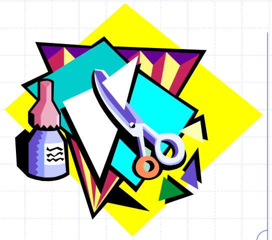
Union-Find Partition Structures

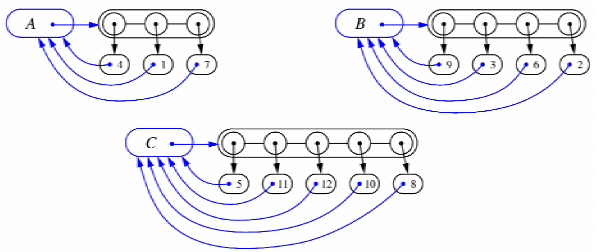


Partitions with Union-Find Operations (§ 12.7)

- makeSet(*x*): Create a singleton set containing the element *x* and return the position storing *x* in this set.
- union(*A,B*): Return the set *A* U *B*, destroying the old *A* and *B*.
- find(*p*): Return the set containing the element in position *p*.

List-based Implementation

- Each set is stored in a sequence represented with a linked-list
- Each node should store an object containing the element and a reference to the set name

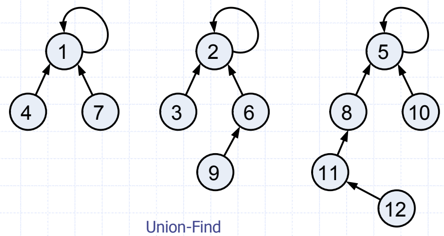


Analysis of List-based Representation

- When doing a union, always move elements from the smaller set to the larger set
  - Each time an element is moved it goes to a set of size at least double its old set
  - Thus, an element can be moved at most O(log n) times
- Total time needed to do n unions and finds is O(n log n).

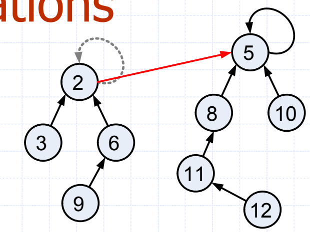
Tree-based Implementation (§ 10.6.3)

- Each element is stored in a node, which contains a pointer to a set name
- A node v whose set pointer points back to v is also a set name
- Each set is a tree, rooted at a node with a self referencing set pointer
- For example: The sets “1”, “2”, and “5”:

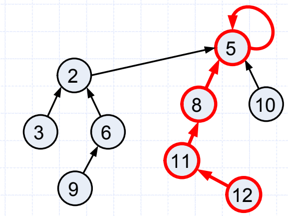


Union-Find Operations

- To do a union, simply make the root of one tree point to the root of the other

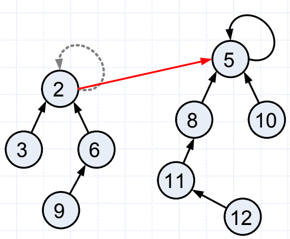


- To do a find, follow set-name pointers from the starting node until reaching a node whose set-name pointer refers back to itself



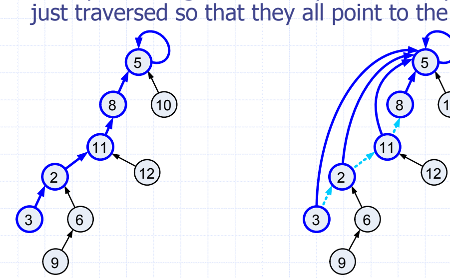
Union-Find Heuristic 1

- Union by size:
  - When performing a union, make the root of smaller tree point to the root of the larger
- Implies O(n log n) time for performing n union-find operations:
  - Each time we follow a pointer, we are going to a subtree of size at least double the size of the previous subtree
  - Thus, we will follow at most O(log n) pointers for any find.



