent bars. Compare the alternate designs on the basis of weight of steel used, construction convenience, and safety.

**5.3** A footbridge is to be built, consisting of a one-way solid slab spanning 16 ft between masonry abutments, as shown in Fig. 5.4. A service load of 100 psf of bridge surface must be carried. In addition, a 2000-lb concentrated load, assumed to be distributed uniformly across the bridge width, may act at any location on the span. A 2-in asphalt wearing surface will be used, weighing 20 psf. Prepare a complete design, using $f'_c = 4000$ psi and $f_y = 60{,}000$ psi, following ACI Code provisions. Concrete curbs are nonstructural and will be added after the slab is poured.



Reinforced Concrete Design II

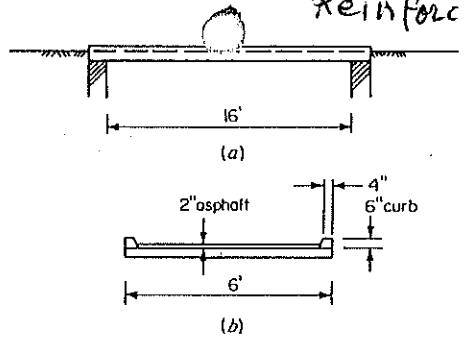



**Figure 5.4** One-way slab footbridge: (*a*) bridge profile, (*b*) cross section.

# TWO-WAY EDGE-SUPPORTED SLABS

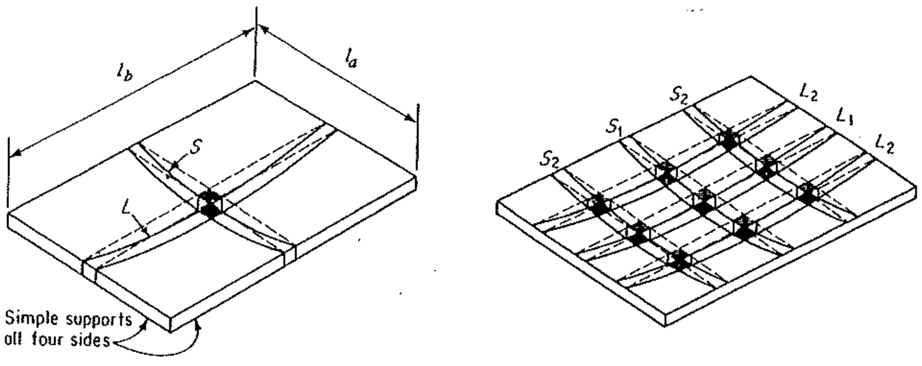
## 5.5 BEHAVIOR

The slabs discussed in Arts. 5.2 to 5.4 deform under load into a cylindrical surface. The main structural action is one-way in such cases, in the direction normal to supports on two opposite edges of a rectangular panel. In many cases, however, rectangular slabs are of such proportions and are supported in such a way that two-way action results. When loaded, such slabs bend into a dished surface rather than a cylindrical one. This means that at any point the slab is curved in both principal directions, and since bending moments are proportional to curvatures, moments also exist in both directions. To resist these moments, the slab must be reinforced in both directions, by two layers of bars perpendicular, respectively, to two pairs of edges. The slab must be designed to take a proportionate share of the load in each direction.

Types of reinforced-concrete construction which are characterized by two-way action include slabs supported by walls or beams on all sides (Fig. 5.1*b*), flat plates (Fig. 5.1*d*), flat slabs (Fig. 5.1*e*), and grid slabs (Fig. 5.1*f*).

The simplest type of two-way slab action is that represented by Fig. 5.1*b*, where the slab, or slab panel, is supported along its four edges by relatively deep, stiff, monolithic concrete beams or by walls or steel girders. If the concrete edge beams are shallow or are omitted altogether, as for flat plates and flat slabs, deformation of the floor system along the column lines significantly alters the distribution of moments in the slab panel itself (Ref. 5.1). Two-way systems of this type are considered separately, in Arts. 5.8 to 5.16. The present discussion pertains to the former type, in which edge supports are stiff enough to be considered unyielding.

Such a slab is shown in Fig. 5.5*a*. To visualize its flexural performance it is convenient to think of it as consisting of two sets of parallel strips, in each of the two directions, intersecting each other. Evidently, part of the load is carried by one set and transmitted to one pair of edge supports, and the remainder by the other.




(a) Bending of center strips of slab

(b) Grid model of slab

**Figure 5.5** Two-way slab on simple edge supports.

Figure 5.5*a* shows the two center strips of a rectangular plate with short span $l_a$ and long span $l_b$. If the uniform load is $w$ per square foot of slab, each of the two strips acts approximately like a simple beam uniformly loaded by its share of $w$. Because these imaginary strips actually are part of the same monolithic slab, their deflections at the intersection point must be the same. Equating the center deflections of the short and long strips gives

$$\frac{5w_a l_a^4}{384EI} = \frac{5w_b l_b^4}{384EI} \qquad (a)$$

where $w_a$ is the share of the load $w$ carried in the short direction and $w_b$ is the share of the load $w$ carried in the long direction. Consequently,

$$\frac{w_a}{w_b} = \frac{l_b^4}{l_a^4} \qquad (b)$$

One sees that the larger share of the load is carried in the short direction, the ratio of the two portions of the total load being inversely proportional to the fourth power of the ratio of the spans.

This result is approximate because the actual behavior of a slab is more complex than that of the two intersecting strips. An understanding of the behavior of the slab itself can be gained from Fig. 5.5*b*, which shows a slab model consisting of two sets of three strips each. It is seen that the two central strips $S_1$ and $L_1$ bend in a manner similar to that of Fig. 5.5*a*. The outer strips $S_2$ and $L_2$, however, are not only bent but also twisted. Consider, for instance, one of the intersections of $S_2$ with $L_2$. It is seen that at the intersection the exterior edge of strip $L_2$ is at a higher elevation than the interior edge, while at the nearby end of strip $L_2$ both edges are at the same elevation; the strip is twisted. This twisting results in torsional stresses and torsional moments which are seen to be most pronounced near the corners. Consequently, the total load on the slab is carried not only by the bending



moments in two directions but also by the twisting moments. For this reason bending moments in elastic slabs are smaller than would be computed for sets of unconnected strips loaded by $w_a$ and $w_b$. For instance, for a simply supported square slab, $w_a = w_b = w/2$. If only bending were present, the maximum moment in each strip would be

$$\frac{(w/2)l^2}{8} = 0.0625wl^2 \qquad (c)$$

The exact theory of bending of elastic plates shows that, actually, the maximum moment in such a square slab is only $0.048wl^2$, so that in this case the twisting moments relieve the bending moments by about 25 percent.

The largest moment occurs where the curvature is sharpest. Figure 5.5*b* shows this to be the case at midspan of the short strip $S_1$. Suppose the load is increased until this location is overstressed, so that the steel at the middle of strip $S_1$ is yielding. If the strip were an isolated beam, it would now fail. Considering the slab as a whole, however, one sees that no immediate failure will occur. The neighboring strips (those parallel as well as those perpendicular to $S_1$), being actually monolithic with it, will take over that share of any additional load which strip $S_1$ can no longer carry until they in turn start yielding. This inelastic redistribution will continue until in a rather large area in the central portion of the slab all the steel in both directions is yielding. Only then will the entire slab fail. From this reasoning, which is confirmed by tests, it follows that slabs need not be designed for the absolute maximum moment in each of the two directions (such as $0.048wl^2$ in the example of the previous paragraph) but only for a smaller average moment in each of the two directions in the central portion of the slab. For instance, one of the several analytical methods in general use permits the above square slab to be designed for a moment of $0.036wl^2$. By comparison with the actual elastic maximum moment $0.048wl^2$, it is seen that, owing to inelastic redistribution, a moment reduction of 25 percent is provided.

The largest moment in the slab occurs at midspan of the short strip $S_1$ of Fig. 5.5*b*. It is evident that the curvature, hence the moment, in the short strip $S_2$ is less than at the corresponding location of strip $S_1$. Consequently, a variation of short-span moment occurs in the long direction of the span. This variation is shown qualitatively in Fig. 5.6. The short-span-moment diagram in Fig. 5.6*a* is valid only along the center strip at 1-1. Elsewhere the maximum-moment value is less, as shown in Fig. 5.6*b*; all other moment ordinates are reduced proportionately. Similarly, the long-span-moment diagram in Fig. 5.6*c* applies only at the longitudinal centerline of the slab; elsewhere ordinates are reduced according to the variation shown in Fig. 5.6*d*. These variations in maximum moment across the width and length of a rectangular slab are accounted for in an approximate way in most practical design methods by designing for a reduced moment in the outer quarters of the slab span in each direction.

It should be noted that only slabs with side ratios less than about 2 need

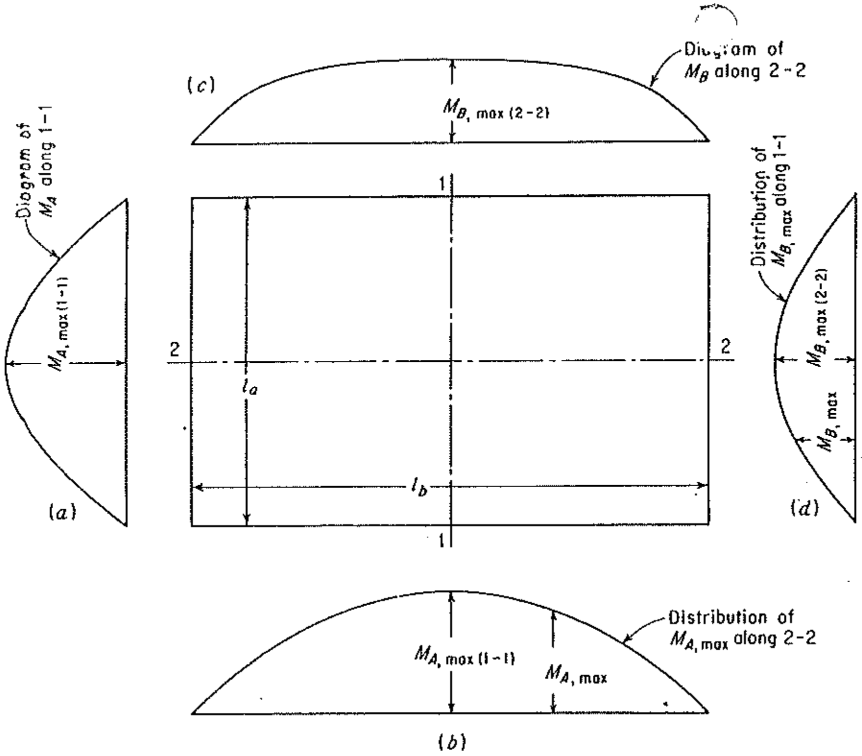



**Figure 5.6** Moments in uniformly loaded, simply supported slab.

be treated as two-way slabs. From Eq. (*b*) above it is seen that for a slab of this proportion the share of the load carried in the long direction is only of the order of one-sixteenth that in the short direction. Such a slab acts almost as if it were spanning in the short direction only. Consequently, rectangular slab panels with aspect ratio of 2 or more may be reinforced for one-way action, with the main steel perpendicular to the long edges. Shrinkage and temperature steel should be provided in the long direction, of course, and auxiliary reinforcement should be provided over, and perpendicular to, the short support beams and at the slab corners to control cracking (see Art. 5.6).

## 5.6 ANALYSIS BY COEFFICIENT METHOD

The precise determination of moments in two-way slabs with various conditions of continuity at the supported edges is mathematically formidable and not suited to design practice. For this reason, various simplified methods have been adopted for determining moments, shears, and reactions of such slabs.



In earlier editions of the ACI Code (Ref. 5.2), three different methods were presented.† The most rational and widely used will be described here.

The method makes use of tables of moment coefficients for a variety of conditions. These coefficients are based on elastic analysis but also account for inelastic redistribution. In consequence, the design moment in either direction is smaller by an appropriate amount than the elastic maximum moment in that direction. The moments in the two directions are computed from

$$M_a = C_a w l_a^2 \tag{5.1a}$$

and

$$M_b = C_b w l_b^2 \tag{5.1b}$$

where $C_a$, $C_b$ = tabulated moment coefficients
$w$ = uniform load, psf
$l_a$, $l_b$ = length of clear span in short and long directions, respectively

The method provides that each panel be divided in both directions into a middle strip whose width is one-half that of the panel and two column strips of one-quarter of the panel width (see Fig. 5.7). As discussed before and shown in Fig. 5.6*b* and *d*, the moments in both directions are larger in the center portion of the slab than in regions close to the edges. Correspondingly, it is provided that the entire middle strip be designed for the full, tabulated design moment. In the column strips this moment is assumed to decrease from its full value at the edge of the middle strip to one-third of this value at the edge of the panel. This distribution is shown in Fig. 5.7*a* and *b*.

The discussion so far has been restricted to a single panel simply supported at all four edges. An actual situation is shown in Fig. 5.8, in which a system of beams supports a two-way slab. It is seen that some panels, such as *A*, have two discontinuous exterior edges, while the other edges are continuous with their neighbors. Panel *B* has one edge discontinuous and three continuous edges, the interior panel *C* has all edges continuous, and so on. At a continuous edge in a slab, moments are negative, just as at interior supports of continuous beams. Also, the magnitude of the positive moments depends on the conditions of continuity at all four edges.

Correspondingly, Table 5.2 gives moment coefficients *C*, for *negative moments at continuous edges*. The details of the tables are self-explanatory. Maximum negative edge moments are obtained when both panels adjacent to the particular edge carry full dead and live load. Hence the moment is

†All were deleted from the 1971 and 1977 editions of the Code so that all types of two-way concrete construction, including edge-supported slabs, flat slabs, and flat plates, could be treated by one unified method. However, the complexity of the generalized method for design of two-way systems in the 1971 and 1977 editions of the Code has led many engineers to use the design method of the 1963 Code for the special case of edge-supported slabs. The method of design described in this article has been used extensively for two-way slabs supported at the edges by walls, steel beams, or concrete beams having total depth not less than about 3 times the slab thickness. Its continued use is endorsed in the ACI Code Commentary 318-77.

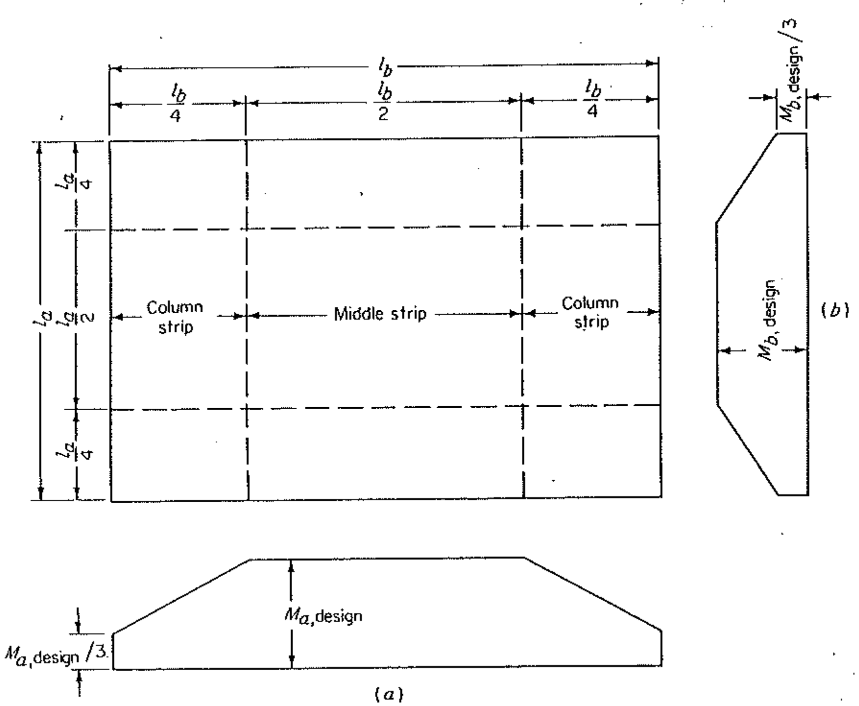



**Figure 5.7** Variation of design moments across width of critical sections for simply supported two-way slab.

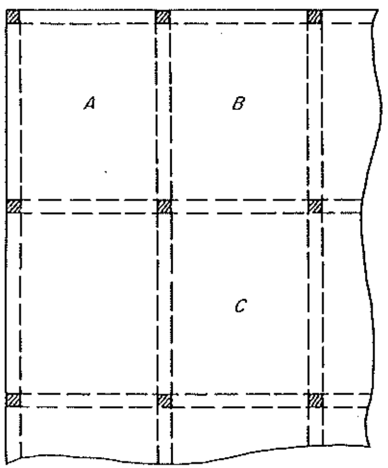



**Figure 5.8** Portion of typical two-way slab floor with beams on column lines.



**Table 5.2 Coefficients for negative moments in slabs†**

$M_{a,neg} = C_{a,neg} w l_a^2$
$M_{b,neg} = C_{b,neg} w l_b^2$ where $w$ = total uniform dead plus live load

| Ratio $m = \frac{l_a}{l_b}$ | | Case 1 | Case 2 | Case 3 | Case 4 | Case 5 | Case 6 | Case 7 | Case 8 | Case 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.00 | $C_{a,neg}$ | | 0.045 | | 0.050 | 0.075 | 0.071 | | 0.033 | 0.061 |
| | $C_{b,neg}$ | | 0.045 | 0.076 | 0.050 | | | 0.071 | 0.061 | 0.033 |
| 0.95 | $C_{a,neg}$ | | 0.050 | | 0.055 | 0.079 | 0.075 | | 0.038 | 0.065 |
| | $C_{b,neg}$ | | 0.041 | 0.072 | 0.045 | | | 0.067 | 0.056 | 0.029 |
| 0.90 | $C_{a,neg}$ | | 0.055 | | 0.060 | 0.080 | 0.079 | | 0.043 | 0.068 |
| | $C_{b,neg}$ | | 0.037 | 0.070 | 0.040 | | | 0.062 | 0.052 | 0.025 |
| 0.85 | $C_{a,neg}$ | | 0.060 | | 0.066 | 0.082 | 0.083 | | 0.049 | 0.072 |
| | $C_{b,neg}$ | | 0.031 | 0.065 | 0.034 | | | 0.057 | 0.046 | 0.021 |
| 0.80 | $C_{a,neg}$ | | 0.065 | | 0.071 | 0.083 | 0.086 | | 0.055 | 0.075 |
| | $C_{b,neg}$ | | 0.027 | 0.061 | 0.029 | | | 0.051 | 0.041 | 0.017 |
| 0.75 | $C_{a,neg}$ | | 0.069 | | 0.076 | 0.085 | 0.088 | | 0.061 | 0.078 |
| | $C_{b,neg}$ | | 0.022 | 0.056 | 0.024 | | | 0.044 | 0.036 | 0.014 |
| 0.70 | $C_{a,neg}$ | | 0.074 | | 0.081 | 0.086 | 0.091 | | 0.068 | 0.081 |
| | $C_{b,neg}$ | | 0.017 | 0.050 | 0.019 | | | 0.038 | 0.029 | 0.011 |
| 0.65 | $C_{a,neg}$ | | 0.077 | | 0.085 | 0.087 | 0.093 | | 0.074 | 0.083 |
| | $C_{b,neg}$ | | 0.014 | 0.043 | 0.015 | | | 0.031 | 0.024 | 0.008 |
| 0.60 | $C_{a,neg}$ | | 0.081 | | 0.089 | 0.088 | 0.095 | | 0.080 | 0.085 |
| | $C_{b,neg}$ | | 0.010 | 0.035 | 0.011 | | | 0.024 | 0.018 | 0.006 |
| 0.55 | $C_{a,neg}$ | | 0.084 | | 0.092 | 0.089 | 0.096 | | 0.085 | 0.086 |
| | $C_{b,neg}$ | | 0.007 | 0.028 | 0.008 | | | 0.019 | 0.014 | 0.005 |
| 0.50 | $C_{a,neg}$ | | 0.086 | | 0.094 | 0.090 | 0.097 | | 0.089 | 0.088 |
| | $C_{b,neg}$ | | 0.006 | 0.022 | 0.006 | | | 0.014 | 0.010 | 0.003 |

†A crosshatched edge indicates that the slab continues across, or is fixed at, the support; an unmarked edge indicates a support at which torsional resistance is negligible.

computed for this total load. *Negative moments at discontinuous edges* are assumed equal to one-third of the positive moments for the same direction. One must provide for such moments because some degree of restraint is provided discontinuous edges by the torsional rigidity of the edge beam or by the supporting wall.

For *positive moments* there will be little, if any, rotation at the continuous

**Table 5.3 Coefficients for dead-load positive moments in slabs†**

$M_{a,pos,dl} = C_{a,dl}wl_a^2$
$M_{b,pos,dl} = C_{b,dl}wl_b^2$ where $w$ = total uniform dead load

| Ratio $m = \frac{l_a}{l_b}$ | | Case 1 | Case 2 | Case 3 | Case 4 | Case 5 | Case 6 | Case 7 | Case 8 | Case 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.00 | $C_{a,dl}$ | 0.036 | 0.018 | 0.018 | 0.027 | 0.027 | 0.033 | 0.027 | 0.020 | 0.023 |
| | $C_{b,dl}$ | 0.036 | 0.018 | 0.027 | 0.027 | 0.018 | 0.027 | 0.033 | 0.023 | 0.020 |
| 0.95 | $C_{a,dl}$ | 0.040 | 0.020 | 0.021 | 0.030 | 0.028 | 0.036 | 0.031 | 0.022 | 0.024 |
| | $C_{b,dl}$ | 0.033 | 0.016 | 0.025 | 0.024 | 0.015 | 0.024 | 0.031 | 0.021 | 0.017 |
| 0.90 | $C_{a,dl}$ | 0.045 | 0.022 | 0.025 | 0.033 | 0.029 | 0.039 | 0.035 | 0.025 | 0.026 |
| | $C_{b,dl}$ | 0.029 | 0.014 | 0.024 | 0.022 | 0.013 | 0.021 | 0.028 | 0.019 | 0.015 |
| 0.85 | $C_{a,dl}$ | 0.050 | 0.024 | 0.029 | 0.036 | 0.031 | 0.042 | 0.040 | 0.029 | 0.028 |
| | $C_{b,dl}$ | 0.026 | 0.012 | 0.022 | 0.019 | 0.011 | 0.017 | 0.025 | 0.017 | 0.013 |
| 0.80 | $C_{a,dl}$ | 0.056 | 0.026 | 0.034 | 0.039 | 0.032 | 0.045 | 0.045 | 0.032 | 0.029 |
| | $C_{b,dl}$ | 0.023 | 0.011 | 0.020 | 0.016 | 0.009 | 0.015 | 0.022 | 0.015 | 0.010 |
| 0.75 | $C_{a,dl}$ | 0.061 | 0.028 | 0.040 | 0.043 | 0.033 | 0.048 | 0.051 | 0.036 | 0.031 |
| | $C_{b,dl}$ | 0.019 | 0.009 | 0.018 | 0.013 | 0.007 | 0.012 | 0.020 | 0.013 | 0.007 |
| 0.70 | $C_{a,dl}$ | 0.068 | 0.030 | 0.046 | 0.046 | 0.035 | 0.051 | 0.058 | 0.040 | 0.033 |
| | $C_{b,dl}$ | 0.016 | 0.007 | 0.016 | 0.011 | 0.005 | 0.009 | 0.017 | 0.011 | 0.006 |
| 0.65 | $C_{a,dl}$ | 0.074 | 0.032 | 0.054 | 0.050 | 0.036 | 0.054 | 0.065 | 0.044 | 0.034 |
| | $C_{b,dl}$ | 0.013 | 0.006 | 0.014 | 0.009 | 0.004 | 0.007 | 0.014 | 0.009 | 0.005 |
| 0.60 | $C_{a,dl}$ | 0.081 | 0.034 | 0.062 | 0.053 | 0.037 | 0.056 | 0.073 | 0.048 | 0.036 |
| | $C_{b,dl}$ | 0.010 | 0.004 | 0.011 | 0.007 | 0.003 | 0.006 | 0.012 | 0.007 | 0.004 |
| 0.55 | $C_{a,dl}$ | 0.088 | 0.035 | 0.071 | 0.056 | 0.038 | 0.058 | 0.081 | 0.052 | 0.037 |
| | $C_{b,dl}$ | 0.008 | 0.003 | 0.009 | 0.005 | 0.002 | 0.004 | 0.009 | 0.005 | 0.003 |
| 0.50 | $C_{a,dl}$ | 0.095 | 0.037 | 0.080 | 0.059 | 0.039 | 0.061 | 0.089 | 0.056 | 0.038 |
| | $C_{b,dl}$ | 0.006 | 0.002 | 0.007 | 0.004 | 0.001 | 0.003 | 0.007 | 0.004 | 0.002 |

†A crosshatched edge indicates that the slab continues across, or is fixed at, the support; an unmarked edge indicates a support at which torsional resistance is negligible.

edges if *dead load* alone is acting, because the loads on both adjacent panels tend to produce opposite rotations which cancel, or nearly so. For this condition, the continuous edges can be regarded as fixed, and the appropriate coefficients for the dead-load moments are given in Table 5.3. On the other hand, the maximum *live-load moments* are obtained when live load is placed only on the particular panel and not on any of the adjacent panels. In this



**Table 5.4 Coefficients for live-load positive moments in slabs†**

$M_{a,pos,ll} = C_{a,ll}wl_a^2$
$M_{b,pos,ll} = C_{b,ll}wl_b^2$ where $w$ = total uniform live load

| Ratio $m = \frac{l_a}{l_b}$ | | Case 1 | Case 2 | Case 3 | Case 4 | Case 5 | Case 6 | Case 7 | Case 8 | Case 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.00 | $C_{a,ll}$ | 0.036 | 0.027 | 0.027 | 0.032 | 0.032 | 0.035 | 0.032 | 0.028 | 0.030 |
| | $C_{b,ll}$ | 0.036 | 0.027 | 0.032 | 0.032 | 0.027 | 0.032 | 0.035 | 0.030 | 0.028 |
| 0.95 | $C_{a,ll}$ | 0.040 | 0.030 | 0.031 | 0.035 | 0.034 | 0.038 | 0.036 | 0.031 | 0.032 |
| | $C_{b,ll}$ | 0.033 | 0.025 | 0.029 | 0.029 | 0.024 | 0.029 | 0.032 | 0.027 | 0.025 |
| 0.90 | $C_{a,ll}$ | 0.045 | 0.034 | 0.035 | 0.039 | 0.037 | 0.042 | 0.040 | 0.035 | 0.036 |
| | $C_{b,ll}$ | 0.029 | 0.022 | 0.027 | 0.026 | 0.021 | 0.025 | 0.029 | 0.024 | 0.022 |
| 0.85 | $C_{a,ll}$ | 0.050 | 0.037 | 0.040 | 0.043 | 0.041 | 0.046 | 0.045 | 0.040 | 0.039 |
| | $C_{b,ll}$ | 0.026 | 0.019 | 0.024 | 0.023 | 0.019 | 0.022 | 0.026 | 0.022 | 0.020 |
| 0.80 | $C_{a,ll}$ | 0.056 | 0.041 | 0.045 | 0.048 | 0.044 | 0.051 | 0.051 | 0.044 | 0.042 |
| | $C_{b,ll}$ | 0.023 | 0.017 | 0.022 | 0.020 | 0.016 | 0.019 | 0.023 | 0.019 | 0.017 |
| 0.75 | $C_{a,ll}$ | 0.061 | 0.045 | 0.051 | 0.052 | 0.047 | 0.055 | 0.056 | 0.049 | 0.046 |
| | $C_{b,ll}$ | 0.019 | 0.014 | 0.019 | 0.016 | 0.013 | 0.016 | 0.020 | 0.016 | 0.013 |
| 0.70 | $C_{a,ll}$ | 0.068 | 0.049 | 0.057 | 0.057 | 0.051 | 0.060 | 0.063 | 0.054 | 0.050 |
| | $C_{b,ll}$ | 0.016 | 0.012 | 0.016 | 0.014 | 0.011 | 0.013 | 0.017 | 0.014 | 0.011 |
| 0.65 | $C_{a,ll}$ | 0.074 | 0.053 | 0.064 | 0.062 | 0.055 | 0.064 | 0.070 | 0.059 | 0.054 |
| | $C_{b,ll}$ | 0.013 | 0.010 | 0.014 | 0.011 | 0.009 | 0.010 | 0.014 | 0.011 | 0.009 |
| 0.60 | $C_{a,ll}$ | 0.081 | 0.058 | 0.071 | 0.067 | 0.059 | 0.068 | 0.077 | 0.065 | 0.059 |
| | $C_{b,ll}$ | 0.010 | 0.007 | 0.011 | 0.009 | 0.007 | 0.008 | 0.011 | 0.009 | 0.007 |
| 0.55 | $C_{a,ll}$ | 0.088 | 0.062 | 0.080 | 0.072 | 0.063 | 0.073 | 0.085 | 0.070 | 0.063 |
| | $C_{b,ll}$ | 0.008 | 0.006 | 0.009 | 0.007 | 0.005 | 0.006 | 0.009 | 0.007 | 0.006 |
| 0.50 | $C_{a,ll}$ | 0.095 | 0.066 | 0.088 | 0.077 | 0.067 | 0.078 | 0.092 | 0.076 | 0.067 |
| | $C_{b,ll}$ | 0.006 | 0.004 | 0.007 | 0.005 | 0.004 | 0.005 | 0.007 | 0.005 | 0.004 |

†A crosshatched edge indicates that the slab continues across, or is fixed at, the support; an unmarked edge indicates a support at which torsional resistance is negligible.

case, some rotation will occur at all continuous edges. As an approximation it is assumed that there is 50 percent restraint for calculating these live-load moments. The corresponding coefficients are given in Table 5.4. Finally, for computing shear in the slab and loads on the supporting beams, Table 5.5 gives the fractions of the total load $w$ which are transmitted in the two directions.

**Table 5.5 Ratio of load $w$ in $l_a$ and $l_b$ directions for shear in slab and load on supports†**

| Ratio $m = \frac{l_a}{l_b}$ | | Case 1 | Case 2 | Case 3 | Case 4 | Case 5 | Case 6 | Case 7 | Case 8 | Case 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1.00 | $W_a$ | 0.50 | 0.50 | 0.17 | 0.50 | 0.83 | 0.71 | 0.29 | 0.33 | 0.67 |
| | $W_b$ | 0.50 | 0.50 | 0.83 | 0.50 | 0.17 | 0.29 | 0.71 | 0.67 | 0.33 |
| 0.95 | $W_a$ | 0.55 | 0.55 | 0.20 | 0.55 | 0.86 | 0.75 | 0.33 | 0.38 | 0.71 |
| | $W_b$ | 0.45 | 0.45 | 0.80 | 0.45 | 0.14 | 0.25 | 0.67 | 0.62 | 0.29 |
| 0.90 | $W_a$ | 0.60 | 0.60 | 0.23 | 0.60 | 0.88 | 0.79 | 0.38 | 0.43 | 0.75 |
| | $W_b$ | 0.40 | 0.40 | 0.77 | 0.40 | 0.12 | 0.21 | 0.62 | 0.57 | 0.25 |
| 0.85 | $W_a$ | 0.66 | 0.66 | 0.28 | 0.66 | 0.90 | 0.83 | 0.43 | 0.49 | 0.79 |
| | $W_b$ | 0.34 | 0.34 | 0.72 | 0.34 | 0.10 | 0.17 | 0.57 | 0.51 | 0.21 |
| 0.80 | $W_a$ | 0.71 | 0.71 | 0.33 | 0.71 | 0.92 | 0.86 | 0.49 | 0.55 | 0.83 |
| | $W_b$ | 0.29 | 0.29 | 0.67 | 0.29 | 0.08 | 0.14 | 0.51 | 0.45 | 0.17 |
| 0.75 | $W_a$ | 0.76 | 0.76 | 0.39 | 0.76 | 0.94 | 0.88 | 0.56 | 0.61 | 0.86 |
| | $W_b$ | 0.24 | 0.24 | 0.61 | 0.24 | 0.06 | 0.12 | 0.44 | 0.39 | 0.14 |
| 0.70 | $W_a$ | 0.81 | 0.81 | 0.45 | 0.81 | 0.95 | 0.91 | 0.62 | 0.68 | 0.89 |
| | $W_b$ | 0.19 | 0.19 | 0.55 | 0.19 | 0.05 | 0.09 | 0.38 | 0.32 | 0.11 |
| 0.65 | $W_a$ | 0.85 | 0.85 | 0.53 | 0.85 | 0.96 | 0.93 | 0.69 | 0.74 | 0.92 |
| | $W_b$ | 0.15 | 0.15 | 0.47 | 0.15 | 0.04 | 0.07 | 0.31 | 0.26 | 0.08 |
| 0.60 | $W_a$ | 0.89 | 0.89 | 0.61 | 0.89 | 0.97 | 0.95 | 0.76 | 0.80 | 0.94 |
| | $W_b$ | 0.11 | 0.11 | 0.39 | 0.11 | 0.03 | 0.05 | 0.24 | 0.20 | 0.06 |
| 0.55 | $W_a$ | 0.92 | 0.92 | 0.69 | 0.92 | 0.98 | 0.96 | 0.81 | 0.85 | 0.95 |
| | $W_b$ | 0.08 | 0.08 | 0.31 | 0.08 | 0.02 | 0.04 | 0.19 | 0.15 | 0.05 |
| 0.50 | $W_a$ | 0.94 | 0.94 | 0.76 | 0.94 | 0.99 | 0.97 | 0.86 | 0.89 | 0.97 |
| | $W_b$ | 0.06 | 0.06 | 0.24 | 0.06 | 0.01 | 0.03 | 0.14 | 0.11 | 0.03 |

†A crosshatched edge indicates that the slab continues across, or is fixed at, the support; an unmarked edge indicates a support at which torsional resistance is negligible.

Since positive-moment steel is placed in two layers, the distance $d$ for the upper layer is smaller than for the lower layer by one bar diameter. Because the moments in the long direction are the smaller ones, it is economical to place the steel in that direction on top of the bars in the short direction.

The twisting moments discussed in Art. 5.5 are usually of consequence only in exterior corners, where they tend to crack the slab along 45° lines at



the corner panels. Special reinforcement should be provided at exterior corners in both bottom and top of the slab, for a distance in each direction from the corner equal to one-fifth the long span. The reinforcement in the top of the slab should be parallel to the diagonal from the corner. The reinforcement in the bottom of the slab should be at right angles to the diagonal, or it may consist of bars in two directions parallel to the sides of the slab. The reinforcement in each band should be of size and spacing equivalent to that required for the maximum positive moment in the slab.

The precise locations of inflection points in two-way slabs are not easily determined, since they depend upon the side ratio, ratio of live to dead load, and continuity conditions at the edges. A reasonable rule is to assume that a line of inflection exists parallel to any continuous edge at a distance from it of one-sixth the span. The rule can be applied to both the long and the short directions.

## 5.7 TWO-WAY EDGE-SUPPORTED SLAB

**Example 5.2:** A monolithic reinforced concrete floor is to be composed of rectangular bays measuring 21 × 26 ft, as shown in Fig. 5.9. Beams of width 12 in and depth 24 in are provided on all column lines; thus the clear-span dimensions for the two-way slab panels are 20 × 25 ft. The floor is to be designed to carry a service live load of 137 psf uniformly distributed over its surface, in addition to its own weight, using concrete of strength $f'_c = 3000$ psi and reinforcement having $f_y = 60{,}000$ psi. Find the required slab thickness and reinforcement for the corner panel shown (bays 6.40 × 7.92 m, $b_w = 305$ mm, $h = 610$ mm, $l_n = 6.10 \times 7.62$ m, $w_L = 6.56$ kN/m², $f'_c = 20.7$ MPa, $f_y = 414$ MPa).

The minimum thickness for slabs of this type is often taken equal to 1/180 times the panel perimeter:

$$h = 2(20 + 25) \times 12/180 = 6 \text{ in (152 mm)}$$

This will be selected for a trial depth. The corresponding dead load is $\frac{1}{2} \times 150 = 75$ psf. Thus the factored loads on which the design is to be based are

$$\text{Live load} = 1.7 \times 137 = 233 \text{ psf}$$
$$\text{Dead load} = 1.4 \times 75 = 105 \text{ psf}$$
$$\text{Total load} = 338 \text{ psf (16.2 kN/m}^2\text{)}$$

With the ratio of panel sides $m = l_a/l_b = 20/25 = 0.8$, the moment calculations for the slab *middle strips* are as follows.

***Negative moments at continuous edges (Table 5.2)***

$$M_{a,neg} = 0.071 \times 338 \times 20^2 = 9600 \text{ ft·lb} = 115{,}000 \text{ in·lb (13.00 kN·m)}$$
$$M_{b,neg} = 0.029 \times 338 \times 25^2 = 6130 \text{ ft·lb} = 73{,}400 \text{ in·lb (8.29 kN·m)}$$

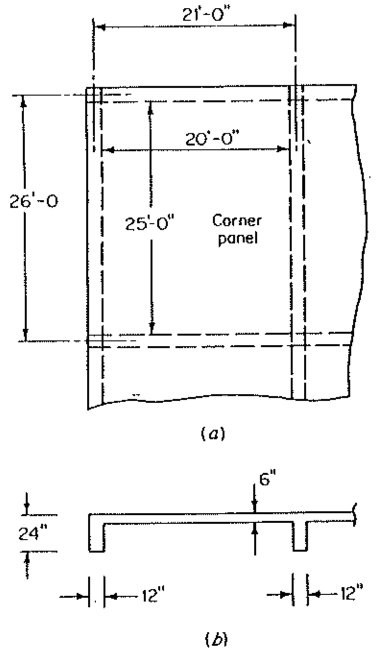



Figure 5.9 Two-way edge-supported slab: (*a*) partial floor plan; (*b*) typical cross section.

*Positive moments (Tables 5.3 and 5.4)*

$$M_{a,pos,dl} = 0.039 \times 105 \times 20^2 = 1638 \text{ ft}\cdot\text{lb} = 19{,}700 \text{ in}\cdot\text{lb } (2.27 \text{ kN}\cdot\text{m})$$

$$M_{a,pos,ll} = 0.048 \times 233 \times 20^2 = 4470 \text{ ft}\cdot\text{lb} = \underline{53{,}700} \text{ in}\cdot\text{lb } (6.07 \text{ kN}\cdot\text{m})$$

$$M_{a,pos,tot} = 73{,}400 \text{ in}\cdot\text{lb } (8.29 \text{ kN}\cdot\text{m})$$

$$M_{b,pos,dl} = 0.016 \times 105 \times 25^2 = 1050 \text{ ft}\cdot\text{lb} = 12{,}600 \text{ in}\cdot\text{lb } (1.42 \text{ kN}\cdot\text{m})$$

$$M_{b,pos,ll} = 0.020 \times 233 \times 25^2 = 2910 \text{ ft}\cdot\text{lb} = \underline{35{,}000} \text{ in}\cdot\text{lb } (2.96 \text{ kN}\cdot\text{m})$$

$$M_{b,pos,tot} = 47{,}600 \text{ in}\cdot\text{lb } (5.38 \text{ kN}\cdot\text{m})$$

*Negative moments at discontinuous edges ($\frac{1}{3}$ × positive moments)*

$$M_{a,neg} = \frac{1}{3}(73{,}400) = 24{,}500 \text{ in}\cdot\text{lb } (2.77 \text{ kN}\cdot\text{m})$$

$$M_{b,neg} = \frac{1}{3}(47{,}600) = 15{,}900 \text{ in}\cdot\text{lb } (1.80 \text{ kN}\cdot\text{m})$$

The required reinforcement in the *middle strips* will be selected with the help of Graph A.1.

*Short direction*

(1) Midspan

$$\frac{M_u}{\phi b d^2} = \frac{73{,}400}{0.90 \times 12 \times 5^2} = 272 \qquad \rho = 0.0048$$

$A_s = 0.0048 \times 12 \times 5 = 0.288$ in²/ft. From Table A.4 No. 4 bars at 7-in spacing are selected, giving $A_s = 0.34$ in²/ft.



(2) Continuous edge

$$\frac{M_u}{\phi b d^2} = \frac{115{,}000}{0.90 \times 12 \times 5^2} = 426 \qquad \rho = 0.0078\dagger$$

$A_s = 0.0078 \times 12 \times 5 = 0.468$ in²/ft. If two of every three positive bars are bent up, and likewise for the adjacent panel, the negative-moment steel area furnished at the continuous edge will be 4/3 times the positive-moment steel in the span, or $A_s = 4/3 \times 0.34 = 0.453$ in²/ft. It is seen that this is 3 percent less than the required amount of 0.468. On the other hand, the positive-moment steel furnished, 0.34 in²/ft, represents about 15 percent more than the required amount. As discussed in Art. 8.7, the Code permits a certain amount of inelastic redistribution, within strictly specified limits. In the case at hand, the negative steel furnished suffices for only 97 percent of the calculated moment, but the positive steel permits about 115 percent of the calculated moment to be resisted. This more than satisfies the conditions for inelastic moment redistribution set by the Code. This situation illustrates how such moment redistribution can be utilized to obtain a simpler and more economical distribution of steel.

(3) Discontinuous edge The negative moment at the discontinuous edge is one-third the positive moment in the span; it would be adequate to bend up every third bar from the bottom to provide negative-moment steel at the discontinuous edge. However, this would result in a 21-in spacing, which is larger than the maximum spacing of $3h = 18$ in permitted by the Code. Hence, for the discontinuous edge, two of every three bars will be bent up from the bottom steel.

*Long direction*

(1) Midspan

$$\frac{M_u}{\phi b d^2} = \frac{47{,}600}{0.90 \times 12 \times 4.5^2} = 218 \qquad \rho = 0.0038$$

(The positive-moment steel in the long direction is placed on top of that for the short direction. This is the reason for using $d = 4.5$ in for the positive-moment steel in the long direction and $d = 5$ in in all other locations.) $A_s = 0.0038 \times 12 \times 4.5 = 0.205$ in²/ft. From Table A.4 No. 3 bars at 6-in spacing are selected, giving $A_s = 0.22$ in²/ft.

(2) Continuous edge

$$\frac{M_u}{\phi b d^2} = \frac{73{,}400}{0.90 \times 12 \times 5^2} = 272 \qquad \rho = 0.0048$$

$A_s = 0.0048 \times 12 \times 5 = 0.288$ in²/ft. Again bending up two of every three bottom bars from both panels adjacent to the continuous edge, one has, at that edge, $A_s = \frac{4}{3} \times 0.22 = 0.29$ in²/ft.

(3) Discontinuous edge For the reasons discussed in connection with the short direction, two out of every three bottom bars will, likewise, be bent up at this edge.

The steel selections above refer to the middle strips in both directions. For the *column strips*, the moments are assumed to decrease linearly from the full calculated value at the inner edge of the column strip to one-third of this value at the edge of the supporting beam. To simplify steel placement, a uniform spacing will be used in the column strips. The

†Note that this value of $\rho$, which is the maximum required anywhere in the slab, is well below the permitted maximum value of $0.75\rho_b = 0.0160$, indicating that a thinner slab might be used. However, use of the minimum possible thickness would require an increase in the tensile-steel area and would be less economical for this reason. In addition, a thinner slab may produce undesirably large deflections. The trial depth of 6 in will be retained for the final design.

average moments in the column strips being two-thirds of the corresponding moments in the middle strips, adequate column strip steel will be furnished if the spacing of this steel is three-halves times that in the middle strip. Maximum spacing limitations should be checked. According to the recommendation of Art. 5.6, inflection points may be assumed a distance $l/6$ from the continuous edges, that is, 3 ft 4 in in the short direction and 4 ft 2 in in the long direction. Two of every three bars are bent up at these locations. The bent-up bars must be carried over the support and beyond the inflection point of the adjacent panel a distance not less than one-sixteenth of the span, or the depth of the member, or 12 bar diameters, whichever is greatest. It is customary and slightly conservative, instead, to extend these bars to the quarter point of the adjacent span. This extension will also satisfy the requirements for development length for these negative-moment bars.

The same pattern of bend and cutoff points will be followed at the discontinuous edge of the panel. At that edge, negative bars will be extended as far as possible into the supporting beams, then bent downward in a 90° bend to provide sufficient anchorage.

The reactions of the slab are calculated from Table 5.5, which indicates that 71 percent of the load is transmitted in the short direction and 29 percent in the long direction. The total load on the panel being $20 \times 25 \times 338 = 169{,}000$ lb, the load per foot on the long beam is $(0.71 \times 169{,}000)/(2 \times 25) = 2400$ lb/ft, and on the short beam is $(0.29 \times 169{,}000)/(2 \times 20) = 1220$ lb/ft. The shear to be transmitted by the slab to these beams is numerically equal to these beam loads. The shear strength of the slab is

$$\phi V_c = 0.85 \times 2\sqrt{3000} \times 12 \times 5 = 5590 \text{ lb}$$

well above the required shear strength at factored loads.

## PROBLEMS

5.4 A concrete slab roof is to be designed to cover a transformer vault. The outside dimensions of the vault are $17 \times 20$ ft and walls are 8-in brick. A service live load of 80 psf, uniformly distributed over the roof surface, will be assumed. Design the roof as a two-way slab, using $f'_c = 4000$ psi and $f_y = 50{,}000$ psi.

5.5 A concrete warehouse floor is framed by beams on the column lines, which are 18 ft on center in one direction and 24 ft on center in the other. Beam webs may be assumed to be 12 in wide. A service live load of 225 psf must be carried. Design a typical interior panel with $f'_c = 5000$ psi and $f_y = 60{,}000$ psi.

