Heaps: Max-Heap, Min-Heap

Building a Heap

COMP 2321
sections : 1 and 3
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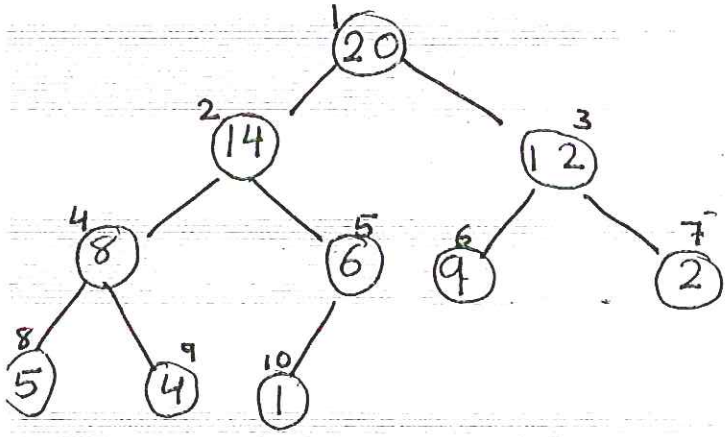
COMP242

Heap: nearly a complete binary tree

tree is completely filled except at the lowest level, which is filled from left to right.



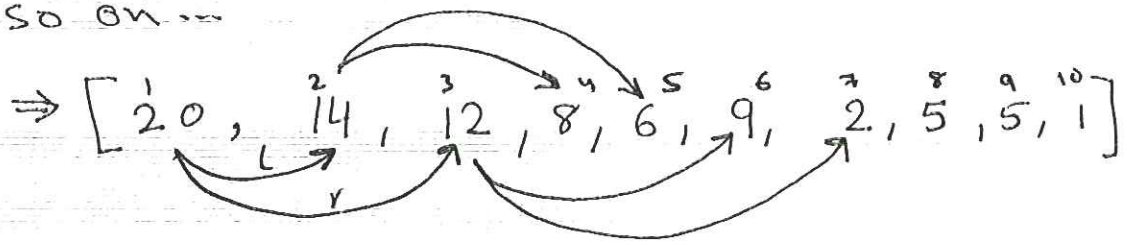
$n = 10$

$parent(i) = \lfloor i/2 \rfloor$

$left(i) = 2i$

$right(i) = 2i + 1$

depending on the complete binary tree property, we add from left to right in level 0 first, then to level 1 untill full and so on...



From this array we can observe that

$parent(i) = \lfloor i/2 \rfloor$

$left(i) = 2i$

$right(i) = 2i + 1$

Max-Heap & Min-Heap

Max-Heap property

$A[parent(i)] \geq A[i]$

maximum element in the Max-Heap is the ROOT

The parent must be greater than (or equals to) its children.

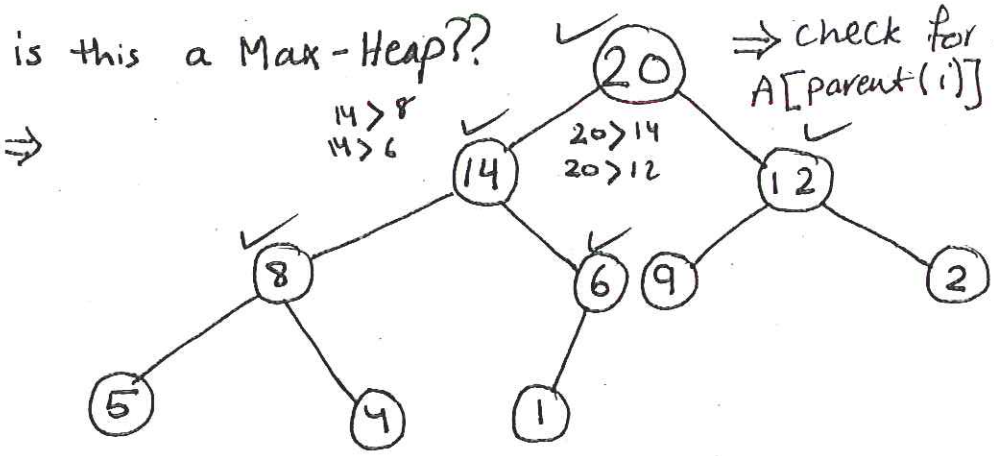
Min-Heap property

$A[parent(i)] \leq A[i]$

minimum element in the Min-Heap is the ROOT

is this a Max-Heap??

