BIRZEIT UNIVERSITY

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Faculty of Engineering and Tecnology

Computer Science Department

Graphs_2

Graphs Search Algorithms

Shortest-Path Algorithms

- Single-Source shortest path: find the shortest path from a source vertex s to all vertices in a graph
- Single-Destination shortest path: find a shorter path to a given destination vertex d from all vertices in a graph
- Single-Pair shortest path: find the shortest path from a source vertex u to a destination vertex v
- All-Pairs shortest path: find the shortest path from a source vertex u to a destination vertex v for all vertices u and v in the graph

Shortest-Path Algorithms

- Shortest-path algorithms aim at finding the shortest path between nodes in a graph
  - The input is a weighted graph: associated with each edge (v_i, v_j) is a cost $c_{i,j}$ to traverse the edge
  - The cost of a path $v_1v_2 \ldots v_N$ is $\sum_{i=1}^{N-1} c_{i,\, i+1}$
  - This is referred to as the **weighted path length**
  - The unweighted path length is the number of edges on the path, namely, N – 1

Single-Source Shortest-Path Algorithms

- Given as input a weighted graph, $G = (V, E)$, and a distinguished vertex, s, find the shortest weighted path from s to every other vertex in G.
- For example, the shortest weighted path from v_1 to v_6 has a cost of 6 and goes from v_1 to v_4 to v_7 to v_6
- The shortest unweighted path between these vertices is 2

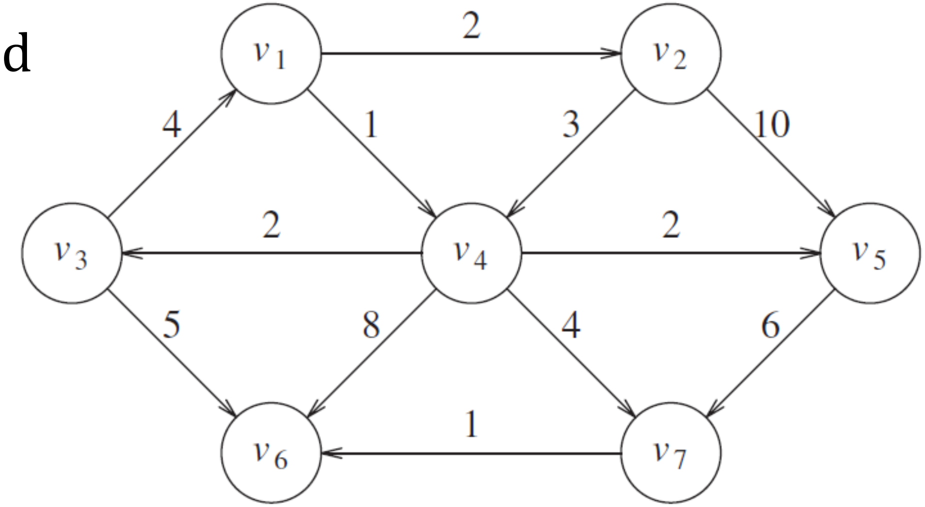


Figure 9.8 A directed graph G

Single-Source Shortest-Path Algorithms

- Having negative weights in the graph may cause some problems.
- The path from v_5 to v_4 has cost 1, but a shorter path exists by following the loop v_5, v_4, v_2, v_5, v_4, which has a cost of −5
- This path is still not the shortest, because we could stay in the loop arbitrarily long.
- Thus, the shortest path between these two points is **undefined**.

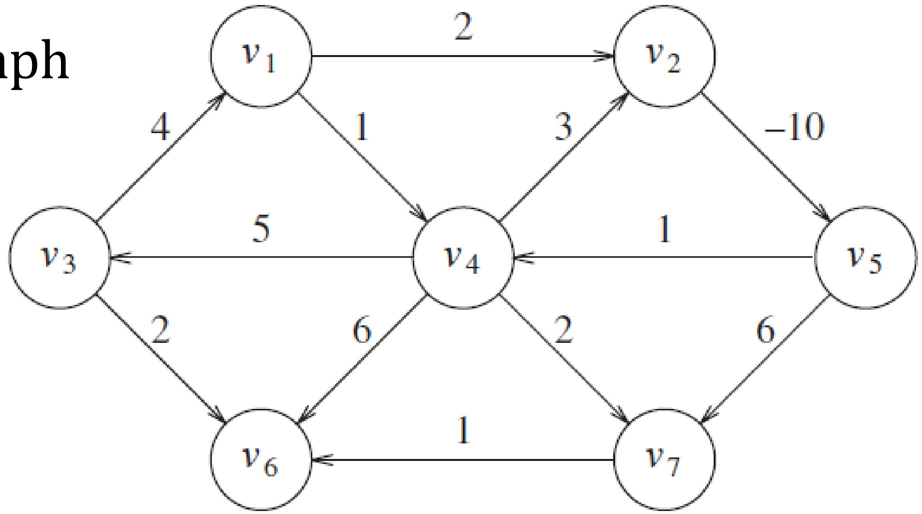


Figure 9.9 A graph with a negative-cost cycle

Single-Source Shortest-Path Algorithms

- Another example, the shortest path
- from v_1 to v_6 is undefined, because we can get into the same loop.
- This loop is known as a **negative-cost cycle;** when one is present in the graph, the shortest paths are not defined.

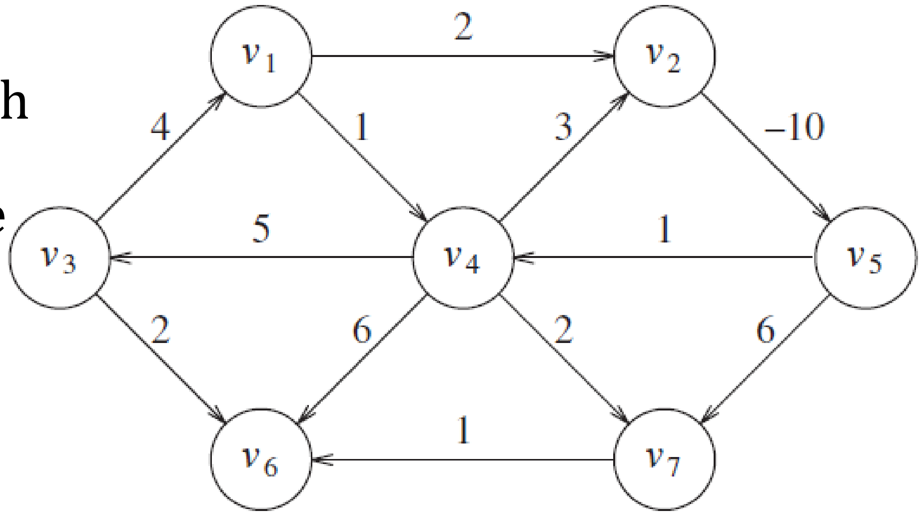


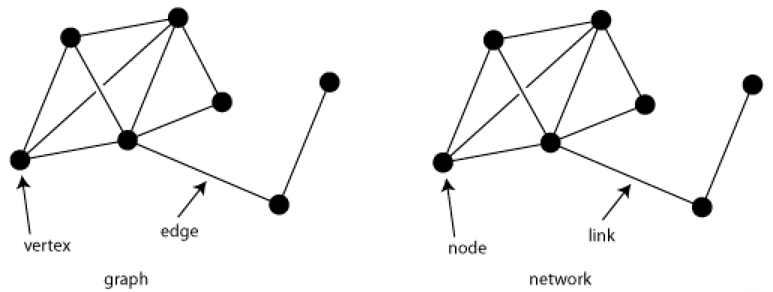
Figure 9.9 A graph with a negative-cost cycle

Single-Source Shortest-Path Algorithms

- Negative-cost edges are not necessarily bad, as the cycles are, but their presence seems to make the problem harder.
- For convenience, in the absence of a negative-cost cycle, the shortest path from *s* to *s* is zero.

Single-Source Shortest-Path Algorithms

- There are many examples where we might want to solve the shortest-path problem.
- If the vertices represent computers; the edges represent a link between computers; and the costs represent communication costs (phone bill per megabyte of data), delay costs (number of seconds required to transmit a megabyte), or a combination of these and other factors, then we can use the shortest-path algorithm to find the cheapest way to send electronic news from one computer to a set of other computers.