# TWO-WAY COLUMN-SUPPORTED SLABS

## 5.8 BEHAVIOR

When two-way slabs are supported by relatively shallow, flexible beams, or if column-line beams are omitted altogether, as for flat plates (Fig. 5.1*d*) or flat slabs (Fig. 5.1*e*), several new considerations are introduced. Figure 5.10*a* shows a portion of a floor system in which a rectangular slab panel is supported by relatively shallow beams on four sides. The beams, in turn, are supported by columns at the intersections of their centerlines. If a surface



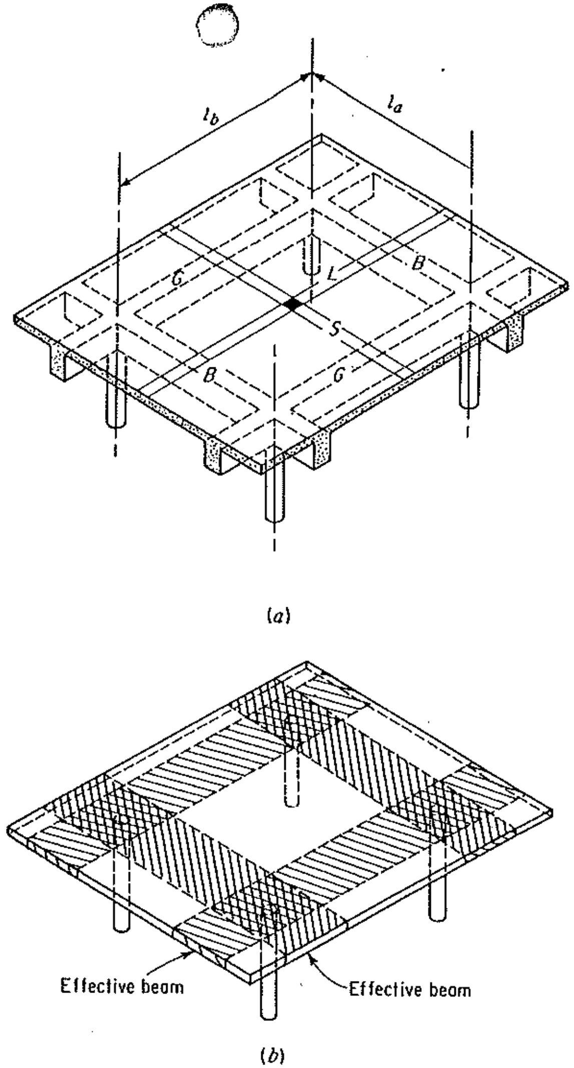


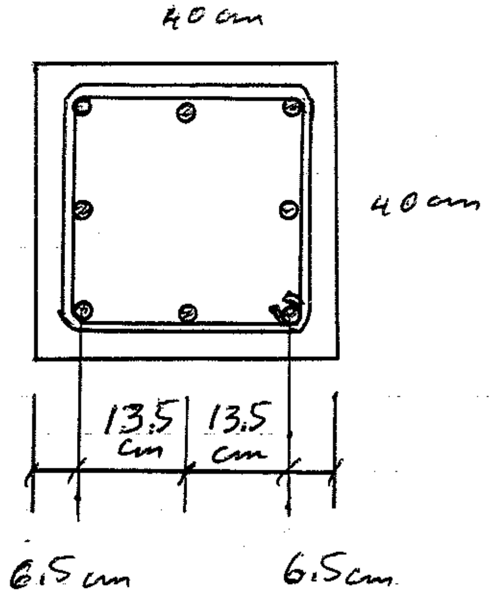

**Figure 5.10** Column-supported two-way slabs: (*a*) two-way slab with beams and columns; (*b*) two-way slab without beams.

load $w$ is applied, that load is shared between imaginary slab strips $S$ in the short direction and $L$ in the long direction, as before. Note that the portion of the load that is carried by the long strips $L$ is delivered to the beams $B$ spanning in the short direction of the panel. This portion carried by the beams $B$ plus that carried directly in the short direction by the slab strips $S$ sums up to 100 percent of the load applied to the panel. Similarly, the short-direction slab strips $S$ deliver a part of the load to long-direction girders $G$. That load plus load carried directly in the long direction by the slab includes 100 percent of the applied load. It is clearly a requirement of statics that for column-supported construction 100 *percent of the applied load must be carried in each direction*, jointly by the slab and its supporting beams.

$f_c' = 20$ MPa

$f_y = 280$ MPa

$E_s = 200000$ MPa




8 ϕ 30 bars

ϕ 30 = 7.07 $cm^2$

$\varepsilon_y = \frac{280}{200000}$

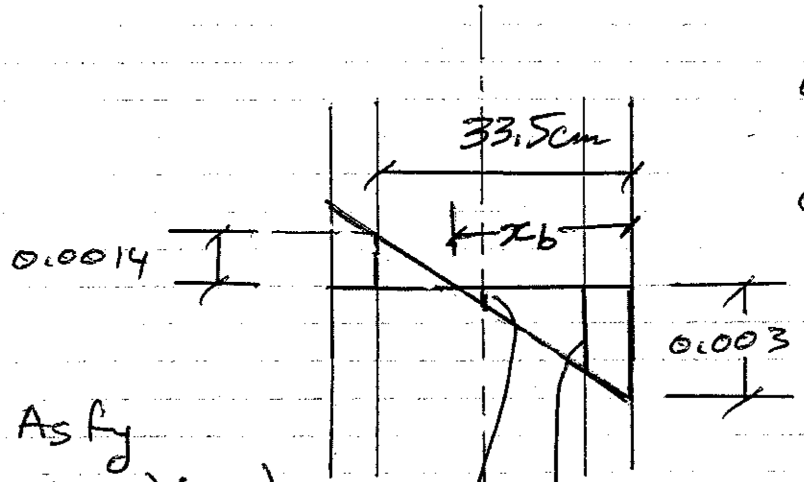
$= 0.0014$

$x_b = \frac{0.003}{0.0044}(33.5)$

$= 22.84$ cm

$a_b = 19.41$ cm

$C_c = 132$ t




$T = A_s f_y$

$= 3(7.07)(2.8)$

$= 59.4$ t

$\varepsilon_{s1}' = \frac{22.84 - 6.5}{22.84} \times 0.003 = 0.00215$

$\therefore f_{s1}' = f_y = 280$ MPa

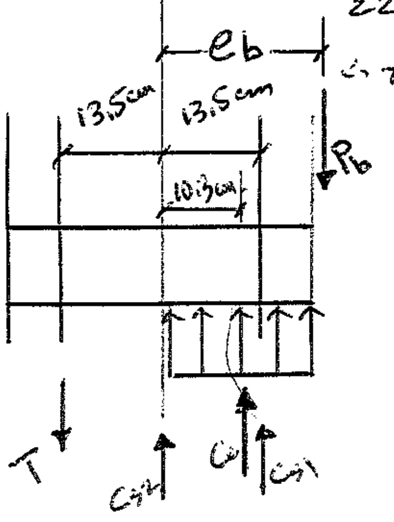
$P_b = C_c + C_{s1} + C_{s2} - T$

$= 132 + 55.8 + 8.1 - 59.4$

$= 136.5$ t

$\varepsilon_{s2}' = \frac{2.84}{22.84} \times 0.003 = 0.000373$

$\therefore f_{s2}' = 74.6$ MPa




$C_{s1} = (3 \times 7.07)(2.8 - 0.85(0.2))$

$= 55.8$ t

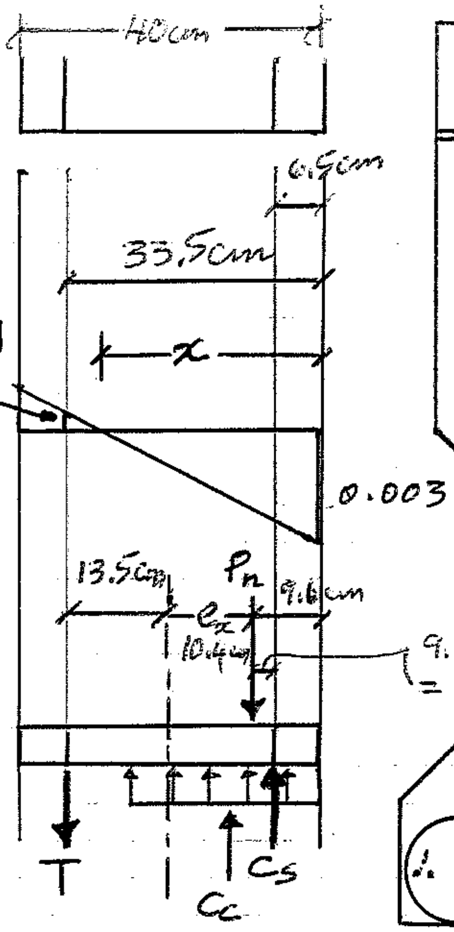
$C_{s2} = (2 \times 7.07)(0.746 - 0.85(0.2))$

$= 8.14$ t

6
Dr. Jamal

For $e_x = 10.4$ cm $< e_b = 21.9$ cm

$\epsilon_s' > \epsilon_y$ and ignoring middle steel




$C_s = A_s'(f_y - 0.85f_c')$
$= (3 \times 7.07)(2.63)$
$= 55.8t$

$T = A_s f_s$

$C_c = 0.85 f_c' b (0.85x)$
$= 0.85(0.2)(40)(0.85x)$
$= 5.78x$

After $x$ has been determined:

$x = 29.9$ cm

$C_s = 55.8t$

$T = 15.3t$

$C_c = 172.8t$

$P_n = C_c + C_s - T$

$\therefore P_y = $ **213.3t**

$$f_s = (2000)(0.003(33.5 - x)/x) = 6(33.5 - x)/x$$

$$T = A_s f_s = (3 \times 7.07)(6(33.5 - x)/x)$$

$$= \frac{4263}{x} - 127.3$$

Taking moments about $P_n$, +↻

$$C_c\left(\frac{0.85x}{2} - 9.6\right) - C_s(3.1) - T(23.9) = 0$$

$$(5.78x)(0.425x - 9.6) - (55.8)(3.1) - \left(\frac{4263}{x} - 127.3\right)(23.9) = 0$$

$$2.46x^2 - 55.49x - 172.98 - \frac{101886}{x} + 3042.5 = 0$$

$$x^3 - 22.56x^2 + 1166.5x - 41417 = 0$$

By trial and error, $x = 29.9$ cm

$(a = 0.85x = 25.4 \text{ cm})$

JZ

For $e_y = 25.4$ cm $> e_b = 21.4$ cm

$\epsilon_s > \epsilon_y$

assume $\epsilon_s' > \epsilon_y$

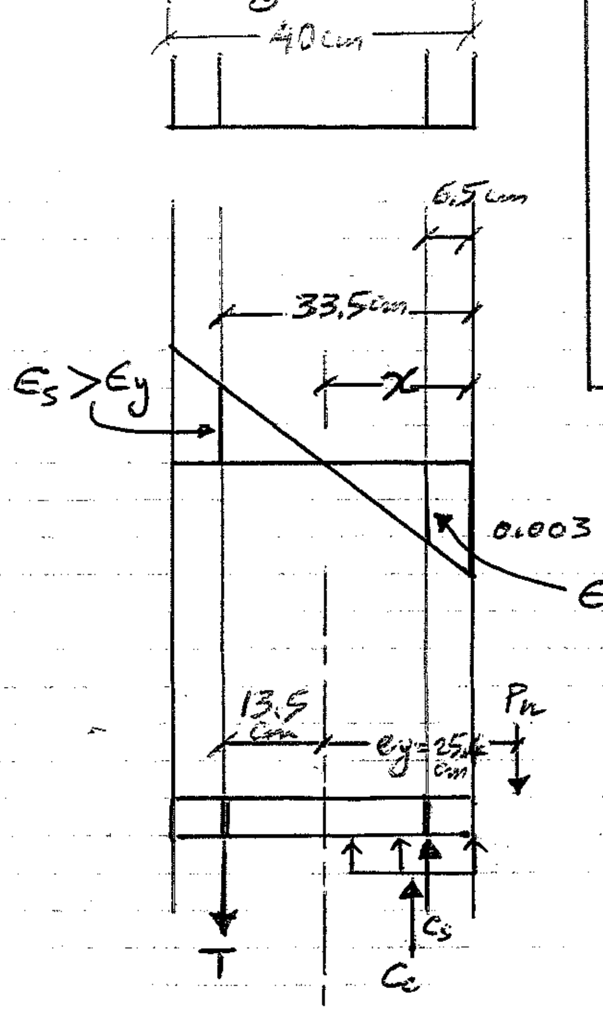
Note that ignoring the middle steel in this case was very reasonable!

$T = A_s f_y$
$= 59.4\,t$

$C_c = 5.78x$

$C_s = 55.8\,t$

$\epsilon_y = 0.0014$




$\epsilon_s' = 0.00204 > \epsilon_y$ as determined after the value of $x$ was found

$P_n = C_c + C_s - T = 5.78x + 55.8 - 59.4 = 5.78x - 3.6$

Taking moments about tension steel, +↻

$$P_n(38.9) - C_c\left(33.5 - \frac{0.85x}{2}\right) - C_s(33.5 - 6.5) = 0$$

$$(38.9)(5.78x - 3.6) - (5.78x)(33.5 - 0.425x) - (55.8)(27) = 0$$

$$224.8x - 140 - 193.6x + 2.46x^2 - 1507 = 0$$

$$x^2 + 12.68x - 670 = 0$$

$\therefore x = 20.31$ cm $\quad (a = 17.3$ cm$)$

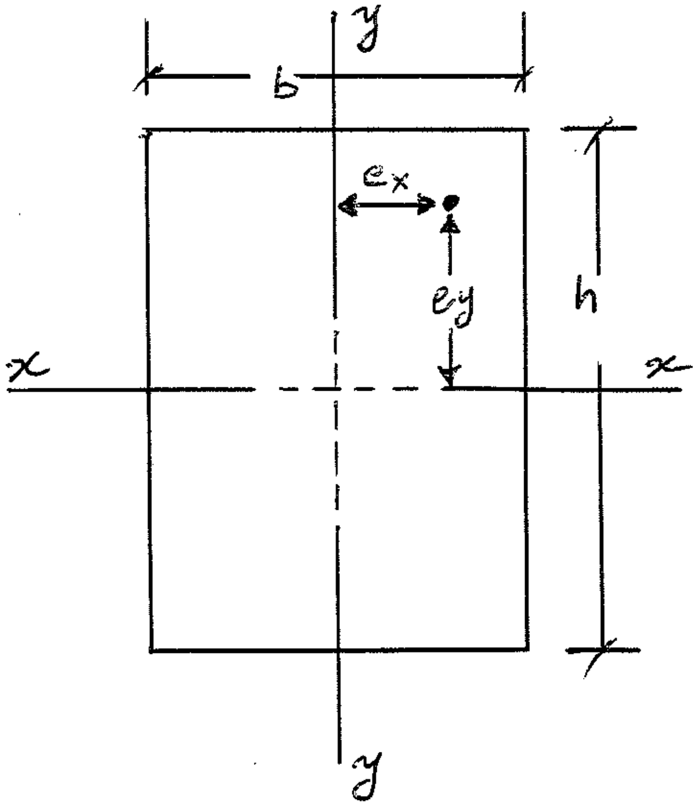
Check assumption: $\epsilon_s' = \frac{(20.31 - 6.5)}{20.31}(0.003) = 0.00204 > \epsilon_y$ ✓

$\therefore$ assumption is correct

$C_c = 117.4\,t$, $P_n = 113.8\,t$

$\therefore P_x = 113.8\,t$

# Biaxial Bending and Compression:




$$M_{nx} = P_n e_y$$
$$M_{ny} = P_n e_x$$

$$\frac{1}{P_i} = \frac{1}{P_x} + \frac{1}{P_y} - \frac{1}{P_o} \qquad \text{(Bresler)}$$

Dr. Jamal Zalatimo

# Transfer of Moment and Shear (Monolithic Joints) Flat Plates

Final Exam

$$M_u = M_{ub} + M_{uv}$$

$M_{ub}$: flexure transfer of $M_u$ $M_{uv}$: shear transfer of $M_u$

(ACI 11.12.6)

$$M_{ub} = \gamma_f M_u = \left[\frac{1}{1+\frac{2}{3}\sqrt{\frac{b_1}{b_2}}}\right] M_u$$

$b_1$ = critical section dimension in the longitudinal direction

$= c_1 + d/2$ for exterior columns

$= c_1 + d$ for interior columns

$b_2$ = critical section dimension in the transverse direction

$= c_2 + d$

Simplifications: (ACI - 13.5.3.3)

if, for exterior supports: $V_u \le 0.75\,\phi V_c$ edge

$V_u \le 0.50\,\phi V_c$ corner

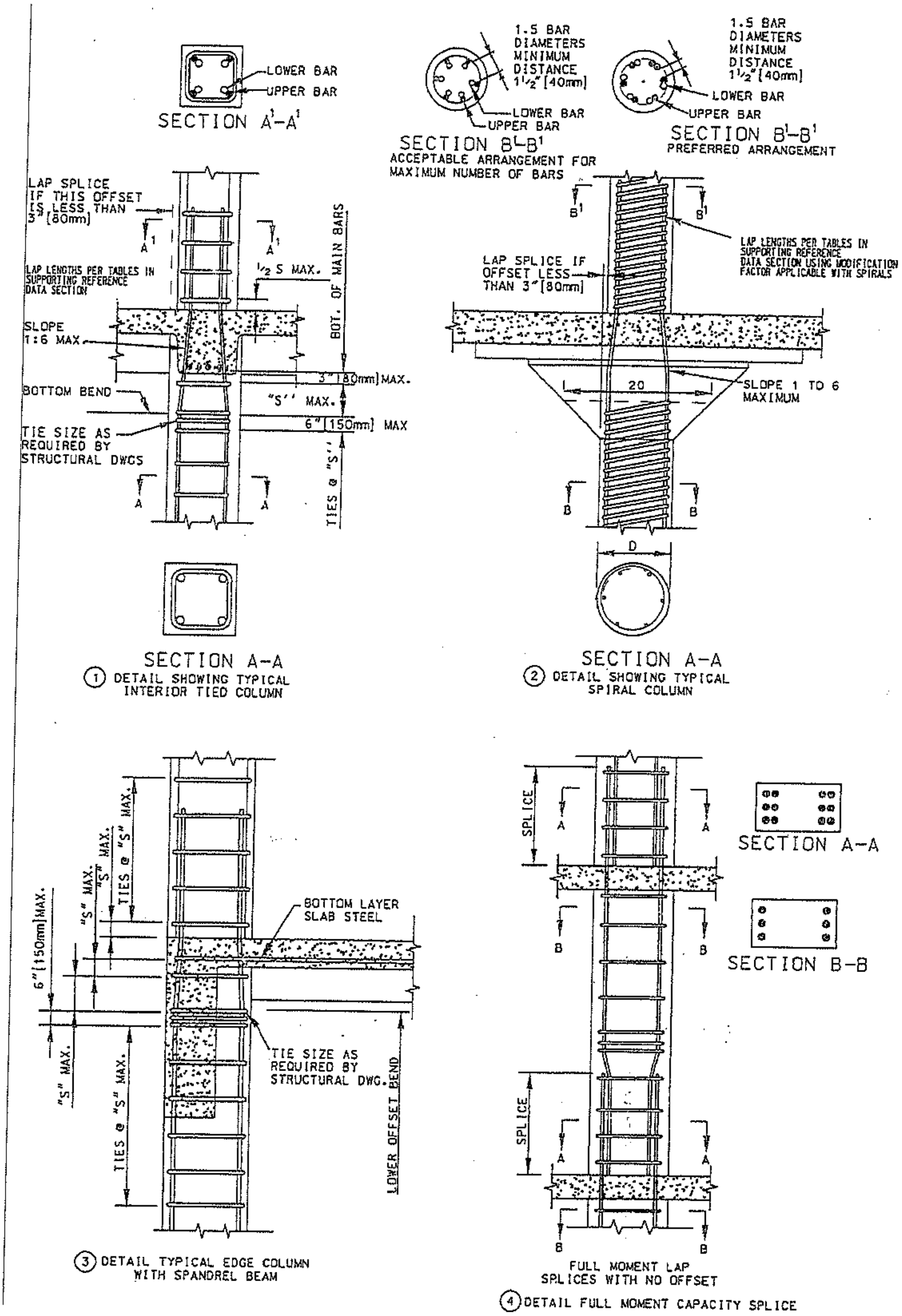
⇒ neglect interaction between shear and moment

i.e., the full exterior moment is transfered through flexure ($\gamma_f = 1.0$)

therefore, consider punching shear only

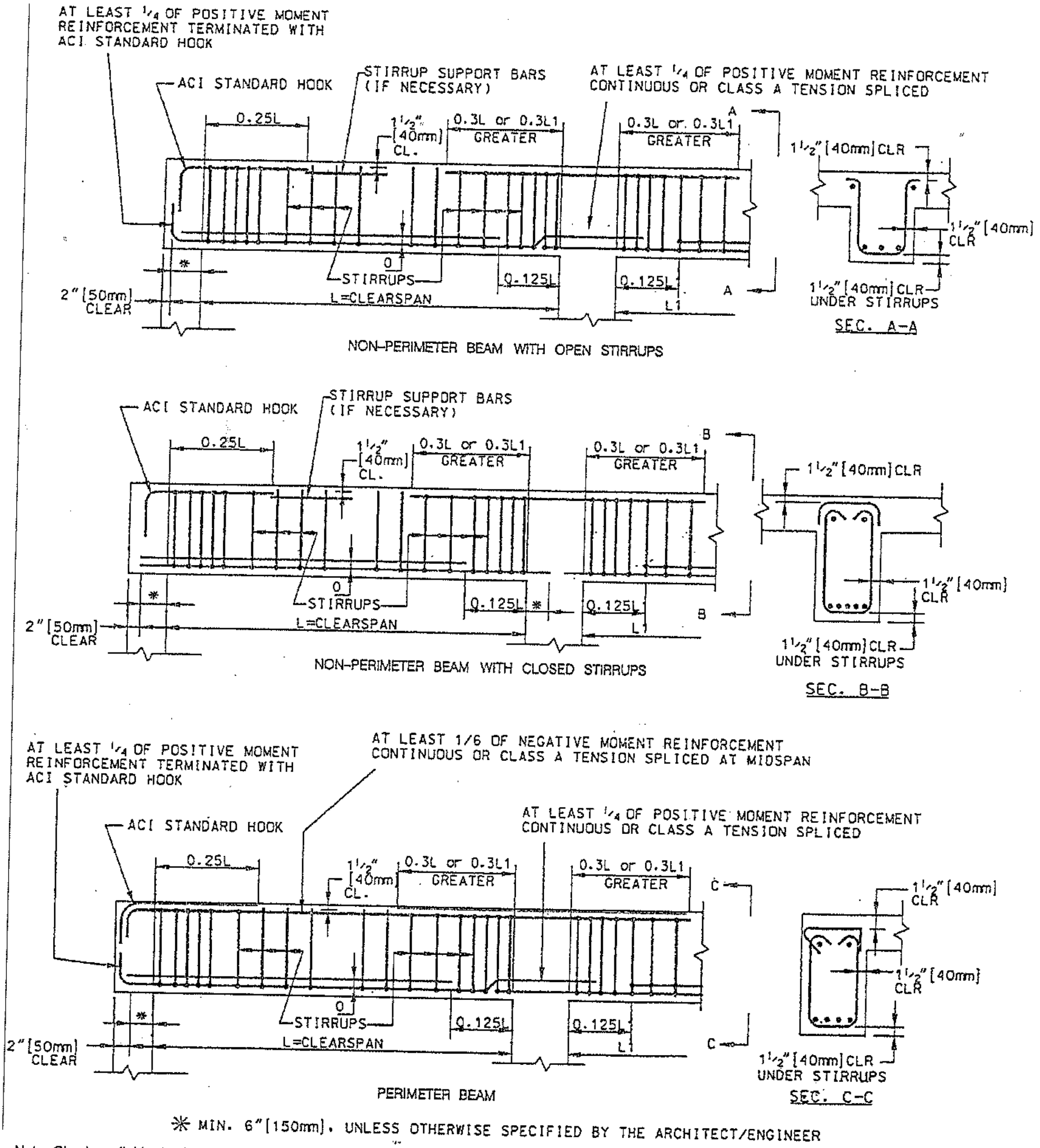
for interior supports: $V_u \le 0.4\,\phi V_c$

⇒ increase $\gamma_f$ by 25%




Note: Where column size above is unchanged from below, "upside down" offset bars are effective in maintaining full moment capacity at end of column. In U.S. practice, this unusual detail is rare, and should be fully illustrated on structural drawings to avoid misunderstandings, whenever its use is deemed necessary. For maximum tie spacing, see table in Supporting Reference Data section.

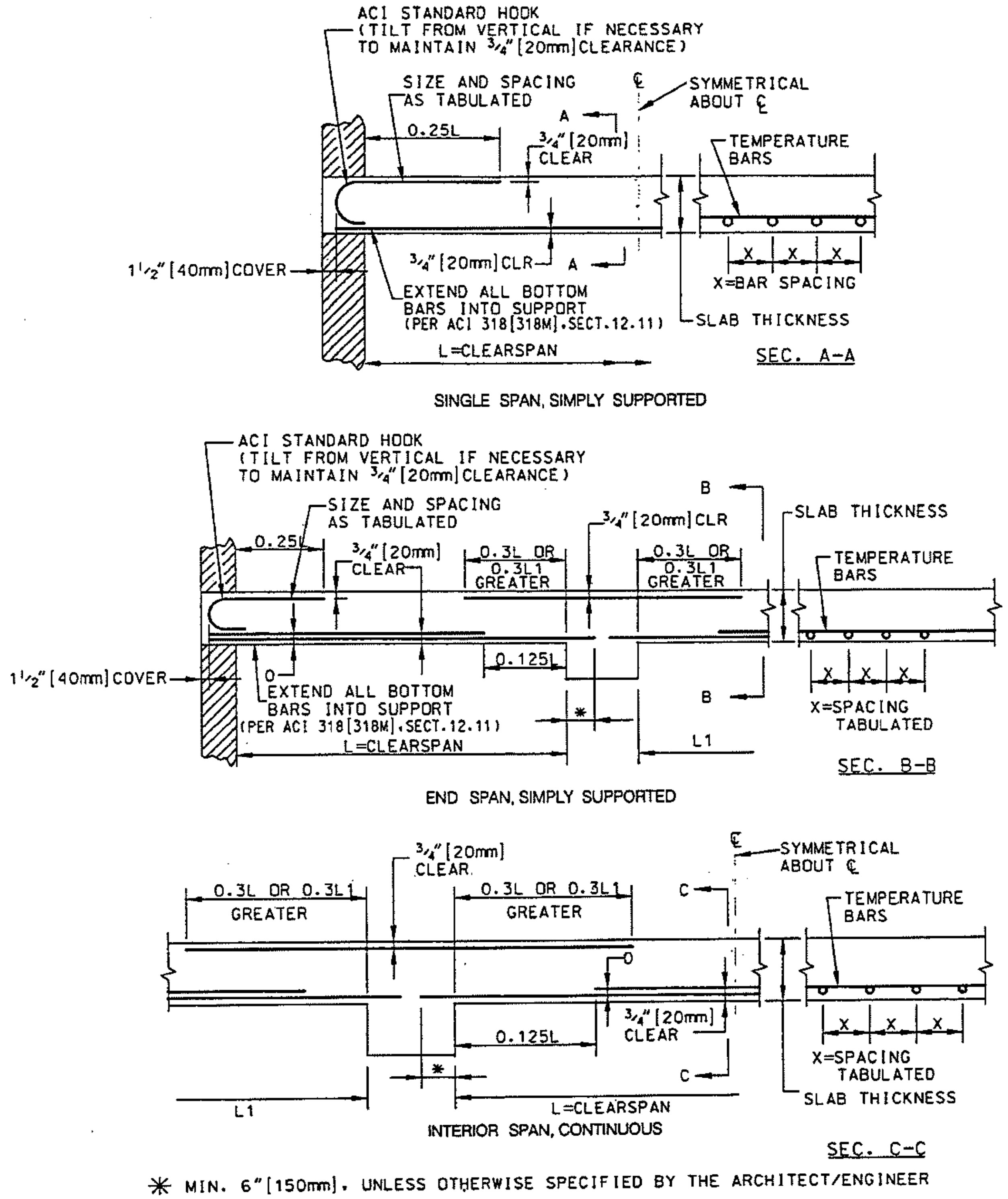
*Fig. 4—Column splice details.*




Note: Check available depth, top and bottom, for required cover on ACI standard hooks. At each end support, add top bar 0.25*L* in length to equal area of bars required. See also Chapter 12 and Chapter 21 of ACI 318 (318M). Bar cutoff details must be verified to provide required development of reinforcement.

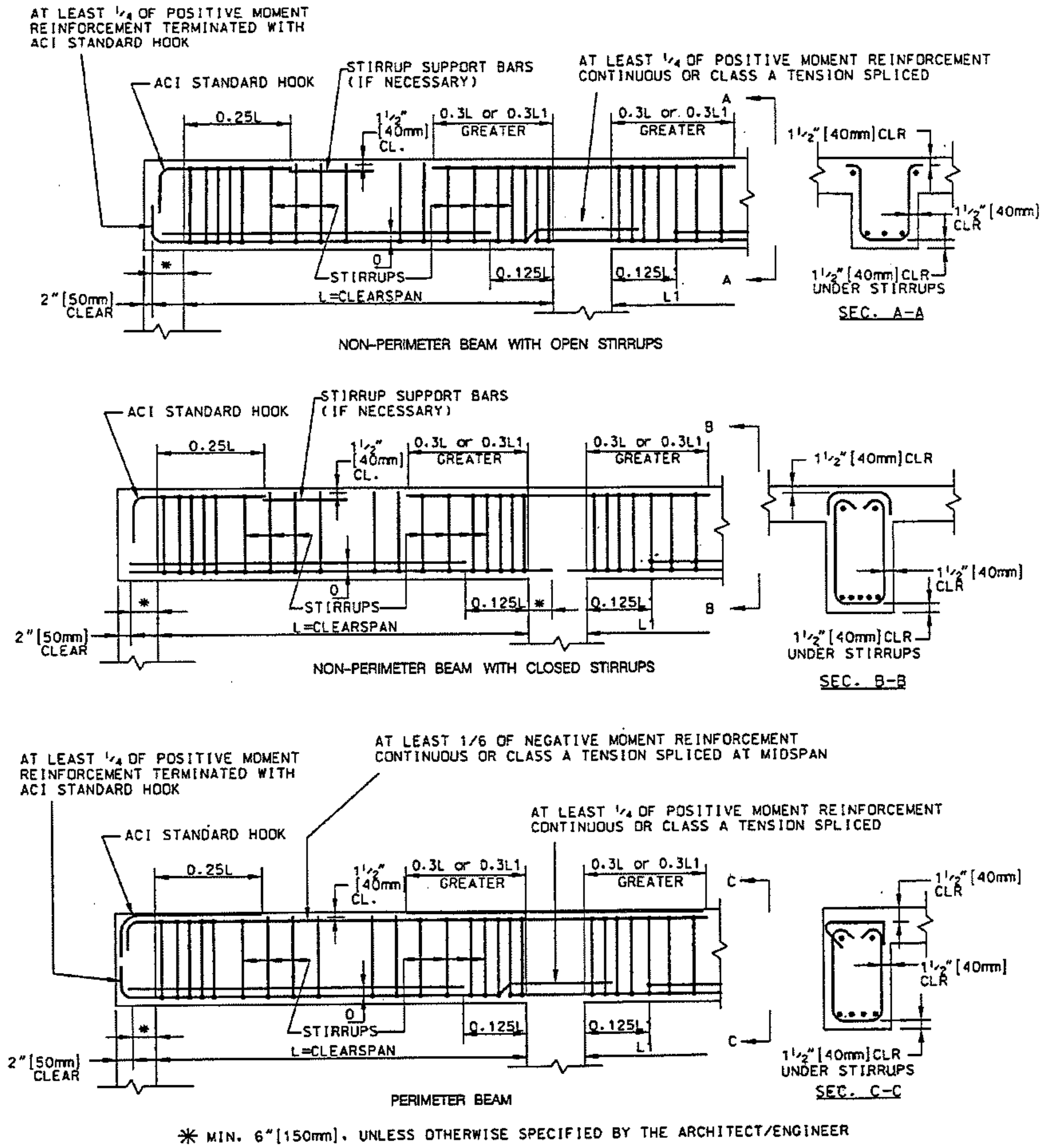
*Fig. 2—Typical details for beams.*

## PART C—FIGURES AND TABLES



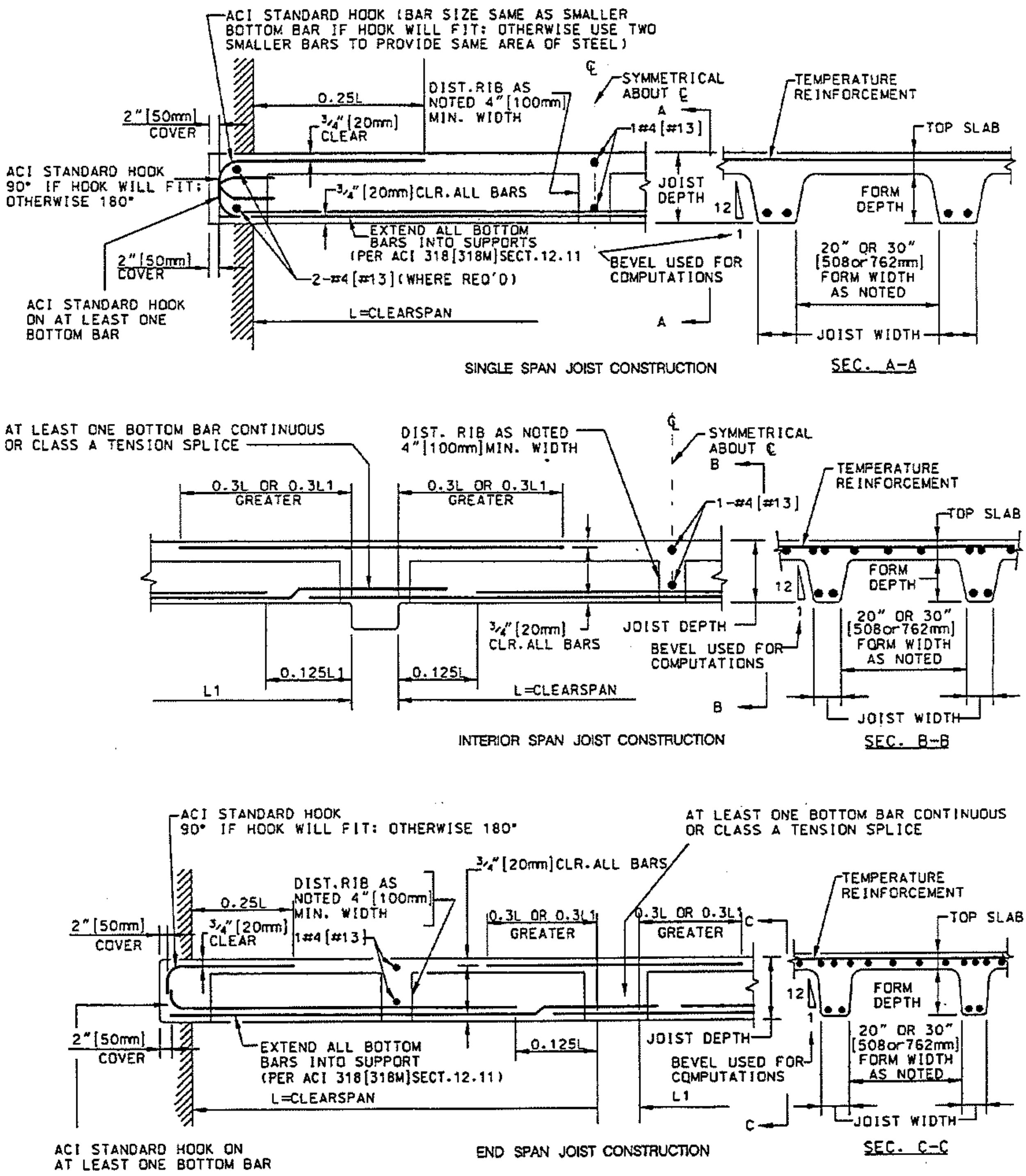

Note: Unless noted otherwise, tables and figures are based on ACI 318 (318M). Concrete cover shown is minimum and should be increased for more severe conditions. Except for single span slabs where top steel is unlikely to receive construction traffic, top bars lighter than No. 4 at 12 in. (No. 13 at 300 mm) are not recommended. For a discussion of bar support spacing, see Section 5.4 of this standard. See also Chapter 12 of ACI 318 (318M). Bar cutoff details must be verified to provide required development of reinforcement.

*Fig. 1—Typical details for one-way solid slabs.*



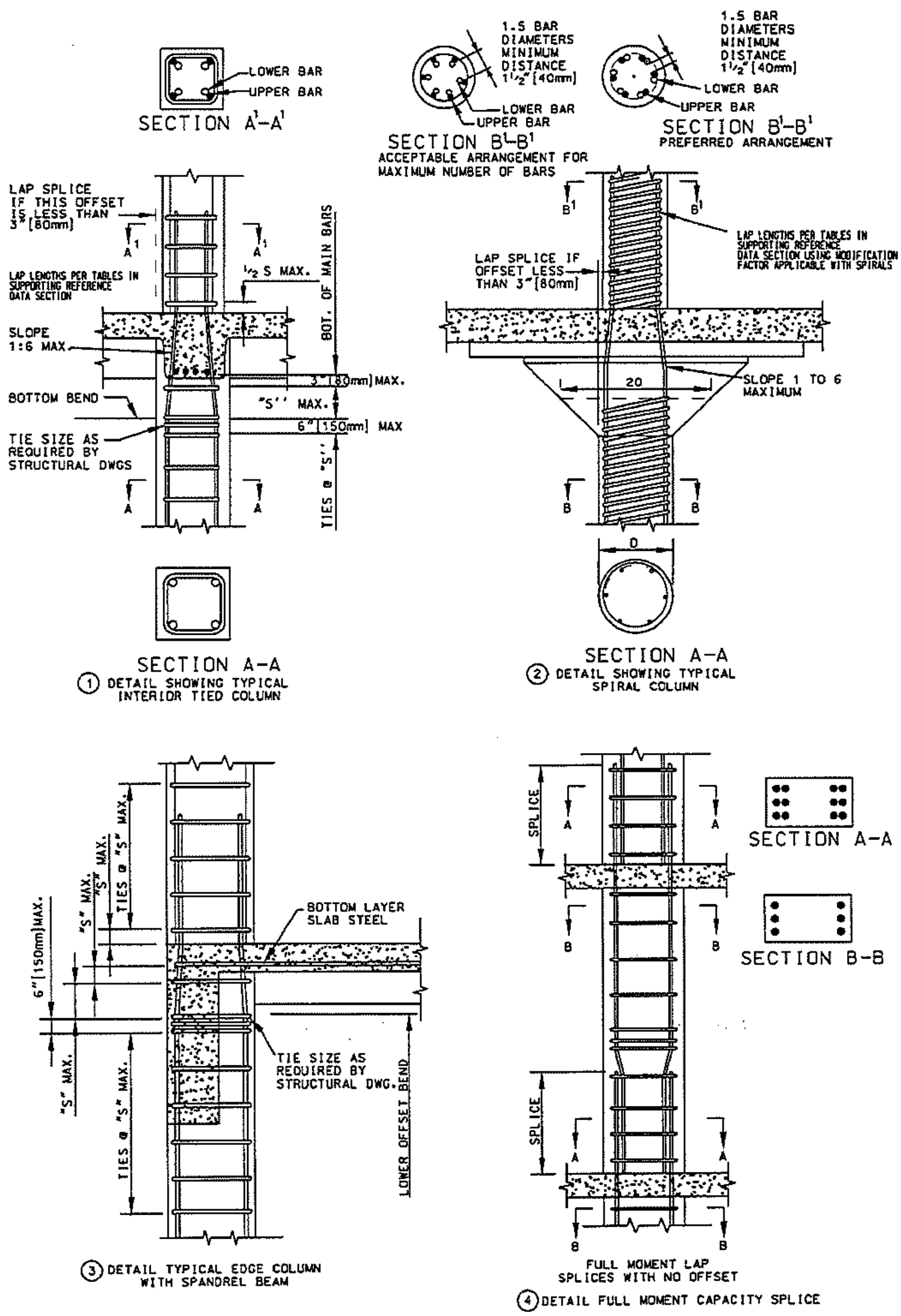

Note: Check available depth, top and bottom, for required cover on ACI standard hooks. At each end support, add top bar 0.25*L* in length to equal area of bars required. See also Chapter 12 and Chapter 21 of ACI 318 (318M). Bar cutoff details must be verified to provide required development of reinforcement.

*Fig. 2—Typical details for beams.*




Note: See also Chapter 12 and Section 7.13 of ACI 318 (318M). Bar cutoff details must be verified to provide required development of reinforcement.

*Fig. 3—Typical details for one-way joist construction.*




Note: Where column size above is unchanged from below, "upside down" offset bars are effective in maintaining full moment capacity at end of column. In U.S. practice, this unusual detail is rare, and should be fully illustrated on structural drawings to avoid misunderstandings, whenever its use is deemed necessary. For maximum tie spacing, see table in Supporting Reference Data section.

*Fig. 4—Column splice details.*

CHAPTER 16

# Design of Two-Way Floor Systems

## 16.1 GENERAL DESCRIPTION

In reinforced concrete buildings, a basic and common type of floor is the slab–beam–girder construction, which has been treated in Chapters 8, 9, and 10. As shown in Fig. 16.1.1(a), the shaded slab area is bounded by the two adjacent beams on the sides and portions of the two girders at the ends. When the length of this area is two or more times its width, almost all of the floor load goes to the beams, and very little, except some near the edge of the girders, goes directly to the girders. Thus the slab may be designed as a one-way slab as treated in Chapter 8, with the main reinforcement parallel to the girder and the shrinkage and temperature reinforcement parallel to the beams. The deflected surface of a one-way slab is primarily one of curvature in its short direction.

When the ratio of the long span $L$ to the short span $S$ as shown in Fig. 16.1.1(b) is less than about 2, the deflected surface of the shaded area becomes one of curvature in both directions. The floor load is carried in both directions to the four supporting beams around the panel; hence the panel is a *two-way slab*. Obviously, when $S$ is equal to $L$, the four beams around a typical interior panel should be identical; for other cases the longer beams take more load than the shorter beams.

Two-way floor systems may also take other forms in practice. Figures 16.1.2(a) and (b) show *flat slab* and *flat plate* floor construction. These are characterized by the absence of beams along the interior column lines, but edge beams may or may not be used at the exterior edges of the floor. Flat slab floors differ from flat plate floors in that slab floors provide adequate shear strength by having either or both of the following: (a) drop panels (i.e., increased thickness of slab) in the region of the columns; or (b) column capitals

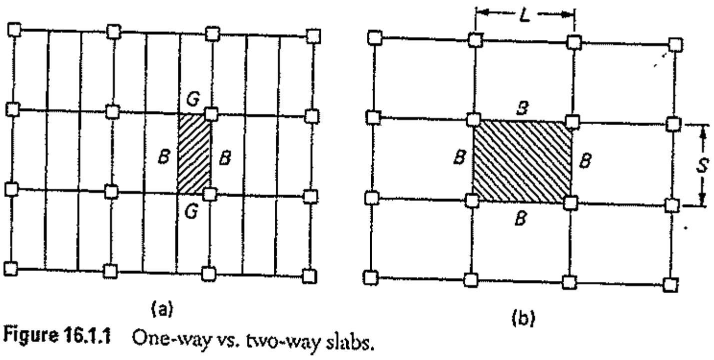
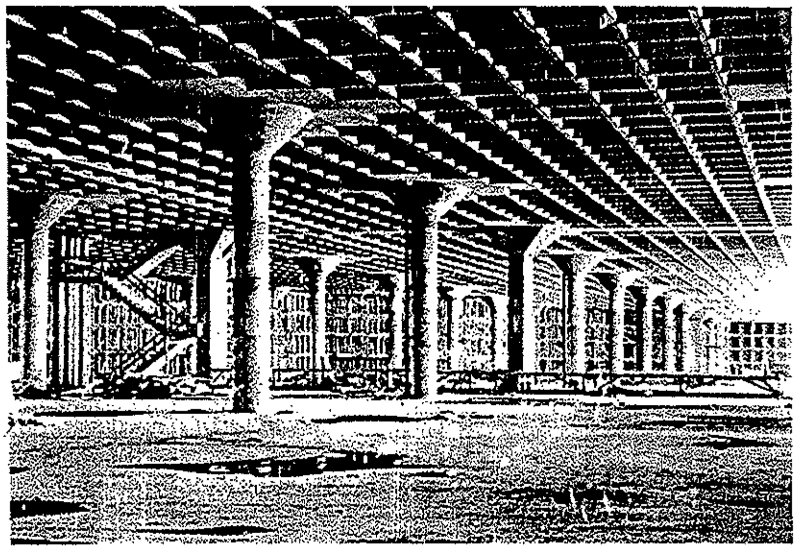
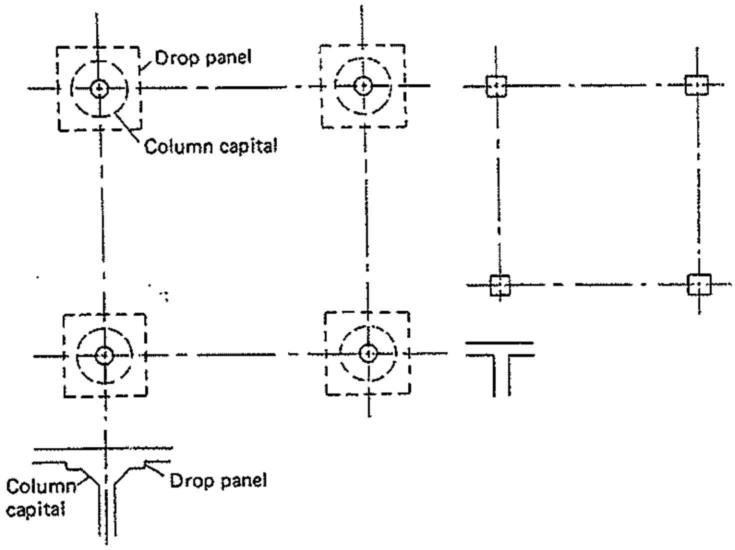



**Figure 16.1.1** One-way vs. two-way slabs.



Flat slab (waffle slab) with capitals in the Fisher Cleveland Plant.
(Courtesy of Portland Cement Association.)