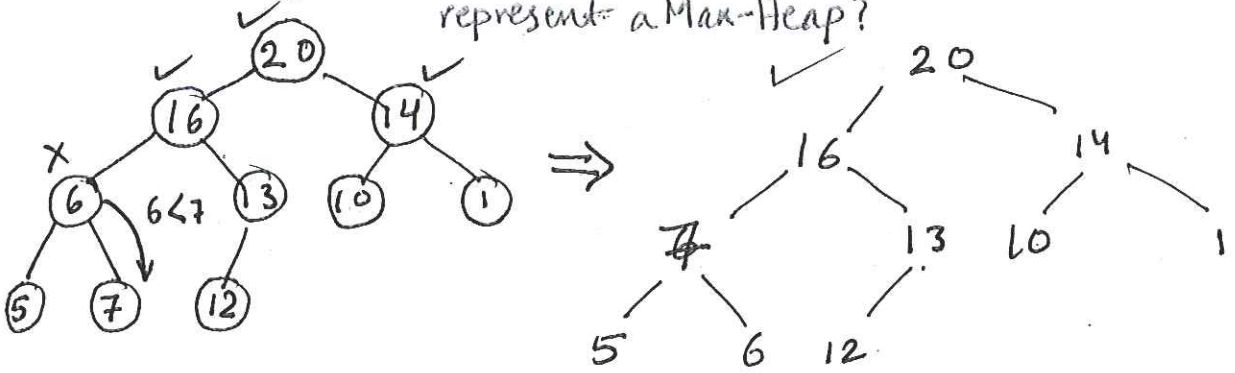
⇒ check for all nodes $A[parent(i)] \geq A[i]$



* * *

Example.

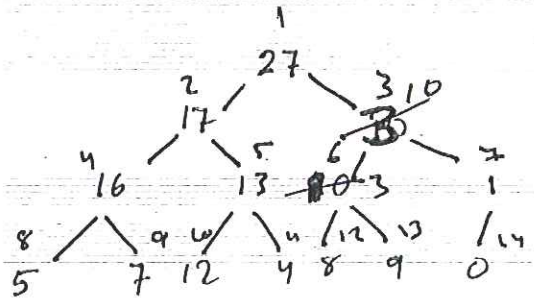
is the sequence A[20, 16, 14, 6, 13, 10, 1, 5, 7, 12] represent a Max-Heap?



Maitaining the Heap property.

```
max-heapify (A, i)
{
    l = left(i);  r = right(i);
    if (l ≤ heap-size(A) && A[l] > A[i])
        max = l;
    else
        max = i;
    if (r ≤ heap-size(A) && A[r] > A[max])
        max = r;
    if (max != i){
        swap(A[i], A[max]);
        max-heapify (A, max);
    }
}
```

Example max-heapify (A, 3) on A = [27, 17, 3, 16, 13, 10, 1, 5, 7, 12, 4, 8, 9, 0]



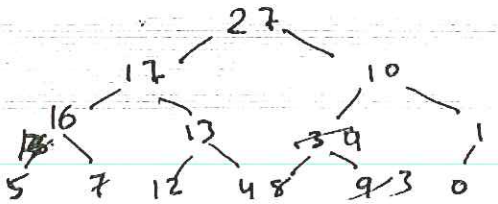
max-heapify (A, 3)

l = 6, r = 7

A[6] ≥? A[3] ? swap max = l

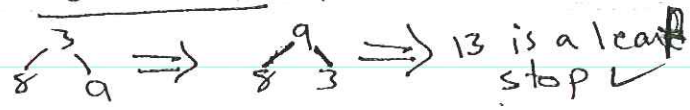
A[r] ≥? A[max] ? X

max ≠ i ⇒ swap 3 → 10 ⇒



check again max-heapify (A, max)

max now is 6 b/s it was the max



Building A heap :: $\Longrightarrow O(n)$

We can use the Max_heapify in bottom up manner to convert an array $A[1 \cdots n]$ to Max_heap

Observations :-

The elements $A\left(\lfloor \frac{n}{2} \rfloor + 1\right), A\left(\lfloor \frac{n}{2} \rfloor + 2\right), A\left(\lfloor \frac{n}{2} \rfloor + 3\right), \cdots A(n)$ are all Leaves.

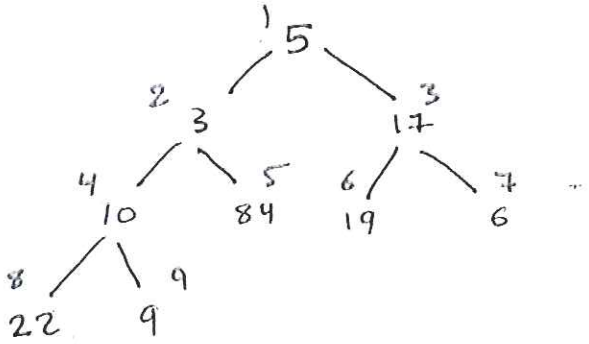
Reason :- n = length of heap = heap_size(A)

any index after $\lfloor \frac{n}{2} \rfloor$ will have left(i) and right(i) > n (does not come in the heap size) so, all are leaves

Example :

$A = \{5, 3, 17, 10, 84, 19, 16, 22, 9\}$ (indices 1 2 3 4 5 6 7 8 9)

left(i) = 2i
right(i) = 2i+1
parent(i) = $\lfloor \frac{i}{2} \rfloor$



Build_max_heap (A)

```
for( i = ⌊size(A)/2⌋ ; i >= 1; i--)
    max_heapify(A, i)
```