Single-Source Shortest-Path Algorithms

- Another example is to model an airplane (or transportation routes) by graphs and use a shortest path algorithm to compute the best route between two points.
- In this and many practical applications, we might want to find the shortest path from one vertex, *s*, to only one other vertex, *t*.
- Currently there are no algorithms in which finding the path from *s* to one vertex is any faster (by more than a constant factor) than finding the path from *s* to all vertices.
- We will solve 4 variations of this problem

Unweighted Shortest Paths

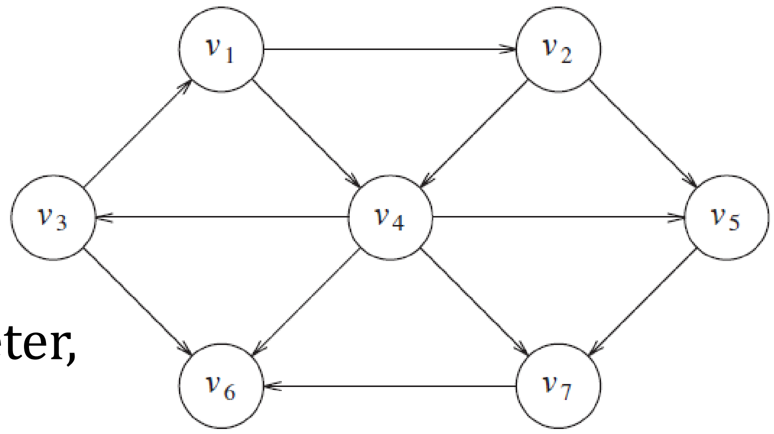


Figure 9.10 An unweighted directed graph G

- Given an unweighted graph, G. Using some vertex, s, which is an input parameter, we want to find the shortest path from s to all other vertices.
- We are only interested in the number of edges contained on the path (because there are no weights).
- This is clearly a special case of the weighted shortest-path problem, since we could assign all edges a weight of 1.

Unweighted Shortest Paths

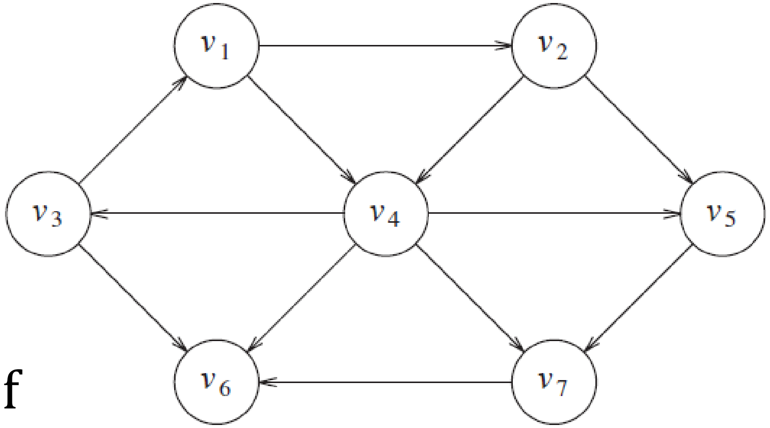


Figure 9.10 An unweighted directed graph G

- Suppose we are interested in the length of the shortest path not in the actual paths themselves. Keeping track of the actual paths will turn out to be a matter of simple bookkeeping.
- Suppose we choose s to be v_3.
- Immediately, we can tell that the shortest path from s to v_3 is then a path of length 0.
- We can mark this information and then obtain the following graph

Unweighted Shortest Paths

- Now look for vertices that are distant by 1 from s (v_3), which are the adjacent vertices of s.
- v_1 and v_6 are the adjacent vertices to s.

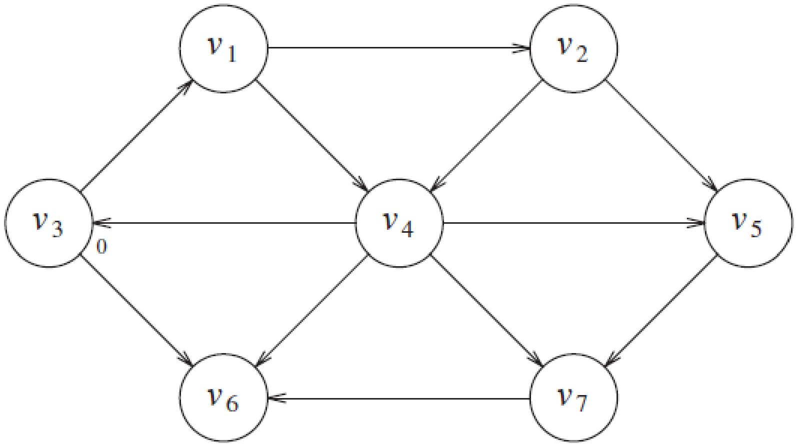


Figure 9.11 Graph after marking the start node as reachable in zero edges

Unweighted Shortest Paths

- Now find vertices whose shortest path from *s* is exactly 2, by finding all the vertices adjacent to v_1 and v_6 (the vertices at distance 1).
- v_2 and v_4 are the adjacent vertices to s.

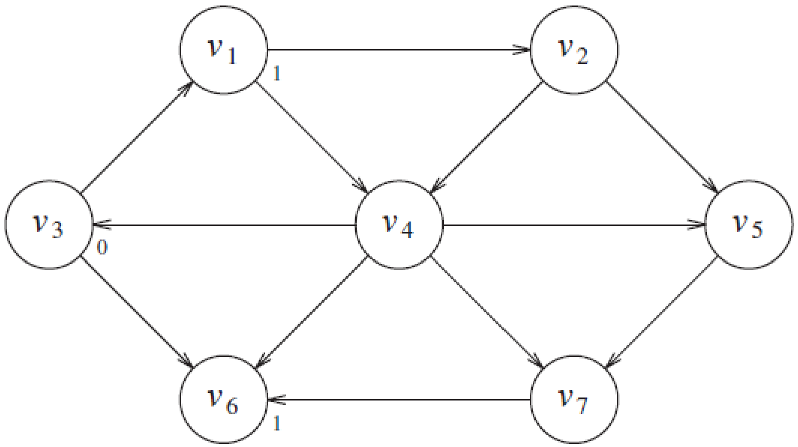


Figure 9.12 Graph after finding all vertices whose path length from *s* is 1

Unweighted Shortest Paths

- Finally we can find, by examining vertices adjacent to the recently evaluated v_2 and v_4, that v_5 and v_7 have a shortest path of three edges.

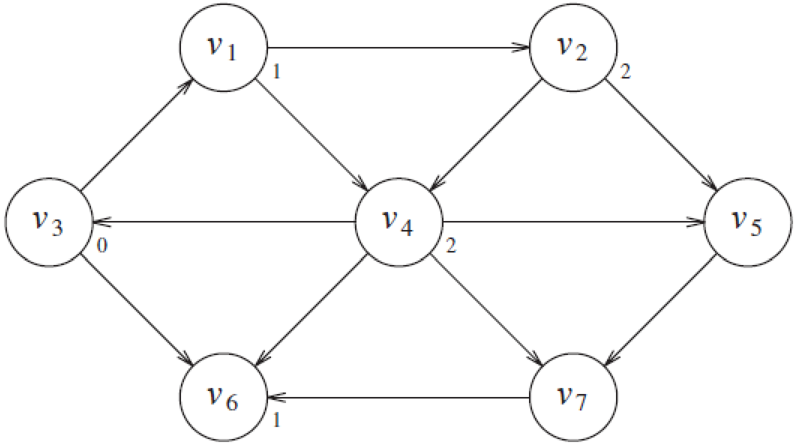


Figure 9.13 Graph after finding all vertices whose shortest path is 2

Unweighted Shortest Paths

- Now all vertices have been calculated.
- This strategy of searching a graph is known as **Breadth-First Search (BFS)**.
- It operates by processing vertices in layers: The vertices closest to the start are evaluated first, and the most distant vertices are evaluated last.
- This is much the same as a level-order traversal for trees.