(i.e., tapered enlargement of the upper ends of columns). In flat plate floors a uniform slab thickness is used and the shear strength is obtained by the embedment of multiple-U stirrups, structural steel devices known as *shearhead reinforcement* [see Fig. 16.16.1(b)], or shear stud reinforcement [see Fig. 16.16.2] within the slab of uniform thickness. Relatively speaking, flat slabs are more suitable for larger panel size or heavier loading than flat plates.




(a) Flat slab floor (b) Flat plate floor

**Figure 16.1.2** Flat slab and flat plate floor construction.

Historically, flat slabs predate both two-way slabs on beams and flat plates. Flat slab floors were originally patented by O. W. Norcross [16.2] in the United States on April 29, 1902. Several systems of placing reinforcement have been developed and patented since then—the four-way system, the two-way system, the three-way system, and the circumferential system. C. A. P. Turner [16.2] was one of the early advocates of a flat slab system known as the "mushroom" system. About 1908, the flat slab began being recognized as an acceptable floor system, but for many years designers were confronted with difficulties of patent infringements.

Actually the terms *two-way slab* [Fig. 16.1.1(b)], *flat slab* [Fig. 16.1.2(a)], and *flat plate* [Fig. 16.1.2(b)] are arbitrary, because there is in fact two-way action in all three types and a flat (usually nearly square) ceiling area usually exists within the panel in all three types. Following tradition, the implication is that there are beams between columns in two-way slabs; but no such beams, except perhaps edge beams along the exterior sides of the entire floor area, are used in flat slabs or flat plates. From the viewpoint of structural analysis, however, the distinction as to whether or not there are beams between columns is not pertinent, because if beams of any relative size could be designed to interact with the slab, use of beams of zero size would be only the limit condition.

If methods of structural analysis and design are developed for two-way slabs with beams, many of these general provisions should apply equally well to flat slabs or flat plates. Until 1971 the design of two-way slabs supported on beams was, historically, treated separately from the flat slabs or flat plates without beams. Various empirical procedures have been proposed and used [16.6–16.8]. Chapter 13 of the present ACI Code takes an integrated view and refers to two-way slab *systems* with or without beams. In addition to solid slabs, hollow slabs with interior voids to reduce dead weight, slabs (such as waffle slabs) with recesses made by permanent or removable fillers between joists in two directions, and slabs with paneled ceilings near the central portion of the panel are also included in this category (ACI-13.1.3).

Thus the term two-way *floor* systems (rather than the term two-way *slab* systems as in the ACI Code) is used in this book to include all three systems: the two-way slab with beams, the flat slab, and the flat plate.

## 16.2 GENERAL DESIGN CONCEPT OF ACI CODE

The basic approach to the design of two-way floor systems involves imagining that vertical cuts are made through the entire building along lines midway between the columns. The cutting creates a series of frames whose width lies between the centerlines of the two adjacent panels as shown in Fig. 16.2.1. The resulting series of rigid frames, taken separately in the longitudinal and transverse directions of the building, may be treated for gravity loading floor by floor, as would generally be acceptable for a rigid frame structure consisting of beams and columns, in accordance with ACI-8.9.1. A typical rigid frame would consist of (1) the columns above and below the floor, and (2) the floor system, with or without beams, bounded laterally between the centerlines of the two panels (one panel for an exterior line of columns) adjacent to the line of columns.

Thus the design of a two-way floor system (including two-way slab, flat slab, and flat plate) is reduced to that of a rigid frame; hence the name "equivalent frame method."

As in the case of design of actual rigid frames consisting of beams and columns, approximate methods of analysis may be suitable for many usual floor systems, spans, and story heights. As treated in Chapter 7, the analysis for actual frames could be (a) approximate using the moment and shear coefficients of ACI-8.3, or (b) more accurate using structural analysis after assuming the relative stiffnesses of the members.

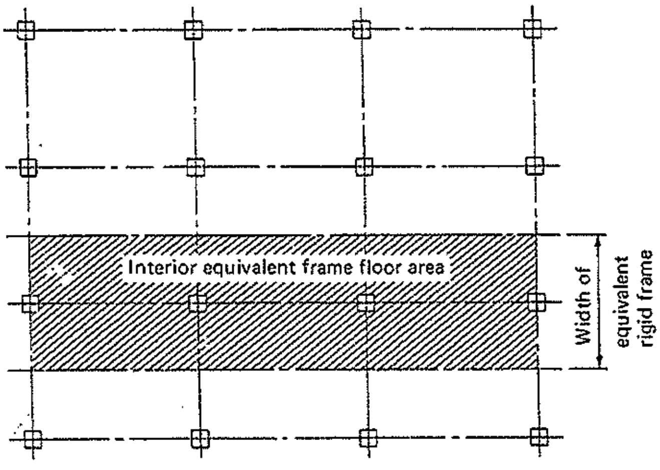



**Figure 16.2.1** Tributary floor area for an interior equivalent rigid frame of a two-way floor system.

For gravity load only and for floor systems within the specified limitations, the moments and shears on these equivalent frames may be determined (a) approximately using moment and shear coefficients prescribed by the "direct design method" of ACI-13.6, or (b) by structural analysis in a manner similar to that for actual frames using the special provisions of the "equivalent frame method" of ACI-13.7. An elastic analysis (such as by the equivalent frame method) must be used for lateral load even if the floor system meets the limitations of the direct design method for gravity load.

The equivalent rigid frame is the structure being dealt with whether the moments are determined by the "direct design method (DDM)" or by the "equivalent frame method (EFM)." These two ACI Code terms describe two ways of obtaining the longitudinal variation of bending moments and shears.

When the "equivalent frame method" is used for obtaining the longitudinal variation of moments and shears, the relative stiffness of the columns, as well as that of the floor system, can be assumed in the preliminary analysis and then reviewed, as is the case for the design of any statically indeterminate structure. Design moment envelopes may be obtained for dead load in combination with various patterns of live load, as described in Chapter 7 (Section 7.2). In lateral load analysis, moment magnification in columns due to sidesway of vertical loads must be taken into account as prescribed in ACI-10.11 through 10.14.

Once the longitudinal variation in factored moments and shears has been obtained, whether by ACI "DDM" or "EFM," the moment across the entire width of the floor system being considered is distributed laterally to the beam, if used, and to the slab. The lateral distribution procedure and the remainder of the design is essentially the same whether "DDM" or "EFM" has been used.

The accuracy of analysis methods utilizing the concept of dividing the structure into equivalent frames has been verified for *gravity load* analysis by tests [16.12–16.25] and analytical studies [16.26–16.35]. For *lateral load* analysis where there is less agreement on procedure, various studies have been made, including those of Pecknold [16.38], Allen and Darvall [16.39, 16.47], Vanderbilt [16.32, 16.40], Elias [16.41–16.43], Fraser [16.44], Vanderbilt and Corley [16.45], Lew and Narov [16.46], Pavlovic and Poulton [16.48], Moehle and Diebold [16.49], Hsu [16.50], and Cano and Klingner [16.51].

## ▸ 16.3 TOTAL FACTORED STATIC MOMENT

Consider two typical interior panels *ABCD* and *CDEF* in a two-way floor system, as shown in Fig. 16.3.1(a). Let $L_1$ and $L_2$ be the panel size in the longitudinal and transverse directions, respectively. Let lines 1–2 and 3–4 be centerlines of panels *ABCD* and *CDEF*, both parallel to the longitudinal direction. Isolate as a free body [see Fig. 16.3.1(b)] the floor slab and the included beam bounded by the lines 1–2 and 3–4 in the longitudinal direction and the transverse lines 1′–3′ and 2′–4′ at the faces of the columns in the transverse direction. The load acting on this free body [see Fig. 16.3.1(c)] is $w_u L_2$ per unit distance in the longitudinal direction. The total upward force acting on lines 1′–3′ or 2′–4′ is $w_u L_2 L_n/2$, where $w_u$ is the factored load per unit area and $L_n$ is the clear span in the longitudinal direction between faces of supports (as defined by ACI-13.6.2.5).

If $M_{neg}$ and $M_{pos}$ are the numerical values of the total negative and positive bending moments along lines 1′–3′ and 5′–6′, then equilibrium of the free body of Fig. 16.3.1(d) requires

$$M_{neg} + M_{pos} = \frac{w_u L_2 L_n^2}{8} \tag{16.3.1}$$

For a typical exterior panel, the negative moment at the interior support would be larger than that at the exterior support, as has been shown in Section 7.5. The maximum positive moment would occur at a section to the left of the midspan, as shown in Fig. 16.3.2(c). In practical design, it is customary to use $M_{pos}$ at midspan for determining

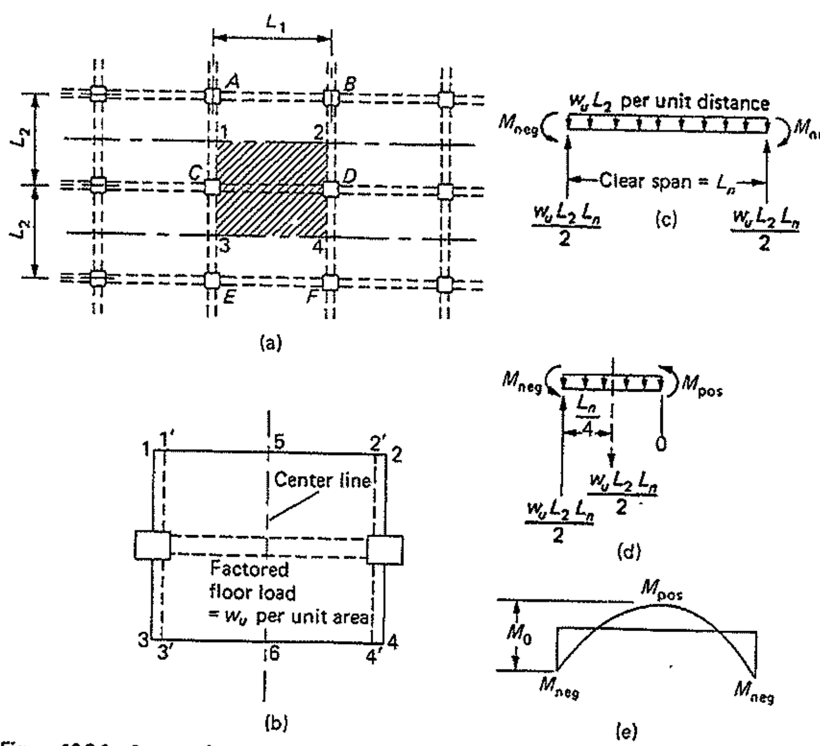



**Figure 16.3.1** Statics of a typical *interior* panel in a two-way floor system.

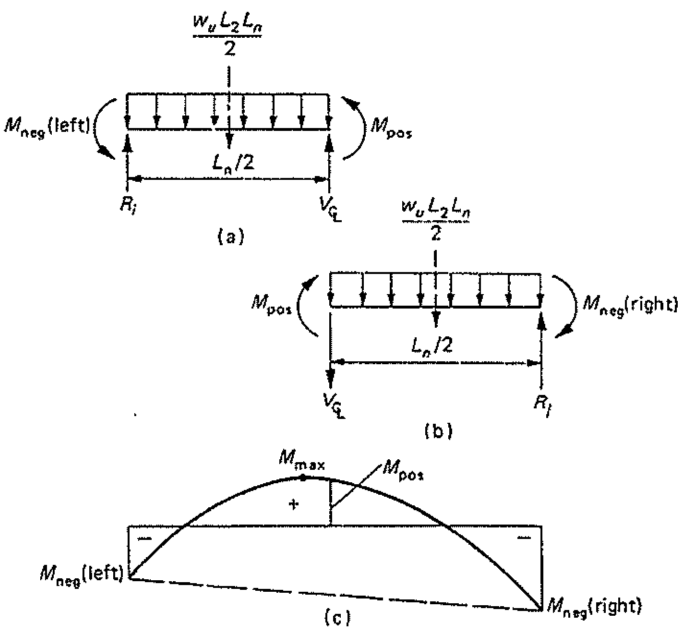



**Figure 16.3.2** Statics of typical *exterior* panel in a two-way floor system.

the required positive moment reinforcement. For this case,

$$\frac{M_{neg}(\text{left}) + M_{neg}(\text{right})}{2} + M_{pos} = \frac{w_u L_2 L_n^2}{8} \tag{16.3.2}$$

A proof for Eq. (16.3.2) can be obtained by writing the moment equilibrium equation about the left end of the free body shown in Fig. 16.3.2(a),

$$M_{neg}(\text{left}) + M_{pos} = \frac{w_u L_2 L_n}{2}\left(\frac{L_n}{4}\right) - V_{\text{midspan}}\left(\frac{L_n}{2}\right)$$

and, by writing the moment equilibrium equation about the right end of the free body shown in Fig. 16.3.2(b),

$$M_{neg}(\text{right}) + M_{pos} = \frac{w_u L_2 L_n}{2}\left(\frac{L_n}{4}\right) + V_{\text{midspan}}\left(\frac{L_n}{2}\right)$$

Equation (16.3.2) is arrived at by adding the two preceding equations and dividing by 2 on each side. Note that Eq. (16.3.2) may also be obtained, as shown in Fig. 16.3.2(c), by the superposition of the simple span uniform loading parabolic positive moment diagram over the trapezoidal negative moment diagram due to end moments.

ACI-13.6.2 uses the symbol $M_0$ to mean $w_u L_2 L_n^2/8$ and calls $M_0$ the *total factored static moment*. It states, "Absolute sum of positive and average negative factored moments in each direction shall not be less than $M_0$"; or

$$\frac{M_{neg}(\text{left}) + M_{neg}(\text{right})}{2} + M_{pos} \geq \left[M_0 = \frac{w_u L_2 L_n^2}{8}\right] \tag{16.3.3}$$

in which

$w_u$ = factored load per unit area

$L_n$ = clear span in the direction moments are being determined, measured face-to-face of supports (ACI-13.6.2.5), but not less than $0.65L_1$

$L_1$ = span length in the direction moment are being determined, measured center-to-center of supports

$L_2$ = transverse span length, measured center-to-center of supports

Equations (16.3.1) and (16.3.2) are theoretically derived on the basis that $M_{neg}$ (left), $M_{pos}$, and $M_{neg}$ (right) occur simultaneously for the same live load pattern on the adjacent panels of the equivalent rigid frame defined in Fig. 16.2.1. If the live load is relatively heavy compared with dead load, then different live load patterns should be used to obtain the critical positive moment at midspan and the critical negative moments at the supports. In such a case, the "equal" sign in Eqs. (16.3.1) and (16.3.2) becomes the "greater" sign. This is the reason why ACI-13.6.2.2 states "absolute sum ... shall not be *less* than $M_0$" as the design requirement. The designer should keep this in mind when steel reinforcement is selected for positive and negative bending moment in two-way floors when the direct design method is used for gravity load. To avoid the use of excessively small values of $M_0$ in the case of short spans and large columns or column capitals, the clear span $L_n$ to be used in Eq. (16.3.3) is not to be less than $0.65L_1$ (ACI-13.6.2.5).

When the limitations for using the direct design method are met, it is customary to divide the value of $M_0$ into $M_{neg}$ into $M_{pos}$, if the restraints at each end of the span are identical (Fig. 16.3.1); or into $[M_{neg}$ (left) + $M_{neg}$ (right)]/2 and $M_{pos}$ if the span end restraints are different (Fig. 16.3.2). Then the moments $M_{neg}$ (left), $M_{neg}$ (right), and $M_{pos}$ must be distributed transversely along the lines 1′–3′, 2′–4′, and 5–6, respectively. This last distribution is a function of the relative flexural stiffness between the slab and the included beam.

### Total Factored Static Moment in Flat Slabs

The ability of flat slab floor systems to carry load has been substantiated by numerous tests of actual structures [16.2]. However, the amount of reinforcement used, say, in a typical interior panel, was less than what it should be to satisfy an analysis by statics, as is demonstrated in this section. This led to some controversy [16.1], but after studies by Westergaard and Slater [16.3], a provision was adopted (about 1921) into the code that a reduction of moment coefficient from the statically required value of 0.125 to 0.09 may be made. This reduction was not regarded as a violation of statics but was used as a way of permitting an increase in the usable strength. The reduction, moreover, was applicable only to flat slabs that satisfied the limitations then specified in the code. Over the years these limitations had been liberalized, but at the same time the moment coefficient was raised to values closer to 0.125. The present ACI Code logically stipulates the use of the full statically required coefficient of 0.125.

The statical analysis of a typical interior panel was first made in 1914 by Nichols [16.1] and further developed later by Westergaard and others [16.3–16.5].

Consider the typical interior panel of a flat slab floor subjected to a factored load of $w_u$ per unit area, as shown in Fig. 16.3.3(a). The total load on the panel area (rectangle minus four quadrantal areas) is supported by the vertical shears at the four quadrantal arcs. Let $M_{neg}$ and $M_{pos}$ be the total negative and positive moments about a horizontal

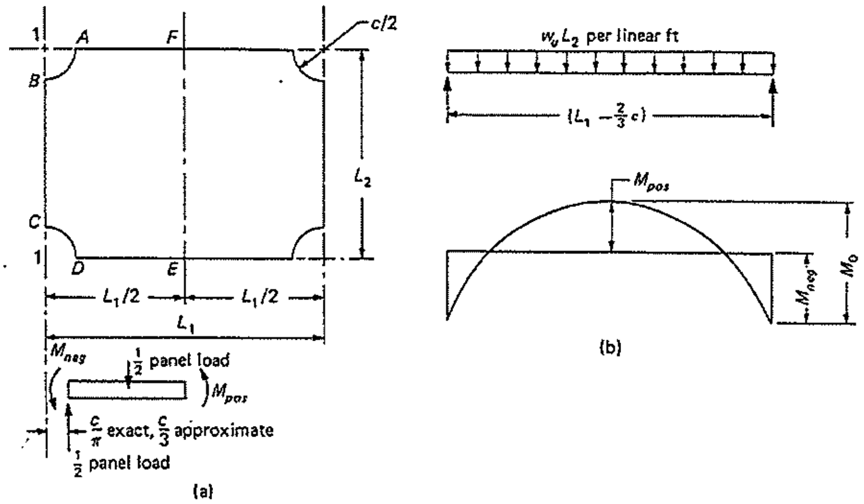



**Figure 16.3.3** Statics of a typical interior panel in a flat slab floor system.

axis in the $L_2$ direction along the edges of $ABCD$ and $EF$, respectively. Then

$$\text{load on area } ABCDEF = \text{sum of reactions at arcs } AB \text{ and } CD$$
$$= w_u\left(\frac{L_1L_2}{2} - \frac{\pi c^2}{8}\right)$$

Considering the half-panel $ABCDEF$ as a free body, recognizing that there is no shear at the edges $BC$, $DE$, $EF$, and $FA$, and taking moments about axis 1-1,

$$M_{neg} + M_{pos} + w_u\left(\frac{L_1L_2}{2} - \frac{\pi c^2}{8}\right)\left(\frac{c}{\pi}\right) - \frac{w_uL_1L_2}{2}\left(\frac{L_1}{4}\right) + \frac{w_u\pi c^2}{8}\left(\frac{2c}{3\pi}\right) = 0$$

Letting $M_0 = M_{neg} + M_{pos}$,

$$M_0 = \tfrac{1}{8}w_uL_2L_1^2\left(1 - \frac{4c}{\pi L_1} + \frac{c^3}{3L_2L_1^2}\right) \approx \tfrac{1}{8}w_uL_2L_1^2\left(1 - \frac{2c}{3L_1}\right)^2 \qquad (16.3.4)$$

Actually, Eq. (16.3.4) may be more easily visualized by inspecting the equivalent interior span as shown in Fig. 16.3.3(b).

ACI-13.6.2.5 states that circular or regular polygon shaped supports shall be treated as square supports having the same area. For flat slabs, particularly with column capitals, the clear span $L_n$ computed from using equivalent square supports should be compared with that indicated by Eq. (16.3.4), which is $L_1$ minus $2c/3$. In some cases the latter value is larger and should be used, consistent with the fact that ACI-13.6.2.2 does express its intent in an inequality.

## Design Examples

In an effort to present, explain, and illustrate the design procedure for the three types of two-way floor systems, identified in this chapter as two-way slabs (with beams), flat slabs, and flat plates, it will be necessary to assume that preliminary dimensions and sizes of the slab (and drop, if any), beams, and columns (and column capitals, if any) are available. In the usual design processes, not only the preliminary sizes may need to be revised as they are found unsuitable, but also designs based on two or three different relative beam sizes (when used) to slab thickness should be made and compared. Preliminary data for the three types of two-way floor systems to be illustrated are as follows.

## Data for Two-Way Slab (with Beams) Design Example

Figure 16.3.4 shows a two-way slab floor with a total area of 12,500 sq ft. It is divided into 25 panels with a panel size of 25 ft × 20 ft. Concrete strength is $f'_c = 3000$ psi and steel yield strength is $f_y = 40{,}000$ psi. Service live load is to be taken as 138 psf. Story height is 12 ft. The preliminary sizes are as follows: slab thickness is $6\frac{1}{2}$ in., long beams are 14 × 28 in. overall; short beams are 12 × 24 in. overall; upper and lower columns are 15 × 15 in. The four kinds of panels (corner, long-sided edge, short-sided edge, and interior) are numbered 1, 2, 3, and 4 in Fig. 16.3.4.

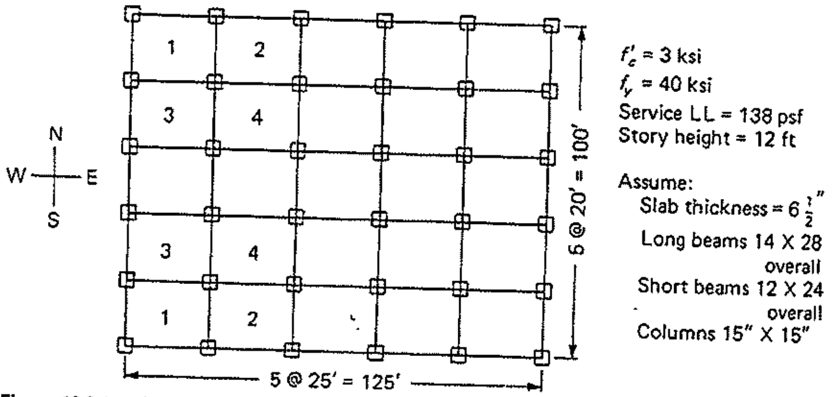



**Figure 16.3.4** Floor plan in design example for two-way slab with beams.

► **EXAMPLE 16.3.1**

For the two-way slab (with beams) design example, determine the total factored static moment in a loaded span in each of four equivalent rigid frames whose widths are designated $A$, $B$, $C$, and $D$ in Fig. 16.3.5.

**SOLUTION** The factored load $w_u$ per unit floor area is

$$w_u = 1.2\,w_D + 1.6\,w_L = 1.2(6.5)(150/12) + 1.6(138)$$
$$= 98 + 221 = 319 \text{ psf}$$

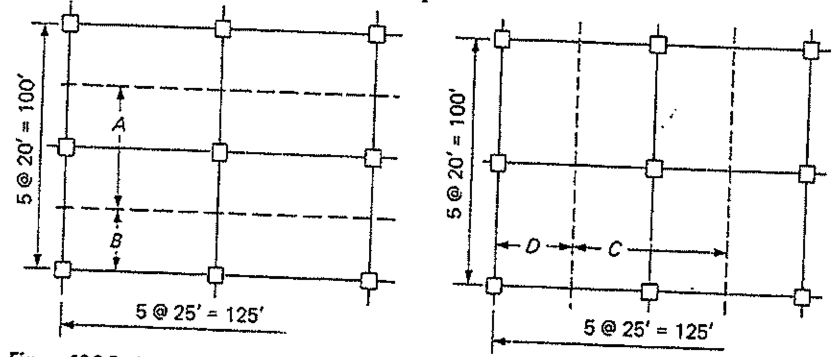



**Figure 16.3.5** Equivalent rigid frame notations in the two-way slab (with beams) design example.

for frame $A$, $M_0 = \frac{1}{8}w_u L_2 L_n^2 = \frac{1}{8}(0.319)(20)(25 - 1.25)^2 = 448$ ft-kips

for frame $B$, $M_0 = 224$ ft-kips

for frame $C$, $M_0 = \frac{1}{8}w_u L_2 L_n^2 = \frac{1}{8}(0.319)(25)(20 - 1.25)^2 = 350$ ft-kips

for frame $D$, $M_0 = 175$ ft-kips ◂

## Data for Flat Slab Design Example

Figure 16.3.6 shows a flat slab floor with a total area of 12,500 sq ft. It is divided into 25 panels with a panel size of 25 × 20 ft. Concrete strength is $f_c' = 3000$ psi and steel yield strength is $f_y = 40{,}000$ psi. Service live load is 140 psf. Story height is 10 ft. Exterior columns are 16 in. square and interior columns are 18 in. round. Edge beams are 14 × 24 in. overall. Thickness of slab is $7\frac{1}{2}$ in. outside of drop panel and $10\frac{1}{2}$ in. through the drop panel. Sizes of column capitals and drop panels are as shown in Fig. 16.3.6.

**EXAMPLE 16.3.2**

Compute the total factored static moment in the long and short directions of an interior panel in the flat slab design example as shown in Fig. 16.3.6. Compare the results obtained by using Eqs. (16.3.3) and (16.3.4).

**SOLUTION** Neglecting the weight of the drop panel, the service dead load is (150/12)(7.5) = 94 psf; thus

$$w_u = 1.2\,w_D + 1.6\,w_L = 1.2(94) + 1.6(140) = 113 + 224 = 337 \text{ psf}$$

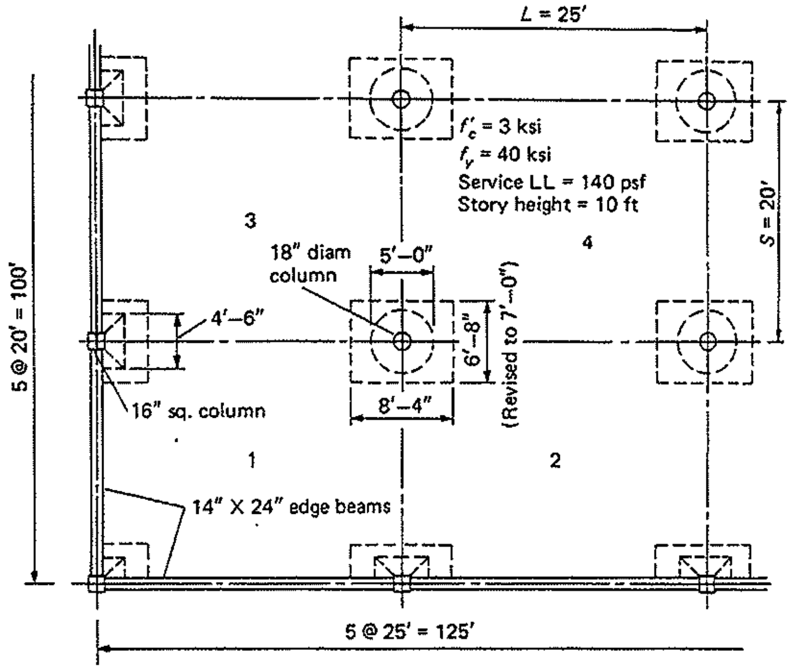



**Figure 16.3.6** Flat slab design example.

Using Eq. (16.3.4),

$$M_0 = \tfrac{1}{8}w_u L_2 L_1^2\left(1 - \frac{2c}{3L_1}\right)^2 = \tfrac{1}{8}(0.337)(20)(25)^2\left[1 - \frac{2(5)}{3(25)}\right]^2 = 396 \text{ ft-kips}$$

(in long direction)

$$M_0 = \tfrac{1}{8}w_u L_2 L_1^2\left(1 - \frac{2c}{3L_1}\right)^2 = \tfrac{1}{8}(0.337)(25)(20)^2\left[1 - \frac{2(5)}{3(20)}\right]^2 = 293 \text{ ft-kips}$$

(in short direction)

The equivalent square area for the column capital (ACI-13.6.2.5) has its side equal to 4.43 ft; then, using Eq. (16.3.3), with $L_n$ measured to the face of capital (i.e., equivalent square),

$$M_0 = \tfrac{1}{8}w_u L_2 L_n^2 = \tfrac{1}{8}(0.337)(20)(25 - 4.43)^2 = 356 \text{ ft-kips}$$

(in long direction)

$$M_0 = \tfrac{1}{8}w_u L_2 L_n^2 = \tfrac{1}{8}(0.337)(25)(20 - 4.43)^2 = 255 \text{ ft-kips}$$

(in short direction)

Insofar as flat slabs with column capitals are concerned, it appears that the larger values of 396 ft-kips and 293 ft-kips should be used because Eq. (16.3.4) is specially suitable; in particular, ACI-13.6.2.2 states that the total factored static moment shall *not be less than* that given by Eq. (16.3.3). ◀

### Data for Flat Plate Design Example

Figure 16.3.7 shows a flat plate floor with a total area of 4500 sq ft. It is divided into 25 panels with a panel size of 15 × 12 ft. Concrete strength is $f'_c = 4000$ psi and steel yield strength is $f_y = 50{,}000$ psi. Service live load is 72 psf. Story height is 9 ft. All columns are rectangular, 12 in. in the long direction and 10 in. in the short direction. Preliminary slab thickness is set at $5\frac{1}{2}$ in. No edge beams are used along the exterior edges of the floor.

▶ **EXAMPLE 16.3.3**

Compute the total factored static moment in the long and short directions of a typical panel in the flat plate design example as shown in Fig. 16.3.7.

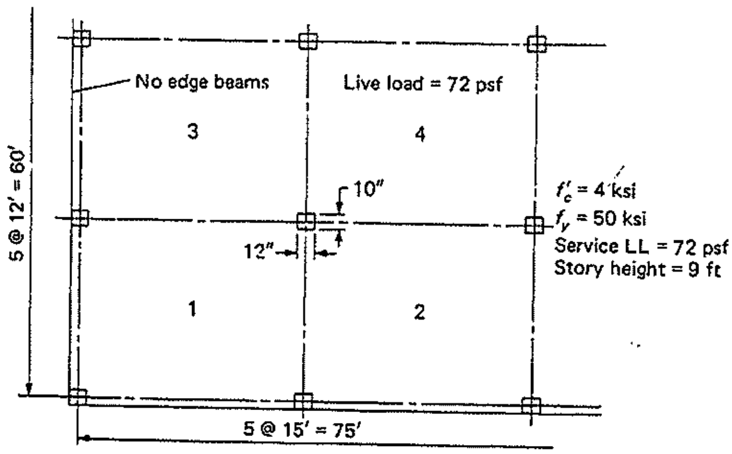



**Figure 16.3.7** Flat plate design example.

▶ 16.4 RA… OF…

**SOLUTION** The dead load for a $5\frac{1}{2}$-in. slab is

$$w_D = (5.5/12)(150) = 69 \text{ psf}$$

The factored load per unit area is

$$w_u = 1.2w_D + 1.6w_L = 1.2(69) + 1.6(72) = 83 + 115 = 198 \text{ psf}$$

Using Eq. (16.3.3), with clear span $L_n$ measured face-to face of columns,

$$M_0 = \tfrac{1}{8}(0.198)(12)(15-1)^2 = 58.2 \text{ ft-kips}$$
(in long direction)

$$M_0 = \tfrac{1}{8}(0.198)(15)(12-0.83)^2 = 46.3 \text{ ft-kips}$$
(in short direction) ◄

## 16.4 RATIO OF FLEXURAL STIFFNESSES OF LONGITUDINAL BEAM TO SLAB

When beams are used along the column lines in a two-way floor system, an important parameter affecting the design is the relative size of the beam to the thickness of the slab. This parameter can best be measured by the ratio $\alpha_f$ of the flexural rigidity (called flexural stiffness by the ACI Code) $E_{cb}I_b$ of the beam to the flexural rigidity $E_{cs}I_s$ of the slab in the transverse cross-section of the equivalent frame shown in Fig. 16.4.1. The separate moduli of elasticity $E_{cb}$ and $E_{cs}$, referring to the beam and slab, provide for different strength concrete (and thus different $E_c$ values) for the beam and slab. The moments of

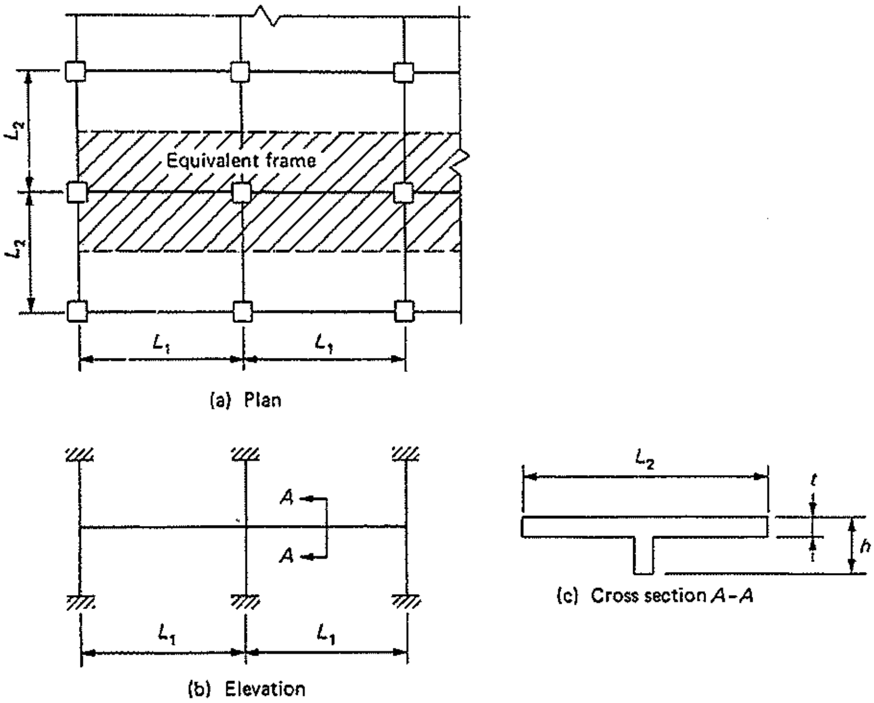



Figure 16.4.1 Plan, elevation, and cross-section of equivalent frame in a two-way floor system.

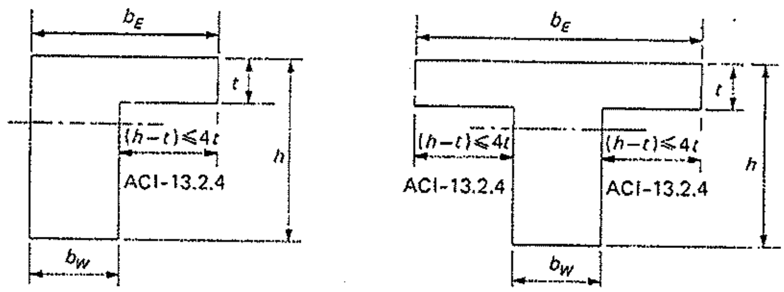



**Figure 16.4.2** Cross-sections for moment of inertia of a flanged section.

inertia $I_b$ and $I_s$ refer to the gross sections of the beam and slab within the cross-section of Fig. 16.4.1(c). ACI-13.2.4 permits the slab on each side of the beam web to act as a part of the beam, this slab portion being limited to a distance equal to the projection of the beam above or below the slab, whichever is greater, but not greater than four times the slab thickness, as shown in Fig. 16.4.2. More accurately, the small portion of the slab already counted in the beam should not be used in $I_s$, but ACI permits the use of the total width of the equivalent frame in computing $I_s$. Thus,

$$\alpha_f = \frac{E_{cb} I_b}{E_{cs} I_s} \tag{16.4.1}$$

The moment of inertia of a flanged beam section about its own centroidal axis (Fig. 16.4.2) may be shown to be

$$I_b = k \frac{b_w h^3}{12} \tag{16.4.2a}$$

in which

$$k = \frac{1 + \left(\frac{b_E}{b_w} - 1\right)\left(\frac{t}{h}\right)\left[4 - 6\left(\frac{t}{h}\right) + 4\left(\frac{t}{h}\right)^2 + \left(\frac{b_E}{b_w} - 1\right)\left(\frac{t}{h}\right)^3\right]}{1 + \left(\frac{b_E}{b_w} - 1\right)\left(\frac{t}{h}\right)} \tag{16.4.2b}$$

where

$h$ = overall beam depth
$t$ = overall slab thickness
$b_E$ = effective width of flange
$b_w$ = width of web

Equation (16.4.2b) expresses the nondimensional constant $k$ in terms of $(b_E/b_w)$ and $(t/h)$. Typical values of $k$ are tabulated in Table 16.4.1 and three curves are plotted in Fig. 16.4.3. The values of $k$ are about 1.4, 1.6, and 1.8, respectively, for $b_E/b_w$ values of 2, 3, and 4, when $t/h$ values are between 0.2 and 0.5. Thus

$$k \approx 1.0 + 0.2\left(\frac{b_E}{b_w}\right) \quad \text{for } 2 < \frac{b_E}{b_w} < 4 \quad \text{and} \quad 0.2 < \frac{t}{h} < 0.5 \tag{16.4.2c}$$

**▶ EXAMPLE 16.4.1**

For the two-way slab (with beams) design example described in Section 16.3, compute the ratio $\alpha_f$ of the flexural stiffness of the longitudinal beam to that of the slab in the equivalent rigid frame, for all the beams around panels 1, 2, 3, and 4 in Fig. 16.4.4.

**TABLE 16.4.1 Values of $k$ in Terms of $(b_E/b_w)$ and $(t/h)$ in Eq. (16.4.2b)**

| | $t/h$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $b_E/b_w$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 2 | 1.222 | 1.328 | 1.366 | 1.372 | 1.375 | 1.396 | 1.454 | 1.565 | 1.743 | 2.000 |
| 3 | 1.407 | 1.564 | 1.605 | 1.608 | 1.625 | 1.694 | 1.844 | 2.098 | 2.477 | 3.000 |
| 4 | 1.564 | 1.744 | 1.777 | 1.781 | 1.825 | 1.956 | 2.212 | 2.621 | 3.209 | 4.000 |

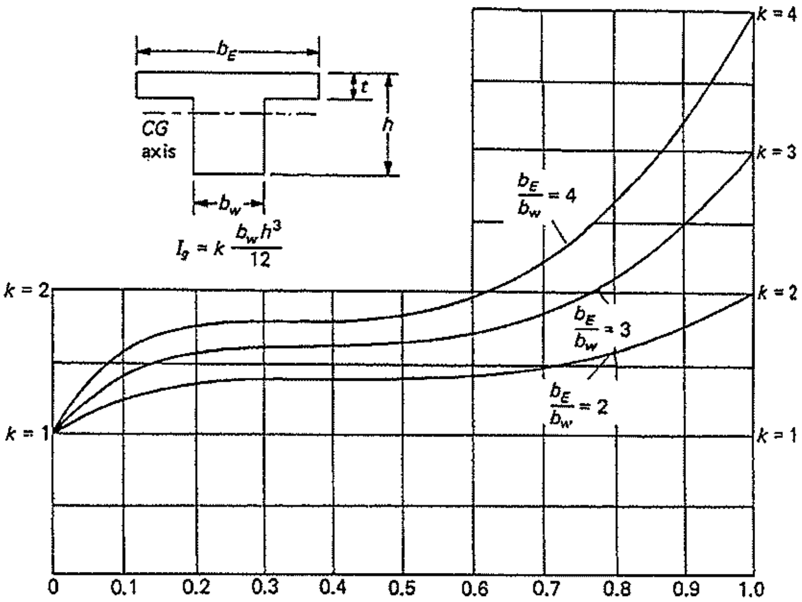



**Figure 16.4.3** Values of $k$ in terms of $b_E/b_w$ and $t/h$.

**SOLUTION** (a) $B1$–$B2$. Referring to Fig. 16.4.4, the effective width $b_E$ for $B1$–$B2$ is the smaller of $14 + 2(21.5) = 57$ in. and $14 + 8(6.5) = 66$ in.; thus $b_E = 57$ in. Using Eq.(16.4.2b),

$$\frac{b_E}{b_w} = \frac{57}{14} = 4.07, \qquad \frac{t}{h} = \frac{6.5}{28} = 0.232$$

$$k = 1.774, \qquad I_b = 1.774\frac{14(28)^3}{12} = 45{,}400 \text{ in.}^4$$

A slightly higher value of $k$ would have been obtained using Eq. (16.4.2c). Using Eq. (16.4.1), where $E_{cb} = E_{cs}$,

$$I_s = \tfrac{1}{12}(240)(6.5)^3 = 5490 \text{ in.}^4, \qquad \alpha_f = \frac{E_{cb}I_b}{E_{cs}I_s} = \frac{45{,}400}{5490} = 8.27$$

(b) $B3$–$B4$. Referring to Fig. 16.4.4, the effective width $b_E$ for $B3$–$B4$ is the smaller of $14 + 21.5 = 35.5$ in. and $14 + 4(6.5) = 40$ in.; thus $b_E = 35.5$ in. Using Eq. (16.4.2b),

$$\frac{b_E}{b_w} = \frac{35.5}{14} = 2.54, \qquad \frac{t}{h} = \frac{6.5}{28} = 0.232$$

$$k = 1.484, \qquad I_b = 1.484\frac{14(28)^3}{12} = 38{,}000 \text{ in.}^4$$

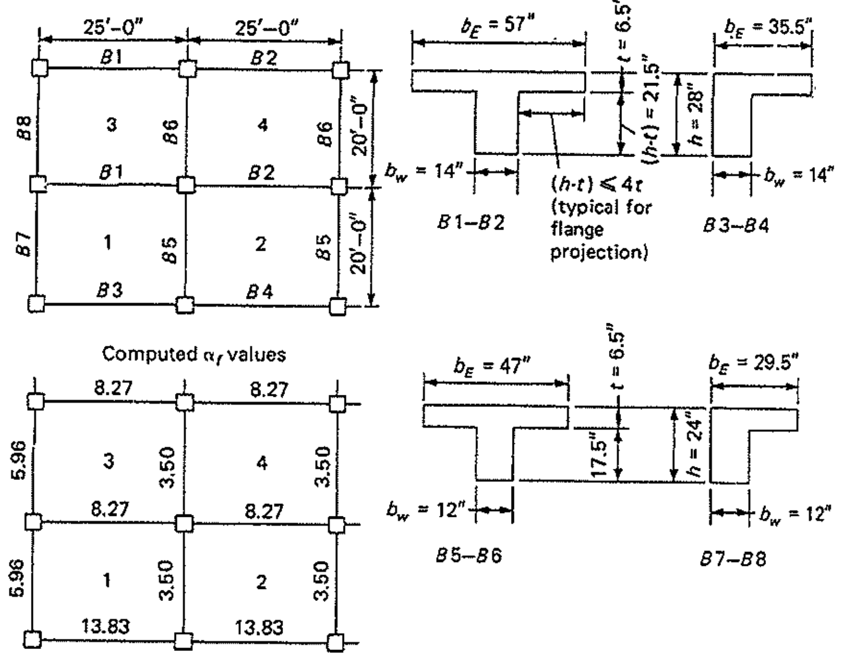



**Figure 16.4.4** Computation of $\alpha_f$ values in Example 16.4.1.

Using Eq. (16.4.1),

$$I_s = \frac{1}{12}(120)(6.5)^3 = 2745 \text{ in.}^4, \qquad \alpha_f = \frac{E_{cb}I_b}{E_{cs}I_s} = \frac{38,000}{2745} = 13.83$$

**(c)** $B5$–$B6$. Referring to Fig. 16.4.4, the effective width $b_E$ for $B5$–$B6$ is the smaller of $12 + 2(17.5) = 47$ in. and $12 + 8(6.5) = 64$ in.; thus $b_E = 47$ in. Using Eq. (16.4.2b),

$$\frac{b_E}{b_w} = \frac{47}{12} = 3.92, \qquad \frac{t}{h} = \frac{6.5}{24} = 0.271$$

$$k = 1.762, \qquad I_b = 1.762\frac{12(24)^3}{12} = 24,400 \text{ in.}^4$$

Using Eq. (16.4.1),

$$I_s = \frac{1}{12}(300)(6.5)^3 = 6870 \text{ in.}^4, \qquad \alpha_f = \frac{E_{cb}I_b}{E_{cs}I_s} = \frac{24,000}{6870} = 3.50$$

**(d)** $B7$–$B8$. Referring to Fig. 16.4.4, the effective width $b_E$ for $B7$–$B8$ is the smaller of $12 + 17.5 = 29.5$ in. and $12 + 4(6.5) = 38$ in.; thus $b_E = 29.5$ in. Using Eq. (16.4.2b),

$$\frac{b_E}{b_w} = \frac{29.5}{12} = 2.46, \qquad \frac{t}{h} = 0.271$$

$$k = 1.480, \qquad I_b = 1.480\frac{12(24)^3}{12} = 20,500 \text{ in.}^4$$

Using Eq. (16.4.1),

$$I_s = \frac{1}{12}(150)(6.5)^3 = 3435 \text{ in.}^4, \qquad \alpha_f = \frac{E_{cb}I_b}{E_{cs}I_s} = \frac{20,500}{3435} = 5.96$$

The resulting $\alpha_f$ values for $B1$ through $B8$ around panels 1, 2, 3, and 4 are shown in Fig. 16.4.4. For this design, the $\alpha_f$ values vary between 3.50 and 13.83; thus the equivalent rigid frames have their substantial portion along or close to the column lines, even though their widths vary from 10 to 25 ft.

## 16.5 MINIMUM SLAB THICKNESS FOR DEFLECTION CONTROL

Control of deflections in two-way floor systems is dealt with in ACI-9.5.3. When deflections are computed according to ACI-9.5.3.4, then ACI-Table 9.5(b) states the maximum permissible computed deflection. To compute deflections, the use of the effective moment of inertia $I_e$, Eq. (14.4.1), is endorsed unless computed deflections using other procedures are "in reasonable agreement with results of comprehensive tests." Various methods for obtaining deflections of two-way floor systems have been proposed [16.52–16.71]; however, no specific procedure is given by the ACI Code or Commentary. Computation of two-way floor system deflections is outside the scope of this book.