Running Time :-

simple upper bound :-

→ Max-heapify costs $O(\log n)$ time

→ There are $O(n)$ such calls

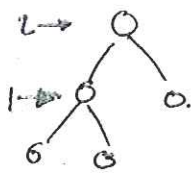
$\Rightarrow O(n \log n)$

However we may produce a tighter bound :-

property one :-

Running time of Max-heapify depends upon the height of the node $O(h)$, and heights of most nodes are small

property Two :-



At most $\lceil n/2^{h+1} \rceil$ nodes are there at any level h

$5/2^2 = \lfloor 5/4 \rfloor = 1$

$\lfloor \log n \rfloor \Rightarrow$ maximum height of a tree. $\Rightarrow$ max heapify

$$\sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h}\right)$$

$$* \sum_{h=0}^{\infty} \frac{h}{2^h} = 2 \qquad S = 0 + \frac{1}{2^1} + \frac{1}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots = 2$$

$$\Rightarrow O\left(n \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h}\right) < O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h}\right) = O(n)$$

$< \quad O(2n) = O(n)$

heape sort

Sorting strategy :-

1- Build Max heap from an unordered array.

2. find the maximum element A[1]

3- Swap elements A[n] and A[1]
⇒ now the max element is at the end of an array.

4. Discard node n from heap
(by decrementing heap size variable)

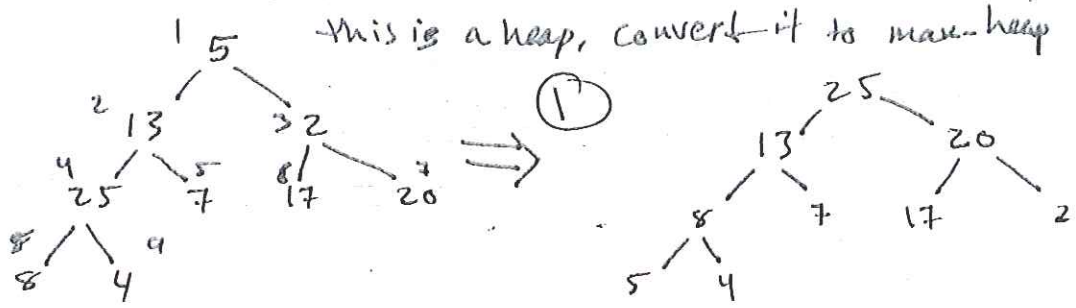
5. New root may violate max heap property, but its children are max heaps. Run max_heapify to fix this.

6. Go to step 2 unless heap is empty.

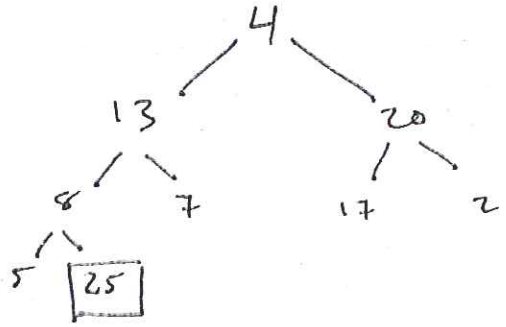
```
heap_sort(A)

Build_max_heap(A);
for(i = size(A); i >= 2 ; i--)
{
    swap(A(1), A(i));
    heap_size(A) = heap_size(A) - 1;
    Max_heapify(A, 1);
}
```

Example: A = {5, 13, 2, 25, 7, 17, 20, 8, 4}

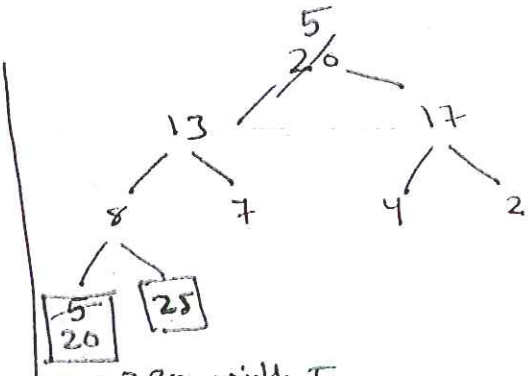


(2) swap 25 with 4

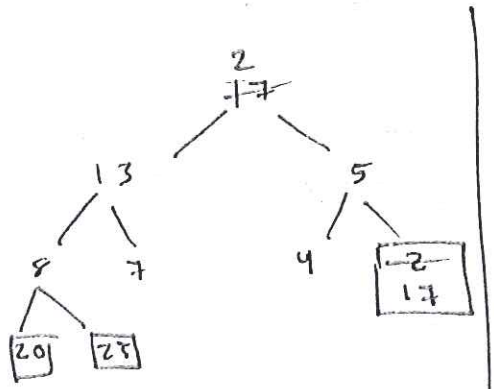


3- discard 25 from the heap.