Union-Find Heuristic 2

- Path compression:
  - After performing a find, compress all the pointers on the path just traversed so that they all point to the root



- Implies $O(n \log^* n)$ time for performing n union-find operations:
  - Proof is somewhat involved… (and not in the book)

Proof of log* n Amortized Time

- For each node *v* that is a root
  - define *n*(*v*) to be the size of the subtree rooted at *v* (including *v*)
  - identified a set with the root of its associated tree.
- We update the size field of *v* each time a set is unioned into *v*. Thus, if *v* is not a root, then *n*(*v*) is the largest the subtree rooted at *v* can be, which occurs just before we union *v* into some other node whose size is at least as large as *v*'s.
- For any node *v*, then, define the ***rank*** of *v*, which we denote as *r*(*v*), as *r*(*v*) = [log *n*(*v*)]:
- Thus, $n(v) \geq 2^{r(v)}$.
- Also, since there are at most *n* nodes in the tree of *v*, *r*(*v*) = [log*n*], for each node *v*.

Proof of log* n Amortized Time (2)

- For each node *v* with parent *w:*
  - $r(v) > r(w)$
- **Claim:** *There are at most* $n/2^s$ *nodes of rank s.*
- **Proof:**
  - Since $r(v) < r(w)$, for any node *v* with parent *w*, ranks are monotonically increasing as we follow parent pointers up any tree.
  - Thus, if $r(v) = r(w)$ for two nodes *v* and *w*, then the nodes counted in *n*(*v*) must be separate and distinct from the nodes counted in *n*(*w*).
  - If a node *v* is of rank *s*, then $n(v) \geq 2^s$.
  - Therefore, since there are at most *n* nodes total, there can be at most $n/2^s$ that are of rank *s*.

Proof of log* n Amortized Time (3)

- Definition: Tower of two's function:
  - $t(i) = 2^{t(i-1)}$
- Nodes *v* and *u* are in the same rank group *g* if
  - *g* = log*(*r*(*v*)) = log*(*r*(*u*)):
- Since the largest rank is log *n*, the largest rank group is
  - log*(log *n*) = (log* *n*)-1

Proof of log* n Amortized Time (4)

- Charge 1 cyber-dollar per pointer hop during a find:
  - If *w* is the root or if *w* is in a different rank group than *v*, then charge the find operation one cyber-dollar.
  - Otherwise (*w* is not a root and *v* and *w* are in the same rank group), charge the node *v* one cyber-dollar.
- Since there are most (log* *n*)-1 rank groups, this rule guarantees that any find operation is charged at most log* *n* cyber-dollars.

Proof of log* n Amortized Time (5)

- After we charge a node *v* then *v* will get a new parent, which is a node higher up in *v*'s tree.
- The rank of *v*'s new parent will be greater than the rank of *v*'s old parent *w*.
- Thus, any node *v* can be charged at most the number of different ranks that are in *v*'s rank group.
- If *v* is in rank group $g > 0$, then *v* can be charged at most $t(g) - t(g-1)$ times before *v* has a parent in a higher rank group (and from that point on, *v* will never be charged again). In other words, the total number, *C*, of cyber-dollars that can ever be charged to nodes can be bound as

$$C \le \sum_{g=1}^{\log^* n - 1} n(g) \cdot (t(g) - t(g-1))$$

Proof of log* n Amortized Time (end)

- Bounding $n(g)$:

$$\begin{aligned} n(g) &\le \sum_{s=t(g-1)+1}^{t(g)} \frac{n}{2^s} \\ &= \frac{n}{2^{t(g-1)+1}} \sum_{s=0}^{t(g)-t(g-1)-1} \frac{1}{2^s} \\ &< \frac{n}{2^{t(g-1)+1}} \cdot 2 \\ &= \frac{n}{2^{t(g-1)}} \\ &= \frac{n}{t(g)} \end{aligned}$$

- Returning to C:

$$\begin{aligned} C &< \sum_{g=1}^{\log^* n - 1} \frac{n}{t(g)} \cdot (t(g) - t(g-1)) \\ &\le \sum_{g=1}^{\log^* n - 1} \frac{n}{t(g)} \cdot t(g) \\ &= \sum_{g=1}^{\log^* n - 1} n \\ &\le n \log^* n \end{aligned}$$