To aid the designer, ACI-9.5.3.2 provides a minimum thickness table [ACI-Table 9.5(c)] for slabs *without* interior beams, though there can be exterior boundary beams. For slabs *with* beams spanning between the supports on all sides, ACI-9.5.3.3 provides minimum thickness equations. If the designer wishes to use lesser thickness than indicated by ACI-9.5.3.2 or 9.5.3.3, ACI-9.5.3.4 permits a lesser thickness "if shown by computation that the deflection will not exceed the limits stipulated in Table 9.5(b)." Computation of deflections must "take into account size and shape of the panel, conditions of support, and nature of restraints at the panel edges." Minimum thickness from ACI-Table 9.5(c) and the formulas of ACI-9.5.3.3 give slab thicknesses that, from experience, are considered satisfactory.

### Slabs Without Interior Beams Spanning Between Supports

The minimum thickness, with the requirement that the ratio of long to short span be not greater than 2, shall be that given by Table 16.5.1 [ACI-Table 9.5(c)], but not less than:

| | |
|---|---|
| For slabs without drop panels | 5 in. |
| For slabs with drop panels | 4 in. |

In the flat slab and flat plate two-way systems, there may or may not be edge beams but there are definitely no interior beams in such systems. Thus, for the flat slab and flat plate, the ACI code *requires* the minimum slab thickness to be obtained from ACI-Table

**TABLE 16.5.1 Minimum Thickness of Slab Without Interior Beams [Adapted from ACI-Table 9.5(c)]**

| $f_y^*$ (ksi) | Without Drop Panels† | | | With Drop Panels† | | |
|---|---|---|---|---|---|---|
| | Exterior Panels | | Interior Panels | Exterior Panels | | Interior Panels |
| | $\alpha_f = 0$ | $\alpha_f \geq 0.8$ | | $\alpha_f = 0$ | $\alpha_f \geq 0.8$ | |
| 40 | $\frac{L_n}{33}$ | $\frac{L_n}{36}$ | $\frac{L_n}{36}$ | $\frac{L_n}{36}$ | $\frac{L_n}{40}$ | $\frac{L_n}{40}$ |
| 60 | $\frac{L_n}{30}$ | $\frac{L_n}{33}$ | $\frac{L_n}{33}$ | $\frac{L_n}{33}$ | $\frac{L_n}{36}$ | $\frac{L_n}{36}$ |
| 75 | $\frac{L_n}{28}$ | $\frac{L_n}{31}$ | $\frac{L_n}{31}$ | $\frac{L_n}{31}$ | $\frac{L_n}{34}$ | $\frac{L_n}{34}$ |

*For $f_y$ between 40 and 60 ksi, min $t$ is to be obtained by linear interpolation.

†Drop panel is defined in ACI-13.3.7.1 and 13.3.7.2.

9.5(c) (i.e., Table 16.5.1), whereas in the past such minimum thickness could also be obtained from ACI Formulas.

### Slabs Supported on Beams

Four parameters affect the equations of ACI-9.5.3.3 for slabs supported on beams on all sides; they are (1) the longer clear span $L_n$ of the slab panel; (2) the ratio $\beta$ of the longer clear span $L_n$ to the shorter clear span $S_n$; (3) the yield strength $f_y$ of the steel reinforcement; (4) the average $\alpha_{fm}$ for the four $\alpha_f$ values for relative stiffness of a panel perimeter beam compared to the slab, as described in Section 16.4.

In terms of these parameters, ACI-9.5.3.3 requires the following for "slabs with beams spanning between the supports on all sides."

### Slabs Supported on Shallow Beams Where $\alpha_{fm} \leq 0.2$

The minimum slab thickness requirements are the same as for slabs without interior beams.

### Slabs Supported on Medium Stiff Beams Where $0.2 < \alpha_{fm} \leq 2.0$

For this case,

$$\text{Min } t = \frac{L_n(0.8 + f_y/200{,}000)}{36 + 5\beta(\alpha_{fm} - 0.2)} \tag{16.5.1}$$

which is ACI Formula (9-12). The minimum is not to be less than 5 in.

### Slabs Supported on Very Stiff Beams Where $\alpha_{fm} > 2.0$

For this case,

$$\text{Min } t = \frac{L_n(0.8 + f_y/200{,}000)}{36 + 9\beta} \tag{16.5.2}$$

which is ACI Formula (9-13). The minimum is not to be less than 3.5 in.

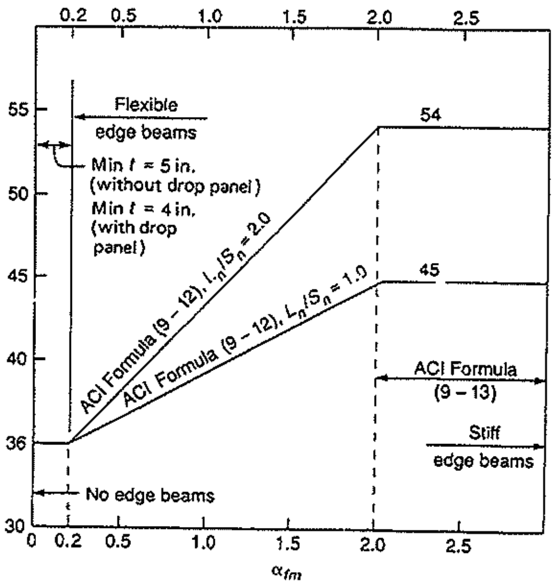



**Figure 16.5.1** ACI minimum slab thickness formulas (for Grade 40 steel). For Grade 60, divide $L_n/t$ by 1.1; for Grade 50, divide $L_n/t$ by 1.05.

The effect of the formulas may be observed from Fig. 16.5.1 where the vertical axis is the ratio of the long direction clear span $L_n$ to the minimum thickness $t$. This approach is similar to the span-to-depth ratio limitations used for beams in Sections 14.10 and 14.11. Figure 16.5.1 includes the full feasible range of parameters: (1) the panel proportions $L_n/S_n$ ranging from square to two-to-one rectangular, and (2) the average edge stiffness parameter $\alpha_{fm}$ ranging from zero with no edge beams to 2.5 or so for very stiff edge beams. The parameter $f_y$ is accounted for by the footnote stating the value for minimum $L_n/t$ must be divided by 1.1 for $f_y = 60{,}000$ psi, or by 1.05 for $f_y = 50{,}000$ psi, which means that the limiting minimum is larger for a given $L_n$ when $f_y$ exceeds 40,000 psi. The parameter $L_n$ is included in Fig. 16.5.1.

### Edge Beams at Discontinuous Edges

For all slabs supported on beams, there must be an edge beam at discontinuous edges having a stiffness ratio $\alpha_f$ not less than 0.80, or the minimum thickness required by Eqs. (16.5.1) or (16.5.2) "shall be increased by at least 10 percent in the panel with the discontinuous edge."

## 16.6 NOMINAL REQUIREMENTS FOR SLAB THICKNESS AND SIZE OF EDGE BEAMS, COLUMN CAPITAL, AND DROP PANEL

Whether the ACI "direct design method" or the "equivalent frame method" is used for determining the longitudinal distribution of bending moments, certain nominal requirements for slab thickness and size of edge beams, column capital, and drop panel must be fulfilled. These requirements are termed "nominal" because they are code-prescribed. It should be realized, of course, that the code provisions are based on a combination of experience, judgment, tests, and theoretical analyses.

### Slab Thickness

As discussed in Section 16.5, ACI Formulas (9-12) and (9-13) [Eqs. (16.5.1) and (16.5.2)], along with ACI-Table 9.5(c) [Table 16.5.1] set minimum slab thickness for two-way floor systems. In addition, ACI-9.5.3.2 and 9.5.3.3 set lower limits for the minimum value based on experience and practical requirements. These lower limits for two-way slab systems are summarized:

| | |
|---|---|
| Flat plates and flat slabs without drop panels | 5 in. |
| Slabs on shallow interior beams having $\alpha_{fm} < 0.2$ | 5 in. |
| Slabs without interior beams but having drop panels | 4 in. |
| Slabs with stiff interior beams having $\alpha_{fm} \geq 2.0$ | 3.5 in. |

### Edge Beams

For slabs supported by interior beams, the minimum thickness requirements assume an edge beam having a stiffness ratio $\alpha_f$ not less than 0.80. If such an edge beam is not provided, the minimum thickness as required by ACI Formulas (9-12) or (9-13) [Eqs. (16.5.1) and (16.5.2)] must be increased by 10% in the panel having the discontinuous edge. For slabs not having interior support beams, the increased minimum thickness in the exterior panel having the discontinuous edge is given by ACI-Table 9.5(c) [Table 16.5.1].

### Column Capital

Used in flat slab construction, the column capital (Fig. 16.6.1) is an enlargement of the top of the column as it meets the floor slab or drop panel. Since no beams are used, the

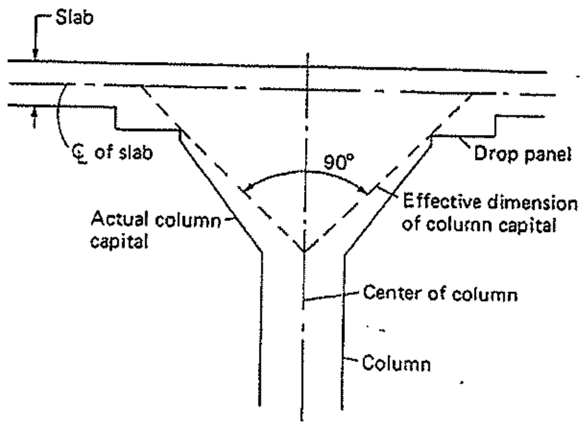



**Figure 16.6.1** Effective dimension of column capital.

purpose of the capital is to gain increased perimeter around the column to transmit shear from the floor loading and to provide increasing thickness as the perimeter decreases near the column. Assuming a maximum 45° line for distribution of the shear into the column, ACI-13.1.2 requires that the effective column capital for strength considerations be within the largest circular cone, right pyramid, or tapered wedge with a 90° vertex that can be included within the outlines of the actual supporting element (see Fig. 16.6.1). The diameter of the column capital is usually about 20 to 25% of the average span length between columns.

## Drop Panel

The drop panel (Fig. 16.1.2) is often used in flat slab and flat plate construction as a means of increasing the shear strength around a column or reducing the negative moment reinforcement over a column. It is an increased slab thickness in the region surrounding a column. A drop panel must comply with the dimensional limitations of ACI-13.2.5. The panel must extend from the centerline of supports a minimum distance of one-sixth of the span length measured from center-to-center in each direction, and the projection of the panel below the slab must be at least one-fourth of the slab thickness outside of the drop (ACI-13.2.5). When a qualifying drop is used, the minimum thickness given by ACI-Table 9.5(c) has been reduced by 10% from the minimum when a drop is not used.

For determining the reinforcement requirement, ACI-13.3.7 stipulates that the thickness of the drop below the slab be assumed no larger than one-quarter of the distance between the edge of the drop panel and the edge of the column or column capital. Because of this limitation, there is little reason to use a drop panel of greater plan dimensions or thickness than enough to satisfy using the reduced thickness for the slab outside the drop panel.

**▶ EXAMPLE 16.6.1**

For the two-way slab (with beams) design example described in Section 16.3, determine the minimum thickness requirement for deflection control; and compare it with the preliminary thickness of $6\frac{1}{2}$ in.

**SOLUTION** The average ratios $\alpha_{fm}$ for panels 1, 2, 3, and 4 may be computed from the $\alpha_f$ values shown in Fig. 16.4.4; thus

$$\alpha_{fm} \text{ for panel } 1 = \tfrac{1}{4}(5.96 + 8.27 + 3.50 + 13.83) = 7.90$$
$$\alpha_{fm} \text{ for panel } 2 = \tfrac{1}{4}(3.50 + 8.27 + 3.50 + 13.83) = 7.29$$

▶ EXAMPLE

$$\alpha_{fm} \text{ for panel 3} = \tfrac{1}{4}(5.96 + 8.27 + 3.50 + 8.27) = 6.50$$

$$\alpha_{fm} \text{ for panel 4} = \tfrac{1}{4}(3.50 + 8.27 + 3.50 + 8.27) = 5.89$$

Since the $\alpha_{fm}$ values for all four panels are well above 2, Fig. 16.5.1 shows that Eq. (16.5.2), which is ACI Formula (9-13), applies. The minimum thickness for all panels, using $L_n = 24$ ft, $S_n = 18.83$ ft, and $f_y = 40{,}000$ psi, becomes

$$\min t = \frac{L_n(0.8 + 0.2f_y/40{,}000)}{36 + 9L_n/S_n} = \frac{24(12)1.0}{36 + 9(24)/18.83} = 6.07 \text{ in.}$$

If a uniform slab thickness for the entire floor area is to be used, the minimum for deflection control is 6.07 in., which compares well with the $6\frac{1}{2}$-in. preliminary thickness. ◂

**EXAMPLE 16.6.2**

Review the slab thickness and other nominal requirements for the dimensions in the flat slab design example described in Section 16.3.

**SOLUTION** (a) Stiffness of edge beams. Before using Table 16.5.1 or ACI-Table 9.5(c), the $\alpha_f$ values for the edge beams are needed. The moment of inertia of the edge beam section shown in Fig. 16.6.2(b) is 22,900 in.[4] Thus the $\alpha_f$ value for the long edge beam is

$$\alpha_f = \frac{I_b}{I_s} = \frac{22{,}900}{120(7.5)^3/12} = \frac{22{,}900}{4220} = 5.42$$

and for the short edge beam, it is

$$\alpha_f = \frac{I_b}{I_s} = \frac{22{,}900}{150(7.5)^3/12} = \frac{22{,}900}{5270} = 4.34$$

These $\alpha_f$ values are entered on Fig. 16.6.2(a).

(b) Minimum slab thickness using Table 16.5.1 or ACI-Table 9.5(c). The long and short clear spans for deflection control are

$$L_n = 25 - 4.43 = 20.57 \text{ ft}; \qquad S_n = 20 - 4.43 = 15.57 \text{ ft}$$

from which

$$\frac{L_n}{S_n} = \frac{20.57}{15.57} = 1.32$$

For $f_y = 40$ ksi, a flat slab with drop panel, and $\alpha_f =$ smaller of 4.34 and 5.42, Table 16.5.1 gives

$$\min t = \frac{L_n}{40} = \frac{20.57(12)}{40} = 6.17 \text{ in.}$$

for both exterior and interior panels.

(c) Nominal requirement for slab thickness. The minimum thickness required is, from part (b), 6.17 in. The $7\frac{1}{2}$ in. slab thickness used is more than ample; $6\frac{1}{2}$ in. should probably have been used.

(d) Thickness of drop panel. Reinforcement within the drop panel must be computed on the basis of the $10\frac{1}{2}$-in. thickness actually used or $7\frac{1}{2}$ in. plus one-fourth of the projection of the drop beyond the column capital, whichever is smaller. In order that the full 3-in. projection of the drop below the $7\frac{1}{2}$-in. slab is usable in computing reinforcement, the 6 ft 8 in. side of the drop is revised to 7 ft so that one-fourth of the distance between the edges of the 5-ft column capital and the 7-ft drop is just equal to (10.5 − 7.5) = 3 in. ◂

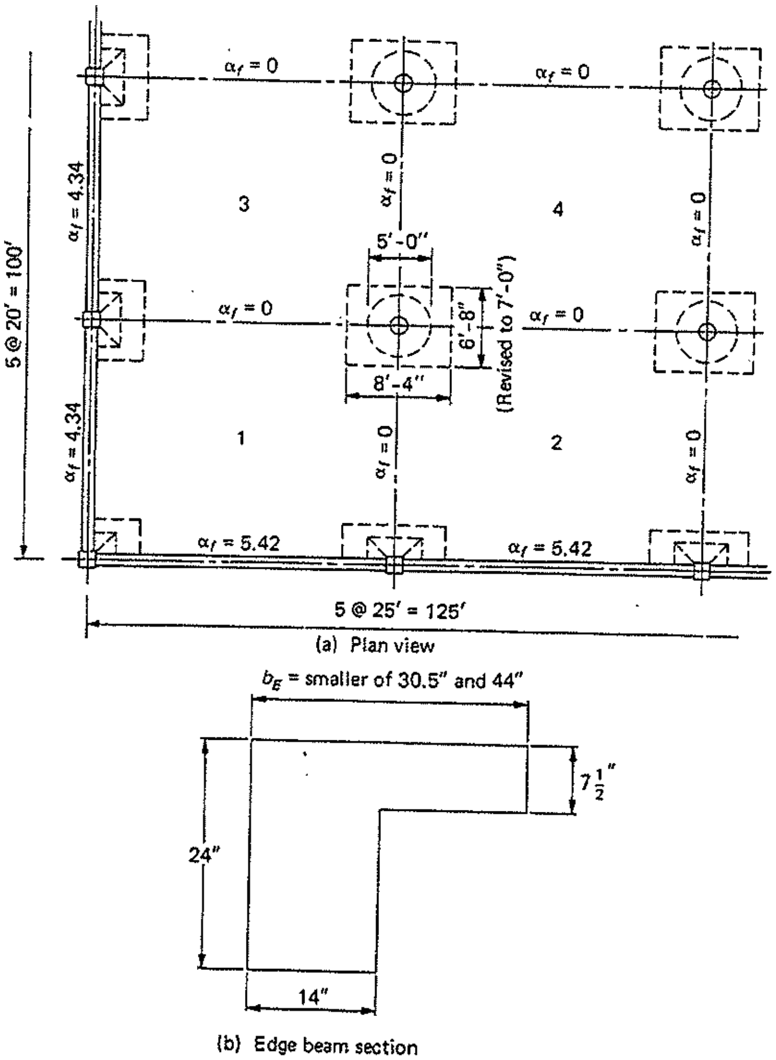



**Figure 16.6.2** Computation of $\alpha_f$ values for the flat slab in Example 16.6.2.

**▶ EXAMPLE 16.6.3**

Review the slab thickness and other nominal requirements for the dimensions in the flat plate design example described in Section 16.3.

**SOLUTION** (a) Minimum slab thickness from ACI-Table 9.5(c). For $f_y = 50$ ksi, for a flat plate which inherently has $\alpha_f = 0$, and $L_n = 15 - 1 = 14$ ft, from Table 16.5.1,

$$\begin{aligned}\text{min } t &= \text{linear interpolation between } f_y = 40 \text{ ksi and } f_y = 60 \text{ ksi}\\ &= \frac{1}{2}\left(\frac{L_n}{33} + \frac{L_n}{30}\right) = \frac{1}{2}(168)\left(\frac{1}{33} + \frac{1}{30}\right) = 5.34 \text{ in.} \quad \text{(exterior panel)}\end{aligned}$$

and

$$\min t = \frac{1}{2}\left(\frac{L_n}{36} + \frac{L_n}{33}\right) = \frac{1}{2}(168)\left(\frac{1}{36} + \frac{1}{33}\right) = 4.88 \text{ in.} \quad \text{(interior panel)}$$

**(b)** General. Prior to the 1995 ACI Code, there was an option to use an ACI Formula to obtain the minimum thickness. Since formulas, no matter how complicated, cannot accurately give the minimum thickness to ensure there will be no deflection problem, a table value seems appropriate and entirely within the accuracy of engineering knowledge regarding deflection. The $5\frac{1}{2}$-in. slab thickness used for all panels satisfies the ACI-Table 9.5(c) minimum and exceeds the nominal minimum of 5 in. for slabs without drop panels and without interior beams. ◄

## 16.7 DIRECT DESIGN METHOD—LIMITATIONS

Over the years the use of two-way floor systems has been extended from one-story or low-rise to medium or high-rise buildings. For the common cases of one-story or low-rise buildings, lateral load (wind or earthquake) is of lesser concern; thus most of the ACI Code refers only to gravity load (dead and live uniform load). In particular, when the dimensions of the floor system are quite regular and when the live load is not excessively large compared to the dead load, the use of a set of prescribed coefficients to distribute longitudinally the total factored static moment $M_0$ seems reasonable. As shown in Figs. 16.3.1 and 16.3.2, for each clear span in the equivalent rigid frame, the equation

$$\frac{M_{neg}(\text{left}) + M_{neg}(\text{right})}{2} + M_{pos} \geq \left[M_0 = \frac{w_u L_2 L_n^2}{8}\right] \qquad [16.3.3]$$

is to be satisfied.

To use the direct design method, in which a set of prescribed coefficients give the negative end moments and the positive moment within the span of the equivalent rigid frame, ACI-13.6.1 imposes the following limitations:

1. There is a minimum of three continuous spans in each direction.
2. Panels must be rectangular with the ratio of longer to shorter span center-to-center of supports within a panel not greater than 2.0.
3. The successive span lengths center to-center of supports in each direction do not differ by more than one-third of the longer span.
4. Columns are not offset more than 10% of the span in the direction of the offset.
5. The load is due to gravity only and is uniformly distributed over an entire panel, and the service live load does not exceed *two* times the service dead load.
6. The relative stiffness ratio of $L_1^2/\alpha_{f1}$ to $L_2^2/\alpha_{f2}$ must lie between 0.2 and 5.0, where $\alpha_f$ is the ratio of the flexural stiffness of the included beam to that of the slab.

Though the design of two-way floor systems is to a large extent empirical, the ACI limitations conform to the experimental results that are available [16.15–16.22] and to many years of experience with slabs in actual structures. The "direct design method" can also be used when it can be demonstrated that variations from any of the six limitations will still produce a slab system that satisfies the conditions of equilibrium and geometric compatibility and provides strength as required by ACI-9.2 and 9.3, and that all serviceability conditions are met, including specified limits on deflection. Van Buren [16.28] has provided such an analysis for staggered columns in flat plates.

**▶ EXAMPLE 16.7.1** Show that for the two-way slab (with beams) design example described in Section 16.3 the six limitations of the direct design method are satisfied.

**SOLUTION** The first four limitations are satisfied by inspection. For the fifth limitation,

$$\text{service dead load } w_D = 6.5\left(\frac{150}{12}\right) = 81 \text{ psf}$$

$$\text{service live load } w_L = 138 \text{ psf}$$

$$\frac{w_L}{w_D} = \frac{138}{81} < 2 \qquad \text{OK}$$

For the sixth limitation, referring to Fig. 16.4.4 and taking $L_1$ and $L_2$ in the long and short directions, respectively,

Panel 1,
$$\frac{L_1^2}{\alpha_{f1}} = \frac{625}{0.5(13.83 + 8.27)} = 56.6$$
$$\frac{L_2^2}{\alpha_{f2}} = \frac{400}{0.5(5.96 + 3.50)} = 84.6$$

Panel 2,
$$\frac{L_1^2}{\alpha_{f1}} = \frac{625}{0.5(13.83 + 8.27)} = 56.6$$
$$\frac{L_2^2}{\alpha_{f2}} = \frac{400}{3.50} = 114.3$$

Panel 3,
$$\frac{L_1^2}{\alpha_{f1}} = \frac{625}{8.27} = 75.6$$
$$\frac{L_2^2}{\alpha_{f2}} = \frac{400}{0.5(5.96 + 3.50)} = 84.6$$

Panel 4,
$$\frac{L_1^2}{\alpha_{f1}} = \frac{625}{8.27} = 75.6$$
$$\frac{L_2^2}{\alpha_{f2}} = \frac{400}{3.50} = 114.3$$

All ratios of $L_1^2/\alpha_{f1}$ to $L_2^2/\alpha_{f2}$ lie between 0.2 and 5. ◀

## ▸ 16.8 DIRECT DESIGN METHOD—LONGITUDINAL DISTRIBUTION OF MOMENTS

In the "direct design method," moment curves in the direction of span length need not be computed by an elastic analysis of the equivalent rigid frame subjected to various pattern loadings; instead they are nominally defined for regular situations.

Figure 16.8.1 shows the longitudinal moment diagram for the typical interior span of the equivalent rigid frame in a two-way floor system as prescribed by ACI-13.6.3.2. Later in Section 16.12, the positive moment $0.35M_0$ or the negative moment $0.65M_0$ is to be distributed transversely to the slab having width $L_2$ and to the included beam (if any) having clear span $L_n$. Note that

$$M_0 = \tfrac{1}{8} w_u L_2 L_n^2 \qquad [16.3.3]$$

For a span that is completely fixed at both ends, the negative moment at the fixed end is twice as large as the positive moment at midspan. For a typical interior span satisfying

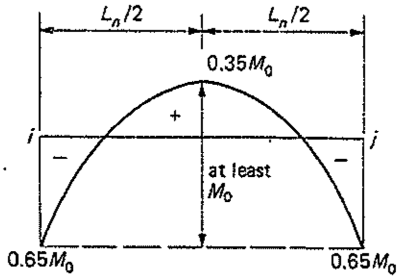



**Figure 16.8.1** Longitudinal moment diagram for interior span.

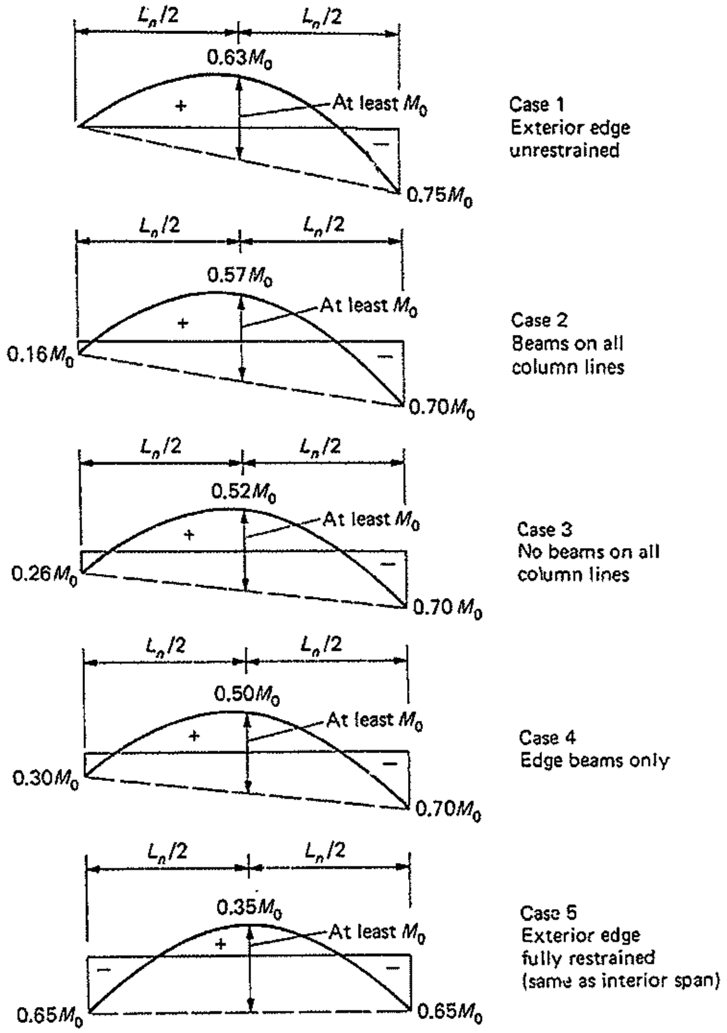



**Figure 16.8.2** Longitudinal moment diagram for exterior span.

the limitations for the direct design method, the specified negative moment of $0.65M_0$ is a little less than twice the specified positive moment of $0.35M_0$, which is fairly reasonable because the restraining effect of the columns and adjacent panels is definitely less than that of a completely fixed-ended beam.

For the exterior span, ACI-13.6.3.3 provides the longitudinal moment diagram for each of the five categories as described in Fig. 16.8.2. On examination of these diagrams, one sees that the negative moment at the exterior support increases from 0 to $0.65M_0$, the positive moment within the span decreases from $0.63M_0$ to $0.35M_0$, and the negative moment at the interior support decreases from $0.75M_0$ to $0.65M_0$, all gradually as the restraint at the exterior support increases from the case of a slab simply supported on a masonry or concrete wall (unrestrained) to that of a reinforced concrete wall built monolithically with the slab (fully restrained). ACI Commentary-R13.6.3.3 states that high positive moments are purposely assigned into the span since design for exterior negative moment will be governed by minimum reinforcement to control cracking.

Regarding the ACI Code suggested moment diagrams of Figs. 16.8.1 and 16.8.2, ACI-13.6.7 permits these moments to be modified by 10% provided the total factored static moment $M_0$ for the panel is statically accommodated.

## ▸ 16.9 DIRECT DESIGN METHOD—EFFECT OF PATTERN LOADINGS ON POSITIVE MOMENT

To understand the effect of pattern loading on the longitudinal moment values in multiple panel two-way floor systems, it is convenient to review some aspects of the continuity analysis of the usual column–beam type of rigid frames discussed earlier in Chapter 7. Some of the findings, which might be visualized using knowledge of influence lines and maximum moment envelopes due to dead and live load combinations, are as follows: (1) the higher the ratio of column stiffness to beam stiffness, the smaller the effect of pattern loadings, because the ends of the span are closer to the fixed condition and less effect is exerted on the span by loading patterns on adjacent spans; (2) the lower the ratio of dead load to live load, the larger the effect of pattern loadings, because dead load exists constantly on all spans and the pattern is related to live load only; and (3) maximum negative moments at supports are less affected by pattern loadings than maximum positive moments within the span.

▸ EXAMPLE

Prior to the 1995 ACI Code, the adjustment of positive moment to account for pattern loading had to be considered. Since 1995, the ACI Code restricts the uses of the direct design method to cases where the service live load does not exceed *two* (instead of three used previously) times the service dead load. With this lower maximum ratio for live load to dead load, the ACI Code committee concluded the number of cases where pattern loading would have a significant effect would be small; thus, adjustment for pattern loading no longer appears in the ACI Code.

## ▸ 16.10 DIRECT DESIGN METHOD—PROCEDURE FOR COMPUTATION OF LONGITUDINAL MOMENTS

The background explanation for the distribution of the total static moment $M_0$ in the longitudinal direction, and the discussion of pattern loading effect, have been discussed in the two preceding sections. Using this information, the procedure for computing the longitudinal moments by the "direct design method" may be summarized:

1. Check limitations 1 through 5 for the "direct design method" listed in Section 16.7. If they comply, and the slab is supported on beams, follow Steps 2 through 6 given below. For slabs not supported on beams, proceed to step 6.
2. Compute the slab moment of inertia $I_s$

$$I_s = \sum L_2 \left(\frac{t^3}{12}\right)$$

3. Compute the longitudinal beam (if any) moment of inertia $I_b$ (ACI-13.2.4).
4. Compute the ratio $\alpha_f$ of the flexural stiffness of beam section to flexural stiffness of a width of slab bounded laterally by centerlines of adjacent panels (if any) on each side of the beam

$$\alpha_f = \frac{E_{cb} I_b}{E_{cs} I_s}$$

5. Check that the ratio $L_1^2/\alpha_{f1}$ to $L_2^2/\alpha_{f2}$ lies between 0.2 and 5.0 for the cases where the slab is supported by beams.
6. Compute the total static moment $M_0 = w_u L_2 L_n^2/8$ as stated by Eq. (16.3.3). Note that $L_n$ is not to be taken less than $0.65L_1$. For flat slabs, Eq. (16.3.4) should preferably be used for computing $M_0$.
7. Obtain the three critical ordinates on the longitudinal moment diagrams for the exterior and interior spans using Figs. 16.8.1 and 16.8.2.

**EXAMPLE 16.10.1**

For the two-way slab (with beams) design example described in Section 16.3, determine the longitudinal moments in frames $A$, $B$, $C$, and $D$, as shown in Figs. 16.3.5 and 16.10.1.

**SOLUTION** (a) Check the six limitations for the direct design method. These limitations have been checked in Example 16.7.1.

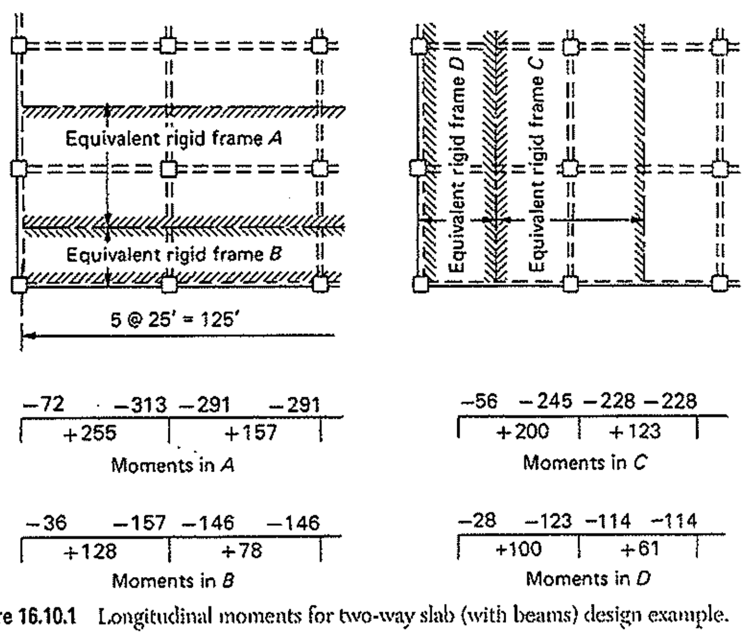



**Figure 16.10.1** Longitudinal moments for two-way slab (with beams) design example.

(b) Totl factored static moment $M_0$. The total factored static moments $M_0$ for the equivalent rigid frames $A$, $B$, $C$, and $D$ have been computed in Example 16.3.1; they are

$$M_0 \text{ (frame A)} = 448 \text{ ft-kips}$$
$$M_0 \text{ (frame B)} = 244 \text{ ft-kips}$$
$$M_0 \text{ (frame C)} = 350 \text{ ft-kips}$$
$$M_0 \text{ (frame D)} = 175 \text{ ft-kips}$$

(c) Longitudinal moments in the frames. The longitudinal moments in frames $A$, $B$, $C$, and $D$ are computed using Case 2 of Fig. 16.8.2 for the exterior span and Fig. 16.8.1 for the interior span. The computations are as shown below, and the results are summarized in Fig. 16.10.1

For Frame $A$, $M_0 = 488$ ft-kips

- $M_{neg}$ at exterior support $= 0.16(448) = 72$ ft-kips
- $M_{pos}$ in exterior span $= 0.57(448) = 255$ ft-kips
- $M_{neg}$ at first interior support $= 0.70(448) = 313$ ft-kips
- $M_{neg}$ at typical interior support $= 0.65(448) = 291$ ft-kips
- $M_{pos}$ in typical interior span $= 0.35(448) = 157$ ft-kips

For Frame $B$, $M_0 = 224$ ft-kips

- $M_{neg}$ at exterior support $= 0.16(224) = 36$ ft-kips
- $M_{pos}$ in exterior span $= 0.57(224) = 128$ ft-kips
- $M_{neg}$ at first interior support $= 0.70(224) = 157$ ft-kips
- $M_{neg}$ at typical interior support $= 0.65(224) = 146$ ft-kips
- $M_{pos}$ in typical interior span $= 0.35(224) = 78$ ft-kips

For Frame $C$, $M_0 = 350$ ft-kips

- $M_{neg}$ at exterior support $= 0.16(350) = 56$ ft-kips
- $M_{pos}$ in exterior span $= 0.57(350) = 200$ ft-kips
- $M_{neg}$ at first interior support $= 0.70(350) = 245$ ft-kips
- $M_{neg}$ at typical interior support $= 0.65(350) = 228$ ft-kips
- $M_{pos}$ in typical interior span $= 0.35(350) = 123$ ft-kips

For Frame $D$, $M_0 = 175$ ft-kips

- $M_{neg}$ at exterior support $= 0.16(175) = 28$ ft-kips
- $M_{pos}$ in exterior span $= 0.57(175) = 100$ ft-kips
- $M_{neg}$ at first interior support $= 0.70(175) = 123$ ft-kips
- $M_{neg}$ at typical interior support $= 0.65(175) = 114$ ft-kips
- $M_{pos}$ in typical interior span $= 0.35(175) = 61$ ft-kips ◀

**▶ EXAMPLE 16.10.2**

For the flat slab design example described in Section 16.3, compute the longitudinal moments in frames $A$, $B$, $C$, and $D$ as shown in Figs. 16.3.6 and 16.10.2.

**SOLUTION** (a) Check the five limitations (the sixth limitation does not apply here) for the direct design method. These five limitations are all satisfied.

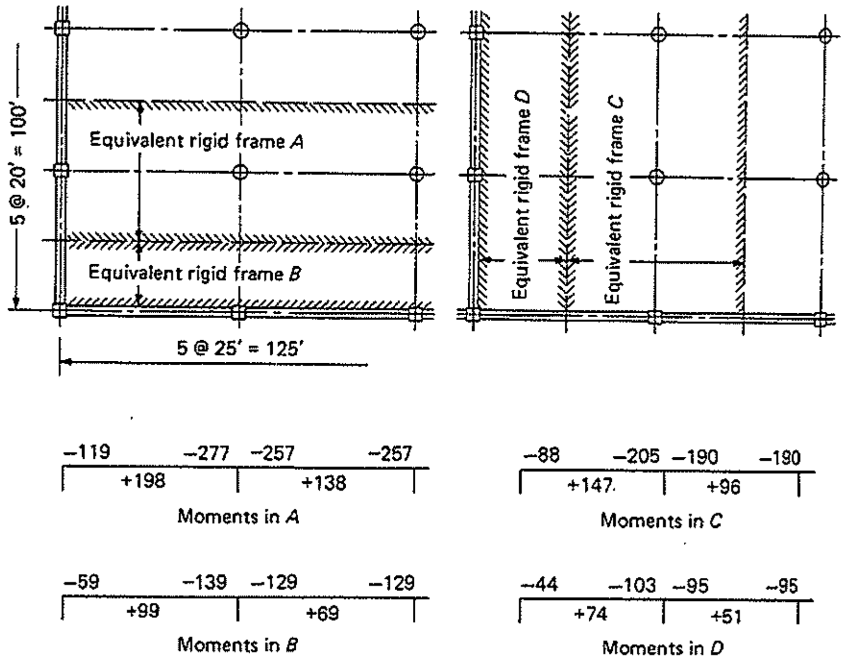



**Figure 16.10.2** Longitudinal moments for flat slab design example.

(b) Total factored static moment $M_0$. Referring to the equivalent rigid frames $A$, $B$, $C$, and $D$ in Fig. 16.10.2, the total factored static moment may be taken from the results of Example 16.3.2; thus

$$M_0 \text{ for } A = 396 \text{ ft-kips}$$
$$M_0 \text{ for } B = \tfrac{1}{2}(396) = 198 \text{ ft-kips}$$
$$M_0 \text{ for } C = 293 \text{ ft-kips}$$
$$M_0 \text{ for } D = \tfrac{1}{2}(293) = 147 \text{ ft-kips}$$

**TABLE 16.10.1 Longitudinal Moments (ft-kips) for the Flat Slab Design Example**

| Frame | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| $M_0$ | 396 | 198 | 293 | 147 |
| $M_{neg}$ at exterior support, $0.30M_0$ | 119 | 59 | 88 | 44 |
| $M_{pos}$ in exterior span, $0.50M_0$ | 198 | 99 | 147 | 74 |
| $M_{neg}$ at first interior support, $0.70M_0$ | 277 | 139 | 205 | 103 |
| $M_{neg}$ at typical interior support, $0.65M_0$ | 257 | 129 | 190 | 95 |
| $M_{pos}$ in typical interior span, $0.35M_0$ | 138 | 69 | 96 | 51 |

(c) Longitudinal moments in the frames. The longitudinal moments in frames $A$, $B$, $C$, and $D$ are computed using Case 4 of Fig. 16.8.2 for the exterior span and Fig. 16.8.1 for the interior span. The computations are shown in Table 16.10.1 and the results are summarized in Fig. 16.10.2. ◄

**▶ EXAMPLE 16.10.3**

For the flat plate design example described in Section 16.3, compute the longitudinal moments in frames $A$, $B$, $C$, and $D$ as shown in Figs. 16.3.7 and 16.10.3.

**SOLUTION** (a) Check the five limitations (the sixth limitation does not apply here) for the direct design method. These limitations are all satisfied.

(b) Total factored static moment $M_0$ from the results of Example 16.3.3.

$$M_0 \text{ for } A = 58.2 \text{ ft-kips}$$
$$M_0 \text{ for } B = \tfrac{1}{2}(58.2) = 29.1 \text{ ft-kips}$$
$$M_0 \text{ for } C = 46.3 \text{ ft-kips}$$
$$M_0 \text{ for } D = 23.1 \text{ ft-kips}$$

(c) Longitudinal moments in the frames. The longitudinal moments in frames $A$, $B$, $C$, and $D$ are computed using Case 3 of Fig. 16.8.2 for the exterior span and Fig. 16.8.1 for the interior span. The computations are as shown in Table 16.10.2 and the results are summarized in Fig. 16.10.3.

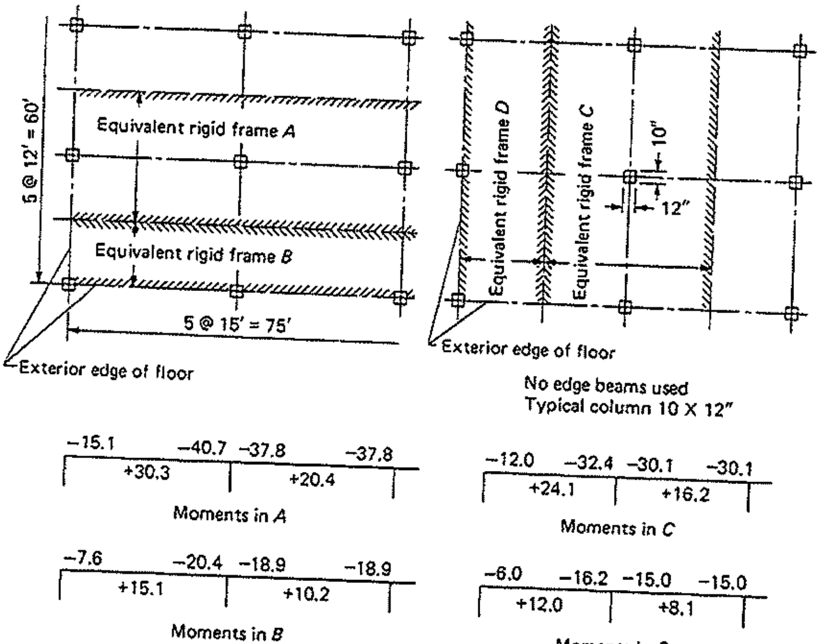



**Figure 16.10.3** Longitudinal moments for flat plate design example.

**TABLE 16.10.2 Longitudinal Moment (ft-kips) for the Flat Plate Design Example**

| Frame | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| $M_0$ | 58.2 | 29.1 | 46.3 | 23.1 |
| $M_{neg}$ at exterior support, $0.26M_0$ | 15.1 | 7.6 | 12.0 | 6.0 |
| $M_{pos}$ in exterior span, $0.52M_0$ | 30.3 | 15.1 | 24.1 | 12.0 |
| $M_{neg}$ at first interior support, $0.70M_0$ | 40.7 | 20.4 | 32.4 | 16.2 |
| $M_{neg}$ at typical interior support, $0.65M_0$ | 37.8 | 18.9 | 30.1 | 15.0 |
| $M_{pos}$ in typical interior span, $0.35M_0$ | 20.4 | 10.2 | 16.2 | 8.1 |

## 16.11 TORSION STIFFNESS OF THE TRANSVERSE ELEMENTS

Up to this point, the stiffness of the equivalent frame has been considered with regard to the members in the plane of the frame only. The transverse members, however, will also contribute to the stiffness of the frame by resisting the in-plane bending through torsion. In the ACI Code, this contribution is considered by the torsional constant $C$ of the transverse beam spanning from column to column. Even if there is no such beam (as defined by projection above or below the slab) actually visible, for the present use one still should imagine that there is a beam made of a portion of the slab having a width equal to that of the column, bracket, or capital in the direction of the span for which moments are being determined (ACI-13.7.5.1a). When there is actually a transverse beam web above or below the slab, the cross-section of the transverse beam should include the portion of slab within the width of column, bracket, or capital described above plus the projection of beam web above or below the slab (ACI-13.7.5.1b). As a third possibility, the transverse beam may include that portion of slab on each side of the beam web equal to its projection above or below the slab, whichever is greater, but not greater than four times the slab thickness (ACI-13.7.5.1c). The largest of the three definitions as shown in Fig. 16.11.1 may be used.

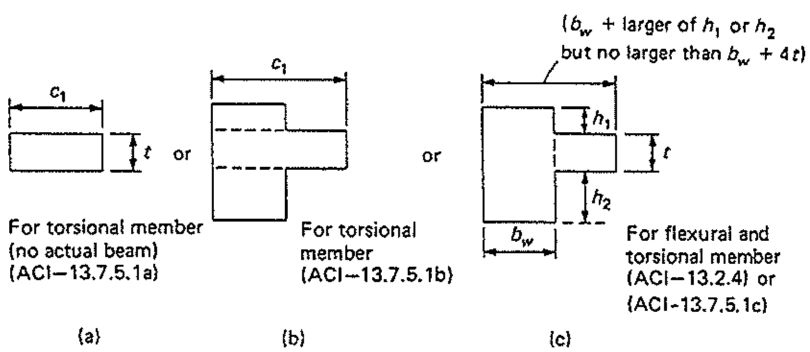



**Figure 16.11.1** Definition of cross-sections for transverse beams in torsion. [Projection of slab beyond beam in Case (c) is allowed on each side for interior beam.]

The torsional constant $C$ of the transverse beam equals

$$C = \sum \left(1 - 0.63\frac{x}{y}\right)\left(\frac{x^3 y}{3}\right) \tag{16.11.1}$$

which is given in ACI-13.6.4.2, where

$x$ = shorter dimension of a component rectangle
$y$ = longer dimension of a component rectangle

and the component rectangle should be taken in such a way that the largest value of $C$ is obtained. Equation (16.11.1) is identical to Eq. (19.3.5), for which there is additional discussion in Chapter 19.

**EXAMPLE 16.11.1** For the two-way slab (with beams) design example, compute the torsional constant $C$ for the edge and interior beams in the short and long directions.