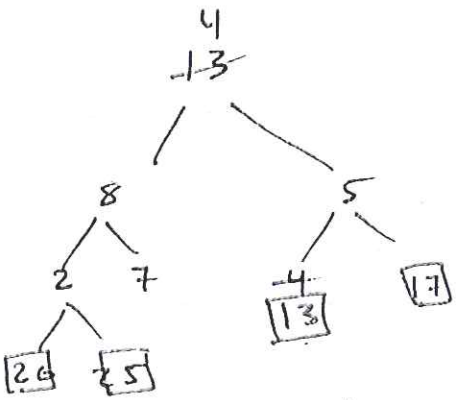
25 is not a part heap now

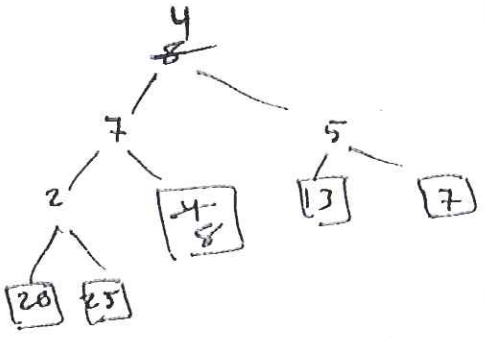


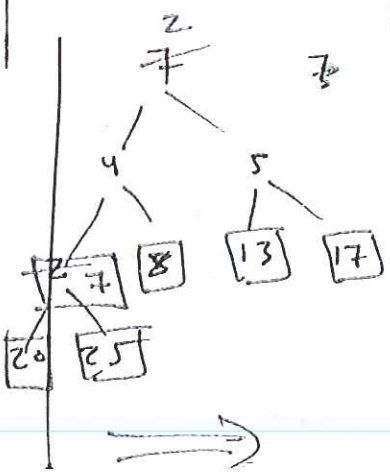
swap 20 with 5

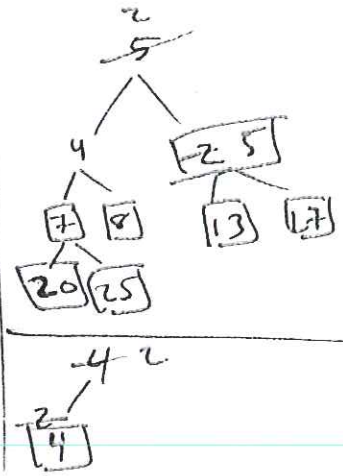


swap 17 and 2









25, 20, 17, 13, 8, 7, 5, 4, 2

{2, 4, 5, 7, 8, 13, 17, 20, 25}

(X) we have n elements, for each of them we call max-heapify log n

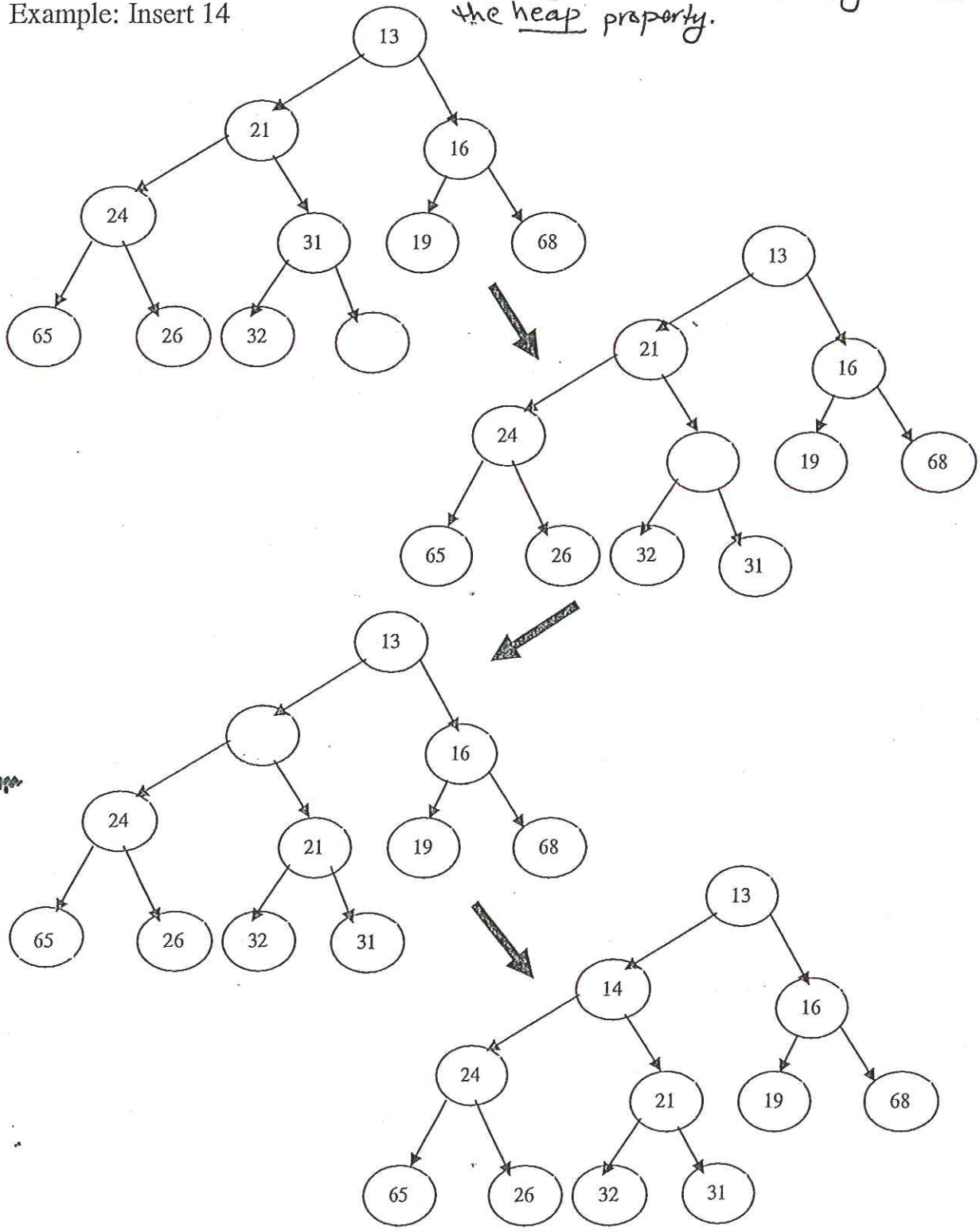
$\Rightarrow O(n \log n)$ time.