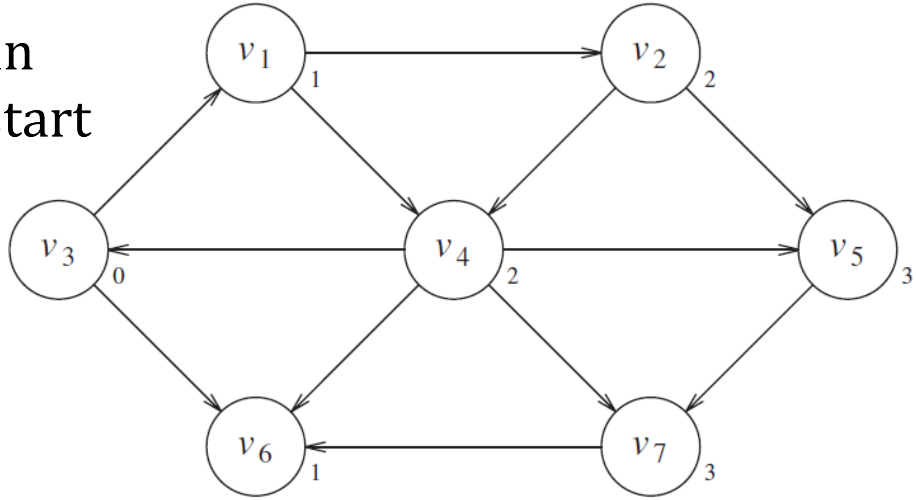


Figure 9.14 Final shortest paths

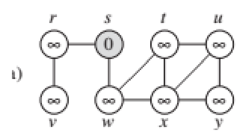
Unweighted Shortest Paths

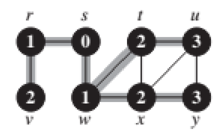
- The BFS can be implemented by adapting the following table
- First, for each vertex, keep its distance from *s* in the entry d_v (initially all vertices are unreachable except for *s*, whose path length is 0).
- Variable p_v is the bookkeeping variable, which will allow us to print the actual paths.
- Variable known is set to true after a vertex is processed.
- Initially, all entries are not known, including the start vertex.
- When a vertex is marked known, we have a guarantee that no cheaper path will ever be found, and so processing for that vertex is essentially complete

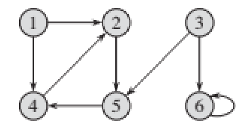
v	*known*	d_v	p_v
v_1	F	∞	0
v_2	F	∞	0
v_3	F	0	0
v_4	F	∞	0
v_5	F	∞	0
v_6	F	∞	0
v_7	F	∞	0

BFS: Unweighted Shortest Paths

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = Ø
ENQUEUE(Q, s)
while Q ≠ Ø
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```





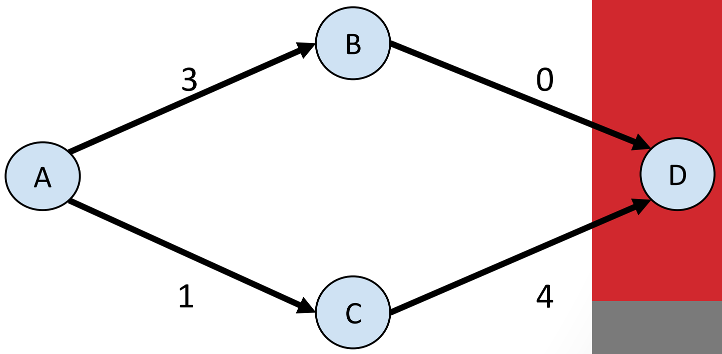


What is the time complexity for BFS?

Dijkstra's Algorithm

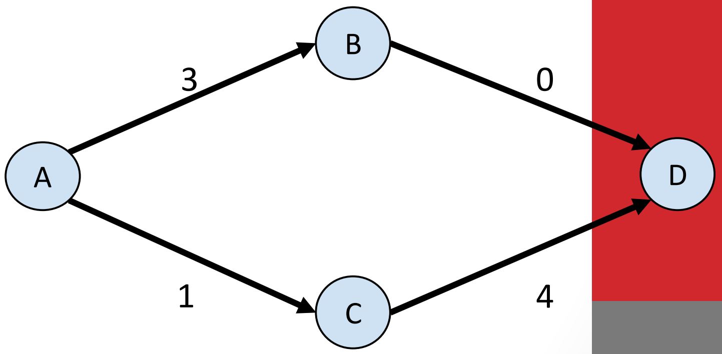
Dijkstra’s Algorithm

- If the graph is weighted, the problem becomes harder, but we can still use the ideas from the unweighted case.
- Dijkstra’s algorithm solves the problem of finding the shortest path from a vertex (source) to another vertex (destination).
- For example, you want to get from one city to another in the fastest possible way?



Dijkstra's Algorithm

- BFS is to find the shortest path between two points.
- "Shortest path" means the path with the fewest segments.
- But in Dijkstra's algorithm, a weight is assigned to each edge.
- Then Dijkstra's algorithm finds the path with the smallest total weight.

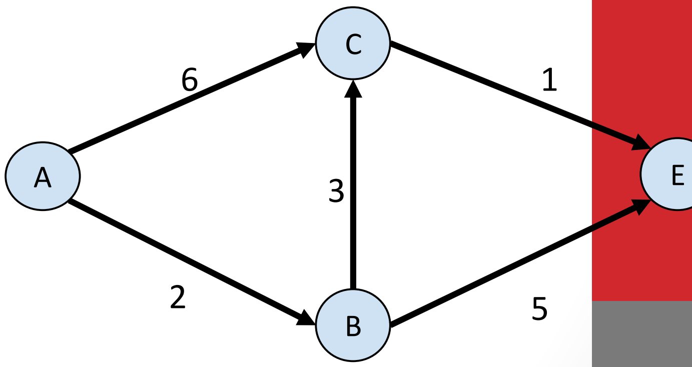


Dijkstra’s Algorithm

- Dijkstra's algorithm computes shortest paths for positive numbers.
- However, if one allows negative numbers, the algorithm will fail.
- Alternatively, the Bellman-Ford algorithm can be used.
- Dijkstra's algorithm is considered as a prime example of a greedy-search algorithm.
- Greedy algorithms generally solve a problem in stages by doing what appears to be the best thing at each stage.

Dijkstra's Algorithm

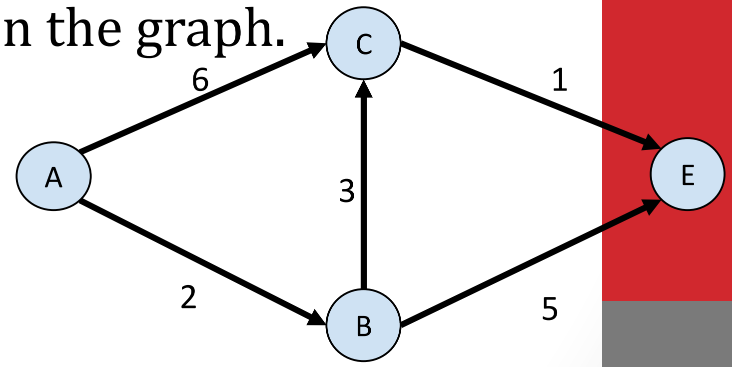
- Dijkstra's algorithm computes the cost of the shortest path from a starting vertex to all other vertices in the graph.
- Consider the following graph: Starting point 'A', destination 'E'.
- If we run this using the BFS, we will end-up with the cost of 7 (6+1)
- We aim at finding the destination is less time! (if exists)



Dijkstra's Algorithm

- 4-basic steps for Dijkstra's algorithm:

1. Find the node with the minimal cost. This is the node you can get to in the least amount of time.
2. Update the costs of the neighbor nodes.
3. Repeat until this is done for every node in the graph.
4. Calculate the final path.



Dijkstra's Algorithm

- At each stage:
  - Select an unknown vertex *v* that has the smallest d_v
  - Declare that the shortest path from *s* to *v* is known.
  - For each vertex *w* adjacent to *v*:
    - Set its distance d_w to the $d_v + \text{cost}_{v,w}$
    - Set its path p_w to *v*.

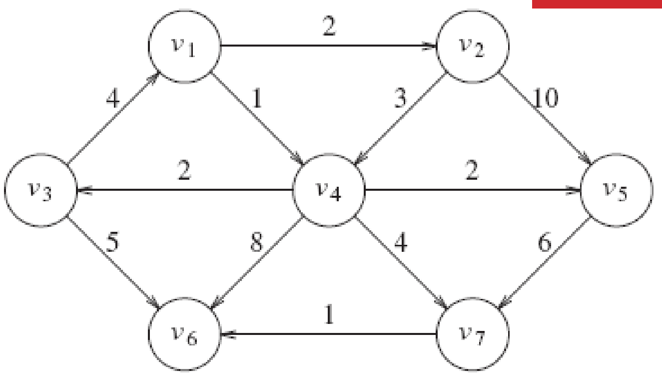
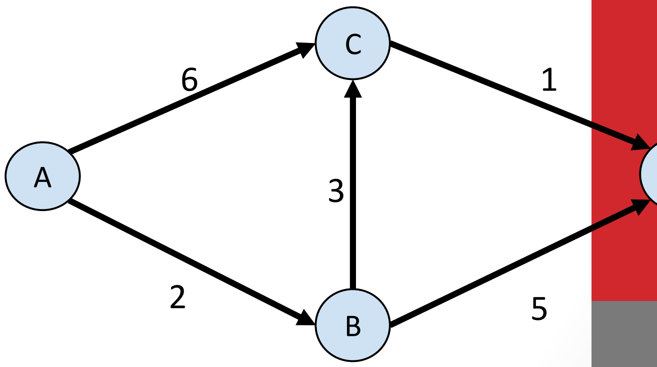


Figure 9.20 The directed graph G (again)

Dijkstra's Algorithm

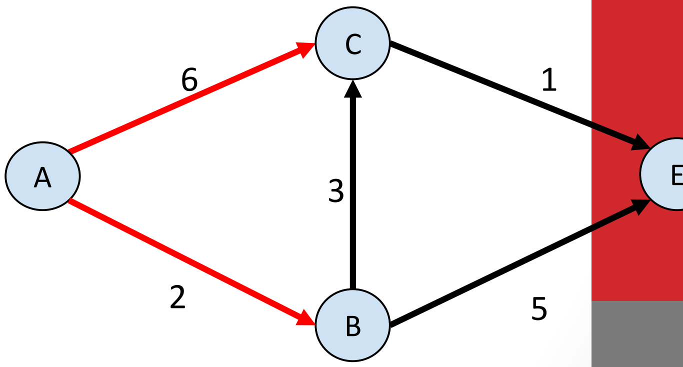
- **Step 1:** Find the node with the minimal cost.
- We are standing at the starting node 'A'. 'B' will take 6; and 'C' will take 2. We don't know the rest yet.
- As we don't know how long it will take to reach the destination, we will put it infinity.