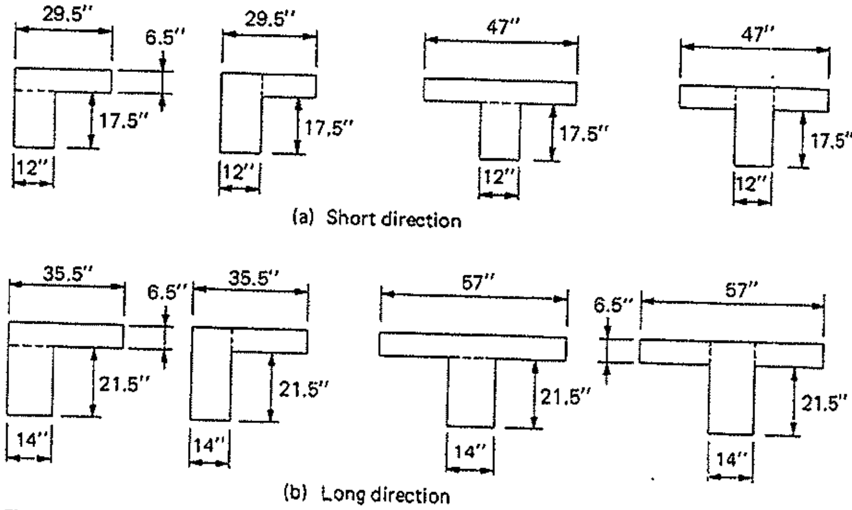



**Figure 16.11.2** Effective cross-sections of transverse beams resisting torsion, in two-way slab (with beams) design example.

▶ EXAMPLE

**SOLUTION** Each cross-section shown in Fig. 16.11.2 may be divided into component rectangles in two different ways and the larger value of $C$ is to be used.

For that short direction,

$$C\begin{pmatrix}\text{edge}\\ \text{beam}\end{pmatrix} = \left[1 - \frac{0.63(6.5)}{29.5}\right]\frac{29.5(6.5)^3}{3} + \left[1 - \frac{0.63(12)}{17.5}\right]\frac{17.5(12)^3}{3}$$

$$= 2325 + 5725 = 8050 \text{ in.}^4$$

or

$$C\begin{pmatrix}\text{edge}\\ \text{beam}\end{pmatrix} = \left[1 - \frac{0.63(6.5)}{17.5}\right]\frac{17.5(6.5)^3}{3} + \left[1 - \frac{0.63(12)}{24}\right]\frac{24(12)^3}{3}$$

$$= 1230 + 9470 = 10{,}700 \text{ in.}^4$$ *Use*

$$C\begin{pmatrix}\text{interior}\\ \text{beam}\end{pmatrix} = \left[1 - \frac{0.63(6.5)}{47}\right]\frac{47(6.5)^3}{3} + \left[1 - \frac{0.63(12)}{17.5}\right]\frac{17.5(12)^3}{3}$$

$$= 3925 + 5725 = 9650 \text{ in.}^4$$

or

$$C\begin{pmatrix}\text{interior}\\ \text{beam}\end{pmatrix} = 2(1230) + 9470 = 11{,}930 \text{ in.}^4$$ *Use*

For the long direction,

$$C\begin{pmatrix}\text{edge}\\ \text{beam}\end{pmatrix} = \left[1 - \frac{0.63(6.5)}{35.5}\right]\frac{35.5(6.5)^3}{3} + \left[1 - \frac{0.63(14)}{21.5}\right]\frac{21.5(14)^3}{3}$$

$$= 2900 + 11{,}600 = 14{,}500 \text{ in.}^4$$

or

$$C\begin{pmatrix}\text{edge}\\ \text{beam}\end{pmatrix} = \left[1 - \frac{0.63(6.5)}{21.5}\right]\frac{21.5(6.5)^3}{3} + \left[1 - \frac{0.63(14)}{28}\right]\frac{28(14)^3}{3}$$

$$= 1600 + 17{,}500 = 19{,}100 \text{ in.}^4$$ *Use*

$$C\begin{pmatrix}\text{interior}\\ \text{beam}\end{pmatrix} = \left[1 - \frac{0.63(6.5)}{57}\right]\frac{57(6.5)^3}{3} + \left[1 - \frac{0.63(14)}{21.5}\right]\frac{21.5(14)^3}{3}$$
$$= 4800 + 11{,}600 = 16{,}400 \text{ in.}^4$$

or

$$C\begin{pmatrix}\text{interior}\\ \text{beam}\end{pmatrix} = 2(1600) + 17{,}500 = 20{,}700 \text{ in.}^4 \qquad \textit{Use} \quad ◄$$

**EXAMPLE 16.11.2** For the flat slab design example, compute the torsional constant $C$ for the edge beam and the interior beam in the short and long directions.

**SOLUTION** For the short or long edge beam [Fig. 16.11.3(a)], the torsional constant $C$ is computed on the basis of the cross-section shown in Fig. 16.11.3(a).

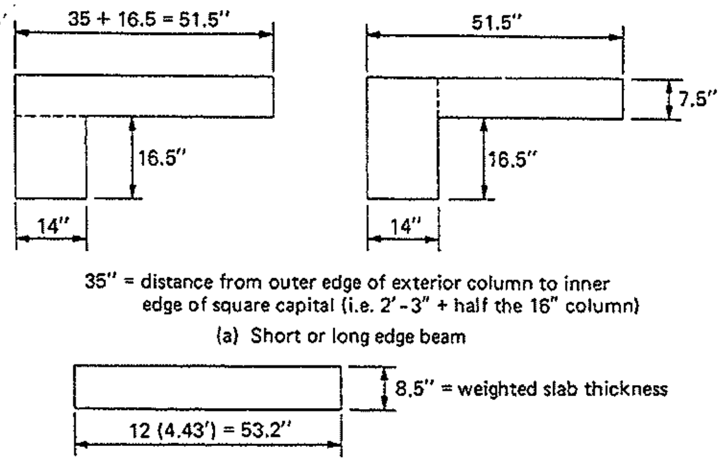



**Figure 16.11.3** Cross-sections of torsional transverse beams in flat slab design example.

$$C = \left[1 - \frac{0.63(7.5)}{51.5}\right]\frac{(7.5)^3(51.5)}{3} + \left[1 - \frac{0.63(14)}{16.5}\right]\frac{(14)^3(16.5)}{3}$$
$$= 6575 + 7025 = 13{,}600 \text{ in.}^4$$

or

$$C = \left[1 - \frac{0.63(14)}{24}\right]\frac{(14)^3(24)}{3} + \left[1 - \frac{0.63(7.5)}{37.5}\right]\frac{(7.5)^3(37.5)}{3}$$
$$= 13{,}890 + 4610 = 18{,}500 \text{ in.}^4 \qquad \textit{Use}$$

For the short or long interior beam [Fig. 16.11.3(b)], a weighted slab thickness of 8.5 in. is used, on the assumption that one-third of the span has a $10\frac{1}{2}$-in. thickness and the remainder has a $7\frac{1}{2}$-in. thickness. (Actually, the ratio is not exactly so because the drop width has been revised from 6 ft 8 in. to 7 ft.)

$$C = \left(1 - 0.63\frac{x}{y}\right)\frac{x^3 y}{3} = \left[1 - \frac{0.63(8.5)}{12(4.43)}\right]\left[\frac{(8.5)^3(12)(4.43)}{3}\right] = 9800 \text{ in.}^4 \quad ◄$$

**▶ EXAMPLE 16.11.3** For the flat plate design example, compute the torsional constant $C$ for the short and long beams.

**SOLUTION** Since no actual edge beams are used, the torsional member is, according to Fig. 16.11.4, equal to slab thickness $t$ by the column width $c_1$.

$$C \text{ for short beams} = \left[1 - \frac{0.63(5.5)}{12}\right]\frac{(5.5)^3(12)}{3} = 474 \text{ in.}^4$$

$$C \text{ for long beams} = \left[1 - \frac{0.63(5.5)}{10}\right]\frac{(5.5)^3(10)}{3} = 362 \text{ in.}^4$$

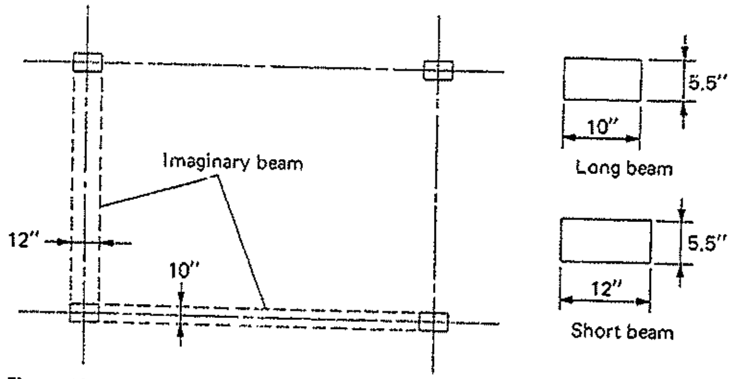



**Figure 16.11.4** Cross-sections of torsional transverse beams in flat plate design example. ◀

## 16.12 TRANSVERSE DISTRIBUTION OF LONGITUDINAL MOMENT

The longitudinal moment values, whether those of the "direct design method" shown in Figs. 16.8.1 and 16.8.2 or those obtained by structural analysis using the "equivalent frame method" (Chapter 17), are for the entire width (sum of the two half panel widths in the transverse direction, for an interior column line) of the equivalent rigid frame. Each of these moments is to be divided, on the basis of studies by Gamble, Sozen, and Siess [16.12], between the column strip and the two half middle strips as defined in Fig. 16.12.1. If the two adjacent transverse spans are each equal to $L_2$, the width of the column strip is then equal to one-half of $L_2$, or one-half of the longitudinal span $L_1$, whichever is smaller (ACI-13.2.1). This seems reasonable, since when the longitudinal span is shorter than the transverse span, a larger portion of the moment across the width of the equivalent frame might be expected to concentrate near the column centerline.

The transverse distribution of the longitudinal moment to column and middle strips is a function of three parameters, using $L_1$ and $L_2$ for the longitudinal and transverse spans, respectively: (1) the aspect ratio $L_2/L_1$; (2) the ratio $\alpha_{f1} = E_{cb}I_b/(E_{cs}I_s)$ of the longitudinal beam stiffness to slab stiffness; and (3) the ratio $\beta_t = E_{cb}C/(2E_{cs}I_s)$ of the torsional rigidity of edge beam section to the flexural rigidity of a width of slab equal to the span length of the edge beam. According to ACI-13.6.4, the column strip is to take the percentage of the longitudinal moment as shown in Table 16.12.1. As may be seen from Table 16.12.1, only the first two parameters affect the transverse distribution of the negative moments at the first and typical interior supports as well as the positive moments in exterior and interior spans, but all three parameters are involved in the transverse distribution of the negative moment at the exterior support.

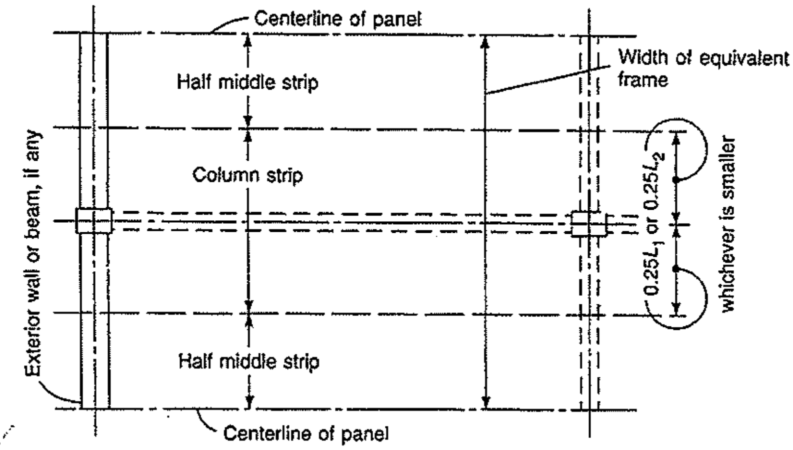



**Figure 16.12.1** Definition of column and middle strips

Regarding the distributing percentages shown in Table 16.12.1, the following observations may be made:

1. In general, the column strip takes more than 50% of the longitudinal moment.
2. The column strip takes a larger share of the negative longitudinal moment than the positive longitudinal moment.
3. When no longitudinal beams are present, the column strip takes the same share of the longitudinal moment, irrespective of the aspect ratio. The reader may note, however, that the column strip width is a fraction of $L_1$ or $L_2$ ($0.25L_1$ or $0.25L_2$ on each side of column line), whichever is smaller.
4. In the presence of longitudinal beams, the larger the aspect ratio, the smaller the distribution to the column strip. This seems consistent because the same reduction in the portion of moment going into the slab is achieved by restricting the column strip width to a fraction of $L_1$ when $L_2/L_1$ is greater than one.
5. The column strip takes a smaller share of the exterior moment as the torsional rigidity of the edge beam section increases.

When the exterior support consists of a column or wall extending for a distance equal to or greater than three-fourths of the transverse width, the exterior negative moment is to be uniformly distributed over the transverse width (ACI-13.6.4.3).

The procedure for distributing the longitudinal moment across a transverse width to the column and middle strips may be summarized as follows:

1. Divide the total transverse width applicable to the longitudinal moment into a column strip width and two half middle strip widths, one adjacent to each side of the column strip. For an exterior column line, the column strip width is $\frac{1}{4}L_1$, or $\frac{1}{4}L_2$, whichever is smaller; for an interior column line, the column strip width is $\sum(\frac{1}{4}L_1$ or $\frac{1}{4}L_2$, whichever is smaller, of the panels on both sides).
2. Determine the ratio $\beta_t = E_{cb}C/(2E_{cs}I_s)$ of edge beam torsional rigidity to slab flexural rigidity. (*Note:* The 2 arises from approximating the shear modulus of elasticity in the numerator as $E_{cb}/2$.)
3. Determine the ratio $\alpha_{f1} = E_{cb}I_b/(E_{cs}I_s)$ of longitudinal beam flexural stiffness to slab flexural stiffness.

4. Divide the longitudinal moment at each critical section into two parts according to the percentage shown in Table 16.12.1: one part to the column strip width; and the remainder to the half middle strip for an exterior column line, or to the half middle strips on each side of an interior column line.

**TABLE 16.12.1 Percentage of Longitudinal Moment in Column Strip (ACI-13.6.4)**

| Aspect Ratio $L_2/L_1$ | | | 0.5 | 1.0 | 2.0 |
|---|---|---|---|---|---|
| Negative moment at exterior support | $\alpha_{f1}L_2/L_1 = 0$ | $\beta_t = 0$ | 100 | 100 | 100 |
| | | $\beta_t \geq 2.5$ | 75 | 75 | 75 |
| | $\alpha_{f1}L_2/L_1 \geq 1.0$ | $\beta_t = 0$ | 100 | 100 | 100 |
| | | $\beta_t \geq 2.5$ | 90 | 75 | 45 |
| Positive moment | $\alpha_{f1}L_2/L_1 = 0$ | | 60 | 60 | 60 |
| | $\alpha_{f1}L_2/L_1 \geq 1.0$ | | 90 | 75 | 45 |
| Negative moment at interior support | $\alpha_{f1}L_2/L_1 = 0$ | | 75 | 75 | 75 |
| | $\alpha_{f1}L_2/L_1 \geq 1.0$ | | 90 | 75 | 45 |

5. If there is an exterior wall instead of an exterior column line, the strip ordinarily called the exterior column strip will not deflect and therefore no moments act. In this case there can be no longitudinal distribution of moments; thus there is no computed moment to distribute laterally to the half middle strip adjacent to the wall. This half middle strip should be combined with the next adjacent half middle strip, which itself receives a lateral distribution in the frame of the first interior column line. The total middle strip in this situation is designed for twice the moment in the half middle strip from the first interior column line (ACI-13.6.6.3).

## Distribution of Moment in Column Strip to Beam and Slab

When a longitudinal beam exists in the column strip along the column centerline, the column strip moment as determined by the percentages in Table 16.12.1 (ACI-13.6.4) should be divided between the beam and the slab. ACI-13.6.5 states that 85% of this moment be taken by the beam if $\alpha_{f1}L_2/L_1$ is equal to or greater than 1.0, and for values of $\alpha_{f1}L_2/L_1$ between 1.0 and 0, the proportion of moment to be resisted by the beam is to be obtained by linear interpolation between 85 and 0%. In additional, any beam must be designed to carry its own weight (projection above and below the slab), and any concentrated or linear load applied directly on it (ACI-13.6.5.3).

► **EXAMPLE 16.12.1**

For the two-way slab (with beams) design example described in Section 16.3, distribute the longitudinal moments computed for Frames $A$, $B$, $C$, and $D$ (see Fig. 16.10.1) into three parts—namely, for the longitudinal beams, for the column strip slab, and for the middle strip slab.

**SOLUTION** The values for the total longitudinal moments in frames $A$, $B$, $C$, and $D$ at the five critical sections are taken from Example 16.10.1 and shown again in Table 16.12.4. The results of transverse distribution of these moments are also shown in this table.

(a) Negative moment at face of exterior support. For Frame $A$, $L_2/L_1 = 0.80$; $\alpha_{f1} = 8.27$ (Fig. 16.4.4); $\alpha_{f1}L_2/L_1 = 6.61$; $C = 10{,}700$ in.[4] (Example 16.11.1); $I_s = 240(6.5)^3/12 = 5490$ in.[4]; and $\beta_t = C/(2I_s) = 10{,}700/[2(5490)] = 0.98$. Table 16.12.2 shows the linear interpolation for obtaining the column strip percentage from the prescribed limits of Table 16.12.1. The total moment of 72 ft-kips is divided into three parts, 92.6% to column strip (of which 85% goes to the beam and 15% to the slab since $\alpha_{f1}L_2/L_1 = 6.61 \geq 1.0$) and 7.4% to the middle strip slab. The results are shown in Table 16.12.4.

For Frame $B$, $L_2/L_1 = 0.80$; $\alpha_{f1} = 13.83$ (Fig. 16.4.4); $\alpha_{f1}L_2/L_1 = 11.1$; $\beta_t = 0.98$, the same as for frame $A$, and column strip moment percentage = 92.6%, the same as for frame $A$.

For Frame $C$, $L_2/L_1 = 1.25$; $\alpha_{f1} = 3.50$ (Fig. 16.4.4); $\alpha_{f1}L_2/L_1 = 4.38$; $C = 19{,}100$ in.$^4$ (Example 16.11.1); $I_s = 300(6.5)^3/12 = 6870$ in.$^4$; and $\beta_t = C/(2I_s) = 19{,}100/[2(6870)] = 1.39$. Table 16.12.3 shows the linear interpolation for obtaining the column strip percentage from the prescribed limits of Table 16.12.1. The total moment of 56 ft-kips is divided into three parts, 81.9% to column strip (of which 85% goes to the beam and 15% to the slab since $\alpha_{f1}L_2/L_1 = 4.38 \geq 1.0$) and 18.1% to the middle strip slab.

For Frame $D$, $L_2/L_1 = 1.25$; $\alpha_{f1} = 5.96$ (Fig. 16.4.4), $\alpha_{f1}L_2/L_1 = 7.45$; $\beta_t = 1.39$, the same as for frame $C$; and column strip moment percentage = 81.9%, the same as for frame $C$.

**TABLE 16.12.2 Linear Interpolation for Column Strip Percentage of Exterior Negative Moment—Frame *A***

| $L_2/L_1$ | | 0.5 | 0.8 | 1.0 |
|---|---|---|---|---|
| | $\beta_t = 0$ | 100% | 100% | 100% |
| $\alpha_{f1}L_2/L_1 = 6.61$ | $\beta_t = 0.98$ | 96.1% | 92.6% | 90.2% |
| | $\beta_t \geq 2.50$ | 90% | 81% | 75% |

**TABLE 16.12.3 Linear Interpolation for Column Strip Percentage of Exterior Negative Moment—Frame *C***

| $L_2/L_1$ | | 1.0 | 1.25 | 2.0 |
|---|---|---|---|---|
| | $\beta_t = 0$ | 100% | 100% | 100% |
| $\alpha_{f1}L_2/L_1 = 4.38$ | $\beta_t = 1.39$ | 86.1% | 81.9% | 69.4% |
| | $\beta_t \geq 2.50$ | 75% | 67.5% | 45% |

(b) Negative moments at exterior face of first interior support and at face of typical interior support. For Frame $A$, $L_2/L_1 = 0.80$ and $\alpha_{f1}L_2/L_1 = 6.61 > 1.0$. Using the prescribed values in Table 16.12.1, the proportion of moment going to the column strip is determined to be 81% by linear interpolation.

| $L_2/L_1$ | 0.5 | 0.8 | 1.0 |
|---|---|---|---|
| $\alpha_{f1}L_2/L_1 = 6.61$ | 90% | 81% | 75% |

For Frame $B$, $L_2/L_1 = 0.80$ and $\alpha_{f1}L_2/L_1 = 11.1$. The proportion of moment is again 81% for the column strip, the same as for Frame $A$.

For Frame $C$, $L_2/L_1 = 1.25$ and $\alpha_{f1}L_2/L_1 = 4.38$. Using the prescribed values in Table 16.12.1, the proportion of moment going to the column strip is determined to be 67.5% by linear interpolation:

| $L_2/L_1$ | 1.0 | 1.25 | 2.0 |
|---|---|---|---|
| $\alpha_{f1}L_2/L_1 = 4.38$ | 75% | 67.5% | 45% |

For Frame $D$, $L_2/L_1 = 1.25$ and $\alpha_{f1}L_2/L_1 = 7.47$. The proportion of moment is again 67.5% for the column strip, the same as for Frame $C$.

(c) Positive moments in exterior and interior spans. Since the prescribed limits for $\alpha_{f1}L_2/L_1 \geq 1.0$ are the same for positive moment and for negative moment at interior

**TABLE 16.12.4 (also see Fig. 16.10.1) Transverse Distribution of Longitudinal Moments (ft-kips) in Two-Way Slab (with Beams) Design Example**

Frame *A*
Total Width = 20 ft, Column Strip Width = 10 ft, Middle Strip Width = 10 ft

| | Exterior Span: Exterior Negative | Exterior Span: Positive | Exterior Span: Interior Negative | Interior Span: Negative | Interior Span: Positive |
|---|---|---|---|---|---|
| Total moment | −72 | +255 | −313 | −291 | +157 |
| Moment in beam | −57 | +176 | −216 | −200 | +108 |
| Moment in column strip slab | −10 | +31 | −38 | −36 | +19 |
| Moment in middle strip slab | −5 | +49 | −60 | −55 | +30 |

Frame *B*
Total Width = 10 ft, Column Strip Width = 5 ft, Half Middle Strip Width = 5 ft

| | Exterior Span: Exterior Negative | Exterior Span: Positive | Exterior Span: Interior Negative | Interior Span: Negative | Interior Span: Positive |
|---|---|---|---|---|---|
| Total moment | −36 | +128 | −157 | −146 | +78 |
| Moment in beam | −28 | +88 | −108 | −101 | +54 |
| Moment in column strip slab | −5 | +16 | −19 | −17 | +9 |
| Moment in middle strip slab | −3 | +25 | −30 | −28 | +15 |

Frame *C*
Total Width = 25 ft, Column Strip Width = 10 ft, Middle Strip Width = 15 ft

| | Exterior Span: Exterior Negative | Exterior Span: Positive | Exterior Span: Interior Negative | Interior Span: Negative | Interior Span: Positive |
|---|---|---|---|---|---|
| Total moment | −56 | +200 | −245 | −228 | +123 |
| Moment in beam | −39 | +115 | −140 | −131 | +71 |
| Moment in column strip slab | −7 | +20 | −25 | −23 | +12 |
| Moment in middle strip slab | −10 | +65 | −80 | −74 | +40 |

Frame *D*
Total Width = 12.5 ft, Column Strip Width = 5 ft, Half Middle Strip Width = 7.5 ft

| | Exterior Span: Exterior Negative | Exterior Span: Positive | Exterior Span: Interior Negative | Interior Span: Negative | Interior Span: Positive |
|---|---|---|---|---|---|
| Total moment | −28 | +100 | −123 | −114 | +61 |
| Moment in beam | −20 | +57 | −71 | −65 | +35 |
| Moment in column strip slab | −3 | +10 | −12 | −12 | +6 |
| Moment in middle strip slab | −5 | +33 | −40 | −37 | +20 |

support, the percentages of column strip moment for positive moments in exterior and interior spans are identical to those for interior negative moments (see Table 16.12.1) as determined in part (b) of this example. ◄

**▶ EXAMPLE 16.12.2**

Divide the five critical moments in each of the equivalent rigid frames $A$, $B$, $C$, and $D$ in the flat slab design example, as shown in Fig. 16.10.2, into two parts: one for the half column strip (for frames $B$ and $D$) or the full column strip (for frames $A$ and $C$), and the other for the half middle strip (for frames $B$ and $D$) or the two half middle strips on each side of the column line (for frames $A$ and $C$).

**SOLUTION** The percentages of the longitudinal moments going into the column strip width are shown in lines 10 to 12 of Table 16.12.5. Note that the column strip width shown in line 2 is one-half of the shorter panel dimension for both frames $A$ and $C$, and one-fourth of this value for frames $B$ and $D$. Note also that the sum of the values on line 2 and 3 should be equal to that on line 1, for each respective frame.

**TABLE 16.12.5 Transverse Distribution of Longitudinal Moment for Flat Slab Design Example**

| Line Number | Equivalent Rigid Frame | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|---|
| 1 | Total transverse width (in.) | 240 | 120 | 300 | 150 |
| 2 | Column strip width (in.) (Fig. 16.12.1) | 120 | 60 | 120 | 60 |
| 3 | Half middle strip width (in.) | 2 @ 60 | 60 | 2 @ 90 | 90 |
| 4 | $C$ (in.$^4$) from Example 16.11.2 | 18,500 | 18,500 | 18,500 | 18,500 |
| 5 | $I_s$ (in.$^4$) in $\beta_t$ | 8440 | 8440 | 10,600 | 10,600 |
| 6 | $\beta_t = E_{cb}C/(2E_{cs}I_s)$ | 1.10 | 1.10 | 0.87 | 0.87 |
| 7 | $\alpha_{f1}$ from Example 16.6.2 | 0 | 5.42 | 0 | 4.34 |
| 8 | $L_2/L_1$ | 0.80 | 0.80 | 1.25 | 1.25 |
| 9 | $\alpha_{f1}L_2/L_1$ | 0 | 4.34 | 0 | 5.43 |
| 10 | Exterior negative moment, percent to to column strip (Table 16.12.1) | 89.0% | 91.6% | 91.3% | 88.7% |
| 11 | Positive moment, percent to column strip (Table 16.12.1) | 60.0% | 81.0% | 60.0% | 67.5% |
| 12 | Interior negative moment, percent to column strip (Table 16.12.1) | 75.0% | 81.0% | 75.0% | 67.5% |

The moment of inertia of the slab equal in width to the transverse span of the edge beam is

$$I_s \text{ in } \beta_t \text{ for } A \text{ and } B = \frac{240(7.5)^3}{12} = 8440 \text{ in.}^4$$

and

$$I_s \text{ in } \beta_t \text{ for } C \text{ and } D = \frac{300(7.5)^3}{12} = 10{,}600 \text{ in.}^4$$

These values are shown in line 5 of Table 16.12.5.

The percentages shown in lines 10 to 12 are obtained from Table 16.12.1, by interpolation (as illustrated in Tables 16.12.2 and 16.12.3) if necessary. Having these percentages, the separation of each of the longitudinal moment values shown in Fig. 16.10.2 into two parts is a simple matter and thus is not shown further. ◄

**▶ EXAMPLE 16.12.3**

Divide the five critical moments in each of the equivalent rigid frames $A$, $B$, $C$, and $D$ in the flat plate design example, as shown in Fig. 16.10.3, into two parts: one for the half column strip (for frames $B$ and $D$) or the full column strip (for frames $A$ and $C$), and the other for the half middle strip (for frames $B$ and $D$) or the two half middle strips on each side of the column line (for frames $A$ and $C$).

**SOLUTION** The percentages of the longitudinal moments going into the column strip width are shown in lines 10 to 12 of Table 16.12.6. Explanations are identical to those for the preceding example.

**TABLE 16.12.6 Transverse Distribution of Longitudinal Moment for Flat Plate Design Example**

| Line Number | Equivalent Rigid Frame | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|---|
| 1 | Total transverse width (in.) | 144 | 72 | 180 | 90 |
| 2 | Column strip width (in.) | 72 | 36 | 72 | 36 |
| 3 | Half middle strip width (in.) | 2 @ 36 | 36 | 2 @ 54 | 54 |
| 4 | $C$(in.$^4$) from Example 16.11.3 | 474 | 474 | 362 | 362 |
| 5 | $I_s$(in.$^4$) in $\beta_t$ | 2000 | 2000 | 2500 | 2500 |
| 6 | $\beta_t = E_{cb}C/(2E_{cs}I_s)$ | 0.119 | 0.119 | 0.073 | 0.073 |
| 7 | $\alpha_{f1}$ | 0 | 0 | 0 | 0 |
| 8 | $L_2/L_1$ | 0.80 | 0.80 | 1.25 | 1.25 |
| 9 | $\alpha_{f1}L_2/L_1$ | 0 | 0 | 0 | 0 |
| 10 | Exterior negative moment, percent to column strip | 98.8% | 98.8% | 99.3% | 99.3% |
| 11 | Positive moment, percent to column strip | 60% | 60% | 60% | 60% |
| 12 | Interior negative moment, percent to column strip | 75% | 75% | 75% | 75% |

◄

## ► 16.13 DESIGN OF SLAB THICKNESS AND REINFORCEMENT

### Slab Thickness

Ordinarily the minimum thickness specified in ACI-9.5.3 controls the thickness for design. Of course, reinforcement for bending moment must be provided, but the reinforcement ratio $\rho$ required is usually well below $0.5\rho_{max}$; thus, it does not dictate slab thickness. For flat slabs, flexural strength must be provided both within the drop panel and outside its limits. In evaluating the strength within a drop panel, the drop width should be used as the transverse width of the compression area, because the drop is usually narrower than the width of the column strip. Also, the effective depth to be used should not be taken greater than what would be furnished by a drop thickness below the slab equal to one-fourth the distance from the edge of drop to the edge of column capital.

The shear requirement for two-way slabs (with beams) may be investigated by observing strips 1-1 and 2-2 in Fig. 16.13.1. Beams with $\alpha_{f1}L_2/L_1$ values larger than 1.0 are assumed to carry the loads acting on the tributary floor areas bounded by 45° lines drawn from the corners of the panel and the centerline of the panel parallel to the long side (ACI-13.6.8.1). If this is the case, the loads on the trapezoidal areas $E$ and $F$ of Fig. 16.13.1 go to the long beams, and those on the triangular areas $G$ and $H$ go to the short beams. The shear per unit width of slab along the beam is highest at the ends of slab strips 1-1 and 2-2, which, considering the increased shear at the exterior face of the first interior support, is approximately equal to

$$V_u = 1.15\left(\frac{w_u S}{2}\right) \tag{16.13.1}$$

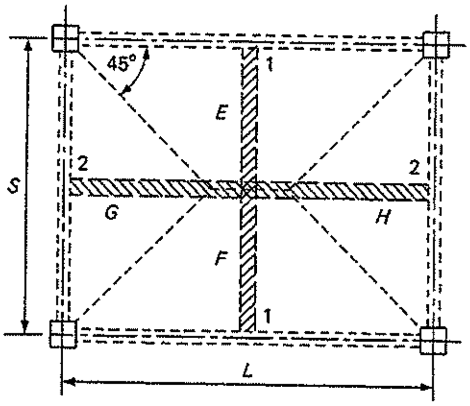



**Figure 16.13.1** Load transfer from floor area to beams.

If $\alpha_{f1} L_2/L_1$ is equal to zero, there is, of course, no load on the beams (because there are no beams). When the value of $\alpha_{f1} L_2/L_1$ is between 0 and 1.0, the percentage of the floor load going to the beams should be obtained by linear interpolation. In such a case, the shear expressed by Eq. (16.13.1) would be reduced, but the shear around the column due to the portion of the floor load going directly to the columns by two-way action should be investigated as for flat plate floors.

The shear strength requirement for flat slab and flat plate systems (without beams) is treated separately in Sections 16.15, 16.16, and 16.18.

## Reinforcement

When the nominal requirements for slab thickness as discussed in Section 16.6 are satisfied, no compression reinforcement will likely be required. The tension steel area required within the strip being considered can then be obtained by the following steps:

1. $$\text{required } M_n = \frac{\text{factored moment } M_u \text{ in the strip}}{(\phi = 0.90\text{-assumed, but reasonable for slabs})}$$

2. $$m = \frac{f_y}{0.85 f_c'}, \quad R_n = \frac{M_n}{bd^2},$$

$$\rho = \frac{1}{m}\left(1 - \sqrt{1 - \frac{2mR_n}{f_y}}\right), \qquad A_s = \rho b d$$

Instead of using the equation for $\rho$ in Step 2, the curves in Fig. 3.8.1 may be used. Note also that the values of $b$ and $d$ to be used in Step 2 for negative moment in a column strip with drop are the drop width for $b$, and for $d$ the smaller of the actual effective depth through the drop and that provided by a drop thickness below the slab at no more than one-fourth the distance between the edges of the column capital and the drop. For positive moment computation, the full column strip width should be used for $b$, and the effective depth in the slab for $d$. After obtaining the steel area $A_s$ required within the strip, a number of bars may be chosen so that they provide either the area required for strength or the area required for shrinkage and temperature reinforcement, which is $0.0018bt$ for Grade 60 steel, but somewhat more for lower grades (see ACI-7.12). The spacing of reinforcing bars must not exceed 2 times the slab thickness (ACI-13.3.2), except in slabs of cellular or ribbed construction where the requirement for shrinkage and temperature reinforcement governs (i.e., 5 times the slab thickness but not greater than 18 in.).

## Reinforcement Details in Slabs Without Beams

ACI-13.3.8, in particular ACI-Fig. 13.3.8, provides detailed dimensions for minimum extensions required for each portion of the total number of bars in the column and middle strips. Since the forces acting in the bars were empirically determined, there is no practical means to evaluate the distances required to develop the reinforcement. Thus, past practice and engineering judgment were used in preparing ACI-Fig. 13.3.8. In 1989, that figure omitted details for bent bars because they are seldom used, although their use is still permitted by ACI-13.3.8.3 when the depth–span ratio allows bends at 45° or less.

For unbraced frames, reinforcement lengths must be determined by analysis but not be less than those prescribed in ACI-Fig. 13.3.8. Also, ACI-13.3.8.5 requires the use of "integrity steel," which consists of a minimum of two of the column strip bottom bars passing continuously (or spliced with Class A splices or anchored within support) through the column core in each direction. The purpose of this integrity steel is to provide some residual strength following a single punching shear failure. Since 2002, the ACI Code also allows the use of mechanical and welded splices as alternative methods of splicing the reinforcement.

## Corner Reinforcement for Two-Way Slab (With Beams)

It is well known from plate bending theory that a transversely loaded slab simply supported along four edges will tend to develop corner reactions as shown in Fig. 16.13.2, for which reinforcement must be provided. Thus in slabs supported on beams having a value of $\alpha_f$ greater than 1.0, special reinforcement (Fig. 16.13.3) shall be provided at exterior corners in both the bottom and top of the slab. This reinforcement (ACI-13.3.6) is to be provided for a distance in each direction from the corner equal to one-fifth the longer span. The reinforcement in both the top and bottom of the slab must be sufficient to resist a moment equal to the maximum positive moment per foot of width in the slab, and it may be placed in a single band parallel to the diagonal in the top of the slab and perpendicular to the diagonal in the bottom of the slab, or in two bands parallel to the sides of the slab.

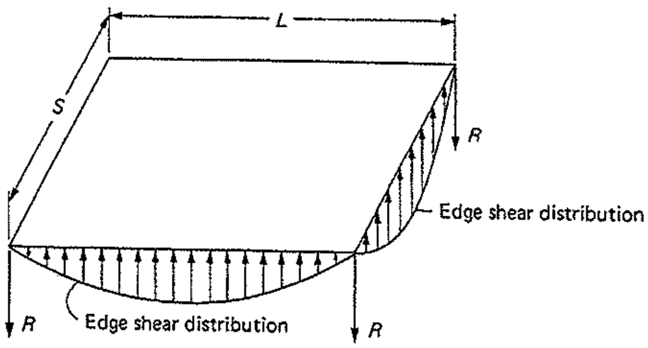



**Figure 16.13.2** Edge reactions for simply supported two-way slab.

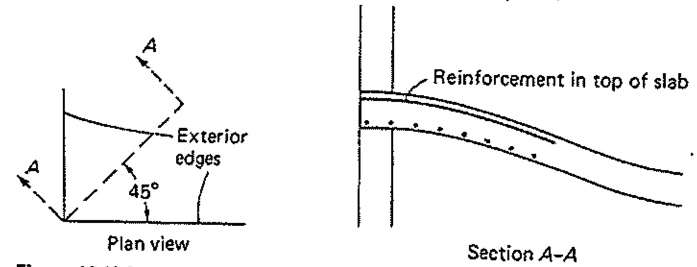



**Figure 16.13.3** Corner reinforcement in two-way slab.

## Crack Control

In addition to deflection control, crack control is the other major serviceability requirement usually considered in the design of flexural members. ACI-10.6 gives criteria for beams and one-way slabs to ensure distribution of flexural reinforcement to minimize crack width under service loads. No ACI Code provisions are given for two-way floor (or roof) systems; however, ACI Committee 224, Cracking, has suggested a formula to predict the possible crack width in two-way acting slabs, flat slabs, and flat plates. The recommendations are based on the work of Nawy et al. [16.72–16.75]. When the predicted crack width is considered excessive (there are no ACI Code limits for slabs), the distribution (size and spacing) of flexural reinforcement may be adjusted [16.75] to decrease predicted crack width. Ordinarily, crack width is not a problem on two-way acting slabs, but when steel with $f_y$ equal to 60,000 psi or higher is used, crack control should be considered.

**EXAMPLE 16.13.1**

Investigate if the preliminary slab thickness of $6\frac{1}{2}$ in. in the two-way slab (with beams) design example described in Section 16.3 is sufficient for resisting flexure and shear.

**SOLUTION** For each of the equivalent frames $A$, $B$, $C$, and $D$, the largest bending moment in the slab occurs at the exterior face of the first interior support in the middle strip slab. From Table 16.12.4, this moment is observed to be 60/10, 30/5, 80/15, or 40/7.5 ft-kips per ft of width in frames $A$, $B$, $C$, and $D$, respectively. Taking the effective depth to the contact level between the reinforcing bars in the two directions, and assuming #5 bars with $\frac{3}{4}$-in. clear cover,

$$\text{average } d = 6.50 - 0.75 - 0.63 = 5.12 \text{ in.}$$

Assuming $\phi = 0.90$, the largest $R_n$ required is

$$R_n = \frac{M_u}{\phi b d^2} = \frac{6000(12)}{0.90(12)(5.12)^2} = 254 \text{ psi}$$

From Fig. 3.8.1, the reinforcement ratio $\rho$ for this value of $R_n$ is about 0.007, which is well below 0.375 $\rho_b = 0.0139$. Hence excessive deflection should not be expected; this is further verification of the minimum thickness formulas given in ACI-9.5.3.

The factored floor load $w_u$ is

$$w_u = 1.2w_D + 1.6w_L = 319 \text{ psf}$$

Since all $\alpha_{f1}L_2/L_1$ values are well over 1.0, take $V$ from Eq. (16.13.1) as

$$V_u = \frac{1.15 w_u S}{2} = \frac{1.15(0.319)(20)}{2} = 3.67 \text{ kips}$$

$$V_c = 2\sqrt{f'_c}\, b_w d = 2\sqrt{3000}(12)(5.12)\tfrac{1}{1000} = 6.73 \text{ kips}$$

$$\phi V_c = 0.75(6.73) = 5.05 \text{ kips} > [V_u = 3.66 \text{ kips}] \qquad \text{OK}$$

Note that the factored shear 3.67 kips is the maximum at strip 1-1 of Fig. 16.13.1; actually, the average for all such strips will be lower. ◄

**EXAMPLE 16.13.2**

Design the reinforcement in the exterior and interior spans of a typical column strip and a typical middle strip in the short direction of the flat slab design example. As described earlier in Section 16.3, $f'_c = 3000$ psi and $f_y = 40,000$ psi.

**TABLE 16.13.1 Factored Moments in a Typical Column Strip and Middle Strip, Example 16.13.2 (Flat Slab)**

| Line Number | Moments at Critical Section (ft-kips) | Exterior Span | | | Interior Span | | |
|---|---|---|---|---|---|---|---|
| | | Negative Moment | Positive Moment | Negative Moment | Negative Moment | Positive Moment | Negative Moment |
| 1 | Total $M$ in column and middle strips (Fig. 16.10.2) (rigid frame C) | −88 | +147 | −205 | −190 | +96 | −190 |
| 2 | Percentage to column strip (Table 16.12.5) | 91.3% | 60% | 75% | 75% | 60% | 75% |
| 3 | Moment in column strip | −80 | +88 | −154 | −143 | +58 | −143 |
| 4 | Moment in middle strip | −8 | +59 | −51 | −47 | +38 | −47 |

**TABLE 16.13.2 Design of Reinforcement in Column Strip, Example 16.13.2 (Flat Slab) ($f_y$ = 40,000 psi, $f'_c$ = 3000 psi)**

| Line No. | Item | Exterior Span | | | Interior Span | | |
|---|---|---|---|---|---|---|---|
| | | Negative Moment | Positive Moment | Negative Moment | Negative Moment | Positive Moment | Negative Moment |
| 1 | Moment, Table 16.13.1, line 3 (ft-kips) | −80 | +88 | −154 | −143 | +58 | −143 |
| 2 | Width $b$ of drop or strip (in.) | 100 | 120 | 100 | 100 | 120 | 100 |
| 3 | Effective depth $d$ (in.) | 8.81 | 6.44 | 8.81 | 8.81 | 6.44 | 8.81 |
| 4 | $M_u/\phi$ (ft-kips) | −89 | +98 | −171 | −159 | +64 | −159 |
| 5 | $R_u$(psi) $= M_u/(\phi bd^2)$ | 138 | 236 | 264 | 246 | 154 | 246 |
| 6 | $\rho$, Eq. (3.8.5) or Fig. 3.8.1 | 0.35% | 0.62% | 0.70% | 0.64% | 0.39% | 0.64% |
| 7 | $A_s = \rho bd$ | 3.08 | 4.79 | 6.17 | 5.64 | 3.01 | 5.64 |
| 8 | $A_s = 0.002bt$* | 2.40 | 1.80 | 2.40 | 2.40 | 1.80 | 2.40 |
| 9 | $N$ = larger of (7) or (8)/0.31 | 9.9 | 15.5 | 19.9 | 18.2 | 9.7 | 18.2 |
| 10 | $N$ = width of strip/$(2t)$ | 5 | 8 | 5 | 5 | 8 | 5 |
| 11 | $N$ required, larger of (9) or (10) | 10 | 16 | 20 | 19 | 10 | 19 |

*$bt$ = 100(10.5) + 20(7.5) = 1200 in.$^2$ for negative moment region.

**TABLE 16.13.3 Design of Reinforcement in Middle Strip, Example 16.13.2 (Flat Slab) ($f_y$ = 40,000 psi, $f'_c$ = 3000 psi)**

| Line No. | Item | Exterior Span | | | Interior Span | | |
|---|---|---|---|---|---|---|---|
| | | Negative Moment | Positive Moment | Negative Moment | Negative Moment | Positive Moment | Negative Moment |
| 1 | Moment, Table 16.13.1, line 4 (ft-kips) | −8 | +59 | −51 | −47 | +38 | −47 |
| 2 | Width of strip, $b$ (in.) | 180 | 180 | 180 | 180 | 180 | 180 |
| 3 | Effective depth $d$ (in.) | 6.44 | 5.81 | 6.44 | 6.44 | 5.81 | 6.44 |
| 4 | $M_u/\phi$ (ft-kips) | −9 | +65 | −57 | −52 | +42 | −52 |
| 5 | $R_u$(psi) $= M_u/(\phi bd^2)$ | 14 | 128 | 92 | 84 | 83 | 84 |
| 6 | $\rho$, Eq. (3.8.5) or Fig. 3.8.1 | 0.04% | 0.32% | 0.23% | 0.22% | 0.21% | 0.22% |
| 7 | $A_s = \rho bd$ | 0.46 | 3.35 | 2.67 | 2.55 | 2.20 | 2.55 |
| 8 | $A_s = 0.002bt$ | 2.70 | 2.70 | 2.70 | 2.70 | 2.70 | 2.70 |
| 9 | $N$ = larger of (7) or (8)/0.31* | 8.7 | 10.8 | 8.7 | 8.7 | 8.7 | 8.7 |
| 10 | $N$ = width of strip/$(2t)$ | 12 | 12 | 12 | 12 | 12 | 12 |
| 11 | $N$ required, larger of (9) or (10) | 12 | 12 | 12 | 12 | 12 | 12 |

*A mixture of #5 and #4 bars could have been selected.



▶ 16.14 B

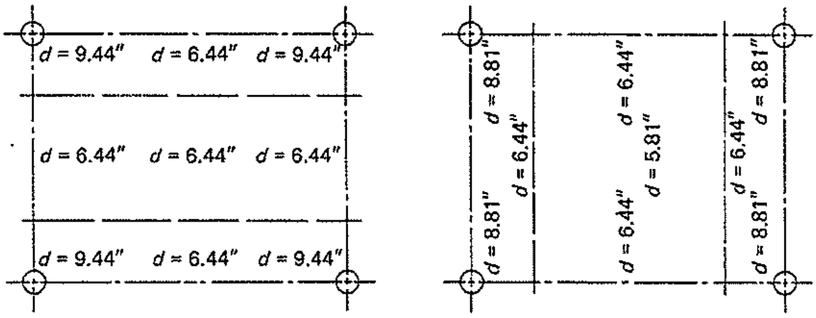



**Figure 16.13.4** Effective depths provided at critical sections in flat slab design example.

**SOLUTION** (a) Moments in column and middle strips. The typical column strip is the column strip of equivalent rigid frame C of Fig. 16.10.2; but the typical middle strip is the sum of two half middle strips, taken from each of the two adjacent equivalent rigid frames C. The factored moments in the typical column and middle strips are shown in Table 16.13.1.