Basic Heap Operation

→ Insert

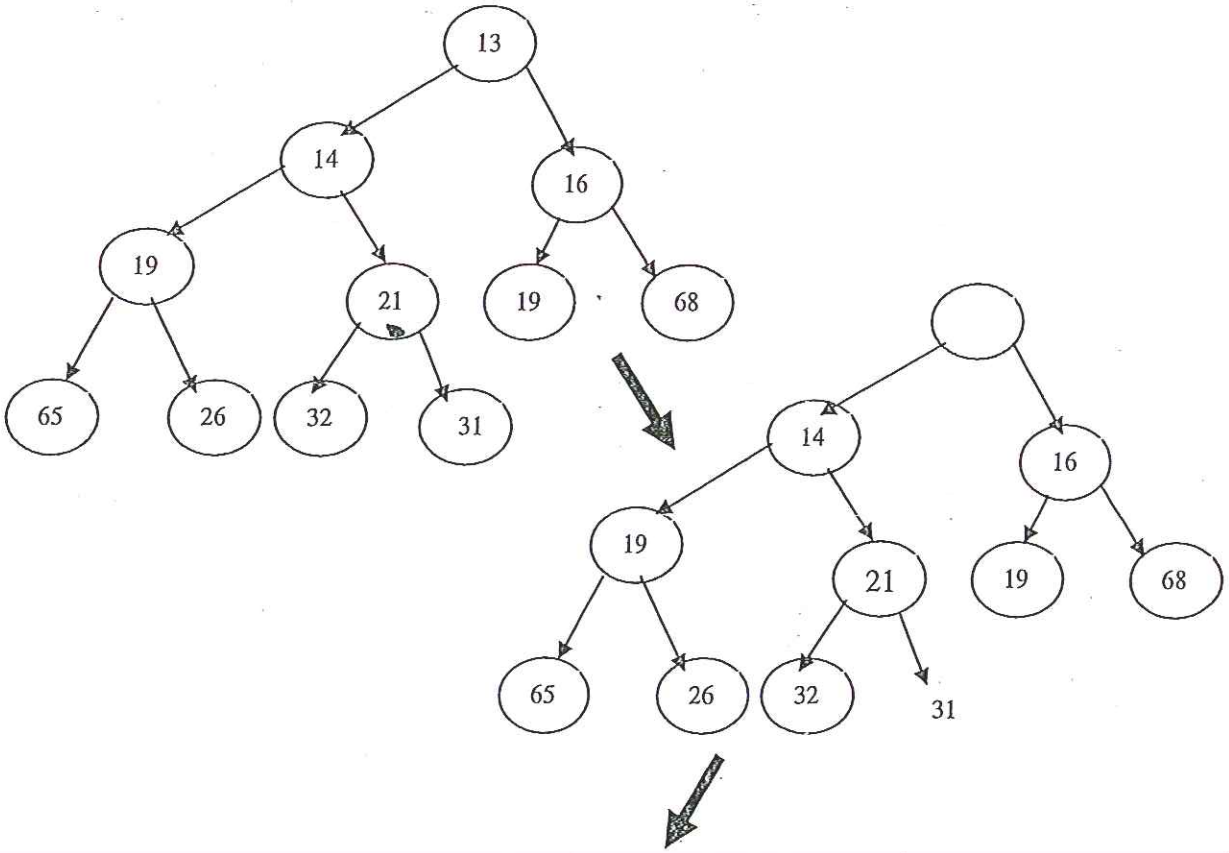
Example: Insert 14

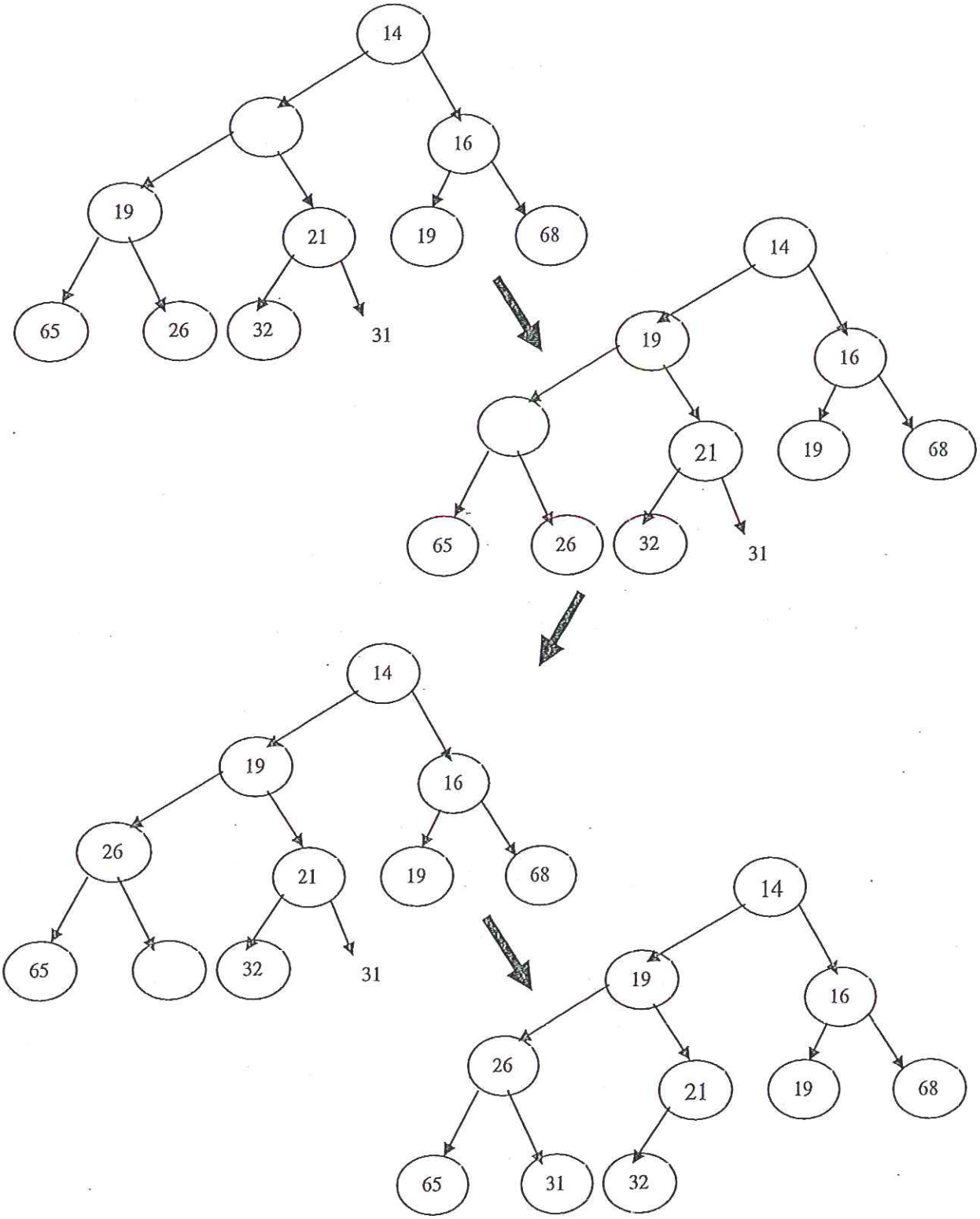
any element can be inserted from the bottom level ~~from~~ starting from the left, then you have to check the heap property.



```
void insert ( element_type x, PRIORITY_QUEUE   H)
{
        unsigned int i;
        if ( is_full (H))
                cout << "Priority Queue is full" << endl;
        else
        {
                i = ++H→size;
                while ( H→element[i/2] > x )
                {
                        H→element[i] = H→element[i/2];
                        i/=2;
                }
                H→element[i] = x;
        }
}
```

→ Delete minimum
Example:

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```
element_type   delete_min  ( PRIORITY_QUEUE  H)
{
        unsigned  int  i,  child;
        element_type    min_element, last_element;
        if ( is_empty(H))
        {
                cout << "Priority Queue is empty" << end;
                return H→element[0];
        }
        min_element = H→element[0];
        last_element = H→element[H→size--];
        for (i=1; i*2 <= H→size ; i = child)
        {
                child = i * 2;
                if ( ( child != H→size )
                        && ( H→element[child+1] < H→element[child]))
                                child++;
                if ( last_element > H→element[child])
                        H→element[i] = H→element[child];
                else
                        break;
        }
        H→element[i] = last_element;
        return min_element;
}
```

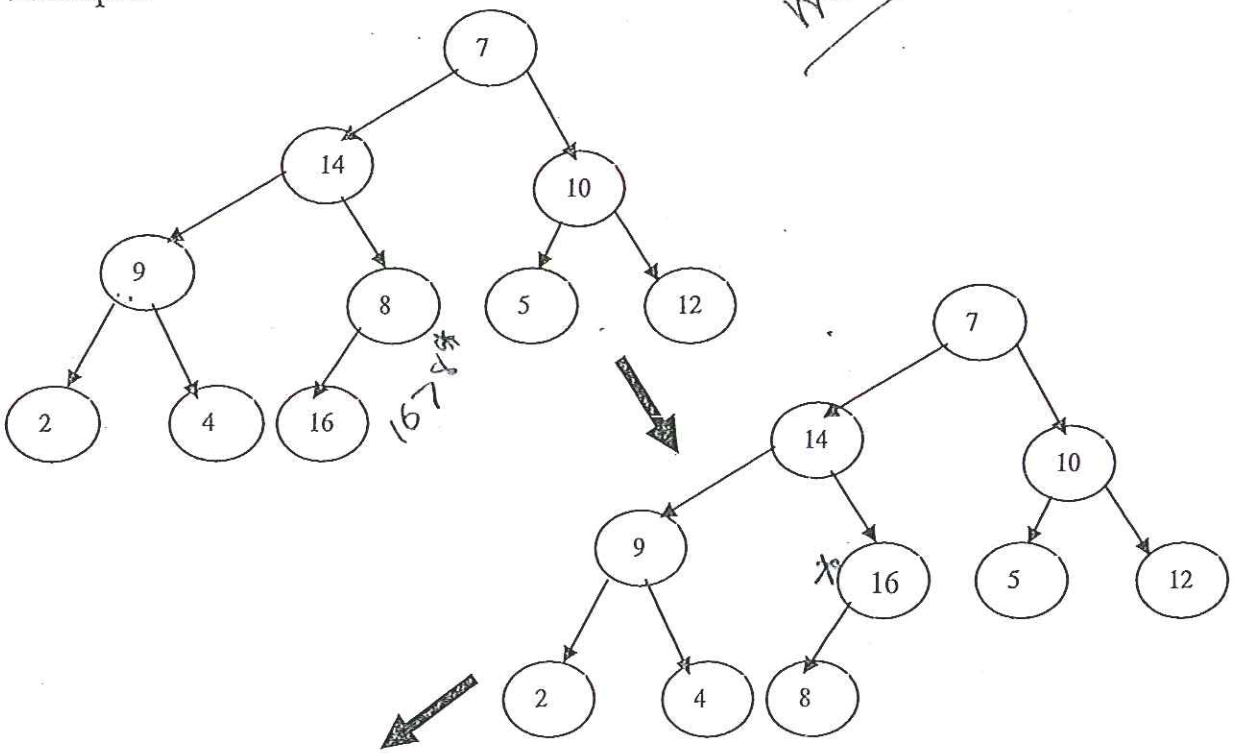
Algorithm to maintain heap property

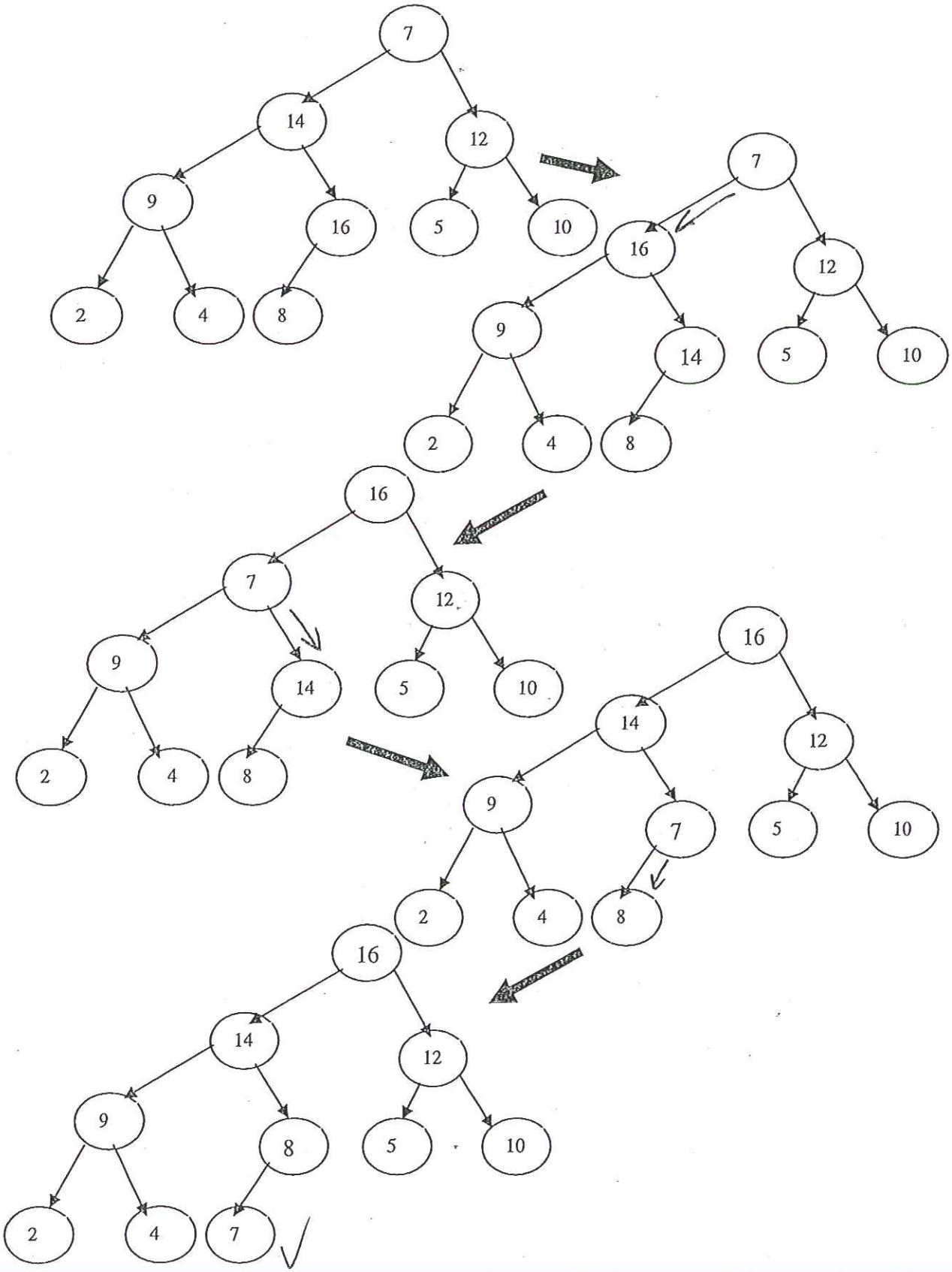
```
Procedure ment_heap_prop ( A, i)
        L  ➔  Left(i)
        R  ➔  Right(i)
If  L <= heap_size   and   A[L] > A[i]  then
        Largest = L;
Else
        Largest = i;
If  R <= heap_size  and   A[R] >= A[Largest])  then
        Largest  = R;
If  Largest  <> i then
        Exchange( A[i] , A[Largest])
Ment_heap_prop( A, Largest)
```

→ Building a heap tree