Dijkstra’s Algorithm

- **Step 1:** Find the node with the minimal cost.
- We are standing at the starting node ‘A’. ‘B’ will take 6; and ‘C’ will take 2. We don’t know the rest yet.
- As we don’t know how long it will take to reach the destination, we will put it infinity.

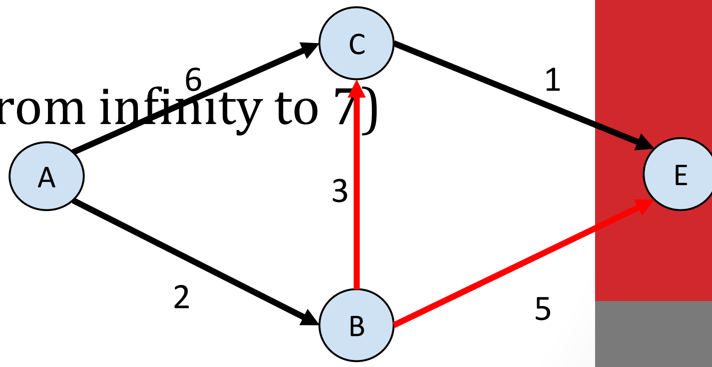
Node	Cost to Node
B	2
C	6
Destination	∞



Dijkstra's Algorithm

- **Step 2:** Calculate how long it takes to get to all of node B's neighbours by following an edge from B.
- Notice that there is a shorter path to C (2 + 3)
- When there is a shorter path for a neighbor of B, update its cost. In this Case
  - A shorter path to C (down from 6 to 5)
  - A shorter path to the destination (down from infinity to 7)

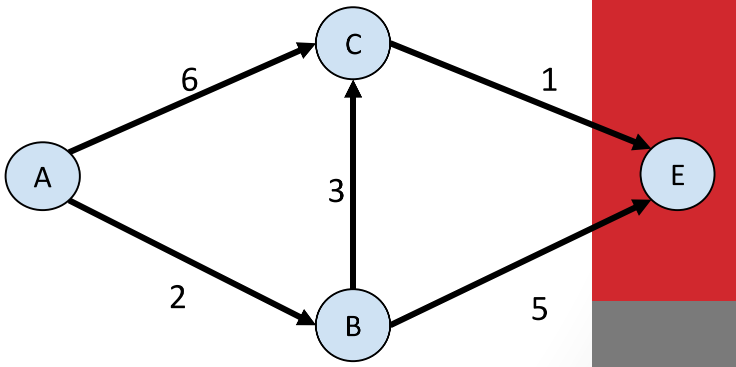
Node	Cost to Node
B	2
C	~~6~~ 5
Destination	~~∞~~ 7



Dijkstra's Algorithm

- **Step 3:** Repeat the steps:
- Step 1 again: Find the node that takes the least cost to get to. We're done with node B, so node C has the next smallest estimate.

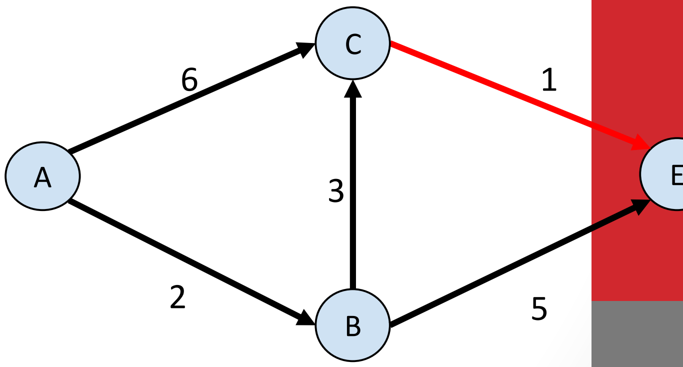
Node	Cost to Node
B	2
C	5
Destination	7



Dijkstra's Algorithm

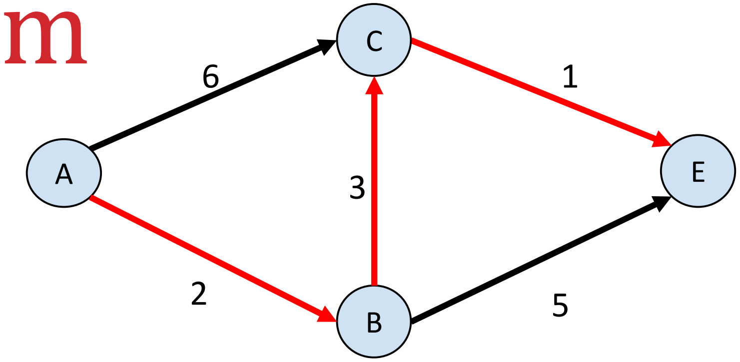
- Step 2 again: Update the cost of C's neighbours.
- We run Dijkstra's algorithm for every node (you don't need to run it
- for the finish node).
- At this point, you know
  - It takes 2 minutes to get to node B.
  - It takes 5 minutes to get to node C.
  - It takes 6 minutes to get to the destination.

Node	Cost to Node
B	2
C	5
Destination	~~7~~ 6

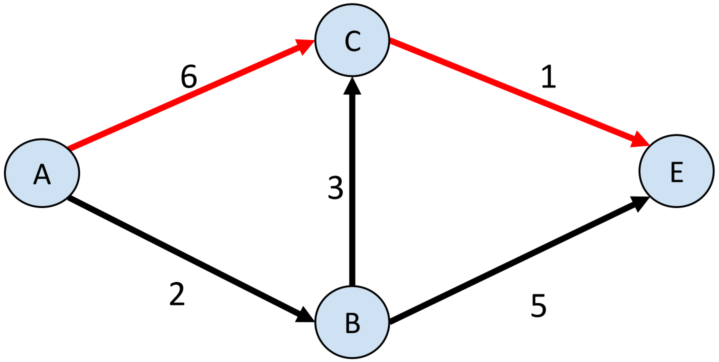


Dijkstra’s Algorithm

- So the final path is

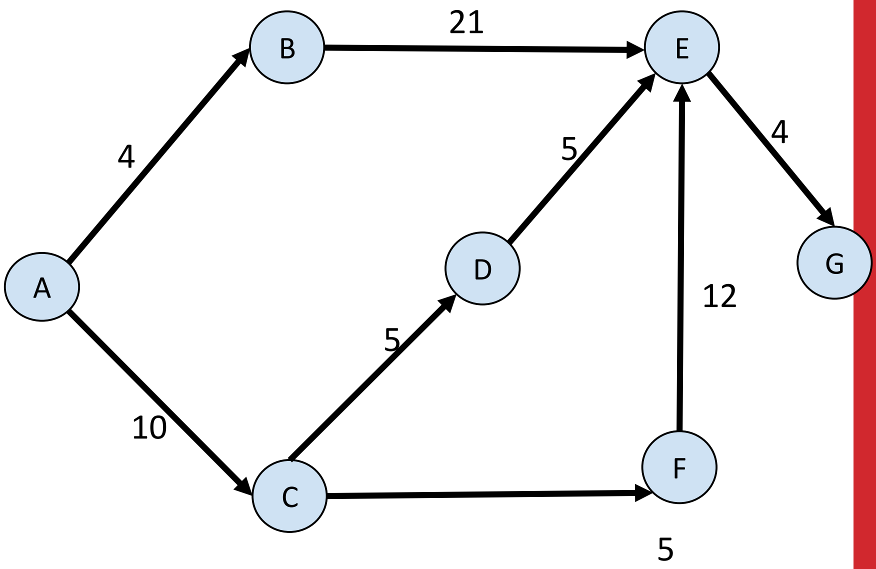


- BFS wouldn’t have found this as the shortest path, because it has three segments.
- And there’s a way to get from the start to the destination in two segments.