(b) Slab thickness for flexure. For $f_c' = 3000$ psi and $f_y = 40{,}000$ psi, the maximum percentage for tension reinforcement only is $\rho_{max} = 0.0232$ (Table 3.6.1). The actual percentages used (line 6 of Tables 16.13.2 and 16.13.3) are nowhere near this maximum. Thus there is ample compressive strength in the slab. This phenomenon is usual because of the deflection control exerted by the minimum slab thickness requirements.

(c) Design of reinforcement. The design of reinforcement for the typical column strip is shown in Table 16.13.2; for the typical middle strip, it is shown in Table 16.13.3. Because the moments in the long direction are larger than those in the short direction, the larger effective depth is assigned to the long direction wherever the two layers of steel are in contact. This contact at crossing occurs in the top steel at the intersection of column strips and in the bottom steel at the intersection of middle strips. Assuming #5 bars and $\frac{3}{4}$-in. clear cover, the effective depths provided at various critical sections of the long and short directions are shown in Fig. 16.13.4. ◀

## 16.14 BEAM (IF USED) SIZE REQUIREMENT IN FLEXURE AND SHEAR

The size of the beams along the column centerlines in a two-way slab (with beams) should be sufficient to provide the bending moment and shear strengths at the critical sections.

For approximately equal spans, the largest bending moment should occur at the exterior face of the first interior column where the available section for strength computation is rectangular in nature because the effective slab projection is on the tension side. Then with the preliminary beam size the required reinforcement ratio $\rho$ may be determined. Deflection is unlikely to be a problem with T-sections, but must be investigated if excessive deflection may cause difficulty.

The maximum shear in the beam should also occur at the exterior face of the first interior column. The shear diagram for the exterior span may be obtained by placing the negative moments already computed for the beam at the face of the column at each end and loading the span with the percentage of floor load interpolated (ACI-13.6.8) between $\alpha_{f1}L_2/L_1 = 0$ and $\alpha_{f1}L_2/L_1 \geq 1.0$. As discussed in Section 10.2, the stem (web) $b_w d$ should for practicality be sized such that nominal shear stress $v_n = V_n/(\phi b_w d)$ does not exceed about $6\sqrt{f_c'}$ at the critical section $d$ from the face of support.

**▶ EXAMPLE 16.14.1** Investigate if the preliminary overall sizes of 14 × 28 in. for the long beam and 12 × 24 in. for the short beam are suitable for the two-way slab (with beams) design example.

**SOLUTION** Since the values of $\alpha_f$, or of $\alpha_{f1} L_2/L_1$, are considerably larger than 1.0 for all beams spans, there is to be no reduction of the floor load going into the beams from the tributary areas (ACI-13.6.8). As shown in Fig. 16.14.1, the most critical span is $B1$ for the long direction and $B5$ for the short direction. Actually, the load acting on the clear span of the beam should include the floor load (including the weight of the beam stem itself or any other load) directly over the beam stem width plus the floor load on the tributary areas bounded by the 45° lines from the corner of the panel. Also for practical purposes it is acceptable to consider the shear due to floor load at the face of column equal to one-half of the floor load on the tributary areas between column centerlines, as shown in Fig. 16.14.1.

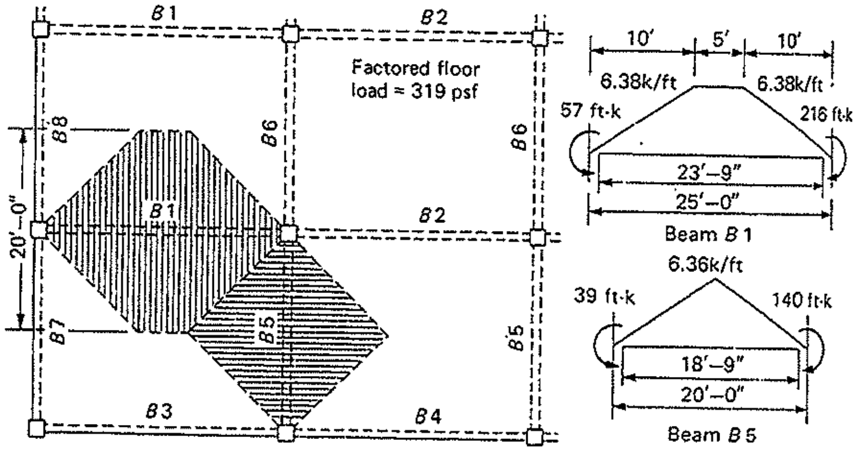



**Figure 16.14.1** Beams around the two-way slab panel.

(a) Size of long beam $B1$. The negative moments at the face of supports, 57 and 216 ft-kips, are taken form Table 16.12.4, frame $A$.

$$\text{weight of beam stem} = \frac{14(21.5)}{144}(150) = 314 \text{ lb/ft}$$

$$\text{maximum negative moment} = \tfrac{1}{10}(1.2)(0.314)(23.75)^2 + 216$$

$$= 21 + 216 = 237 \text{ ft-kips}$$

$$b = 14 \text{ in.} \quad d = 28 - 2.5 \text{ (assume one layer of steel)} = 25.5 \text{ in.}$$

$$R_n = \frac{M_u}{\phi b_w d^2} = \frac{237(12{,}000)}{0.90(14)(25.5)^2} = 347 \text{ psi}$$

From Fig. 3.8.1, $\rho = 0.010$; which is well below $\rho_{max} = 0.0232$. Perhaps the beam size should be reduced. From Fig. 16.14.1,

$$\text{total factored floor load on } B1 = 6.38(15) = 95.7 \text{ kips}$$

$$\max V_u = 1.15(1.2)(0.314)\frac{23.75}{2} + \frac{1}{2}(95.7) + \frac{216 - 57}{23.75}$$

$$= 5.1 + 47.9 + 6.7 = 59.7 \text{ kips}$$

$$v_n = \frac{V_u}{\phi b_w d} = \frac{59{,}700}{0.75(14)(25.5)} = 223 \text{ psi} = 4.1\sqrt{f'_c} < 6\sqrt{f'_c} \qquad \text{OK}$$

**(b)** Size of short beam $B5$. The negative moments at the face of supports, 39 and 140 ft-kips, are taken from Table 16.12.4, frame $C$.

$$\text{weight of beam stem} = \frac{12(17.5)}{144}(150) = 219 \text{ lb/ft}$$

$$\text{maximum negative moment} = \tfrac{1}{10}(1.2)(0.219)(18.75)^2 + 140$$

$$= 9 + 140 = 149 \text{ ft-kips}$$

$$b = 12 \text{ in.} \qquad d = 24 - 2.5 \text{ (assume one layer of steel)} = 21.5 \text{ in.}$$

$$R_n = \frac{M_u}{\phi b_w d^2} = \frac{149(12{,}000)}{0.90(12)(21.5)^2} = 358 \text{ psi}$$

From Fig. 3.8.1, $\rho = 0.0105$, which is well below $\rho_{max} = 0.0232$. From Fig. 16.14.1,

$$\text{total factored floor load on } B5 = 6.38(10) = 63.8 \text{ kips}$$

$$\max V_u = 1.15(1.2)(0.219)\frac{18.75}{2} + \frac{1}{2}(63.8) + \frac{140 - 39}{18.75}$$

$$= 2.8 + 31.9 + 5.4 = 40.1 \text{ kips}$$

$$v_n = \frac{V_u}{\phi b_w d} = \frac{40{,}100}{0.75(12)(21.5)} = 207 \text{ psi} = 3.8\sqrt{f'_c} < 6\sqrt{f'_c} \qquad \text{OK}$$

As mentioned above, the size of beams in both the long and short directions should probably be reduced; the nominal stress $v_n$ is well below $6\sqrt{f'_c}$ at the face of support and is even lower at $d$ therefrom. ◄

## 16.15 SHEAR STRENGTH IN TWO-WAY FLOOR SYSTEMS

The shear strength of a flat slab or flate plate floor around a typical interior column under dead and full live loads is analogous to that of a square or rectangular spread footing subjected to a concentrated column load, except each is an inverted situation of the other. The area enclosed between the parallel pairs of centerlines of the adjacent panels of the floor is like the area of the footing, because there is no shear force along the panel centerline of a typical interior panel in a floor system. Consequently the discussion here is essentially identical to what is included in Chapter 20 on footings.

The shear strength of two-way slab systems without shear reinforcement has been studied by many investigators [16.76–16.92, 16.142]. An excellent summary is provided by ASCE-ACI Task Committee 426 [16.83].

### Wide-Beam Action

The shear strength of the flat slab or flat plate should be first investigated for wide-beam action and then for two-way action (ACI-11.12). In the wide-beam action, the critical section is parallel to the panel centerline in the transverse direction and extends across the full distance between two adjacent longitudinal panel centerlines. As in beams, this critical section of width $b_w$ times the effective depth $d$ is located at a distance $d$ from the face of the equivalent square column capital or from the face of the drop panel, if any. The nominal strength in usual cases where no shear reinforcement is used is

$$V_n = V_c = 2\sqrt{f'_c}\, b_w d \qquad (16.15.1)$$

according to the simplified method of ACI-11.3.1.1

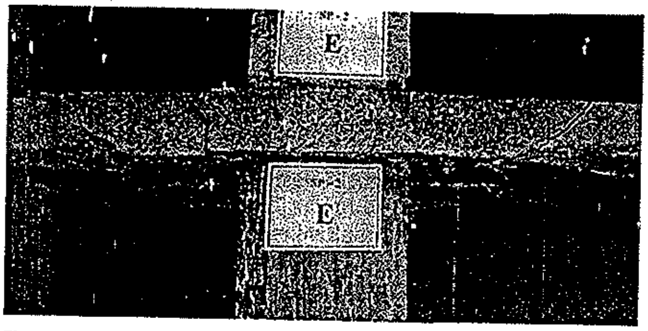



**Figure 16.15.1** Punching shear failure along a truncated pyramid around the column. (Photo courtesy of James O. Jirsa.)

## Two-Way Action

A second failure mode may occur by diagonal cracking along a truncated cone or pyramid around columns, concentrated loads, or reactions (see Fig. 16.15.1). This failure mode is commonly called "punching" shear. The critical section is located so that its periphery $b_0$ is at a distance $d/2$ (that is, one half the effective depth) outside a column, concentrated load, or reaction.

The ACI Code states that $b_0$ is a "minimum but need not approach closer than $d/2$." Some confusion may arise as to whether the "minimum" at $d/2$ would require using a curved-corner perimeter around a square or rectangular column. Since exactness is neither improved nor reduced by calculating the critical section $b_0$ by such elaborate procedures, ACI-11.12.1.3 permits the critical section for square or rectangular loaded areas to have "four straight sides". For slabs with changes in thickness, such as slabs with capitals or drop panels, shear must be checked at several sections to determine the critical section.

When shear reinforcement is not used, the nominal shear strength $V_n = V_c$, which is given by ACI-11.12.2.1 as the smallest of

$$V_c = \left(2 + \frac{4}{\beta_c}\right)\sqrt{f'_c}\, b_0 d, \qquad \text{ACI Formula (11-33)} \qquad \textbf{(16.15.2a)}$$

$$V_c = \left(\frac{\alpha_s}{b_0/d} + 2\right)\sqrt{f'_c}\, b_0 d, \qquad \text{ACI Formula (11-34)} \qquad \textbf{(16.15.2b)}$$

and

$$V_c = 4\sqrt{f'_c}\, b_0 d, \qquad \text{ACI Formula (11-35)} \qquad \textbf{(16.15.2c)}$$

where

$b_0$ = perimeter of critical section

$\beta_c$ = ratio of long side to short side of the column

$\alpha_s$ = 40 for interior columns, 30 for edge columns, and 20 for corner columns.

Equation (16.15.2a) recognizes that there should be a transition between, say, a square column ($\beta_c = 1$) where $V_c$ might be based on $4\sqrt{f'_c}$ for two-way action, and a wall ($\beta_c = \infty$) where $V_c$ should be based on the $2\sqrt{f'_c}$ used for one-way action as for beams. However, unless $\beta_c$ is larger than 2.0, Eq. (16.15.2a) does not control.

Equation (16.15.2b) was new with the 1989 ACI Code. Though designers have generally been investigating critical sections around the perimeter at changes in slab thickness, such as at the edges of capitals and drop panels, it was not generally realized that the strength at such locations may be less than that based on $4\sqrt{f'_c}$. The two-way action strength may reduce, even for a square concentrated load area, both (1) as the distance to the critical section from the concentrated load increases, such as for drop panels, and (2) as the perimeter becomes large compared to the slab thickness, such as, for example, a 6-in. slab supported by a 10-ft square column ($b_0/d \approx 80$). The new equation accounts for this reduced strength.

In the application of Eqs. (16.15.2), $b_0$ is the perimeter of the critical section at a distance $d/2$ from the edge of column capital or drop panel. For Eq. (16.15.2b), $\alpha_s$ for an "interior column" applies when the perimeter is four-sided, for an "edge column" when the perimeter is three-sided, and for a "corner column" when the perimeter is two-sided. As shown by Fig. 16.15.2, Eq. (16.15.2b) will give a $V_c$ smaller than $4\sqrt{f'_c}\, b_0 d$ for large columns (or very thin slabs), such as a square interior column having side larger than $4d$, a square edge column having side larger than $4.33d$, and a square corner column having side larger than $4.5d$. Thus, the nominal shear strength $V_c$ in a two-way system is generally set by Eq. (16.15.2c), that is $V_c = 4\sqrt{f'_c}\, b_0 d$, unless either of Eq. (16.15.2a) or Eq. (16.15.2b) gives a lesser value.

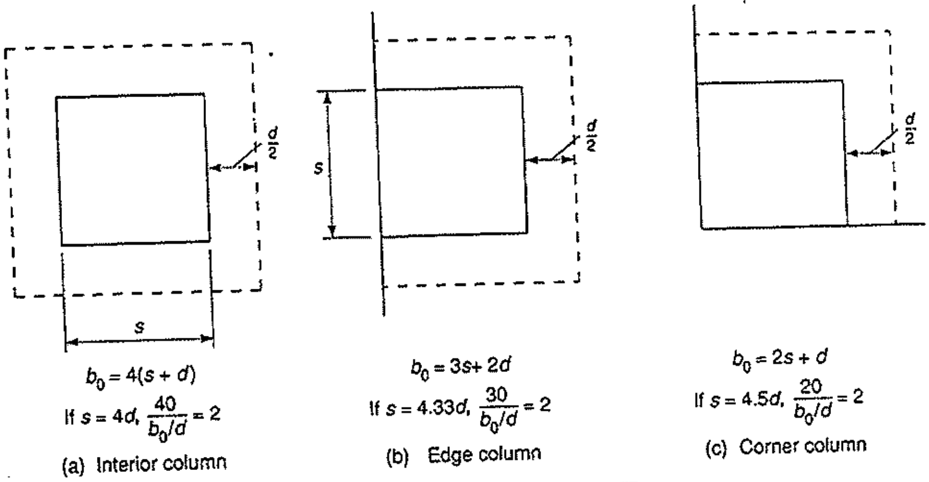



**Figure 16.15.2** Minimum size of square columns for $V_c = 4\sqrt{f'_c}\, b_0 d$.

### Shear Reinforcement

Even when shear reinforcement is used (ACI-11.12.3.2), the nominal strength is limited to a maximum of

$$V_n = V_c + V_s \le 6\sqrt{f'_c}\, b_0 d \qquad (16.15.3)$$

Further, in the design of any shear reinforcement, the portion of the strength $V_c$ may not exceed $2\sqrt{f'_c}\, b_0 d$ (ACI-11.12.3.1). If shearhead reinforcement such as described in Section 16.16 is used (ACI-11.12.4.8), the maximum $V_n$ in Eq. (16.15.3) is $7\sqrt{f'_c}\, b_0 d$.

Unlike the design for beams, a minimum amount of shear reinforcement is not required for slabs (ACI-11.5.6.1) because there is the possibility of load sharing between the weak and strong areas. For deep, lightly reinforced one-way slabs, however, shear failure may occur at loads less than $V_c$, especially if made of high-strength concrete

(ACI-R11.5.6) and it would be prudent to provide a minimum amount of shear reinforcement even if it is not required by the code in these cases.

The investigation for concentric shear (without moment) transfer from slab to column is shown in the following two examples, for the flat slab and flat plate design examples, respectively. When there must be transfer of both shear and moment for the slab to the column, ACI-11.12.6 applies, as will be discussed in Section 16.18.

**▶ EXAMPLE 16.15.1**

Investigate the shear strength in wide-beam and two-way actions in the flat slab design example for an interior column with no bending moment to be transferred. Note that $f'_c = 3000$ psi.

**SOLUTION** (a) Wide-beam action. Investigation for wide-beam action is made for sections 1-1 and 2-2 in the long direction, as shown in Fig. 16.15.3(a). The short direction has a wider critical section and shorter span; thus it does not control. For section 1-1, if the entire width of 20 ft is conservatively assumed to have an effective depth of 6.12 in.,

$$V_u = 0.337(20)(9.52) = 64 \text{ kips} \quad \text{(section 1-1)}$$

$$V_n = V_c = 2\sqrt{f'_c}\,(240)(6.12)\frac{1}{1000} = 161 \text{ kips}$$

$$\phi V_n = 0.75(161) = 121 \text{ kips} > V_u \qquad \text{OK}$$

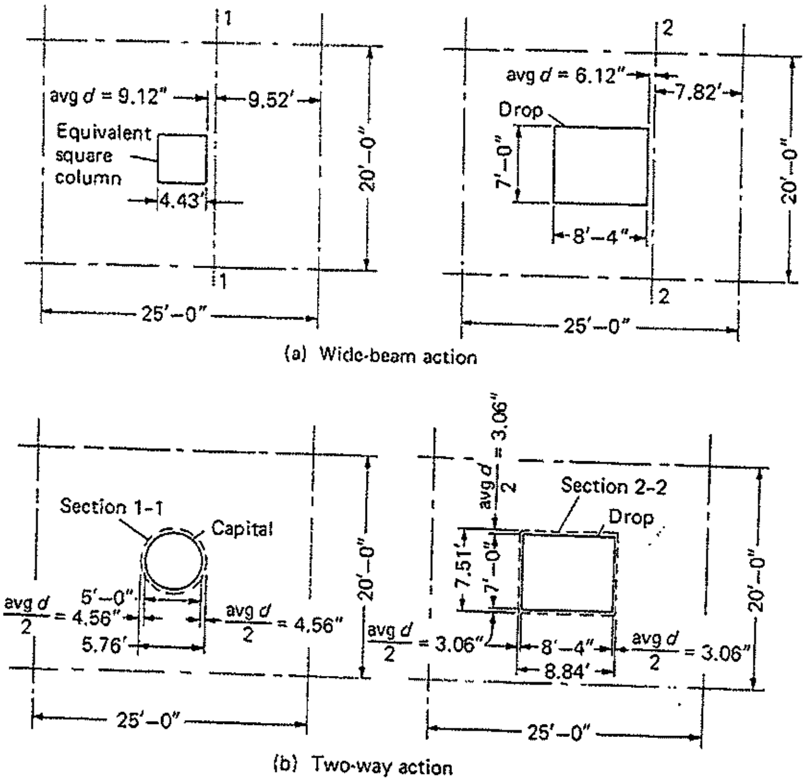



Figure 16.15.3 Critical sections for shear in flat slab design example.

If, however, $b_w$ is taken as 84 in. and $d$ as 9.12 in. on the contention that the increased depth $d$ is only over a width of 84 in.,

$$V_n = V_c = 2\sqrt{f_c'}(84)(9.12)\tfrac{1}{1000} = 84 \text{ kips}$$

This latter value is probably unrealistically low. For section 2-2, the shear resisting section has a constant $d$ of 6.12 in.; thus

$$V_u = 0.337(20)(7.82) = 53 \text{ kips} \quad \text{(section 2-2)}$$

$$\phi V_n = 121 \text{ kips} > V_u \qquad \text{OK}$$

It will be rare that wide-beam (one-way) action will govern.

**(b)** Two-way action. The critical sections for two-way action are the circular section 1-1 at $d/2 = 4.56$ in. from the edge of the column capital and the rectangular section 2-2 at $d/2 = 3.06$ in. from the edge of the drop, as shown in Fig. 16.15.3(b). Since there are no shearing forces at the centerlines of the four adjacent panels, the shear forces around the critical sections 1-1 and 2-2 in Fig. 16.15.3(b) are

$$V_u = 0.337\left[500 - \frac{\pi(5.76)^2}{4}\right] + 1.2(0.038)\left[7(8.33) - \frac{\pi(5.76)^2}{4}\right]$$

$$= 159.2 + 1.5 = 161 \text{ kips} \quad \text{(section 1-1)}$$

In the second term, the 0.038 is the weight of the 3-in. drop in ksf.

$$V_u = 0.337[500 - 8.84(7.51)] = 146 \text{ kips} \quad \text{(section 2-2)}$$

Compute the shear strength at section 1-1 around the perimeter of the capital [Fig. 16.15.3(b)],

$$b_0 = \pi(5.76)12 = 217.1 \text{ in.}; \quad \frac{b_0}{d} = \frac{217.1}{9.12} = 23.8$$

Since $b_0/d > 20$, and $\beta_c = 1$, Eq. (16.15.2b) controls. Thus,

$$\phi V_n = \phi V_c = \phi\left(\frac{40}{23.8} + 2\right)\sqrt{f_c'}\,b_0 d = \phi(3.68\sqrt{f_c'}\,b_0 d)$$

$$= 0.75(3.68\sqrt{f_c'})(217.1)(9.12)\tfrac{1}{1000} = 299 \text{ kips}$$

At section 2-2, Fig. 16.15.3(b),

$$b_0 = [2(8.84) + 2(7.51)]12 = 392.4 \text{ in.}; \quad \frac{b_0}{d} = \frac{392.4}{6.12} = 64.1$$

and since $b_0/d > 20$, Eq. (16.15.2b) controls. Thus,

$$\phi V_n = \phi V_c = \phi\left(\frac{40}{64.1} + 2\right)\sqrt{f_c'}\,b_0 d = \phi(2.62\sqrt{f_c'}\,b_0 d)$$

$$= 0.75\left(2.62\sqrt{f_c'}\right)(392.4)(6.12)\tfrac{1}{1000} = 258 \text{ kips}$$

Though both sections 1-1 and 2-2 have $\phi V_n$ significantly greater than $V_u$, the section around the drop panel is loaded to a slightly higher percentage of its strength (50% for section 2-2 vs 47% for section 1-1). Prior to the 1989 ACI Code, using $4\sqrt{f_c'}\,b_0 d$, the shear strength at the drop panel perimeter would rarely have been of concern. Shear reinforcement is not required at this interior location. ◄

**EXAMPLE 16.15.2** Investigate the shear strength in wide-beam and two-way actions in the flat plate design example for an interior column with no bending moment to be transferred. Note that $f'_c = 4000$ psi.

**SOLUTION** (a) Wide-beam action. Assuming $\frac{3}{4}$-in. clear cover and #4 bars, the average effective depth when bars in two directions are in contact is

$$\text{avg } d = 5.50 - 0.75 - 0.50 = 4.25 \text{ in.}$$

Referring to Fig. 16.15.4(a),

$$V_u = 0.198(12)6.65 = 15.8 \text{ kips}$$

$$\phi V_n = \phi V_c = 0.75(2\sqrt{4000})(12)(12)4.25\frac{1}{1000} = 58.1 \text{ kips} \qquad \text{OK}$$

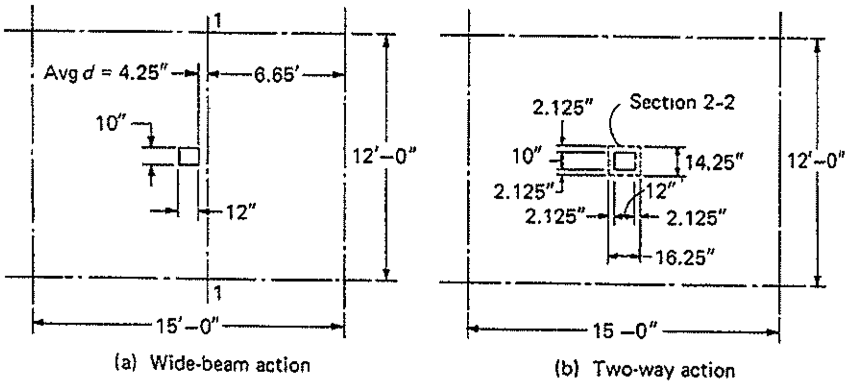



**Figure 16.15.4** Critical sections for shear in flat plate design example.

(b) Two-way action. Referring to Fig. 16.15.4(b),

$$V_u = 0.198[15(12) - 1.35(1.19)] = 35.3 \text{ kips}$$

The perimeter of the critical section at $d/2$ around the column is

$$b_0 = 2(16.25) + 2(14.25) = 61.0 \text{ in.}; \quad \frac{b_0}{d} = \frac{61.0}{4.25} = 14.3 < 20$$

Since $b_0/d < 20$, and $\beta_c = 1.2$, Eq. (16.15.2c) controls. Thus,

$$V_c = 4\sqrt{f'_c}\, b_0 d \qquad [16.15.2c]$$

$$\phi V_n = \phi V_c = 0.75(4\sqrt{4000})(61.0)4.25\frac{1}{1000} = 49.2 \text{ kips} \qquad \text{OK}$$

Shear reinforcement is not required at this interior location.

## 16.16 SHEAR REINFORCEMENT IN FLAT PLATE FLOORS

In flat plate floors where neither column capitals nor drop panels are used, shear reinforcement is frequently necessary. In such cases, two-way action usually controls. The shear reinforcement may take the form of properly anchored bars or wires placed in vertical sections around the column [Fig. 16.16.1(a)], or consist of shearheads, which are steel I- or channel-shaped sections fabricated by welding into four (or three for an exterior column) identical arms at right angles and uninterrupted within the column section [Fig. 16.16.1(b)].

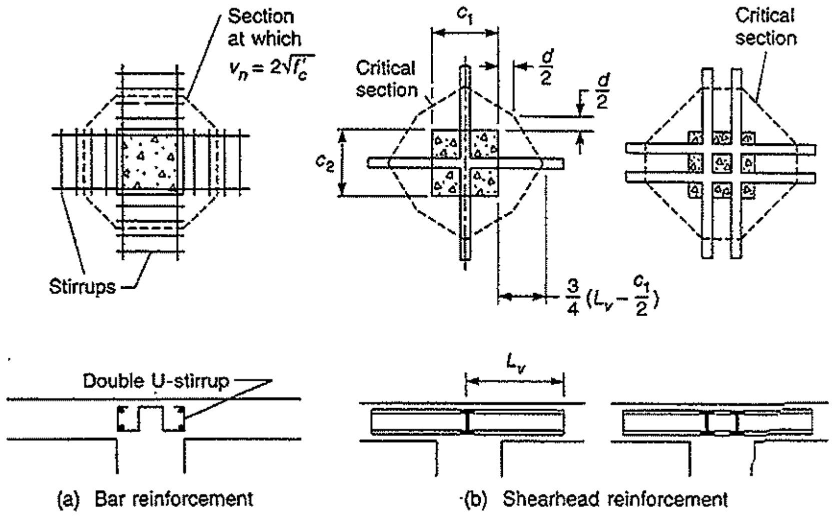



**Figure 16.16.1** Bar and shearhead reinforcement in flat plate floors.

Although widely used, stirrups are often difficult to install in the slab around the column because the region is commonly congested with the column and slab reinforcement. Shearheads can be used instead, but they can be more expensive to fabricate and install. Alternatively, shear studs have been used in Canada and Europe as shear reinforcement in slabs [16.101, 16.144], and they are also widely used on the West Coast in the United States. They consist of headed steel studs welded to a steel strip as shown in Fig. 16.16.2.

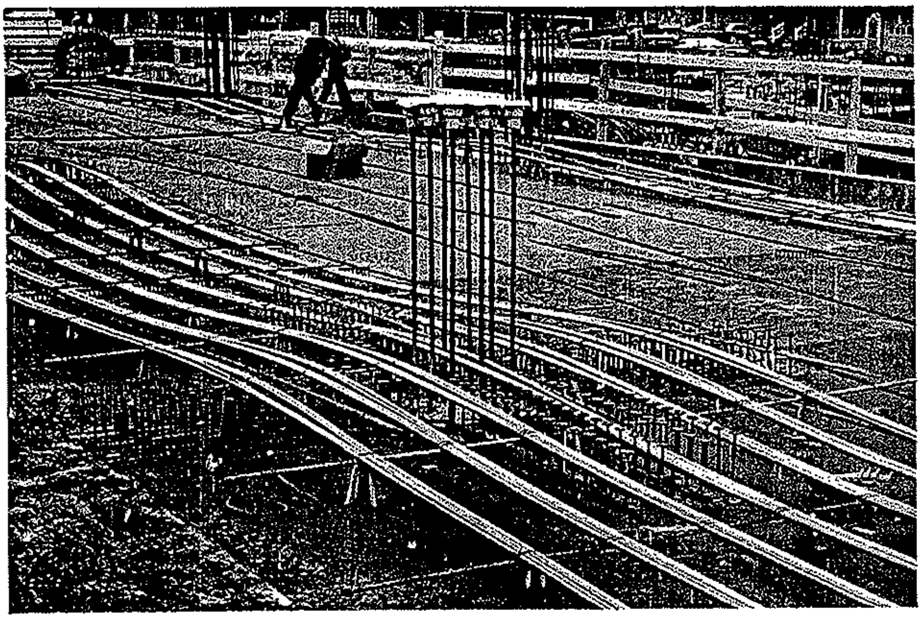


**Figure 16.16.2** Shear stud reinforcement.
(Photo by José A. Pincheira.)

The strips may be arranged in orthogonal directions for rectangular or square columns (Fig. 16.16.2) or in radial directions for circular columns. The ACI Code contains no specific provisions for this system, however, ACI–ASCE Committee 421 provides specific design recommendations for the use of shear studs as shear reinforcement in slabs.

A summary of shear reinforcement for flat plates has been provided by Dilger and Ghali [16.99]. The strength of two-way slab systems with shear reinforcement has been summarized by Hawkins [16.94]. Corley and Hawkins [16.93, 16.110] have studied shearhead reinforcement. Other studies of shear reinforcement in flat plates have been made by Ghali, Dilger, et al. [16.95–16.102], and Pillai, Kirk, and Scavuzzo [16.115].

When *bar or wire shear reinforcement* is used, the nominal strength is

$$V_n = V_c + V_s = 2\sqrt{f'_c}\, b_0 d + \frac{A_v f_y d}{s} \tag{16.16.1}$$

where $b_0$ is the periphery around the critical section for two-way shear action and $A_v$ is the total stirrup bar area around $b_0$. Such bar or wire reinforcement is required wherever $V_u$ exceeds $\phi V_c$ based on $V_c$ of Eqs. (16.15.2a to c). However, in the design of shear reinforcement, $V_c$ for Eq. (16.16.1) may not be taken greater than $2\sqrt{f'_c}\, b_0 d$, and the maximum nominal strength $V_n$ (i.e., $V_c + V_s$) when shear reinforcement is used may not exceed $6\sqrt{f'_c}\, b_0 d$ according to ACI-11.12.3.2.

Shear strength may be provided by *shearheads* under ACI-11.12.4 whenever $V_u/\phi$ at the critical section is between that permitted by Eqs. (16.15.2a to c) and $7\sqrt{f'_c}\, b_0 d$. These provisions, based on the tests of Corley and Hawkins [16.93], apply only where shear alone (i.e., no bending moment) is transferred at an interior column. When there is moment transfer to columns, ACI-11.12.6.3 applies, as is discussed in Section 16.18.

With regard to the size of the shearhead, it must furnish a ratio $\alpha_v$ of 0.15 or larger (ACI-11.12.4.5) between the stiffness for each shearhead arm ($E_s I_x$) and that for the surrounding composite cracked slab section of width $(c_2 + d)$, or

$$\min \alpha_v = \frac{E_s I_x}{E_c\ (\text{composite } I_s)} = 0.15 \tag{16.16.2}$$

where $c_2$ is the dimension of the column measured perpendicular to the span for which the moments are being calculated. The steel shape used must not be deeper than 70 times its web thickness, and the compression flange must be located within 0.3 $d$ of the compression surface of the slab (ACI-11.12.4.2 and 11.12.4.4). In addition, the plastic moment capacity $M_p$ of the shearhead arm must be at least (ACI-11.12.4.6).

$$\min M_p = \frac{V_u}{2\eta\phi}\left[h_v + \alpha_v\left(L_v - \frac{c_1}{2}\right)\right] \tag{16.16.3}$$

where

- $\eta$ = number (usually 4) of identical shearhead arms
- $V_u$ = factored shear around the periphery of column face
- $h_v$ = depth of shearhead
- $L_v$ = length of shearhead measured from column centerline
- $c_1$ = dimension of the column measured in the direction of the span for which the moments are being calculated
- $\phi$ = 0.90, strength reduction factor for tension-controlled members

Equation (16.16.3) is to ensure that the required shear strength of the slab is reached before the flexural strength of the shearhead is exceeded.

The length of the shearhead should be such that the nominal shear strength $V_n$ will not exceed $4\sqrt{f'_c}\, b_0 d$ computed at a peripheral section located at $\frac{3}{4}(L_v - c_1/2)$ along the shearhead but no closer elsewhere than $d/2$ from the column face (ACI-11.12.4.7 and 11.12.4.8). This length requirement is shown in Fig. 16.16.1(b).

When a shearhead is used, it may be considered to contribute a resisting moment (ACI-11.12.4.9).

$$M_v = \frac{\phi \alpha_v V_u}{2\eta}\left(L_v - \frac{c_1}{2}\right) \tag{16.16.4}$$

to each column strip, but not more than 30% of the total moment resistance required in the column strip, nor the change in column strip moment over the length $L_v$, nor the required $M_p$ given by Eq. (16.16.3).

**EXAMPLE 16.16.1**

Using the dimensions of the flat plate design example but changing the live load to 190 psf, investigate the shear strength for wide-beam and two-way actions around an interior column. If the required nominal shear strength $V_n$ for two-way action is between that permitted by Eqs. (16.15.2) and $6\sqrt{f'_c}\, b_0 d$, determine the $A_v/s$ requirement for shear reinforcement at the peripheral critical section and show the nominal shear stress (which is factored shear $V_u$ divided by $\phi b_0 d$) variation from the critical section to the panel centerline. Use $f'_c = 4000$ psi and $f_y = 50{,}000$ psi; assume #5 bars for slab reinforcement.

**SOLUTION** (a) Wide-beam action.

$$w_u = 1.2w_D + 1.6w_L = 1.2(150)(5.5/12) + 1.6(190)$$
$$= 83 + 304 = 387 \text{ psf}$$
$$\text{avg } d \text{ in column strip} = 5.50 - 0.75 - 0.63 = 4.12 \text{ in.}$$

For a 12-in.-wide strip along section 1-1 of Fig. 16.16.3,

$$v_n = \frac{V_u}{\phi b_w d} = \frac{387(6.66)}{0.75(12)(4.12)} = 70 \text{ psi} < (2\sqrt{f'_c} = 126 \text{ psi}) \qquad \text{OK}$$

(b) Two-way action. Referring to Section 2-2 of Fig. 16.16.3,

$$b_0 = 2(16.12) + 2(14.12) = 60.48 \text{ in.}; \quad \frac{b_0}{d} = \frac{60.48}{4.12} = 14.7 < 20$$

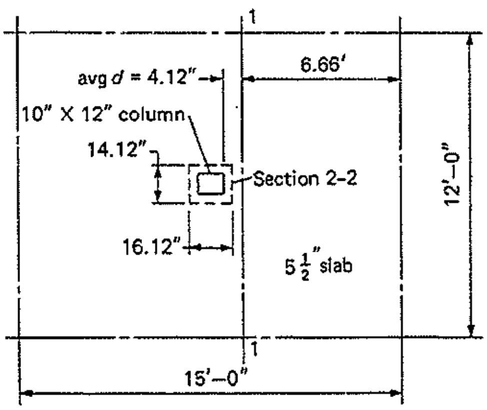



**Figure 16.16.3** Critical sections for shear, Example 16.16.1.

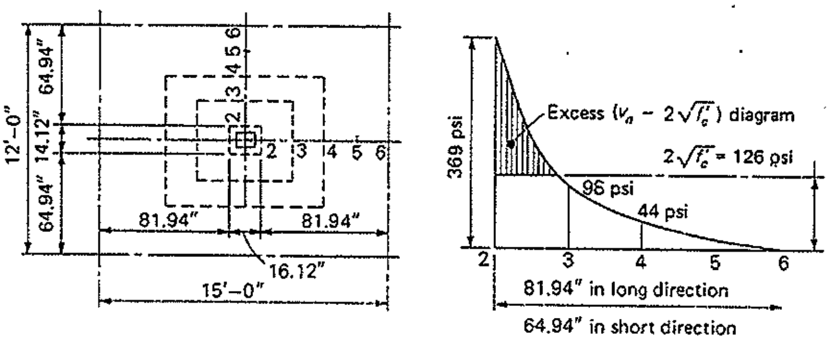



**Figure 16.16.4** Variation of two-way nominal shear stress ($V_u/\phi\ b_0d$), Example 16.16.1.

With a rectangular perimeter $b_0$ having long to short side ratio less than 2 (meaning $\beta_c$ less than 2), and $b_0/d$ less than 20 for an interior column, Eq. (16.15.2c) controls; thus, the strength without shear reinforcement is $V_c = 4\sqrt{f'_c}\, b_0d$. Using nominal stress $v_n = V_u/\phi\ b_0d$,

$$v_n = \frac{V_u}{\phi b_0 d} = \frac{387[180 - 1.34(1.18)]}{0.75(60.48)(4.12)} = 369 \text{ psi}$$

Since the maximum nominal shear stress of 369 psi exceeds $4\sqrt{f'_c} = 253$ psi but not the maximum $6\sqrt{f'_c} = 380$ psi permitted when bar or wire shear reinforcement is used, shear reinforcement is required to take the excess stress $v_n$ which exceeds $2\sqrt{f'_c} = 126$ psi. The shear reinforcement in this case may consist of properly anchored bars or wires and need not be a shearhead. The $A_v/s$ requirement around the critical section of 60.48 in. periphery is, from applying Eq. (16.16.1) with required $V_n = V_u/\phi$,

$$\frac{A_v}{s} = \frac{V_u/\phi - (2\sqrt{f'_c})b_0d}{f_yd} = \frac{(v_n - 2\sqrt{f'_c})b_0}{f_y}$$

$$= \frac{(369 - 126)(60.48)}{50{,}000} = 0.294 \text{ in.}$$

Assuming $s = d/2 \approx 2$-in. spacing,

$$A_v = 0.59 \text{ sq in.}$$

If two double #3 U stirrups are used at each of the four sides,

$$\text{provided } A_v = 4(4)(0.11) = 1.76 \text{ sq in.}$$

The variation of the nominal shear stress $v_n$ from the maximum value of 369 psi to zero at the panel centerline over the equally spaced points 2 to 6 is shown in Fig. 16.16.4. The nominal shear stress ($V_u/\phi\ b_0d$) drops to 126 psi in a rather steep manner so that the number and spacing of these U stirrups can be laid out by the aid of the excess ($v_n - 2\sqrt{f'_c}$) diagram. ◀

**▶ EXAMPLE 16.16.2**

Redesign the connection using shearhead reinforcement for the two-way shear action of Example 16.16.1.

**SOLUTION** (a) Two-way action. Since the maximum nominal shear stress $v_n = V_u/\phi\ b_0d$ of 369 psi is between $4\sqrt{f'_c} = 253$ psi and the maximum of $7\sqrt{f'_c} = 443$ psi when a

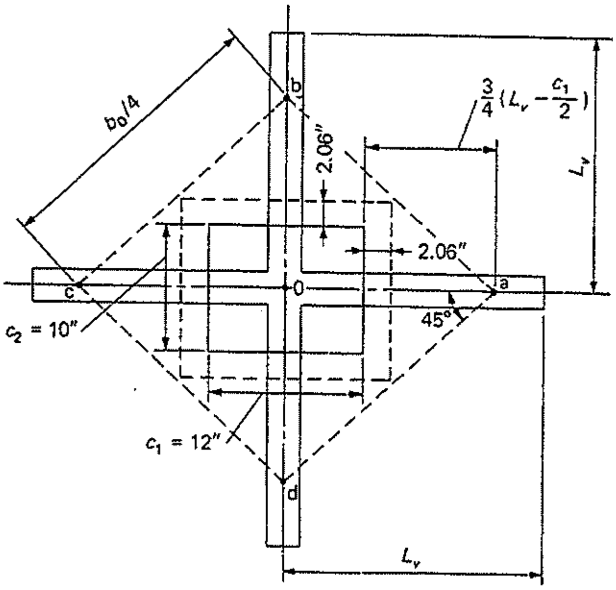



**Figure 16.16.5** Required length of shearhead, Example 16.16.2.

shearhead is used, shearhead reinforcement for the interior column (having no moment transfer to column) is permitted to be designed according to ACI-11.12.4.

(b) Length of shearhead. The length of shearhead should be such that the nominal shear stress is less than $4\sqrt{f'_c}$, computed around a periphery passing through points at $\frac{3}{4}(L_v - c_1/2)$ from, but no closer than, $d/2$ to the column faces. Assuming a square as the critical periphery since the shearhead is to have four identical arms (ACI-11.12.4.1), the required $b_0$ (ft) may be computed from Fig. 16.16.5; thus,

$$4\sqrt{4000} = \frac{387[180 - (b_0/4)^2]}{0.75(b_0)\,4.12} \cdot \frac{1}{12}$$

Neglecting the $(b_0/4)^2$ in the numerator,

$$b_0 = \frac{387(180)}{253(0.75)4.12} \cdot \frac{1}{12} = 7.4 \text{ ft (88.5 in.)}$$

The required distance $L_v$ may be computed from the following geometric considerations. From right triangle $oab$,

$$\left[\frac{3}{4}\left(L_v - \frac{c}{2}\right) + \frac{c}{2}\right]\sqrt{2} = \frac{b_0}{4}$$

which gives, based on leg $ob$, $c = c_2 = 10$ in.,

$$L_v = \left(\frac{88.5}{4\sqrt{2}} - 5\right)\frac{4}{3} + 5 = 19.2 \text{ in.}$$

and, based on leg $oa$, $c = c_1 = 12$ in.,

$$L_v = \left(\frac{88.5}{4\sqrt{2}} - 6\right)\frac{4}{3} + 6 = 18.9 \text{ in.}$$

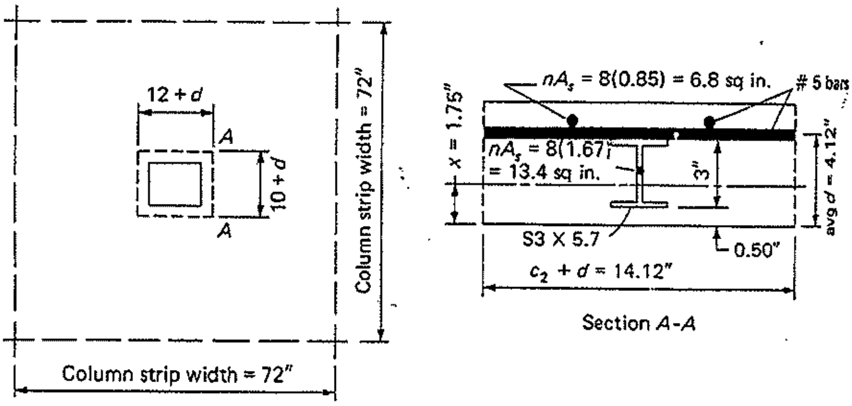



**Figure 16.16.6** Cracked slab section of width $(c_2 + d)$, Example 16.16.2.

For the periphery *abcd* not to approach closer than $d/2$ to the periphery of the column section,

$$oa = ob = 8.06 + 7.06 = 15.12 \text{ in.}$$

But,

$$oa = 6 + \tfrac{3}{4}(L_v - 6) \quad \text{and} \quad ob = 5 + \tfrac{3}{4}(L_v - 5)$$

which gives

$$L_v = (15.12 - 6)\tfrac{4}{3} + 6 = 18.2 \text{ in.}$$
$$L_v = (15.12 - 5)\tfrac{4}{3} + 5 = 18.5 \text{ in.}$$

Use $L_v = 20$ in.

(c) Size of shearhead. The shearhead stiffness must be at least 0.15 of that of the composite cracked slab section of width $(c_2 + d)$. It can be shown that 14-#5 bars and 10-#5 bars are required for negative slab reinforcement in the 72-in.-wide column strips of the long and short directions, respectively. The composite cracked section across width *A-A* in Fig. 16.16.6 should be used because there is more steel in the slab in the long direction. The steel area $A_s$ in section *A-A*, of width $c_2 + d$ is

$$A_s = \frac{10 + d}{72}(14)(0.31) = \frac{14.12}{72}(14)(0.31) = 0.85 \text{ sq in.}$$

Assume an S3×5.7 section for the shearhead placed as shown in Fig. 16.16.6. The S3×5.7 is the shallowest available rolled steel I- or channel-shaped section. With $\tfrac{3}{4}$-in. cover at the top face of slab and #5 bars for top reinforcement in the two orthogonal directions, average $d$ will be $4\tfrac{1}{8}$ in., but the cover to the compression face (bottom) of the rolled shape will be only $\tfrac{1}{2}$ in. Even $\tfrac{3}{4}$-in. cover at the compression face would require that all bottom slab steel be cut short. If the $\tfrac{1}{2}$-in. cover over the rolled shape is not deemed adequate, either a thicker slab must be used or a shallower shearhead fabricated (welded) from three plates would have to be used.