```
heap_size = length_Array
for ( i= heap_size/2; i >= 1; i--)
        ment_heap_prop(A, i);
```

Example:

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7

Sorting Using heap tree

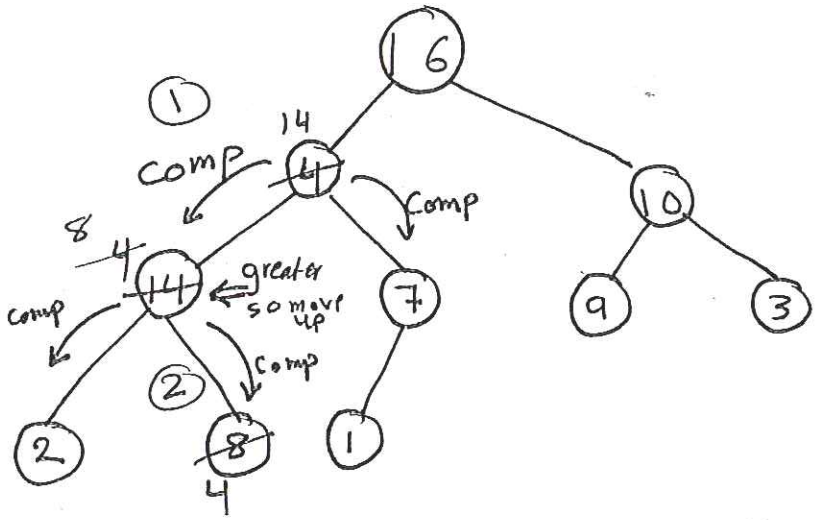
```
Procedure heap_sort(A)
        Build_heap;
        For(i=length(A); i==2; i--)
        {
                Exchange(A[i], A[1]);
                Dec(heap_size);
                Ment_heap_prop(A,i);
        }
```

→Analysis

ment_heap_prop ➔ O(log n)

for statement ➔ O(n)

Heap sort ➔ O(n log n)



this is not a max heap, node with key 4 violates the max heap property.