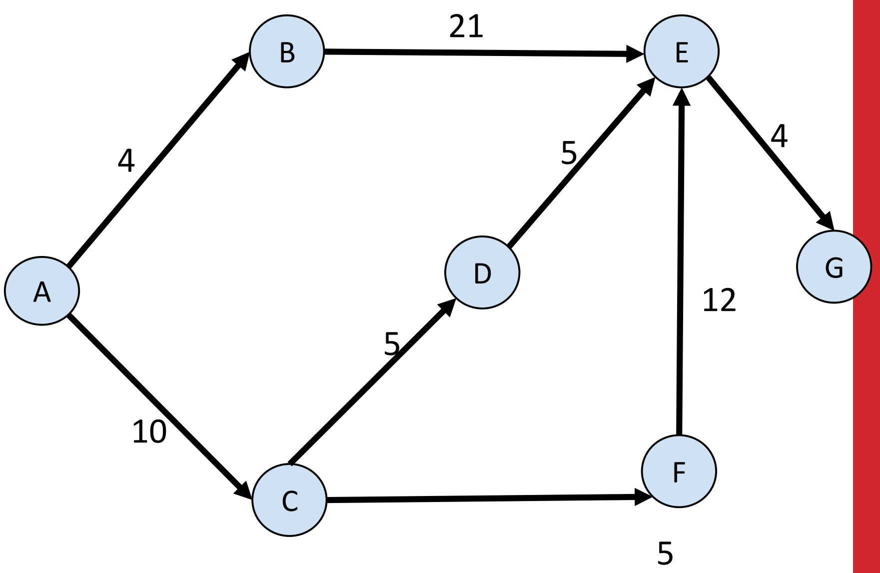
Dijkstra's Algorithm - Example

Node	Cost to Node
A	0
B	∞
C	∞
D	∞
E	∞
F	∞
Destination	∞



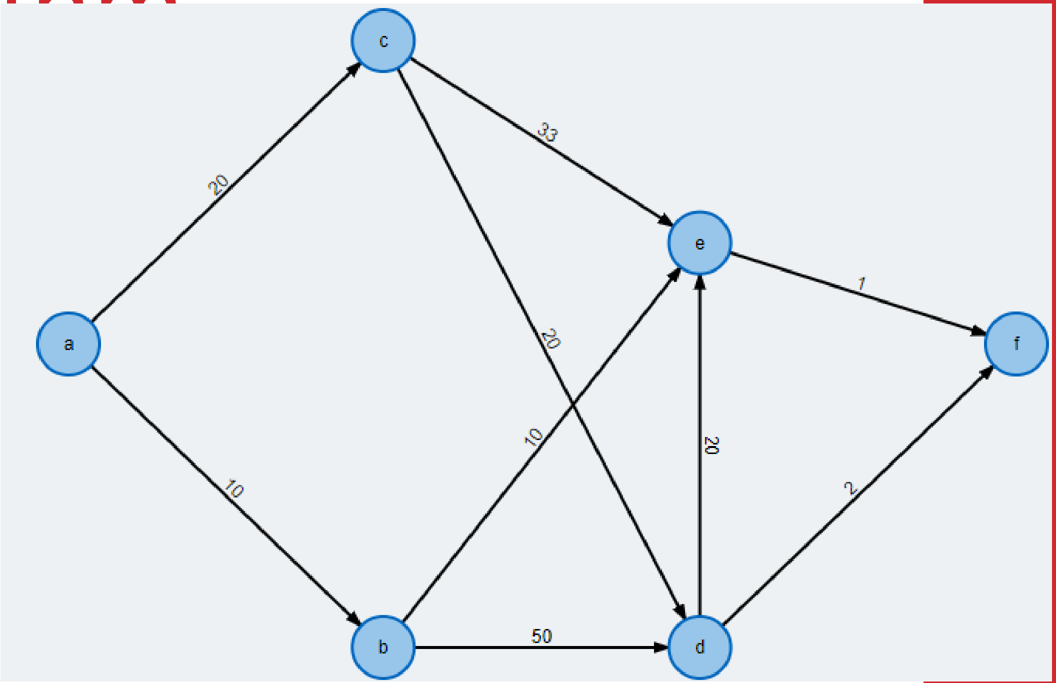
Dijkstra’s Algorithm - Example

Node	Cost to Node
A	0
B	4
C	10
D	15
E	~~25~~ 20
F	18
Destination	~~∞~~ ~~34~~ 24



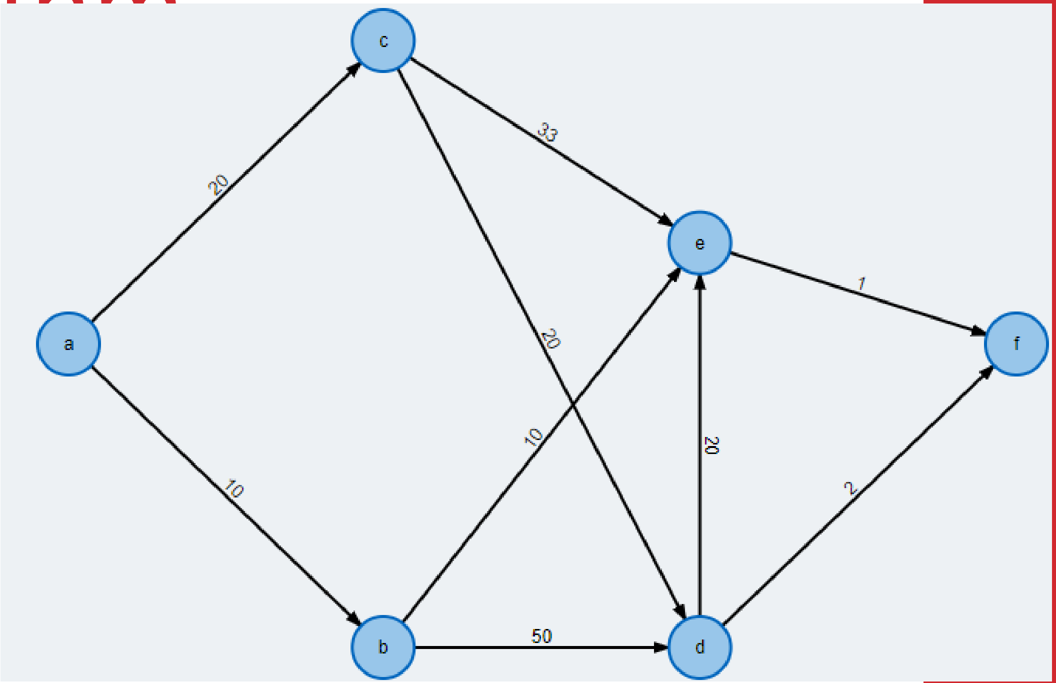
Dijkstra's Algorithm

Node	Initial.
A	0
B	∞
C	∞
D	∞
E	∞
Dest.	∞



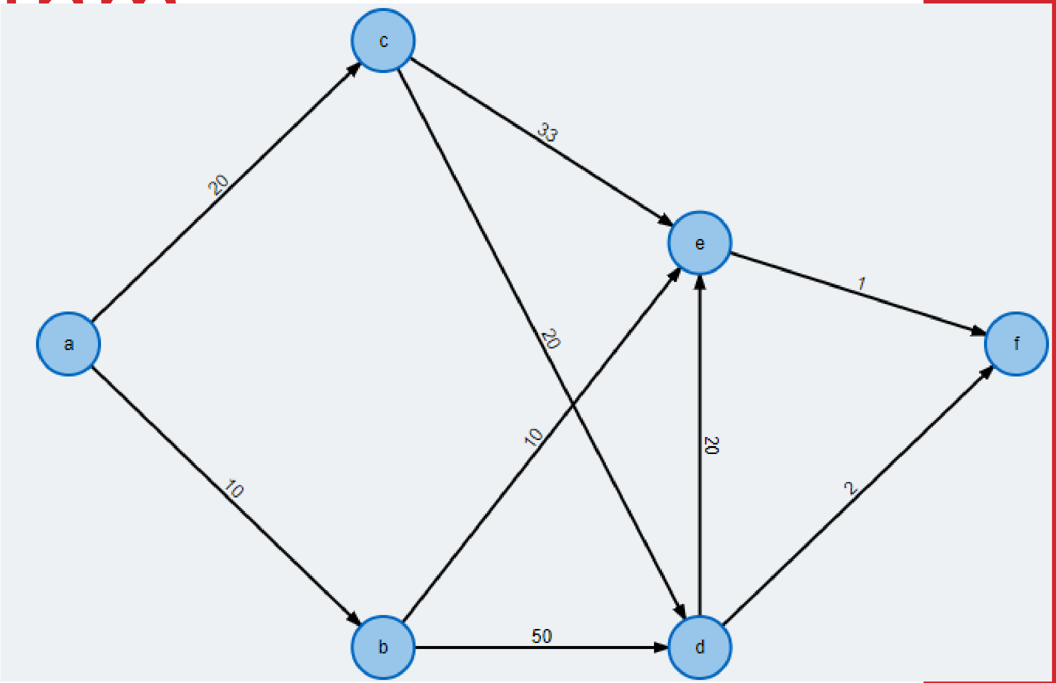
Dijkstra's Algorithm

Node	Initial.	Step1
A	0	0
B	∞	10
C	∞	20
D	∞	∞
E	∞	∞
Dest.	∞	∞

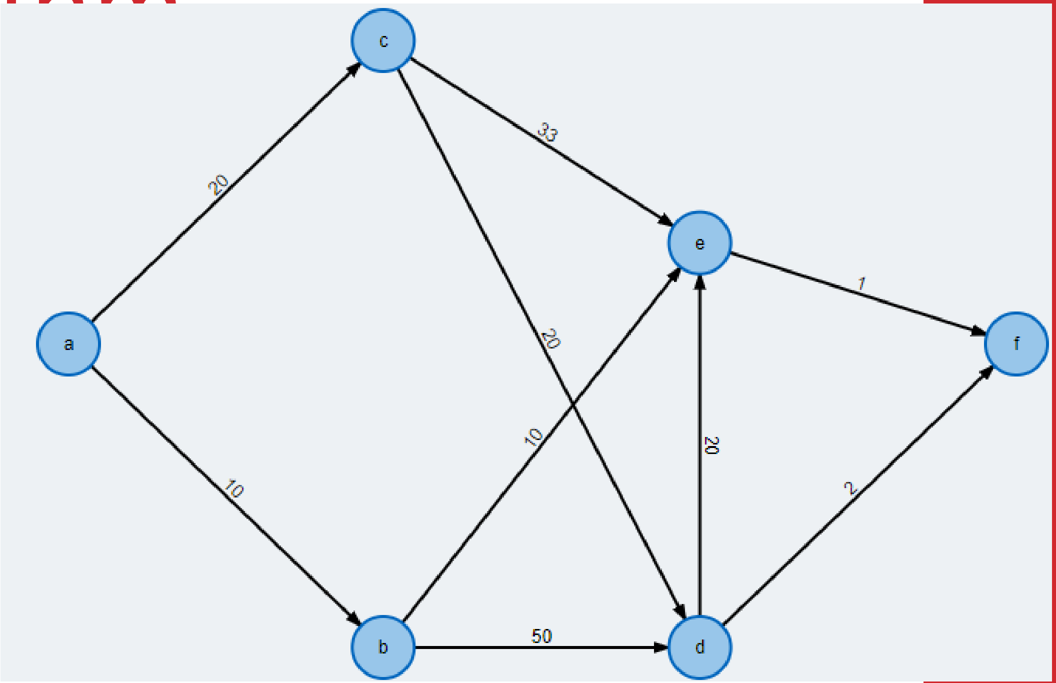


Dijkstra's Algorithm

Node	Initial.	Step1	Step2 (C)
A	0	0	0
B	∞	10	10
C	∞	20	20
D	∞	∞	40
E	∞	∞	53
Dest.	∞	∞	56



Dijkstra's Algorithm