The centroidal axis of the composite cracked section may be obtained by equating the static moments of the compression and tension transformed areas,

$$\frac{14.12x^2}{2} = 13.4(2.0 - x) + 6.8(4.44 - x)$$

$$x = 1.75 \text{ in.}$$

$$\text{composite } I_s = \frac{14.12(1.75)^3}{3} + n(I_x \text{ of steel section}) + 13.4(0.25)^2 + 6.8(2.69)^2$$

$$= 25.2 + 8(2.50) + 0.8 + 49.2 = 95.4 \text{ in.}^4$$

$$\text{provided } \alpha_v = \frac{E_s(2.50)}{E_c(\text{composite } I_s)} = \frac{8(2.50)}{95.4} = 0.21 > 0.15 \qquad \textbf{OK}$$

The plastic section modulus of the S3×5.7 is given by the *AISC Manual** as 1.94 in.$^3$ Using A36 steel, the provided $M_p$ is

$$\text{provided } M_p = 36(1.94) = 69.8 \text{ in.-kips}$$

The required $M_p$ is computed from Eq. (16.16.3) as

$$\text{required } M_p = \frac{V_u}{8\phi}\left[h_v + \alpha_v\left(\text{required } L_v - \frac{c_1}{2}\right)\right]$$

$$= \frac{0.387(180)}{8(0.90)}[3 + 0.21(19.2 - 5)]$$

$$= 57.9 \text{ in.-kips} < 69.8 \text{ in.-kips} \qquad \textbf{OK}$$

**(d)** Shearhead contribution to resist negative moment in slab. The negative moments at the face of column in the 72-in. column strip width in the long and short directions are (387/198)(0.75) times those for equivalent rigid frames *A* and *C* in Fig. 16.10.3, wherein (387/198) is the ratio of factored loads (using 190 psf compared to using 72 psf live load) on the slab and 0.75 is the factor for transverse distribution shown in line 12 of Table 16.12.6. Thus

$$\text{column strip moment in long direction} = \frac{387}{198}(0.75)(37.8) = 55.4 \text{ ft-kips}$$

$$\text{column strip moment in short direction} = \frac{387}{198}(0.75)(30.1) = 44.1 \text{ ft-kips}$$

The resisting moment of the shearhead may be computed from Eq. (16.16.4),

$$M_v = \frac{\phi\alpha_v V_u}{2\eta}\left(L_v - \frac{c_1}{2}\right)$$

$$= \frac{0.90(0.21)(0.387)(180)}{8}[(20 - 6) \quad \text{or} \quad (20 - 5)]\frac{1}{12}$$

$$= 1.92 \text{ or } 2.06 \text{ ft-kips}$$

Thus the contribution is rather small and the revision of slab reinforcement is unnecessary. ◀

*See *Manual of Steel Construction, Load and Resistance Factor Design* (3rd ed.), 2001. Chicago: American Institute of Steel Construction.

## 16.17 DIRECT DESIGN METHOD—MOMENTS IN COLUMNS

The moments in columns due to unbalanced loads on adjacent panels are readily available when an elastic analysis is performed on the equivalent rigid frame for the various pattern loadings. In the "direct design method," wherein the limitations listed in Section 16.7 are satisfied, the longitudinal moments in the slab are prescribed by the provisions of ACI-13.6.3. In a similar manner, the code prescribes the unbalanced moment at an interior column as follows [ACI Formula (13-7)]:

$$M = 0.07\left[\left(w_D + \frac{1}{2}w_L\right)L_2L_n^2 - w'_DL'_2(L'_n)^2\right] \tag{16.17.1}$$

where

$w_D$ = factored dead load per unit area

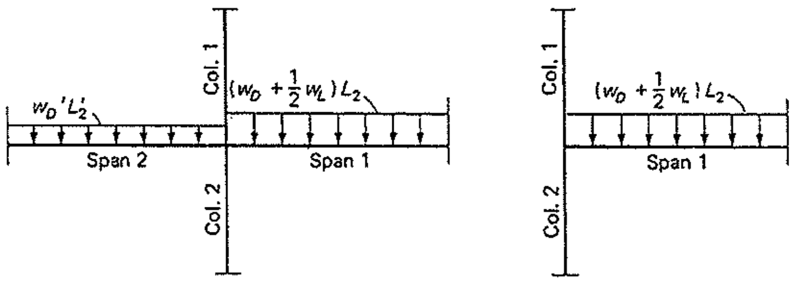
$w_L$ = factored live load per unit area

$w'_D, L'_2, L'_n$ = quantities referring to shorter span

The moment is yet to be distributed between the two ends of the upper and lower columns meeting at the joint.

The rational for Eq. (16.17.1) may be observed from the stiffness ratios at a typical interior joint shown in Fig. 16.17.1(a), wherein the distribution factor for the sum of the column end moments is taken as $\frac{7}{8}$ and the unbalanced moment in the column strip is taken to be 0.080/0.125 times the difference in the total static moments due to dead plus half live load on the longer span and dead load only on the shorter span.

For the edge column, ACI-13.6.3.6 requires using $0.3M_0$ as the moment to be transferred between the slab and an edge column.




(a) Interior joint (b) Exterior joint

**Figure 16.17.1** Direct design method—moments in columns.

**▶ EXAMPLE 16.17.1**

Obtain the factored moments in the interior and exterior columns in each direction for the flat plate design example.

**SOLUTION** (a) Exterior column, long direction (Frame $A$). The factored moment $M_u$ to be transferred to the exterior column is (ACI-13.6.3.6) $0.3M_0$,

$$M_u = 0.3M_0 = 0.3(58.2) = 17.5 \text{ ft-kips}$$

where the 58.2 ft-kips was obtained from Table 16.10.2. The moment $M_u$ is to be divided between upper and lower columns in proportion to their stiffnesses (in this case, equally).

On flat plate construction, nearly all (98.8% for Frame $A$ and 99.3% for Frame $C$) of the exterior frame moment is taken by the column strip; the arbitrary use of $0.3M_0$ to be taken by the column strip seems appropriate.

**(b)** Interior column, long direction. The factored moment to be transferred to the column is empirically the amount obtained from ACI Formula (13-7) [Eq. 16.17.1],

$$M_u = 0.07(0.083 + 0.058)(12)(15 - 1)^2 - 0.083(12)(15 - 1)^2$$
$$= 0.07(0.058)(12)(14)^2 = 9.5 \text{ ft-kips}$$

The moment $M_u$ is to be divided between upper and lower columns.

**(c)** Exterior column, short direction (Frame C). The factored moment to be transferred is

$$M_u = 0.3M_0 = 0.3(46.3) = 13.9 \text{ ft-kips}$$

where the 46.3 ft-kips was obtained from Table 16.10.2. The moment $M_u$ is to be divided between upper and lower columns.

**(d)** Interior column, short direction. The factored moment to be transferred is

$$M_u = 0.07(0.058)(15)(12 - 0.83)^2 = 7.6 \text{ ft-kips}$$

The moment $M_u$ is to be divided between upper and lower columns. ◄

## 16.18 TRANSFER OF MOMENT AND SHEAR AT JUNCTION OF SLAB AND COLUMN

Inasmuch as the columns meet the slab at monolithic joints, there must be moment as well as shear transfer between the slab and the column ends. The moments may arise out of lateral loads due to wind or earthquake effects acting on the multistory frame, or they may be due to unbalanced gravity loads as considered in Section 16.17. In addition, the shear forces at the column ends and throughout the columns must be considered in the design of lateral reinforcement (ties or spiral) in the columns (ACI-11.11). The transfer of moment and shear at the slab–column interface is extremely important in the design of flat plates and has been the subject of numerous research studies [16.103–16.138, 16.143]. Particularly, the current status is presented by ACI–ASCE Committee 352 in its *Recommendations for Design of Slab–Column Connections in Monolithic Reinforced Concrete Structures* [16.128], and the background explanation by Moehle, Kreger, and Leon [16.127].

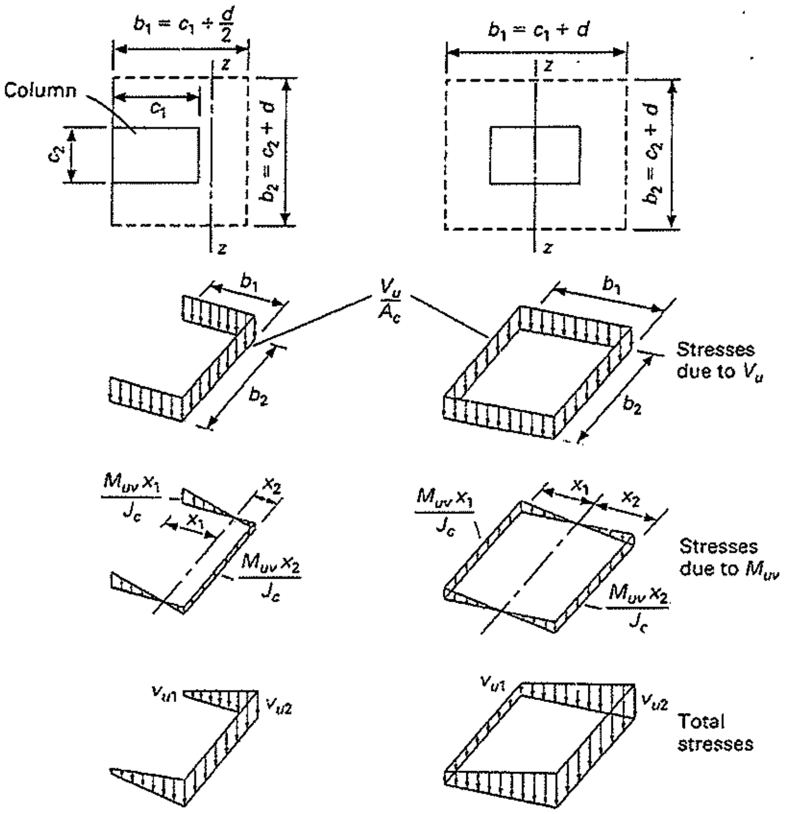
Let $M_u$ be the total factored moment that is to be transferred to both ends of the columns meeting at an exterior or an interior joint. Test results by Hanson and Hanson [16.104] have shown that about 60% of the moment is transferred by flexure and the remainder by unbalanced shear stresses around the critical periphery located at $d/2$ from the column faces. The ACI Code requires the total factored moment $M_u$ to be divided into $M_{ub}$ "transferred by flexure" (ACI-13.5.3) and $M_{uv}$ "transferred by shear" (ACI-11.12.6) such that

$$M_{ub} = \gamma_f M_u = \left( \frac{1}{1 + \frac{2}{3}\sqrt{\frac{b_1}{b_2}}} \right) M_u \qquad (16.18.1)$$

where

$b_1$ = critical section dimension in the longitudinal direction
$= c_1 + d/2$ for exterior columns [Fig. 16.18.1(a)]
$= c_1 + d$ for interior columns [Fig. 16.18.1(b)]

$b_2$ = critical section dimension in the transverse direction
$= c_2 + d$ (Fig. 16.18.1)




(a) Exterior column (b) Interior column

**Figure 16.18.1** Shear transfer of moment to columns.

and

$$M_{uv} = M_u - M_{ub} = M_u(1 - \gamma_f) \tag{16.18.2}$$

The moment $M_{ub}$ is considered to be transferred within an effective slab width equal to $(c_2 + 3t)$ at the column (ACI-13.5.3.2), where $t$ is the slab or drop panel thickness. The moment strength for $M_{ub}$ is achieved by using additional reinforcement and closer spacing within the width $(c_2 + 3t)$.

If $b_2 = b_1$, Eq. (16.18.1) becomes

$$M_{ub} = 0.60 M_u$$

If $b_2 = 1.5b_1$, Eq. (16.18.1) becomes

$$M_{ub} = 0.648 M_u$$

It appears reasonable that when $b_2$ in the transverse direction is larger than $b_1$ in the longitudinal direction, the moment transferred by flexure is greater because the effective slab width $(c_2 + 3t)$ resisting the moment is larger.

Because the aspect ratio $b_2/b_1$ affects only slightly the proportion of the exterior support moment "transferred by flexure," the ACI Code has simplified the procedure for many situations.

## Simplified Procedure

For unbalanced moments *about an axis parallel to the edge* at exterior supports, where the factored shear $V_u$ does not exceed $0.75\phi V_c$ at an edge support, or does not exceed $0.5\phi V_c$ at a corner support, ACI-13.5.3.3 permits neglect of the interaction between shear and moment. In other words, for such situations, the full exterior moment can be considered transferred through flexure (i.e., $\gamma_f = 1.0$), and the exterior support factored shear $V_u$ can be considered independently.

For unbalanced moments at interior supports and for unbalanced moments *about an axis transverse to the edge* at exterior supports, where factored shear $V_u$ does not exceed $0.4\phi V_c$, ACI-13.5.3.3 permits increasing by as much as 25% the proportion $\gamma_f$ of the full exterior moment transferred by flexure.

When using the simplified procedure, the reinforcement $\rho$, within the effective slab width defined in ACI-13.5.3.2, is not permitted to exceed $0.375\rho_b$. The simplified procedure is not permitted for prestressed concrete systems.

## Stresses Representing Interaction Between Flexure and Shear

The moment $M_{uv}$ transferred by shear acts in addition to the associated shear force $V_u$ at the centroid of the shear area around the critical periphery located at $d/2$ from the column faces, as shown in Fig. 16.18.1. Referring to that figure, the factored shear *stress* is

$$v_{u1} = \frac{V_u}{A_c} - \frac{M_{uv}x_1}{J_c} \qquad (16.18.3)$$

$$v_{u2} = \frac{V_u}{A_c} + \frac{M_{uv}x_2}{J_c} \qquad (16.18.4)$$

By using a section property $J_c$ analogous to the polar moment of inertia of the shear area along the critical periphery taken about the $z$-$z$ axis, it is assumed that there are both horizontal and vertical shear stresses on the shear areas having dimensions $b_1$ by $d$ in Fig. 16.18.1. The $z$-$z$ axis is perpendicular to the longitudinal axis of the equivalent frame; that is, in the transverse direction, and located at the centroid of the shear area.

For an exterior column, $x_1$ and $x_2$ are obtained by locating the centroid of the channel-shaped vertical shear area represented by the dashed line $(b_1 + b_2 + b_1)$ shown in Fig. 16.18.1(a), and

$$A_c = (2b_1 + b_2)d \qquad (16.18.5)$$

$$x_2 = \frac{b_1^2 d}{A_c} \qquad (16.18.6)$$

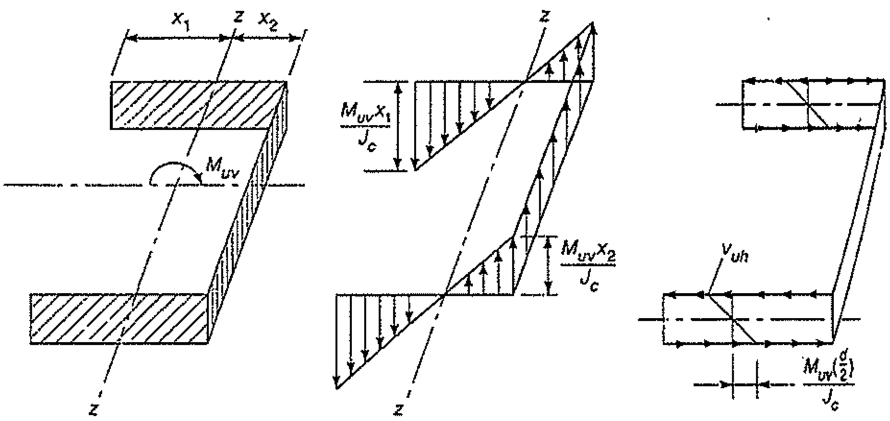
$$J_c = d\left[2\frac{b_1^3}{3} - (2b_1 + b_2)x_2^2\right] + \frac{b_1 d^3}{6} \qquad (16.18.7)$$

For an interior column, referring to Fig. 16.18.1(b),

$$A_c = 2(b_1 + b_2)d \qquad (16.18.8)$$

$$J_c = d\left[\frac{b_1^3}{6} + \frac{b_2 b_1^2}{2}\right] + \frac{b_1 d^3}{6} \qquad (16.18.9)$$

Equations (16.18.5) to (16.18.9) are derivable by letting the shear stress at any location resulting from $M_{uv}$ alone be proportional to the distance from the centroidal axis $z$-$z$ to the shear areas $b_1$ by $d$, and either (a) to the one shear area $b_2$ by $d$ for an exterior column as shown in Fig. 16.18.2, or (b) to the two shear areas $b_2$ by $d$ for the interior column.




(a) Shear areas to resist $M_{uv}$ (b) Vertical resisting shear stresses (c) Horizontal resisting shear stresses

**Figure 16.18.2** Resisting shear stresses due to $M_{uv}$ acting on an exterior column.

According to ACI-11.12.6.2, the larger factored shear stress $v_{u2}$ shown in Fig. 16.18.1 must not exceed the stress $\phi v_n = \phi V_c / b_0\, d$ obtained from ACI Formulas (11-33) to (11-35), that is, Eqs. (16.15.2a, b, and c), otherwise shear reinforcement as described in Section 16.16 is required.

**▶ EXAMPLE 16.18.1**

For the plate design example, investigate the transfer of unbalanced gravity load moments in the long direction, as already computed in Example 16.17.1, to the exterior and interior columns, respectively.

**SOLUTION** (a) Exterior column (long direction), transfer by flexure. From Example 16.17.1, the moment to be transferred is

$$M_u = 17.5 \text{ ft-kips}$$

The factored shear $V_u$ is taken as $w_u$ times the floor area, 12 ft × 7.5 ft, tributary to the exterior column.

$$V_u = 0.198(12)7.5 = 17.8 \text{ kips}$$

The nominal shear strength $V_c$ in accordance with ACI-11.12.2.1 is the smallest of

$$V_c = \left[\left(2 + \frac{4}{\beta_c}\right)\sqrt{f'_c}\, b_0 d = \left(2 + \frac{4}{12/10}\right)\sqrt{f'_c}\, b_0 d\right] = 5.3\sqrt{f'_c}\, b_0 d$$

$$V_c = \left[\left(\frac{\alpha_s}{b_0/d} + 2\right)\sqrt{f'_c}\, b_0 d = \left(\frac{30}{42.5/4.25} + 2\right)\sqrt{f'_c}\, b_0 d\right] = 5.0\sqrt{f'_c}\, b_0 d$$

$$V_c = 4\sqrt{f'_c}\, b_0 d \qquad \textbf{Controls!}$$

which means

$$V_c = 4\sqrt{f'_c}\, b_0 d = 4\sqrt{4000}(42.5)4.25\frac{1}{1000} = 45.7 \text{ kips}$$

According to ACI-13.5.3.3, the simplified procedure may be used when

$$[0.75\phi V_c = 0.75(0.75)(45.7) = 0.75(34.3) = 25.7 \text{ kips}] > [V_u = 17.8 \text{ kips}]$$

Thus, ACI permits all of the exterior moment to be taken as flexure; or

$$M_{ub} = \gamma_f M_u = 1.0(17.5) = 17.5 \text{ ft-kips}$$

The shear can be considered independently.

(b) Exterior column, long direction, transfer by flexure using the shear–flexure interaction procedure. This procedure involves more calculations and is more conservative than treating the flexure and shear independently. From Eq. (16.18.1), using the average effective depth $d = 4.25$ in. for #4 slab reinforcement,

$$M_{ub} = \gamma_f M_u = \left( \frac{1}{1 + \frac{2}{3}\sqrt{\frac{b_1}{b_2}}} \right) M_u$$

$$= \left( \frac{1}{1 + \frac{2}{3}\sqrt{\frac{12 + 2.125}{10 + 4.25}}} \right) 17.5 = 0.601(17.5) = 10.5 \text{ ft-kips}$$

As shown by Fig. 16.18.3, this moment is be carried in a slab width (ACI-13.5.3.2) equal to the column width plus three times the slab thickness, that is, 26.5 in. From Table 16.12.6 and Fig. 16.10.3 (Frame $A$), the total moment in the 72-in.-wide column strip is

$$M \text{ in column strip} = 0.988(15.1) = 15 \text{ ft-kips}$$

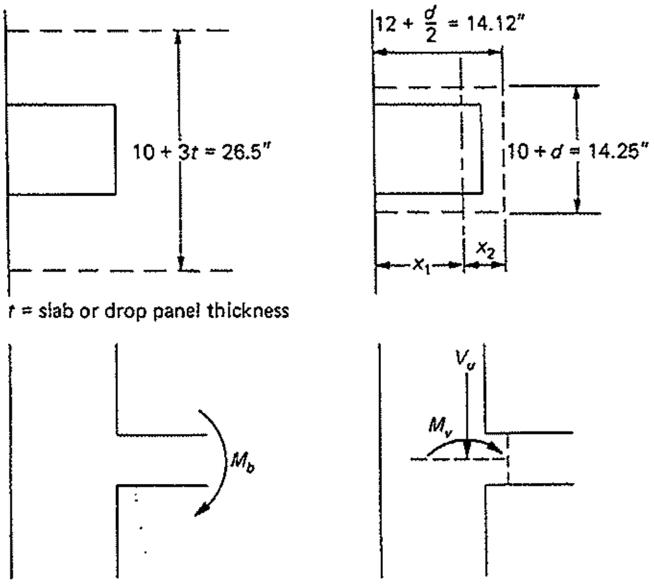



**Figure 16.18.3** Transfer of moments at exterior column, Example 16.18.1.

If the slab reinforcement is placed at equal spacing in the column strip, additional reinforcement is needed in the 26.5-in. width for a bending moment of

$$M_{ub} - 15\left(\frac{26.5}{72}\right) = 10.5 - 5.5 = 5 \text{ ft-kips}$$

(c) Exterior column, long direction, transfer by eccentricity of shear using the shear-flexure interaction procedure. From part (a),

$$V_u = 0.198(12)7.5 = 17.8 \text{ kips}$$
$$M_{uv} = M_u - M_{ub} = 17.5 - 10.5 = 7 \text{ ft-kips}$$

From Fig. 16.18.3,

$$x_2 = \frac{2(14.12)7.06}{28 + 14.25} = 4.70 \text{ in.}$$
$$A_c = 4.25(28.24 + 14.25) = 181 \text{ sq in.}$$
$$J_c = 4.25\left[\frac{2(14.12)^3}{3} - 42.49(4.70)^2\right] + \frac{14.12(4.25)^3}{6}$$
$$= 4004 + 181 = 4185 \text{ in.}^4$$
$$v_{u1} = \frac{17{,}800}{180} - \frac{7000(12)9.42}{4185} = 98 - 189 = -91 \text{ psi}$$
$$v_{u2} = \frac{17{,}800}{180} + \frac{7000(12)4.70}{4185} = 98 + 94 = +192 \text{ psi}$$

The nominal stress limit based on strength in shear was determined in part (a) to be that based on $4\sqrt{f'_c}$. Thus, the limit to the above stresses is

$$\text{limit } v_u = \phi v_c = \phi(4\sqrt{f'_c}) = 0.75(253) = 190 \text{ psi}$$

when no shear reinforcement is provided. In this example, the shear strength is still adequate based on the shear–flexure interaction procedure.

The horizontal shear stress $v_{uh}$ at the upper or lower edge of the two shear areas $b_1$ by $d$ is

$$v_{uh} = \frac{7000(12)(4.25/2)}{4185} = 43 \text{ psi}$$

The $v_{uh}$ of 43 psi, $v_{u1}$ of $-91$ psi, and $v_{u2}$ of $+192$ psi may be drawn on a sketch like that of Fig. 16.18.2, and by basic statics computation of the resultant upward force should equal $V_u$ of 17.8 kips and the resultant moment about the $z$-$z$ axis should equal $M_{uv}$ of 7 ft-kips.

(d) Interior column (long direction), transfer by flexure. Investigate whether the simplified procedure is permitted. The factored shear $V_u$ is computed as $w_u$ times the tributary floor area of 12 × 15 ft,

$$V_u = 0.198(12)15 = 35.6 \text{ kips}$$

Applying ACI Formulas (11-33), (11-34), and (11-35), as shown in part (a) for the exterior column, will indicate that the shear strength based on $4\sqrt{f'_c}$ controls. Since the interior and exterior columns are the same size, the ACI Formula (11-33) involving the aspect ratio $\beta_c$ gives the same value as in part (a). Regarding ACI Formula (11-34), $\alpha_s$ is 40 for interior columns, and

$$b_0 = 2(16.25) + 2(14.25) = 61.0 \text{ in.}$$

Thus, ACI Formula (11-34) gives

$$V_c = \left[\left(\frac{\alpha_s}{b_0/d} + 2\right)\sqrt{f_c'}\, b_0 d = \left(\frac{40}{61.0/4.25} + 2\right)\sqrt{f_c'}\, b_0 d\right] = 4.8\sqrt{f_c'}\, b_0 d$$

The strength $V_c$ cannot exceed that based on $4\sqrt{f_c'}$ from ACI Formula (11-35). Thus,

$$V_c = 4\sqrt{f_c'}\, b_0 d = 4\sqrt{4000}(61.0)4.25\tfrac{1}{1000} = 65.6 \text{ kips}$$

Applying the simplified procedure authorized by ACI-13.5.3.3,

$$[0.4\phi V_c = 0.4(0.75)(65.6) = 0.4(49.2) = 19.7 \text{ kips}] < [V_u = 35.6 \text{ kips}]$$

Since the factored shear $V_u$ is *not* less than $0.4\phi V_c$, the regular shear–flexure interaction procedure must be used!

From Example 16.17.1, the moment to be transferred [computed by ACI Formula (13-7)] is

$$M_u = 9.5 \text{ ft-kips}$$

$$M_{ub} = \gamma_f M_u = \left(\frac{1}{1 + \frac{2}{3}\sqrt{\frac{b_1}{b_2}}}\right) M_u$$

$$= \left(\frac{1}{1 + \frac{2}{3}\sqrt{\frac{12 + 4.25}{10 + 4.25}}}\right) 9.5 = 0.584(9.5) = 5.5 \text{ ft-kips}$$

From Table 16.12.6 and Fig. 16.10.3, the total moment in the 72-in.-wide column strip is

$$M \text{ in column strip} = 0.75(37.8) = 28.4 \text{ ft-kips}$$

Since the column strip moment in the 26.5-in. width of 26.5(28.4)/72 = 10.5 ft-kips is larger than 5.5 ft-kips, no additional reinforcement is needed.

(e) Interior column (long direction), transfer by eccentricity of shear. From part (d), the factored shear $V_u$ is 35.6 kips.

$$M_{uv} = M_u - M_{ub} = 9.5 - 5.5 = 4.0 \text{ ft-kips}$$

From Fig. 16.18.1(b),

$$A_c = 4.25(32.50 + 28.50) = 259 \text{ sq in.}$$

$$J_c = 4.25\left[\frac{(16.25)^3}{6} + \frac{14.25(16.25)^2}{2}\right] + \frac{16.25(4.25)^3}{6}$$

$$= 11{,}040 + 210 = 11{,}250 \text{ in.}^4$$

$$v_{u1} = \frac{35{,}600}{259} - \frac{4000(12)8.12}{11{,}250} = 137 - 35 = -102 \text{ psi}$$

$$v_{u2} = \frac{35{,}600}{259} + \frac{4000(12)8.12}{11{,}250} = 137 + 35 = +172 \text{ psi}$$

The capacity $\phi v_n = \phi v_c = \phi(4\sqrt{f_c'}) = 0.75(253) = 190$ psi when no shear reinforcement is provided.

Again, by basic statics the sum of the factored load vertical shear stresses on the two areas $b_1$ by $d$ plus that on the two areas $b_2$ by $d$ should add to give $V_u$ of 35.6 kips. Likewise, the moment of these shear stress resultants along with that of the horizontal shear stresses on the two faces $b_1$ by $d$ should add up to 4.0 ft-kips. The horizontal shear stress at the upper or lower edge of the faces $b_1$ by $d$ is

$$v_{uh} = \frac{4000(12)(4.25/2)}{11,250} = 9 \text{ psi}$$ ◀

## Moment Transfer from Flat Plate to Column When Shearheads Are Used

Tests [16.93, 16.110] have indicated that shear stresses computed for factored loads at the critical section distance $d/2$ from the column face are appropriate for transfer of $M_{uv} = M_u - M_{ub}$ as described above, even when shearheads are used. However, the critical section for $V_u$ is at a periphery passing through points at $\frac{3}{4}(L_v - c_1/2)$ from, but no closer than, $d/2$ to the column faces. When there are both $V_u$ and $M_u$ to be transferred, ACI-11.12.6.3 requires that the sum of the shear stresses computed for $M_{uv}$ and $V_u$ at their respective locations not exceed $\phi(4\sqrt{f'_c})$. The reason for this apparent inconsistency (ACI Commentary-R11.12.6.3) is that these two critical sections are in close proximity at the column corners where the failures initiate.

**▶ EXAMPLE 16.18.2**

Recompute the periphery length $b_0$ required, located at $\frac{3}{4}(L_v - c_1/2)$ but no closer than $d/2$ from the column face for the shearhead in Example 16.16.2 when there is unbalanced moment at the interior column equal to that of Eq. (16.17.1), ACI Formula (13-7).

**SOLUTION** (a) Determine whether or not additional reinforcement is necessary for moment transfer. Referring to Example 16.16.1, part (a) of solution,

$$w_u = 387 \text{ psf}$$

Referring to Example 16.16.2, part (d) of solution,

$$\text{column strip moment in long direction} = 55.4 \text{ ft-kips}$$

The moment to be transferred to the column is, using Eq. (16.17.1),

$$\begin{aligned} M_u &= 0.07[(0.083 + 0.152)(12)(15-1)^2 - 0.083(12)(15-1)^2] \\ &= 0.07(0.152)(12)(14)^2 = 25.0 \text{ ft-kips} \end{aligned}$$

Column strip moment in 26.5-in. width of Fig. 16.18.4 is

$$55.4\left(\frac{26.5}{72}\right) = 20.4 \text{ ft-kips}$$

Additional reinforcement is needed to take $(25.0 - 20.4) = 4.6$ ft-kips within the 26.5-in. width unless $25.0/55.4 = 45\%$ of the total column strip reinforcement is concentrated in the 26.5-in. width. ▶ 16.1

(b) Compute factored load shear stress at critical section of Fig. 16.18.4 due to $M_{uv}$ only. Referring to Example 16.18.1, part (d) of solution,

$$M_{ub} = 0.584\, M_u = 0.584(25.0) = 14.6 \text{ ft-kips}$$

$$M_{uv} = 25.0 - 14.6 = 10.4 \text{ ft-kips}$$

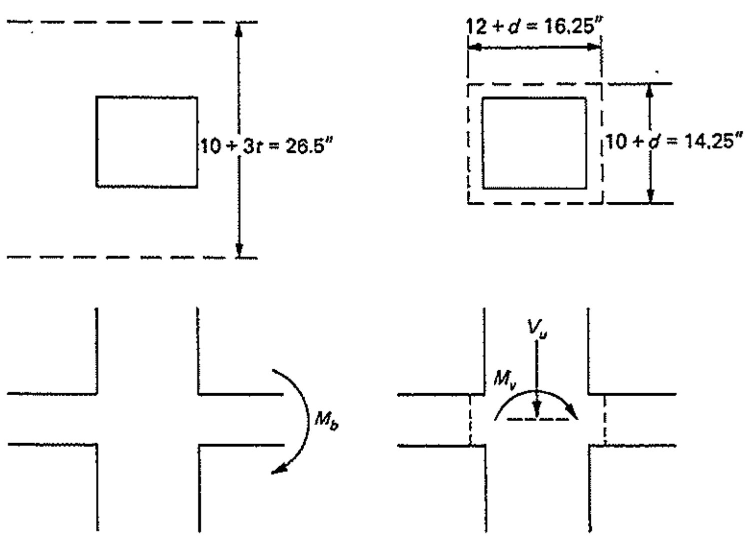



**Figure 16.18.4** Transfer of moments in the long direction at interior column, Examples 16.18.1 and 16.18.2.

Using critical section properties in Example 16.18.1, part (d) of solution,

$$v_u = \frac{10{,}400(12)8.12}{11{,}250} = 90 \text{ psi}$$

(c) Compute the required periphery $b_0$. Referring to Example 16.16.2, part (b) of solution,

$$\phi(4\sqrt{4000}) = [90 \text{ from part (b)}] + \frac{387[180 - (b_0/4)^2]}{b_0(4.12)} \cdot \tfrac{1}{12}$$

Neglecting the $(b_0/4)^2$ in the numerator,

$$b_0 = \frac{387(180)}{100(4.12)} \cdot \tfrac{1}{12} = 14.1 \text{ ft}$$

Placing $b_0 = 14.1$ ft in the numerator and solving for $b_0$ again

$$b_0 = \frac{387[180 - (3.53)^2]}{100(4.12)} \cdot \tfrac{1}{12} = 3.1 \text{ ft}$$

(d) Discussion. This example is for illustration of the procedure only. Actually, when the service live load is increased from 60 psf in the original flat plate design example to 140 psf, the slab thickness of $5\frac{1}{2}$ in. would have to be increased if the spans are not reduced. The requirement of ACI-11.12.6.3 is expected to be more controlling for the shearhead in the exterior column. ◄

## ▶ 16.19 OPENINGS AND CORNER CONNECTIONS IN FLAT SLABS

When openings and corner connections are present in flat slabs floors, designers must make sure that adequate provisions are made for them. The ASCE-ACI Joint Task Committee [16.83] has summarized available information. Tests by Roll, Zaidi, Sabnis, and Chuang [16.79] have provided additional data for treating openings, while Zaghlool and de Paiva [16.107, 16.108] have provided data for corner connections.

ACI-13.4.1 first prescribes in general that openings of any size may be provided if it can be shown by analysis that all strength and serviceability conditions, including the limits on the deflections, are satisfied. However, in common situations (ACI-13.4.2) a special analysis need not be made for slab systems not having beams when (1) openings are within the middle half of the span in each direction, provided the total amount of reinforcement required for the panel without the opening is maintained; (2) openings in the area common to two column strips do not interrupt more than one-eighth of the column strip width in either span, and the equivalent of reinforcement interrupted is added on all sides of the openings; (3) openings in the area common to one column strip and one middle strip do not interrupt more than one-fourth of the reinforcement in either strip, and the equivalent of reinforcement interrupted is added on all sides of the openings.

In regard to nominal shear strength in two-way action, the critical section for slabs without shearhead is not to include that part of the periphery which is enclosed by radial projections of the openings to the center of the column (ACI-11.12.5). For slabs with shearhead, the critical periphery is to be reduced only by one-half of what is cut away by the radial lines from the center of the column to the edges of the opening. Some critical sections with cutaways by openings are shown in Fig. 16.19.1.

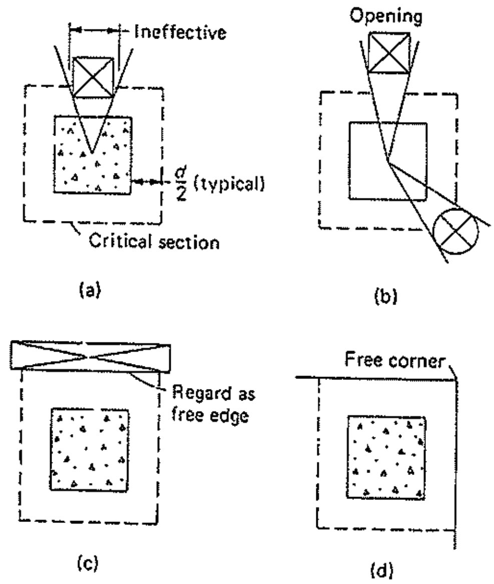



**Figure 16.19.1** Effect of openings and free edges on critical periphery of two-way shear action.
(From ACI Commentary-R11.12.5.)

## 16.20 EQUIVALENT FRAME METHOD FOR GRAVITY AND LATERAL LOAD ANALYSIS

For gravity load analysis the "equivalent frame method" prescribed by the ACI Code differs from the "direct design method" *only* in the way by which the longitudinal moments along the spans of the equivalent rigid frame (as defined in Section 16.2) are obtained. In either method of analysis, the transverse distribution of the longitudinal moments may be carried out as described in Section 16.12, except in the case of the two-way slab *supported on beams*, where the beams must be sufficiently stiff (limitation No. 6 of the direct design method discussed in Section 16.7) (ACI-13.7.7.5) to serve as boundary supports for the slab.

When lateral (wind) load needs to be considered, an elastic analysis must be made of the structure under lateral load and the results combined with those due to gravity load. Consistent with the tradition under ACI Code of using an equivalent frame for gravity load analysis, a logical extension is to use an equivalent frame approach to lateral load analysis. ACI-13.5.1.2 does not prescribe an equivalent frame method for lateral load analysis, but only requires taking into account "effects of cracking and reinforcement on stiffness of frame members." It is suggested in ACI Commentary-R13.5.1.2 that one-quarter to one-half of the uncracked bending stiffness may be appropriate.

The maximum positive moments (and reversals) within the span and negative moments at the supports should be obtained for various combinations of gravity load patterns with lateral load as indicated by ACI-9.2. When the equivalent frame method *is* used for gravity load analysis of two-way floor systems meeting the limitations of the direct design method, the computed moments may be reduced such that the absolute sum of the positive and average negative moments is at least equal to the total static moment $w_u L_2 L_n^2/8$ (ACI-13.7.7.4).

The elastic analysis for the equivalent rigid frame is treated separately in Chapter 17. In this section, considerations are given to the determination of the flexural stiffnesses of the columns and of the slab-beam within the width of the equivalent rigid frame, the torsional stiffness of the transverse beam, and the fixed-end moments due to gravity load. These values are the required input data for the analysis procedure presented in Chapter 17.

## Torsional Stiffness of the Transverse Beam

The structure enclosed between the two parallel centerlines of two adjacent panels in a multistory two-way floor system is a three-dimensional structure. The equivalent rigid frame described in Section 16.2 approximates the three-dimensional structure by a series of two-dimensional ones. But the columns stand on or provide support for only a small portion of the width of the equivalent rigid frame. Hence, either the column stiffness has to be spread thinly over the entire width (denoted by $L_2$) of the equivalent rigid frame, or the slab-beam has to be shrunk to the narrow transverse width (denoted by $c_2$) of the columns. Corley, Sozen, and Siess [16.26] first developed the idea of attaching a torsional member to the column (a cutaway from the three-dimensional structure) in the transverse direction and in essence shifting the flexural stiffness of the slab-beam to the end of the torsional member away from the columns (see Fig. 17.3.1). Thus the effectiveness of the column to restrain the ends of the slab-beam is reduced; hence the name of a less effective "equivalent column." As shown by Fig. 16.20.1, under gravity loading, the restraint at the column is more like a fixed end to the slab-beam but the restraint away from the column tends to approach that of a simple support.

Corley and Jirsa [16.27] developed a formula for the torsional stiffness $K_t$ of the attached torsional member so that results of the equivalent frame analysis are close to those of tests, as follows [ACI Commentary-R13.7.5]:

$$K_t = \sum \frac{9 E_{cs} C}{L_2 \left(1 - \dfrac{c_2}{L_2}\right)^3} \left(\frac{I_{sb}}{I_s}\right) \tag{16.20.1}$$

in which

$C$ = torsional constant of the transverse beam (see Section 16.11)
$E_{cs}$ = modulus of elasticity of slab concrete
$I_s$ = moment of inertia of slab over width of equivalent frame
$I_{sb}$ = moment of inertia of entire T-section (if so) within the width of the equivalent rigid frame
$L_2$ = span of member subject to torsion
$c_1, c_2$ = defined as in Fig. 16.20.1

The summation sign is for the transverse spans (denoted by $L_2$) on each side of the column.

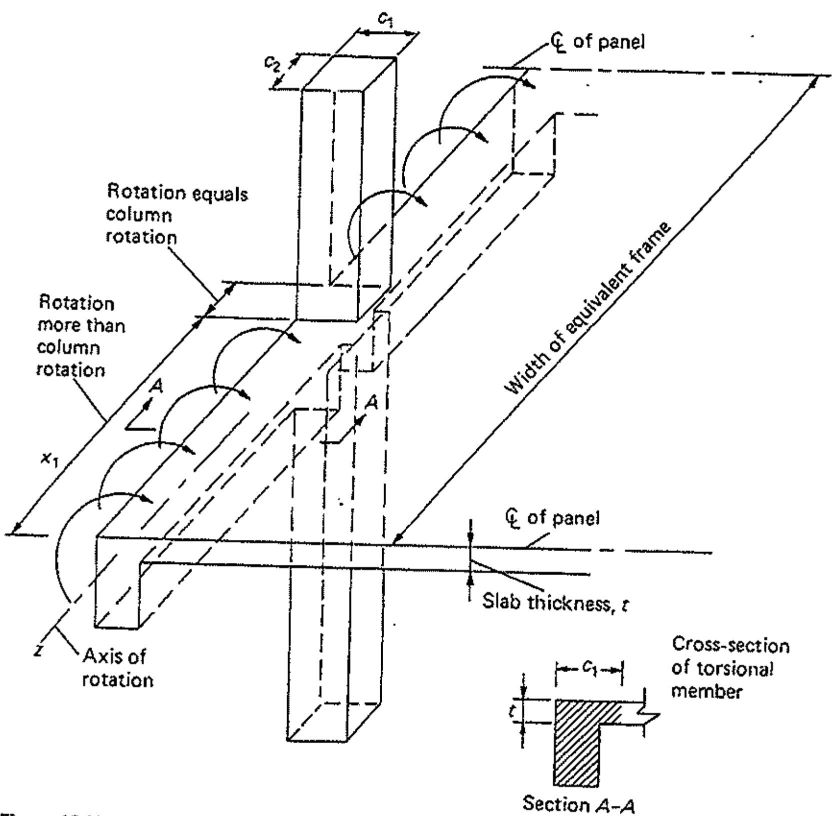



**Figure 16.20.1** Attached torsional member for the columns.

The parameter $K_t$ is the most influential parameter to relate results of the theoretical elastic analysis to those of tests for gravity (or lateral) load. As test results become more available, especially for lateral load, the coefficient "9" in Eq. (16.20.1) might be adjusted in the future.

## Treatment of Flexural Element Having Variable Moment of Inertia

The flexural stiffness of a flexural element $ij$ having variable moment of inertia can be expressed by two near-end stiffnesses $S_{ii}$ and $S_{jj}$ and a cross stiffness $S_{ij}$. In applying the moment distribution method, the carry-over factor from $i$ to $j$ is $S_{ji}/S_{ii}$ and the carry-over factor from $j$ to $i$ is $S_{ij}/S_{jj}$. In applying the matrix displacement method the element stiffness matrix $[S]$ is

$$[S] = \begin{bmatrix} S_{ii} & S_{ij} \\ S_{ji} & S_{jj} \end{bmatrix} \quad \text{(in which } S_{ij} = S_{ji}\text{)} \tag{16.20.2}$$

In applying the slope deflection method [16.11],

$$\begin{aligned} M_i &= M_{0i} + S_{ii}\theta_i + S_{ij}\theta_j - (S_{ii} + S_{ij})R \\ M_j &= M_{0j} + S_{ji}\theta_i + S_{jj}\theta_j - (S_{ii} + S_{ij})R \end{aligned} \tag{16.20.3}$$

in which $M_{0i}$ and $M_{0j}$ are the fixed-end moments, $\theta_i$ and $\theta_j$ are the slopes at ends $i$ and $j$ of the flexural element, and $R$ is the clockwise rotation of the element axis.

The column analogy method may be the best way to compute the fixed-end moments, the near-end stiffnesses $S_{ii}$ and $S_{jj}$, and the cross stiffness $S_{ij}$ or $S_{ji}$ of a flexural element having variable moment of inertia, thus avoiding using the flexibility analysis for pin-end rotations due to applied loads or unit values of end moments. Wang [16.11] has presented a description of this method, as well as the elastic analysis of rigid frames by moment distribution, slope deflection, and matrix displacement methods.

There are tables available for fixed-end moments and flexural stiffness values for slabs having various combinations of column widths, column capitals, and drop panels [16.9, 16.10, 16.139]

## Flexural Stiffness of Columns

ACI Commentary-R13.7.4 states that the height of the column is to be measured from middepth of slab above to middepth of slab below, as shown in Fig. 16.20.2. The moment of inertia is to be taken as infinite from the top to the bottom of the slab-beam at the joint (ACI-13.7.4.3). In flat slabs having column capitals, the authors suggest that the $1/I$ value be assumed to vary linearly from zero at the top to its value based on gross cross-section at the bottom of the capital.

## Flexural Stiffness of Slab–Beam

ACI-13.7.3.3 states that the moment of inertia of slab-beams from center of column to face of column, bracket, or capital shall be assumed equal to that of the slab-beam at face of column, bracket, or capital divided by the quantity $(1 - c_2/L_2)^2$, where $c_2$ and $L_2$ are measured transverse to the direction of the span in which the moments are being computed. Although ACI-13.7.3.1 permits the use of gross concrete area for computation of the moment of inertia in gravity load analysis, at the same time ACI-13.5.1.2 requires taking into account the effects of cracking and reinforcement in lateral load analysis. When computers are used, loadings for different gravity load patterns to be combined with lateral load can be listed together in an input load matrix; thus, it would be convenient to use only one set of assumptions for flexural stiffness properties, especially in the final analysis after the finished design.

There is not definitive agreement on what constitutes the appropriate assumptions for stiffness, either for gravity load analysis or lateral load analysis. For gravity load analysis, the use of gross section is reasonable because it is the simplest assumption and the results are acceptable. For lateral load analysis, particularly for the unbraced frame where the entire lateral resistance is provided by the flexural stiffness of the slab-beams and columns, the use of gross section for stiffness overemphasizes the resistance to lateral

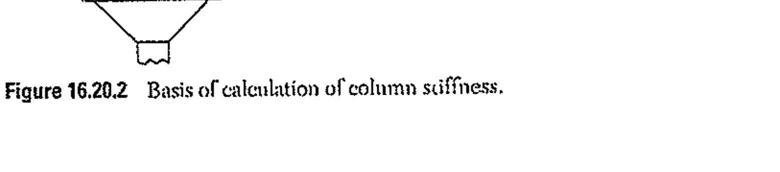



**Figure 16.20.2** Basis of calculation of column stiffness.

loads. In Chapter 17, devoted to the analysis of two-way floor systems, several analytical models are presented for possible use in lateral load analysis, or in combined gravity and lateral load analysis. For an extended discussion of lateral load analysis, the reader should consult the study by Vanderbilt [16.32]. Perhaps the acceptance of an equivalent beam or reduced beam method of analysis would make possible the use of gross section in lateral load analysis. In 1995 Luo and Durrani [16.140, 16.141] developed formulas for the effective slab width as functions of column and slab aspect ratios and the magnitude of the gravity load; the proposed model was based on coordination with 40 previous test results.

