

**(Lecture 25) Priority Queues (Heaps)**

A **priority queue** is a data structure that allows at least the following two operations:

- **Insert**: which does the obvious thing;
- **deleteMin (or deleteMax)**: which finds, returns, and removes the minimum (or maximum) element in the priority queue.

**Simple Implementations:**

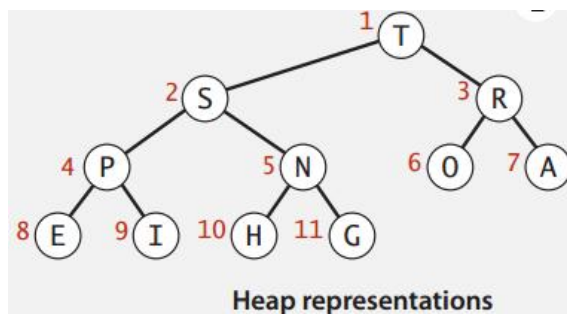
- **Unsorted Linked list**, performing insertions at the front in **O(1)** and traversing the list, which requires **O(N)** time, to delete the minimum/maximum.
- **Sorted Linked list**, performing insertions in **O(N)** and **O(1)** to delete the minimum/maximum.
- **Binary search tree**: this gives an **O(log N)** average running time for both operations.

**Binary Heap**

A heap is a binary tree that is completely filled, with the possible exception of the bottom level, which is filled from left to right.

Such a tree is known as a **complete binary tree**.

A complete binary tree of height  $h$  has between  $2^h$  and  $2^{h+1} - 1$  nodes.



As complete binary tree is so regular, it can be represented as an array:

$i$	0	1	2	3	4	5	6	7	8	9	10	11
$a[i]$	-	T	S	R	P	N	O	A	E	I	H	G

- Parent of node at  $i$  is at  $i/2$ .
- Children of node at  $i$  are at  $2i$  (left child) and  $2i+1$  (right child).

**Heap-order property:**

- In a **min heap**, for every node  $X$ , the key in the parent of  $X$  is smaller than (or equal to) the key in  $X$ , with the exception of the root (which has no parent). Therefore, the minimum element can always be found at the root.
- In a **max heap**, for every node  $X$ , the key in the parent of  $X$  is larger than (or equal to) the key in  $X$ , with the exception of the root (which has no parent). Therefore, the maximum element can always be found at the root.

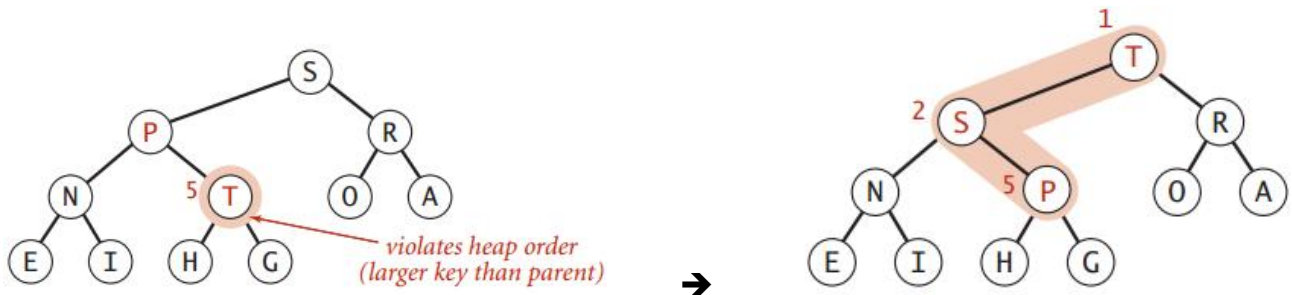
### Promotion in a heap

**Scenario 1:** Child's key becomes larger than its parent's key.

To eliminate the violation:

- Exchange key in child with key in parent.
- Repeat until heap order restored.

Example:



```
private void swim(int k)
{
    while (k > 1 && less(k/2, k))
    {
        exch(k, k/2);
        k = k/2;
    }
}
```