Node	Initi al.	Step 1	Step2 (C)	Step3 (B)
A	0	0	0	0
B	∞	10	10	10
C	∞	20	20	20
D	∞	∞	40	40
E	∞	∞	53	20
Dest.	∞	∞	56	21

Dijkstra's Algorithm

Maintain 2 sets (arrays) of vertices:

S: a set of vertices whose shortest path from vertex s has been determined

Q: a set of vertices in V-S (uses Heaps)

*keys in Q are estimates of shortest path weights.

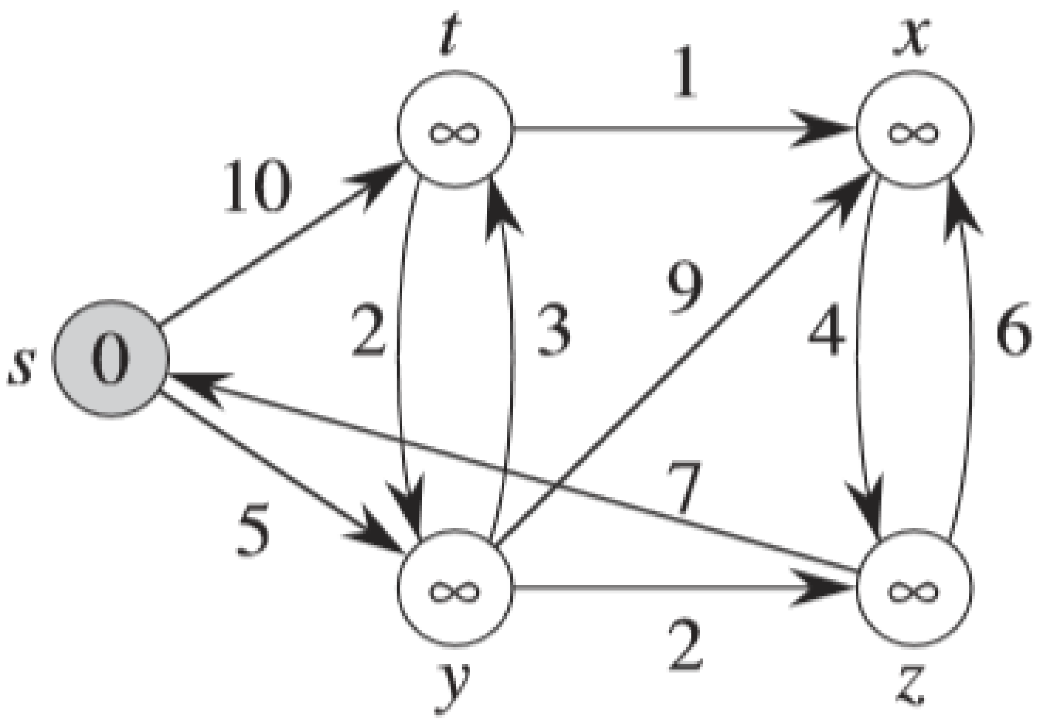
Dijkstra's Algorithm

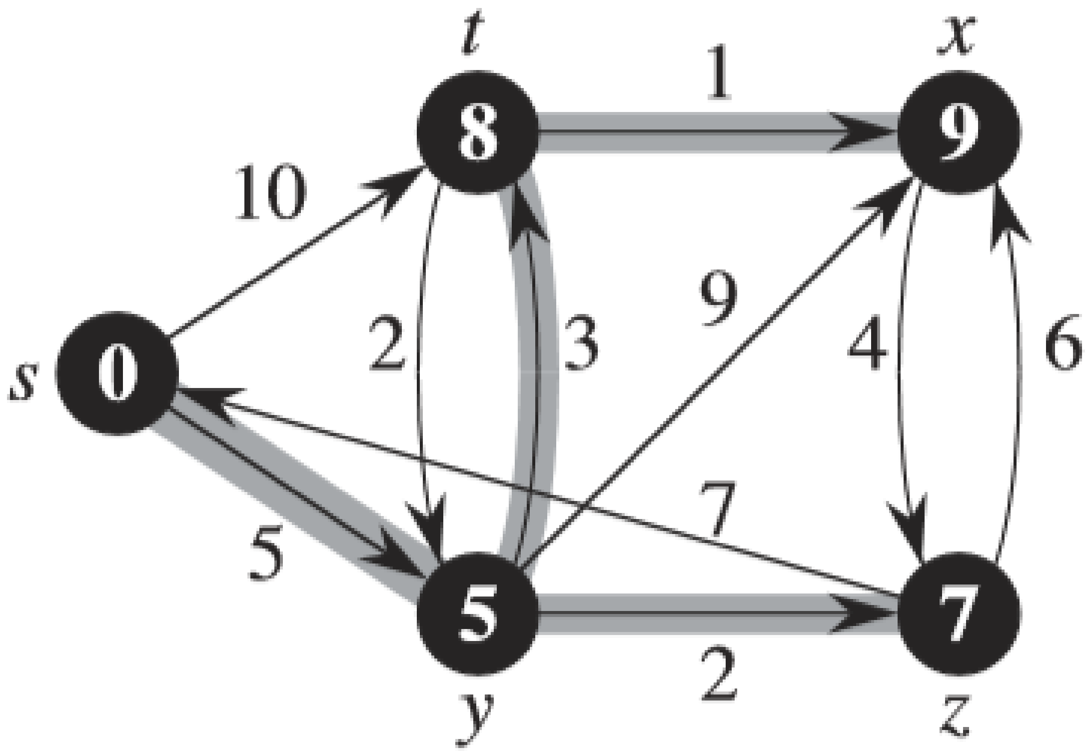
1. Store S in a heap with distance = 0
2. While there are vertices in the queue
   1. Delete Min a vertex v from queue
   2. For all adjacent vertices w:
      1. Compute new distance
      2. Update distance table
      3. Insert/update heap