### Arrangement of Live Load

When there is a definitely known load pattern, of course, analysis should be made for it (ACI-13.7.6.1). When service live load does not exceed $\frac{3}{4}$ of the service dead load, analysis needs to be made only for full factored dead and live load on all spans (ACI-13.7.6.2). When load patterns in accordance with influence line concepts are used, only $\frac{3}{4}$ of the full factored live load needs to be used (ACI-13.7.6.3); however, factored moments used in design should not be less than those due to full factored dead and live loads on all panels (ACI-13.7.6.4).

### Reduction of Negative Moments Obtained at Column Centerlines from Structural Analysis

Negative moments obtained at interior column centerlines may be reduced to the face of rectilinear or equivalent square (for circular or polygon-shaped supports) supports but not greater than $0.175L_1$ from the column centerline (ACI-13.7.7.1). For exterior columns, having capitals or brackets, reduction of negative moments can be made only to a section no greater than halfway between the face of column and edge of the capital or bracket (ACI-13.7.7.2).

### Deflections

When the deflection must be calculated for a two-way slab system, the ACI Code (ACI-9.5.3.4) provides little guidance other than that one should take into account "size and shape of the panel, conditions of support, and nature of restraints at panel edges." The effective moment of inertia $I_e$ [Eq. (14.4.1)] is required to be used in such calculations. Although a number of techniques have been proposed [16.52–16.71], adoption of the equivalent frame concept seems to have the most promise of being relatively simple to apply and giving reasonable results. This equivalent frame application has been developed by Nilson and Walters [16.53] for essentially uncracked systems and extended by Kripanarayanan and Branson [16.55] for partially cracked load ranges. More recently, Scanlon et al. [16.58, 16.63, 16.65, 16.66, 16.68, and 16.70] have treated the subject in detail.

It is noted that the equivalent frame method was originally derived to be used with the method of moment distribution. The method, however, can be also be used with other analysis procedures (e.g., matrix displacement method [16.11]) and standard frame analysis computer programs by specifying appropriate values for the stiffness of the slab-beam, column, and torsional elements as discussed in Chapter 17.

**▶ EXAMPLE 16.20.1**

Assuming the equivalent frame method is to be applied to the two-way slab (with beams) design example described in Section 16.3, obtain the stiffnesses necessary for the analysis of the equivalent rigid frames *A*, *B*, *C*, and *D* as shown by the notations in Fig. 16.3.5. Also obtain the fixed-end moments for gravity load and the carry-over factors *COF* to be used with the method of moment distribution.

**SOLUTION** (a) Compute flexure properties of slab–beam. The variations in the moment of inertia of the slab–beam in the long and short directions are shown in Fig. 16.20.3. For

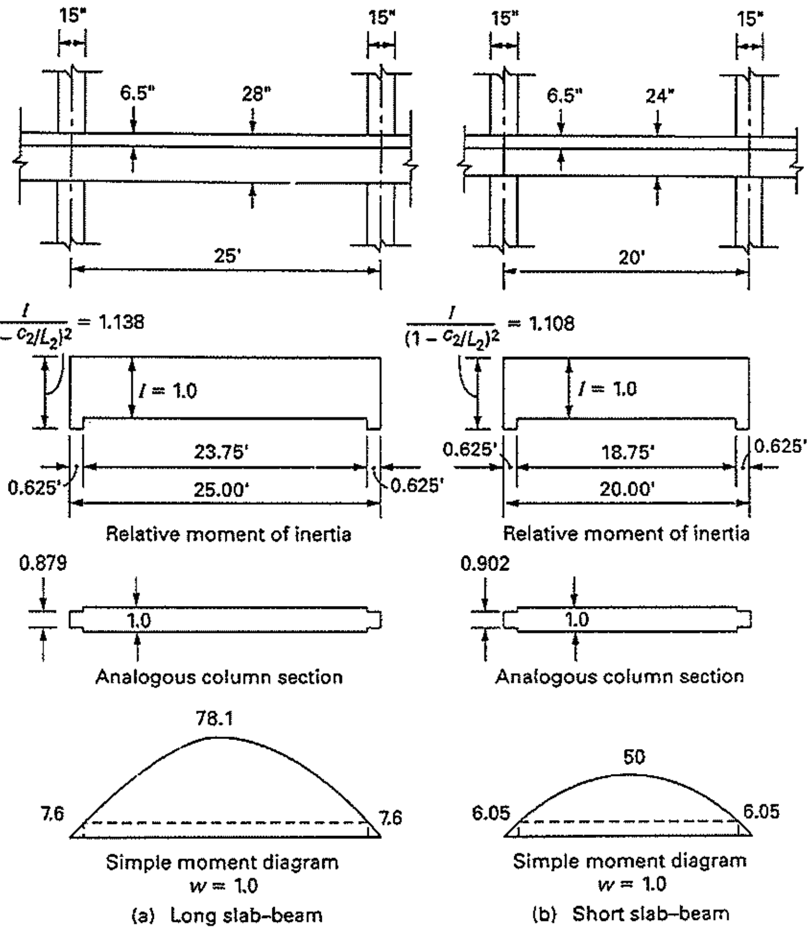



**Figure 16.20.3** Flexure properties of slab–beam in two-way slab (with beams) design example.

the long slab–beam, the ratio of moment of inertia between the center and the face of the column to the moments of inertia of the rest of the span is $1.0/(1 - 15/240)^2 = 1.138$; and it is $1.0/(1 - 15/300)^2 = 1.108$ for the short slab–beam (ACI-13.7.3.3). The stiffness $K$, carry-over factor COF, and fixed-end moment FEM coefficients may be computed by the column analogy method [16.11]. The variation in the width of the analogous column section is $1/I$, as shown in Fig. 16.20.3.

For the long direction, the area and moment of inertia of the analogous column section [see Fig. 16.20.3(a)] are

$$A = 23.75 + 2(0.879)(0.625) = 23.75 + 1.10 = 24.85$$

$$I = \tfrac{1}{12}(23.75)^3 + 1.10(12.1875)^2 = 1116 + 163 = 1279$$

$$\text{stiffness factor } s_{ii} = L\left(\frac{1}{A} + \frac{Mc}{I}\right)$$

$$s_{ii} = \frac{25}{24.85} + \frac{25(12.5)^2}{1279} = 1.006 + 3.054 = 4.06$$

$$\text{stiffness factor } s_{ij} = L\left(\frac{Mc}{I} - \frac{1}{A}\right)$$

$$s_{ij} = -(1.006 - 3.054) = 2.05$$

$$\text{COF} = \frac{2.05}{4.06} = 0.505$$

load on analogous column for uniform load ($w = 1.0$)

$$= \tfrac{2}{3}(78.1 - 7.6)(23.75) + 7.6(23.75) + 0.879(7.6)(0.625)$$
$$= 1116 + 180 + 4 = 1300$$

$$\text{FEM coefficient} = \frac{1300}{24.85L_1^2} = \frac{1300}{24.85(625)} = 0.084$$

For the short direction, referring to Fig. 16.20.3(b),

$$A = 18.75 + 2(0.902)(0.625) = 18.75 + 1.13 = 19.88$$
$$I = \tfrac{1}{12}(18.75)^3 + 1.13(9.6875)^2 = 549 + 106 = 655$$
$$s_{ii} = \frac{20}{19.88} + \frac{20(10)^2}{655} = 1.006 + 3.053 = 4.06$$
$$s_{ij} = -(1.006 - 3.053) = 2.05$$
$$\text{COF} = \frac{2.05}{4.06} = 0.505$$

load on analogous column for uniform load ($w = 1.0$)

$$= \tfrac{2}{3}(50 - 6.05)(18.75) + 6.05(18.75) + 0.902(6.05)(0.625)$$
$$= 549 + 113 + 3 = 665$$

$$\text{FEM coefficient} = \frac{665}{19.88L_1^2} = \frac{665}{19.88(400)} = 0.084$$

The flexural stiffness of the slab–beams in frames $A$, $B$, $C$, and $D$ are, using the $I_{sb}$ values shown in Fig. 16.20.4,

$$\text{Frame } A, \quad K_{sb} = \frac{4.06E(66{,}540)}{300} = 901E$$
$$\text{Frame } B, \quad K_{sb} = \frac{4.06E(57{,}730)}{300} = 781E$$
$$\text{Frame } C, \quad K_{sb} = \frac{4.06E(39{,}530)}{240} = 669E$$
$$\text{Frame } D, \quad K_{sb} = \frac{4.06E(33{,}980)}{240} = 575E$$

Note that the moment of inertia values used above are based on the gross cross-sections as shown in Fig. 16.20.4, an acceptable procedure for gravity load analysis. These stiffness values are likely to be too high for lateral load analysis, since cracking reduces flexural stiffness. ACI-13.5.1.2 states that for lateral load analysis, effects of cracking and reinforcement must be taken into account.

(b) Compute flexure properties of columns. The variations in the moment of inertia of the column section in the long and short directions are shown in Fig. 16.20.5. The stiffness coefficients and carry-over factors may be computed by the column analogy method.

For the long direction, referring to Fig. 16.20.5(a),

$$A = 9.67, \quad I = \tfrac{1}{12}(9.67)^3 = 75.3$$
$$s_{TT} = \frac{12}{9.67} + \frac{12(6.90)^2}{75.3} = 1.24 + 7.59 = 8.83$$

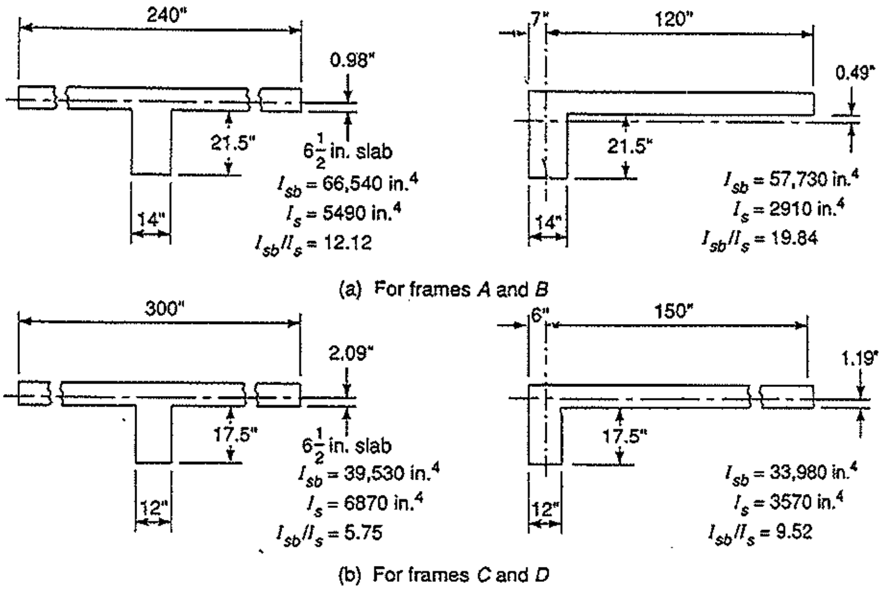



**Figure 16.20.4** Slab–beam cross-sections in two-way slab (with beams) design example.

$$s_{BB} = \frac{12}{9.67} + \frac{12(5.10)^2}{75.3} = 1.24 + 4.15 = 5.39$$

$$s_{TB} = s_{BT} = -\left[\frac{12}{9.67} - \frac{12(6.90)(5.10)}{75.3}\right] = -1.24 + 5.61 = 4.37$$

$$(\text{COF})_{TB} = \frac{4.37}{8.83} = 0.495$$

$$(\text{COF})_{BT} = \frac{4.37}{5.39} = 0.811$$

$$\text{stiffness at top, } K_{cT} = s_{TT}\frac{EI}{L} = \frac{8.83E(15)^4/12}{144} = 259E$$

$$\text{stiffness at bottom, } K_{cB} = s_{BB}\frac{EI}{L} = \frac{5.39E(15)^4/12}{144} = 158E$$

For the short direction, referring to Fig. 16.20.5(b),

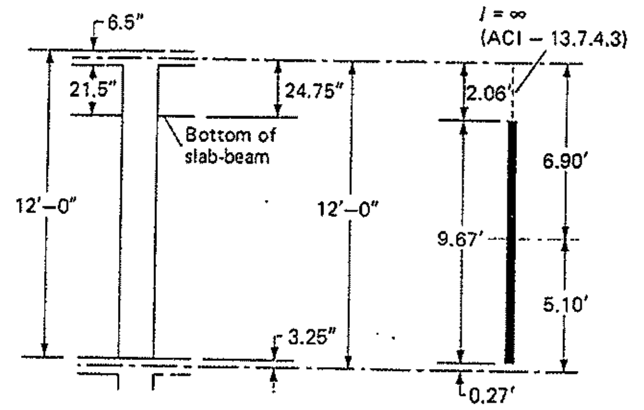
$$A = 10.00, \quad I = \tfrac{1}{12}(10)^3 = 83.3$$

$$s_{TT} = \frac{12}{10} + \frac{12(6.73)^2}{83.3} = 1.20 + 6.53 = 7.73$$

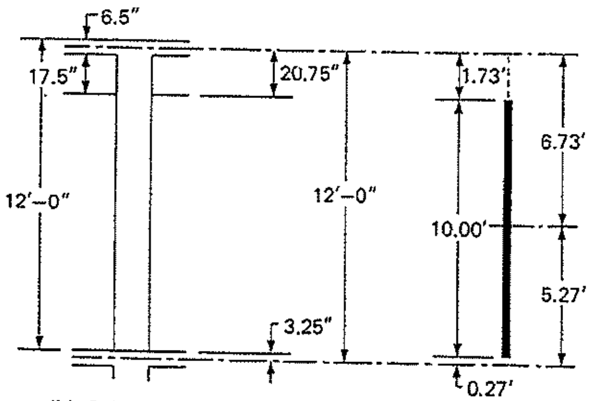
$$s_{BB} = \frac{12}{10} + \frac{12(5.27)^2}{83.3} = 1.20 + 4.00 = 5.20$$

$$s_{TB} = s_{BT} = -\left[\frac{12}{10} - \frac{12(6.73)(5.27)}{83.3}\right] = -1.20 + 5.11 = 3.91$$

$$(\text{COF})_{TB} = \frac{3.91}{7.73} = 0.506$$




(a) Column section in long direction (for frames *A* and *B*)




(b) Column section in short direction (for frames *C* and *D*)

**Figure 16.20.5** Flexure properties of columns in two-way slab (with beams) design example.

$$(\text{COF})_{BT} = \frac{3.91}{5.20} = 0.752$$

$$\text{stiffness at top, } K_{cT} = \frac{7.73E(15)^4/12}{144} = 226E$$

$$\text{stiffness at bottom, } K_{cB} = \frac{5.20(15)^4/12}{144} = 152E$$

(c) Compute torsional stiffness of transverse torsional members. The torsional constants $C$ for the transverse members shown in Fig. 16.20.6 are taken from Example 16.11.1. The values for the ratio of $I_{sb}$ to $I_s$ needed to increase the torsional stiffness $K_t$ [Eq. (16.20.1)] for each direction are shown in Fig. 16.20.4.

For Frame *A*, using $I_{sb}/I_s = 12.12$ for $14 \times 21.5$ projection below $240 \times 6.5$ slab,

$$\text{exterior } K_t = \frac{18E(10,700)}{240(1 - 15/240)^3}(12.12) = 974E(12.12) = 11,800E$$

$$\text{interior } K_t = \frac{18E(11,930)}{240(1 - 15/240)^3}(12.12) = 1086E(12.12) = 13,200E$$

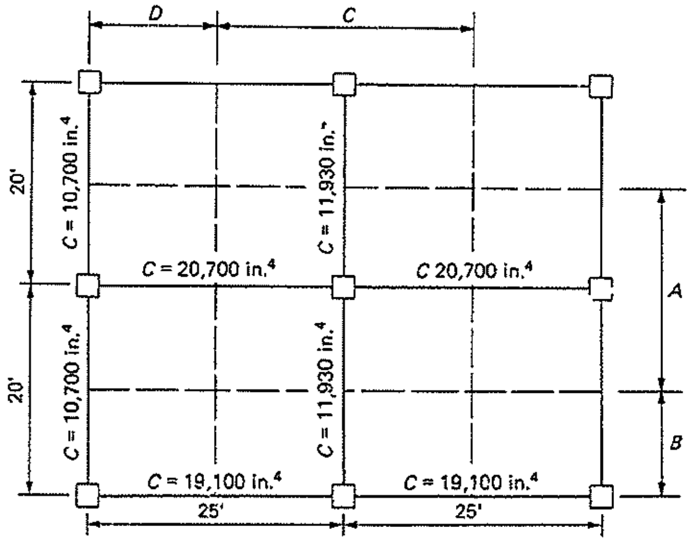



**Figure 16.20.6** Torsional constants in two-way slab (with beams) design example (from Example 16.11.1).

For Frame $B$, using $I_{sb}/I_s = 19.84$ for 14 × 21.5 projection below 127 × 6.5 slab,

$$\text{exterior } K_t = 487E(19.84) = 9660E$$
$$\text{interior } K_t = 543E(19.84) = 10{,}800E$$

For Frame $C$, using $I_{sb}/I_s = 5.75$ for 12 × 17.5 projection below 300 × 6.5 slab,

$$\text{exterior } K_t = \frac{18E(19{,}100)}{300(1 - 15/300)^3}(5.75) = 1340E(5.75) = 7700E$$

$$\text{interior } K_t = \frac{18E(20{,}700)}{300(1 - 15/300)^3}(5.75) = 1450E(5.75) = 8340E$$

For Frame $D$, using $I_{sb}/I_s = 9.52$ for 12 × 17.5 projection below 156 × 6.5 slab,

$$\text{exterior } K_t = 670E(9.52) = 6380E$$
$$\text{interior } K_t = 725E(9.52) = 6900E$$

◀

**EXAMPLE 16.20.2**

Assuming the equivalent frame method is to be applied to the flat slab design example described in Section 16.3, obtain the stiffnesses and carry-over factors necessary for the analysis of equivalent rigid Frame $A$ in the long direction. Also obtain the fixed-end moments for gravity load.

**SOLUTION** (a) Compute flexure properties of slab strip. The stiffnesses, carry-over factors, and fixed-end moments may be determined by various analysis methods. The column analogy method [16.11] is used in this example. Simmonds and Misic [16.9] have provided design aids to meet the ACI Code assumptions of the "equivalent frame method."

The variation in the moment of inertia along the interior span of the slab strip is shown in Fig. 16.20.7(a). Taking the moment of inertia through the $7\frac{1}{2}$-in. slab as the reference value of 1, the moment of inertia through the drop, where there is a T-section of a 240 × 7.5 in. flange and an 84 × 3 in. web, is 1.745. The moment of inertia between

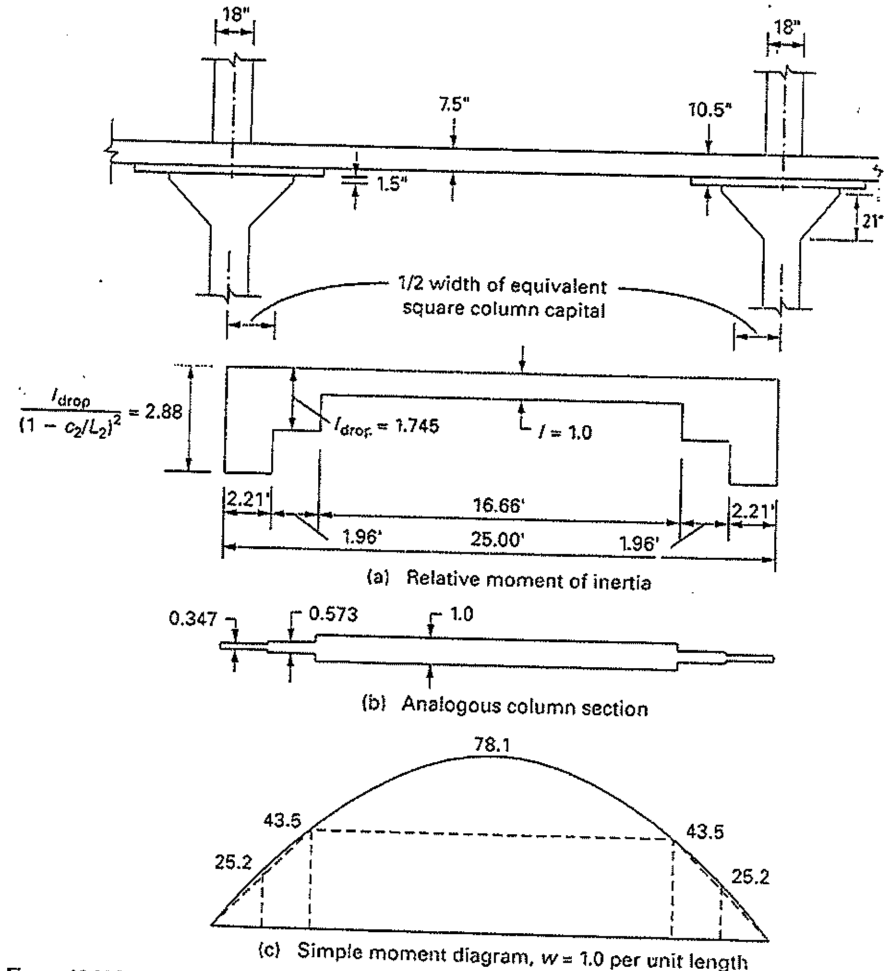



**Figure 16.20.7** Flexure properties of slab strip in flat slab design example.

the column centerline and the face of the equivalent square column capital is $1.745/(1 - 4.43/20)^2 = 1.745/0.606 = 2.88$.

The variation in the width of the analogous column section is $1/I$, which is shown in Fig. 16.20.7(b). The area of the analogous column section is

$$\begin{aligned} A &= 16.66 + 2(0.573)(1.96) + 2(0.347)(2.21) \\ &= 16.66 + 2.25 + 1.53 = 20.44 \end{aligned}$$

The moment of inertia about the midspan, neglecting the moments of inertia of the short segments about their own centroidal axes, is

$$I = \tfrac{1}{12}(16.66)^3 + 2.25(9.31)^2 + 1.53(11.40)^2 = 385 + 195 + 199 = 779$$

$$\text{stiffness factor } s_{ii} = L\left(\frac{1}{A} + \frac{Mc}{I}\right) = 25\left[\frac{1}{20.44} + \frac{12.5(12.5)}{779}\right]$$

$$s_{ii} = 1.22 + 5.01 = 6.23$$

$$s_{ij} = -1.22 + 5.01 = 3.79$$

$$\text{COF} = \frac{3.79}{6.23} = 0.608$$

$$\text{stiffness } K \text{ at end of 20-ft-wide slab strip} = \frac{6.23E(7.5)^3/12}{300}(240) = 175E$$

The load on the analogous column is equal to the summation of the product of the width of the analogous column section and the area of the simple beam moment diagram of Fig. 16.20.7(c). Considering the moment areas over the short segments as being trapezoidal, the load on the analogous column is

$$\begin{aligned} P &= \tfrac{2}{3}(78.1 - 43.5)(16.66) + 43.5(16.66) + 2(0.573)(\tfrac{1}{2})(43.5 + 25.2)(1.96) \\ &\quad + 2(0.347)(\tfrac{1}{2})(25.2)(2.21) \\ &= 384 + 725 + 77 + 19 = 1205 \end{aligned}$$

$$\text{FEM coefficient} = \frac{P}{AL^2} = \frac{1205}{20.44(25)^2} = 0.0943$$

Since the edge column capital is almost equal in size to the equivalent square of the interior column capital, the FEM coefficient, stiffness, and carry-over factor obtained above for the interior span will also be used for the exterior span.

**(b)** Compute flexure properties of columns. For the interior column, the length is measured between the centerlines of slab thickness, as shown in Fig. 16.20.8(a). The moment of inertia is assumed to be infinite from the top of the slab to the bottom of the drop panel, and then the $1/I$ value is taken to vary linearly to the base of the column capital.

The $0.188L$ of the analogous column representing the column capital is divided into four parts, $\Delta L = 0.188L/4$, with $1/I$ of 1/8, 3/8, 5/8, and 7/8.

$$A = 0.725L + \left(\frac{1}{8} + \frac{3}{8} + \frac{5}{8} + \frac{7}{8}\right)\left(\frac{0.188L}{4}\right) = 0.819L$$

$$\begin{aligned} \textstyle\sum Ay \text{ from top} &= \frac{0.188L}{4}\left[\frac{1}{8}(0.080L) + \frac{3}{8}(0.126L) + \frac{5}{8}(0.174L) + \frac{7}{8}(0.220L)\right] \\ &\quad + 0.725L(0.6065L) = 0.4566L^2 \end{aligned}$$

$$\bar{y} \text{ from top} = \frac{0.4566L^2}{0.819L} = 0.558L$$

$$\begin{aligned} I &= \frac{1}{12}(0.725L)^3 + 0.725L(0.0485L)^2 \\ &\quad + \frac{0.188L}{4}\left[\frac{1}{8}(0.478L)^2 + \frac{3}{8}(0.432L)^2 + \frac{5}{8}(0.384L)^2 + \frac{7}{8}(0.338L)^2\right] \\ &= 0.047L^3 \end{aligned}$$

The stiffness factors at the top and bottom are

$$s_{TT} = \frac{1}{0.819} + \frac{(0.558)^2}{0.0471} = 1.22 + 6.61 = 7.83$$

$$s_{BB} = \frac{1}{0.819} + \frac{(0.442)^2}{0.0471} = 1.22 + 4.15 = 5.37$$

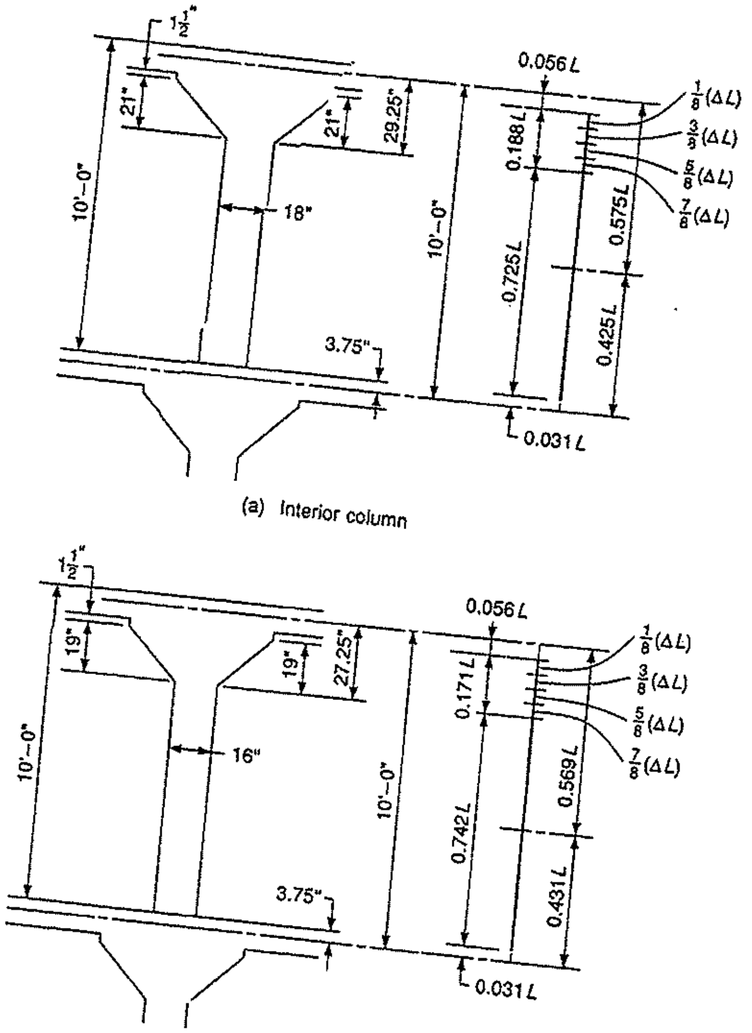



**Figure 16.20.8** Flexure properties of columns in flat slab design example.

for which the stiffnesses are

$$K_{cT} = \frac{s_{TT}EI}{L} = \frac{7.83E\pi(9)^4/4}{120} = 336E$$

$$K_{cB} = \frac{s_{BB}EI}{L} = \frac{5.37E\pi(9)^4/4}{120} = 230E$$

The carry-over factors are

$$(\text{COF})_{TB} = \frac{0.558(0.442)/(0.0471) - 1.22}{7.83} = \frac{4.02}{7.83} = 0.513$$

$$(\text{COF})_{BT} = \frac{4.02}{5.37} = 0.749$$

For the exterior column [Fig. 16.20.8(b)],

$$A = 0.742L + \left(\frac{1}{8} + \frac{3}{8} + \frac{5}{8} + \frac{7}{8}\right)\left(\frac{0.171L}{4}\right) = 0.827L$$

$$\sum Ay \text{ from top} = \frac{0.171L}{4}\left[\frac{1}{8}(0.077L) + \frac{3}{8}(0.120L) + \frac{5}{8}(0.163L) + \frac{7}{8}(0.206L)\right]$$
$$+ 0.742L(0.598L) = 0.4581L^2$$

$$\bar{y} \text{ from top} = \frac{0.4581L^2}{0.827L} = 0.555L$$

$$I = \frac{1}{12}(0.742L)^3 + 0.742L(0.044L)^2$$
$$+ \frac{0.171L}{4}\left[\frac{1}{8}(0.477L)^2 + \frac{3}{8}(0.434L)^2 + \frac{5}{8}(0.391L)^2 + \frac{7}{8}(0.348L)^2\right]$$
$$= 0.0482L^3$$

$$s_{TT} = \frac{1}{0.827} + \frac{(0.554)^2}{0.0482} = 1.21 + 6.37 = 7.58$$

$$s_{BB} = \frac{1}{0.827} + \frac{(0.446)^2}{0.0482} = 1.21 + 4.13 = 5.34$$

$$K_{cT} = \frac{s_{TT}EI}{L} = \frac{7.58E(16)^4/12}{120} = 345E$$

$$K_{cB} = \frac{s_{BB}EI}{L} = \frac{5.34E(16)^4/12}{120} = 243E$$

The carry-over factors are

$$(\text{COF})_{TB} = \frac{0.554(0.446)/0.0482 - 1.21}{7.58} = \frac{3.92}{7.58} = 0.517$$

$$(\text{COF})_{BT} = \frac{3.92}{5.34} = 0.734$$

(c) Compute torsional stiffness of transverse torsional members. From Example 16.11.2,

$$C \text{ (edge beam)} = 18{,}500 \text{ in.}^4$$
$$C \text{ (interior beam)} = 9800 \text{ in.}^4$$

For the two members, one framing in from each side,

$$K_t(\text{edge}) = \frac{2(9E)C}{L_2\left(1 - \dfrac{c_2}{L_2}\right)^3} = \frac{2(9E)(18{,}500)}{240\left(1 - \dfrac{4.5}{20}\right)^3} = 2980E$$

$$K_t(\text{interior}) = \frac{2(9E)9800}{240\left(1 - \dfrac{4.43}{20}\right)^3} = 1560E$$

◀

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## Equivalent Frame Method

a) Basis of Analysis :

- derived with the assumption that the analysis would be done using the moment distribution method.
- For vertical loading, each floor with its columns may be analyzed separately, with the columns assumed to be fixed at the floors above and below.

b) Moment of Inertia :

- based on the concrete x-section
- starting with the moment of inertia at midspan, changes occur at:
  * edge of drop panels
  * edge of the column or capital

from the center of the column to the face of the column (or capital)

$$I_{slab} = I_{\text{slab at the face}} / (1 - c_2/l_2)^2$$

∴ moment of inertia varies in a stepwise manner

c) The equivalent column

Torsional deformation of the transverse beams reduces the effective flexural stiffness provided by the actual column

⇒ equivalent column is assumed with a stiffness less than that of the actual column, for which:

total flexibility of the equivalent column (inverse of stiffness) equals the sum of the flexibilities of the actual column and beam

$$\frac{1}{K_{ec}} = \frac{1}{\Sigma K_c} + \frac{1}{K_t}$$

In computing $K_c$, the moment of inertia of the actual column is assumed to be infinite from the top of the slab to the bottom of the slab beam
⇒ stiffness values are tabulated

$$K_t = \Sigma \frac{9 E_{cs} C}{l_2 (1 - c_2/l_2)^3} \quad \left( \times \frac{I_{sb}}{I_s} \right)$$

($I_{sb}$: slab beam) ($I_s$: slab)
if a beam is used in the longitudinal direction

summation = slab beams on both sides of the column

$l_2$ measured (transverse) center-to-center of the supports and may have different values in each of the summation terms

$I_{sb}$ = Moment of inertia of the <u>total</u> slab width including the beam drop

$I_s$ = moment of inertia of the slab alone

d. Moment analysis

if $LL \le \frac{3}{4} DL$ (service)

maximum moment assumed to occur at all critical sections when the full load acts everywhere (ACI 13.7.6)

⇒ otherwise, use cases of loading when using cases of loading, only ($\frac{3}{4}$ LL) factored is used for maximum load on a panel.

Note! factored moments must not be taken less than that obtained when full factored LL is placed on ALL panels.

e. Computers

General purpose programs based on the direct stiffness method.

Variable I values require nodal points (continuous joints) between sections with constant I.

Also, compute $K_{ec}$ for each columns, then compute I equivalent

Alternately, a 3-dimensional frame analysis may be used in which

## Transfer of Moment and Shear (Monolithic Joints) Flat Plates

$$M_u = M_{ub} + M_{uv}$$

$M_{ub}$: flexure transfer of $M_u$ — $M_{uv}$: shear transfer of $M_u$

(ACI 11.12.6)

$$M_{ub} = \gamma_f M_u = \left[\frac{1}{1+\frac{2}{3}\sqrt{\frac{b_1}{b_2}}}\right] M_u$$

$b_1$ = critical section dimension in the longitudinal direction

$= c_1 + d/2$ for exterior columns

$= c_1 + d$ for interior columns

$b_2$ = critical section dimension in the transverse direction

$= c_2 + d$

Simplifications: (ACI 13.5.3.3)

if, for exterior supports: $V_u \leq 0.75\,\phi V_c$ edge

$V_u \leq 0.50\,\phi V_c$ corner

⇒ neglect interaction between shear and moment

i.e., the full exterior moment is transfered through flexure ($\gamma_f = 1.0$)

therefore, consider punching shear only

for interior supports: $V_u \leq 0.4\,\phi V_c$

⇒ increase $\gamma_f$ by 25%

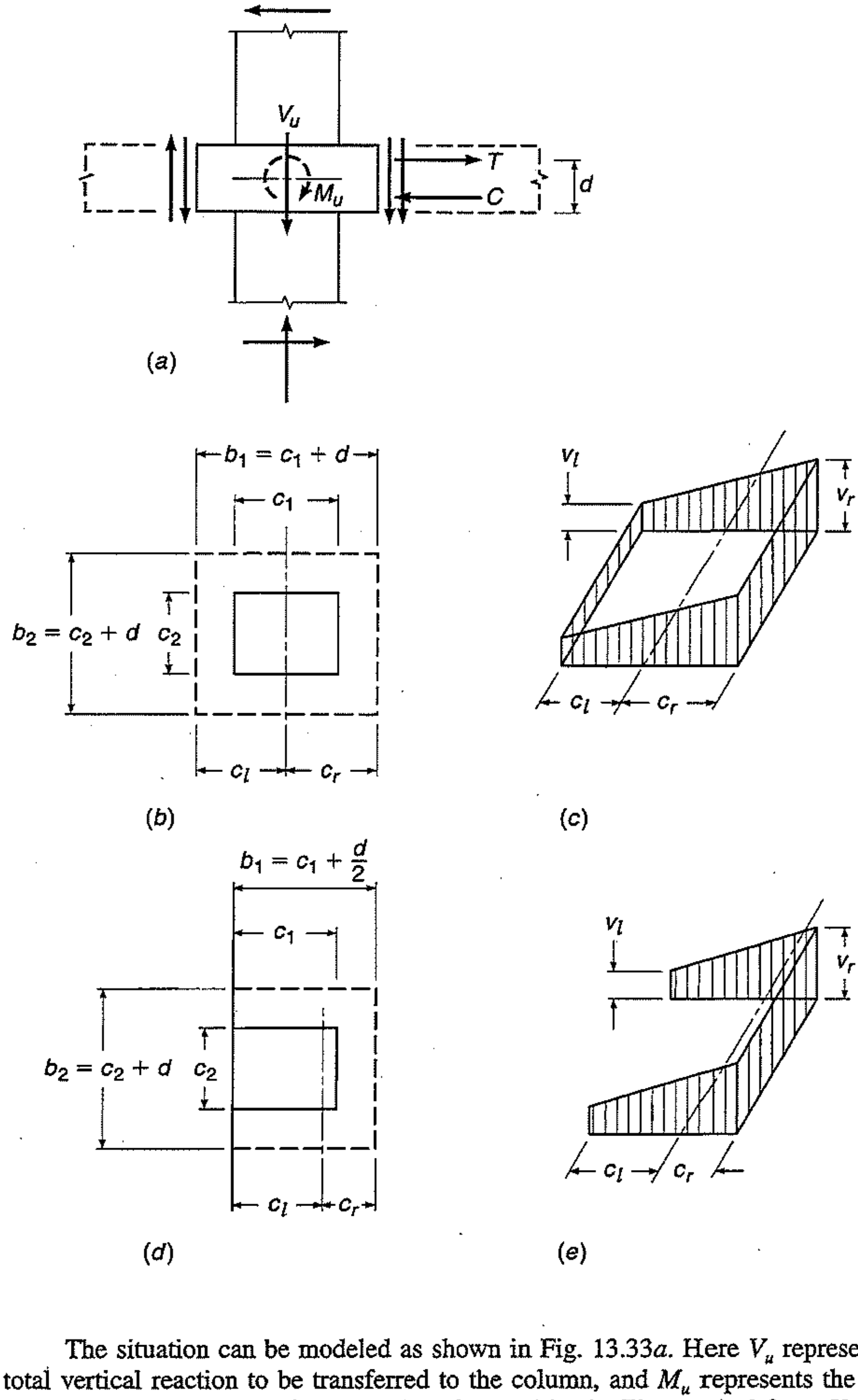



**FIGURE 13.33**
Transfer of moment from slab to column: (*a*) forces resulting from vertical load and unbalanced moment; (*b*) critical section for an interior column; (*c*) shear stress distribution for an interior column; (*d*) critical section for an edge column; (*e*) shear stress distribution for an edge column.

The situation can be modeled as shown in Fig. 13.33*a*. Here $V_u$ represents the total vertical reaction to be transferred to the column, and $M_u$ represents the unbalanced moment to be transferred, both at factored loads. The vertical force $V_u$ causes shear stress distributed more or less uniformly around the perimeter of the critical section as assumed earlier, represented by the inner pair of vertical arrows, acting downward. The unbalanced moment $M_u$ causes additional loading on the joint, represented by the outer pair of vertical arrows, which add to the shear stresses otherwise present on the right side, in the sketch, and subtract on the left side.

Tests indicate that for square columns about 60 percent of the unbalanced moment is transferred by flexure (forces $T$ and $C$ in Fig. 13.33*a*) and about 40 percent by shear stresses on the faces of the critical section (Ref. 13.24). For rectangular columns, it is reasonable to suppose that the portion transferred by flexure increases as the width of the critical section that resists the moment increases, i.e., as $c_2 + d$ becomes larger relative to $c_1 + d$ in Fig. 13.33*b*. According to ACI Code 13.5.3, the moment considered to be transferred by flexure is

$$M_{ub} = \gamma_f M_u \tag{13.16a}$$

where

$$\gamma_f = \frac{1}{1 + \frac{2}{3}\sqrt{b_1/b_2}} \tag{13.16b}$$

and $b_1$ = width of critical section for shear measured in direction of span for which moments are determined

$b_2$ = width of critical section for shear measured in direction perpendicular to $b_1$

The value of $\gamma_f$ may be modified if certain conditions are met: For unbalanced moments about an axis parallel to the edge of exterior supports, $\gamma_f$ may be increased to 1.0, provided that the factored shear $V_u$ at the edge support does not exceed $0.75\phi V_c$ or at a corner support does not exceed $0.5\phi V_c$. For unbalanced moments at interior supports and about an axis perpendicular to the edge at exterior supports, $\gamma_f$ may be increased up to 1.25 times the value in Eq. (13.16*b*), provided that $V_u \leq 0.4\phi V_c$. In all of these cases, the net tensile strain $\epsilon_t$ calculated for the section within $1.5h$ on either side of the column or column capital must be at least 0.010.

The moment assumed to be transferred by shear, by ACI Code 11.11.7, is

$$M_{ub} = (1 - \gamma_f)M_u = \gamma_v M_u \tag{13.16c}$$

It is seen that for a square column Eqs. (13.16*a*), (13.16*b*), and (13.16*c*) indicate that 60 percent of the unbalanced moment is transferred by flexure and 40 percent by shear, in accordance with the available data. If $b_2$ is very large relative to $b_1$, nearly all of the moment is transferred by flexure.

The moment $M_{ub}$ can be accommodated by concentrating a suitable fraction of the slab column-strip reinforcement near the column. According to ACI Code 13.5.3, this steel must be placed within a width between lines $1.5h$ on each side of the column or capital, where $h$ is the total thickness of the slab or drop panel.

The moment $M_{uv}$, together with the vertical reaction delivered to the column, causes shear stresses assumed to vary linearly with distance from the centroid of the critical section, as indicated for an interior column by Fig. 13.33*c*. The stresses can be calculated from

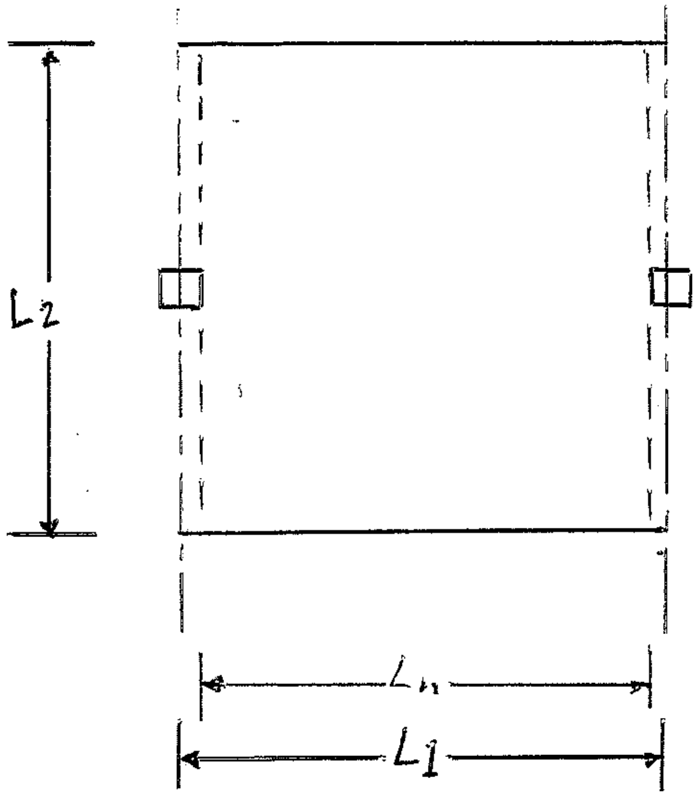
$$v_l = \frac{V_u}{A_c} - \frac{M_{uv}c_l}{J_c} \tag{13.17a}$$

$$v_r = \frac{V_u}{A_c} + \frac{M_{uv}c_r}{J_c} \tag{13.17b}$$

where $A_c$ = area of critical section = $2d[(c_1 + d) + (c_2 + d)]$

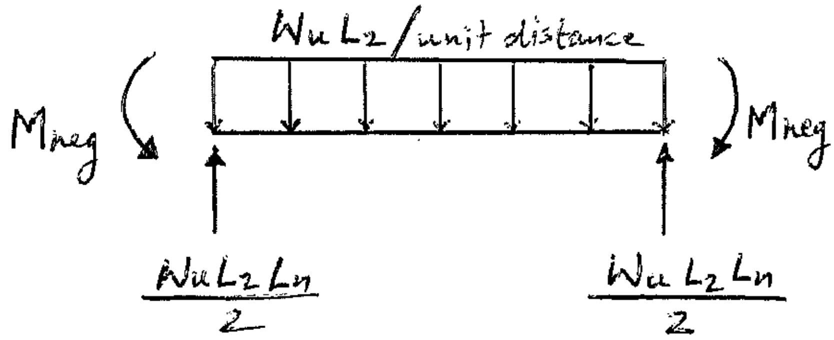
$c_l$, $c_r$ = distances from centroid of critical section to left and right faces of section, respectively

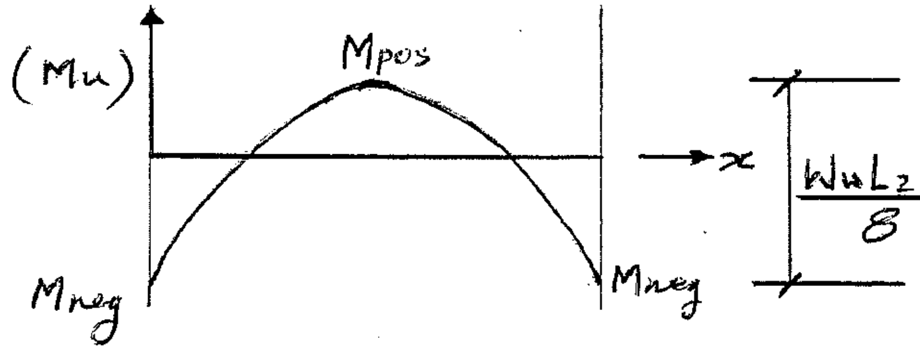
$J_c$ = property of critical section analogous to polar moment of inertia



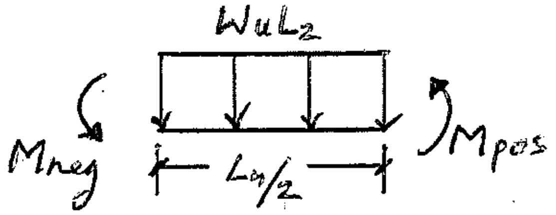

$W_u$ = factored floor load per unit area

$L_n \geq 0.65 L_1$




$\frac{W_u L_2 L_n^2}{8} = M_o$ Total factored static moment




$$M_{neg} + M_{pos} = W_u L_2 \left(\frac{L_n}{2}\right)\left(\frac{L_n}{4}\right)$$

$$= \frac{W_u L_2 L_n^2}{8}$$