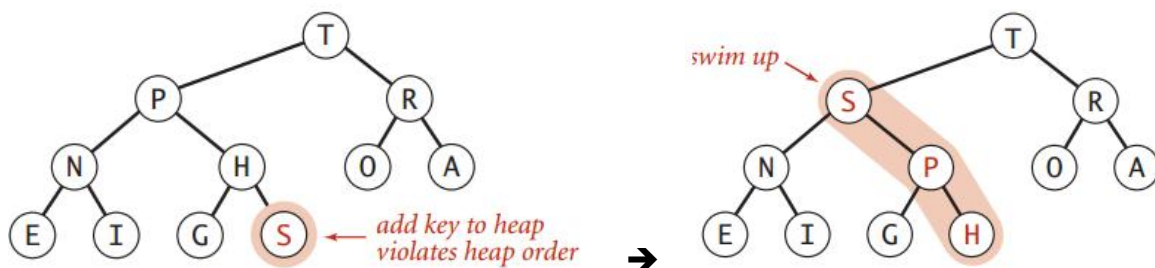
parent of node at k is at k/2

### Insertion in a heap

**Insert:** Add node at end, then swim it up.

**Cost:** At most  $1 + \lg N$  compares.

Example: Insert **S**



```
public void insert(Key x)
{
    pq[++N] = x;
    swim(N);
}
```

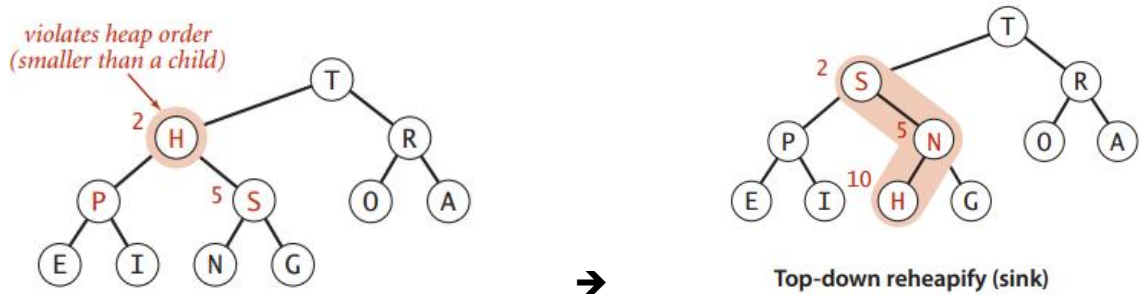
## Demotion in a heap

**Scenario 2:** Parent's key becomes smaller than one (or both) of its children's.

To eliminate the violation:

- Exchange key in parent with key in larger child.
- Repeat until heap order restored.

Example:



```
private void sink(int k)
{
    while (2*k <= N)
    {
        int j = 2*k;
        if (j < N && less(j, j+1)) j++;
        if (!less(k, j)) break;
        exch(k, j);
        k = j;
    }
}
```

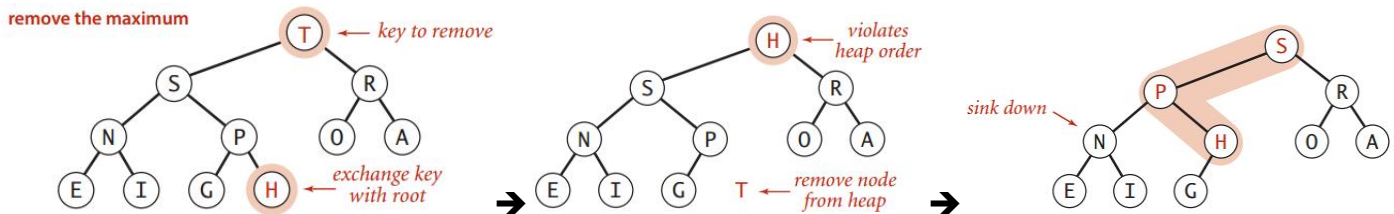
children of node at k are 2k and 2k+1

## Delete the maximum in a heap

**Delete max:** Exchange root with node at end, and then sink it down.

**Cost:** At most  $2 \lg N$  compares.

Example: delete T



```
public Key delMax()
{
    Key max = pq[1];
    exch(1, N--);
    sink(1);
    pq[N+1] = null;
    return max;
}
```

prevent loitering

## Binary heap: Java implementation

```
public class MaxPQ<Key extends Comparable<Key>>
{
    private Key[] pq;
    private int N;

    public MaxPQ(int capacity)
    { pq = (Key[]) new Comparable[capacity+1]; }

    public boolean isEmpty()
    { return N == 0; }
    public void insert(Key key)
    public Key delMax()
    { /* see previous code */ }

    private void swim(int k)
    private void sink(int k)
    { /* see previous code */ }

    private boolean less(int i, int j)
    { return pq[i].compareTo(pq[j]) < 0; }
    private void exch(int i, int j)
    { Key t = pq[i]; pq[i] = pq[j]; pq[j] = t; }
}
```

fixed capacity  
(for simplicity)

PQ ops

heap helper functions

array helper functions