Dijkstra's Algorithm

```
DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
S = ∅
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    S = S ∪ {u}
    for each vertex v ∈ G.Adj[u]
        RELAX(u, v, w)
```

Extra Example




t
x
1
8
9
10
2
3
9
4
6
s
0
7
5
5
7
2
y
z

Dijkstra's Algorithm - complexity

1. Each vertex is stored in the queue O(V)
2. Delete Min O(V log V)
3. Updating the queue (search and insert) O(log V)
   1. Performed at most for each edge O(E log V)

4. O(E log V + V log V) = O((E + V) log V)

Graphs with Negative Edge Costs

- If the graph has negative edge costs, then Dijkstra's algorithm does not work.
- Bellman-Ford algorithm solves the single-source shortest path when there may be negative weights in the graph.
- It checks if there is a negative-weight cycle that is reachable from a source vertex
  - If exists; it indicates there is no solution exists
  - If no cycle; then the algorithm produces the shortest paths and their weights

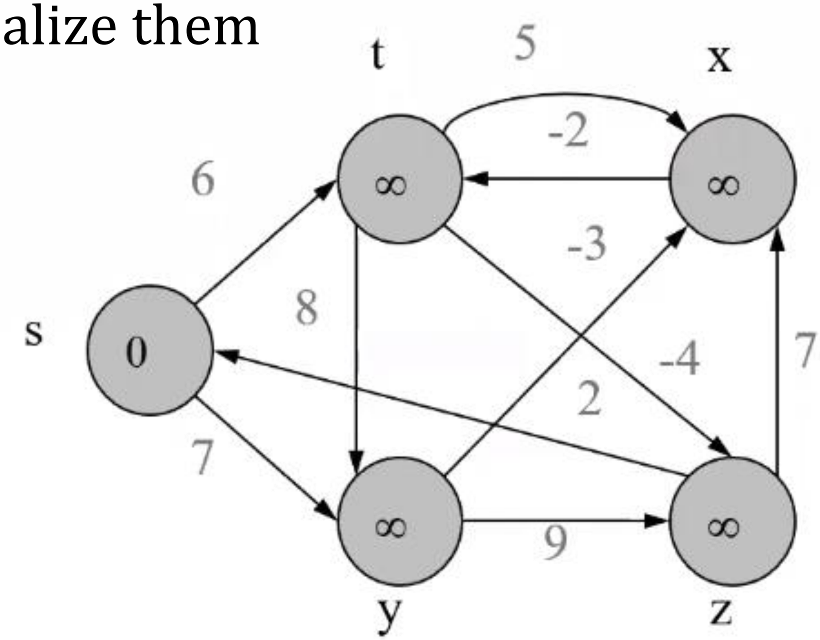
Graphs with Negative Edge Costs

- N-1 iterations should ensure that the shortest path is reached.
- The run-time is O(V.E)

Graphs with Negative Edge Costs

- We will visit all vertices and initialize them
- s is the source node

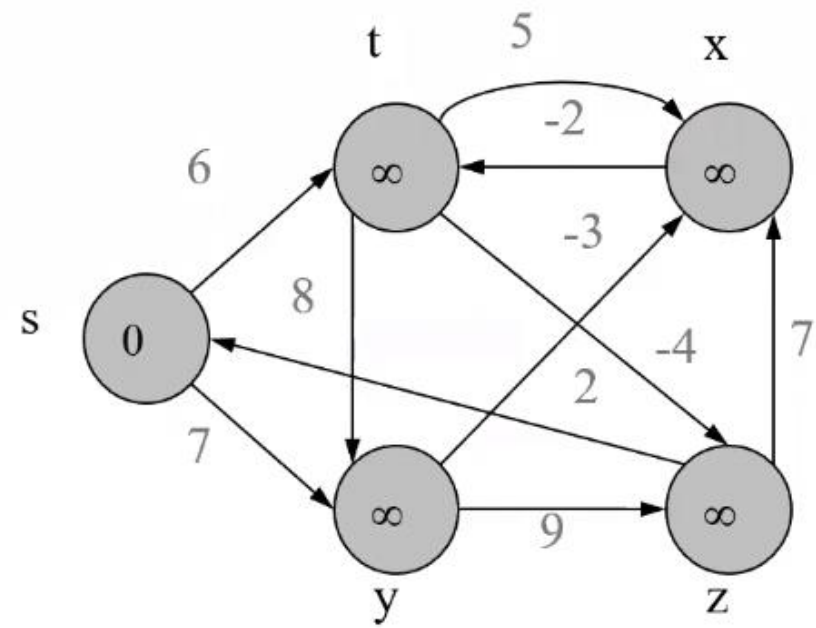
Node	Initial.
s	0
t	∞
y	∞
x	∞
z	∞



Graphs with Negative Edge Costs

- The adjacent of s are y and t.

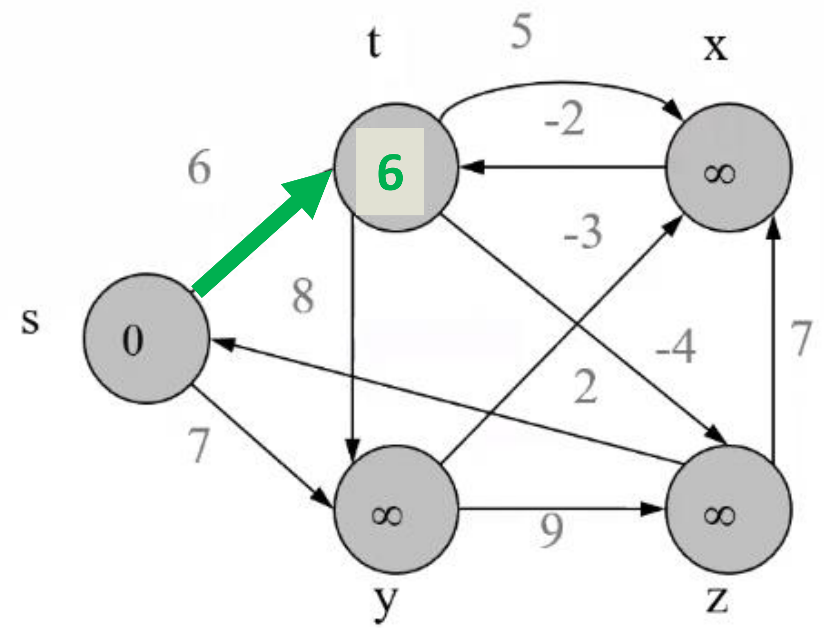
Node	Initial.	Iter. 1
s	0	
t	∞	
y	∞	
x	∞	
z	∞	



Graphs with Negative Edge Costs

- The adjacent of s are y and t.

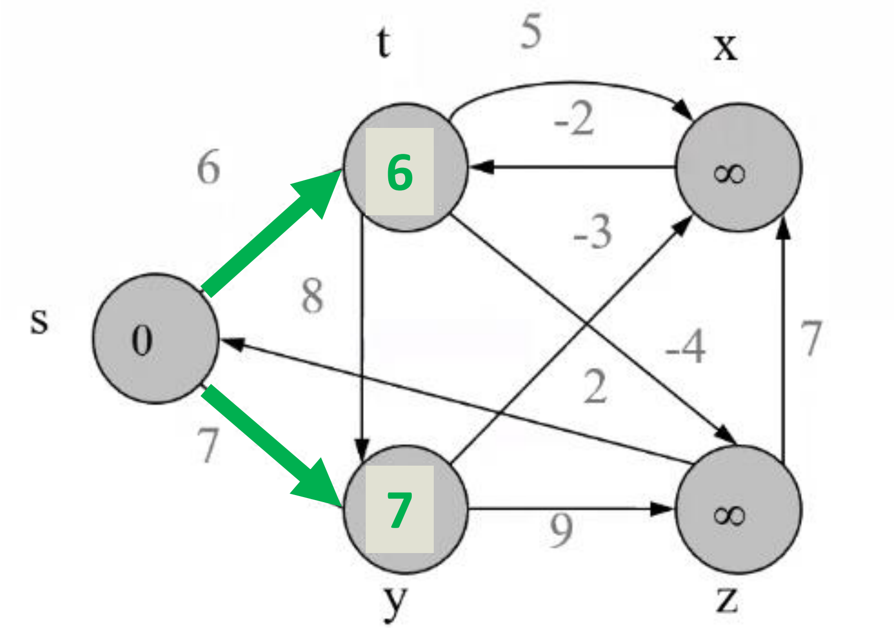
Node	Initial.	Iter. 1
s	0	
t	∞	6
y	∞	
x	∞	
z	∞	



Graphs with Negative Edge Costs

- The adjacent of s are y and t.

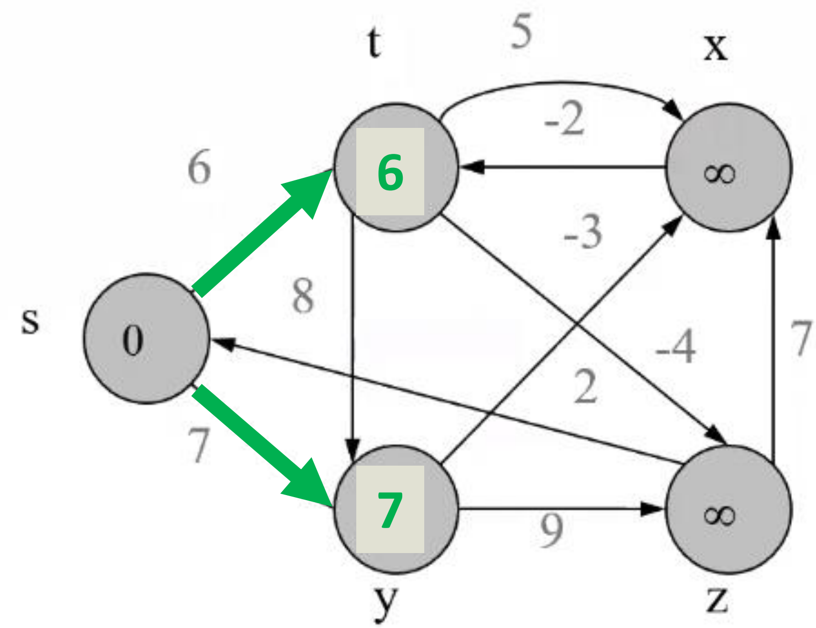
Node	Initial.	Iter. 1
s	0	
t	∞	6
y	∞	7
x	∞	
z	∞	



Graphs with Negative Edge Costs

- Now we can reach x & z.
- We will check for all edges.
- Check for X

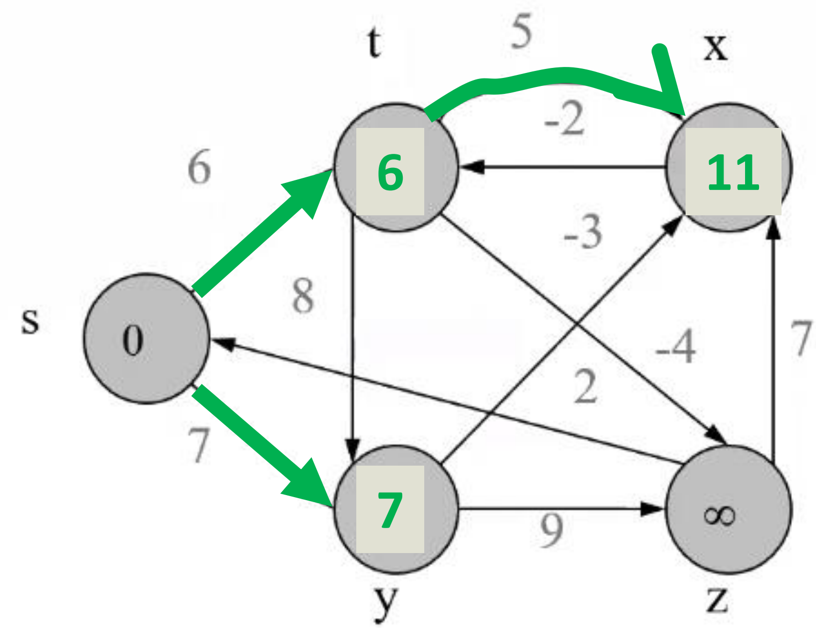
Node	Initial.	Iter. 1	Iter. 2
s	0	0	
t	∞	6	
y	∞	7	
x	∞	∞	
z	∞	∞	



Graphs with Negative Edge Costs

- Now we can reach x & z.
- We will check for all edges.
- Check for X

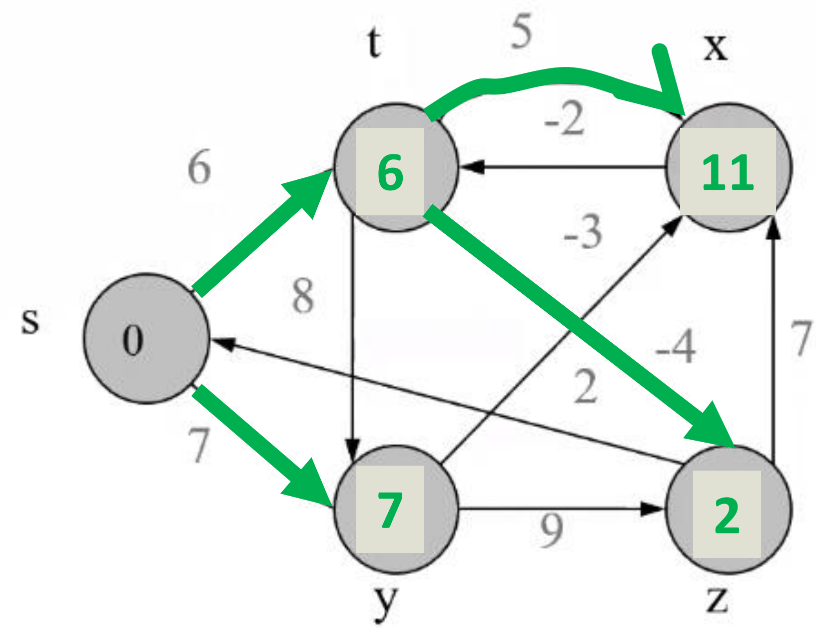
Node	Initial.	Iter. 1	Iter. 2
s	0	0	
t	∞	6	6
y	∞	7	7
x	∞	∞	11
z	∞	∞	



Graphs with Negative Edge Costs

- Now we can reach x & z.
- We will check for all edges.
- Check for X

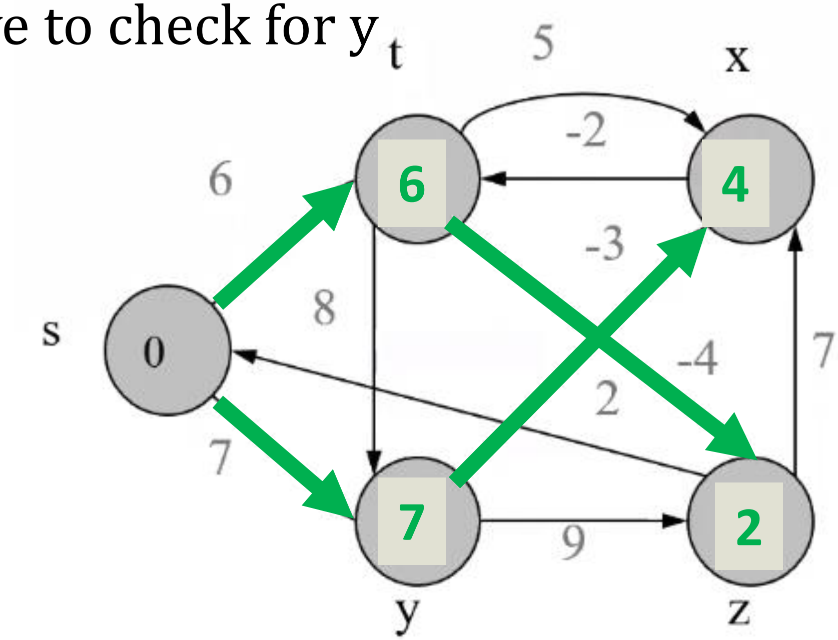
Node	Initial.	Iter. 1	Iter. 2
s	0	0	0
t	∞	6	6
y	∞	7	7
x	∞	∞	11
z	∞	∞	2



Graphs with Negative Edge Costs

- Now we are done with t, we have to check for y
- y to z = 16. y to x = 4.

Node	Initial.	Iter. 1	Iter. 2
s	0	0	0
t	∞	6	6
y	∞	7	7
x	∞	∞	~~11~~ 4
z	∞	∞	2



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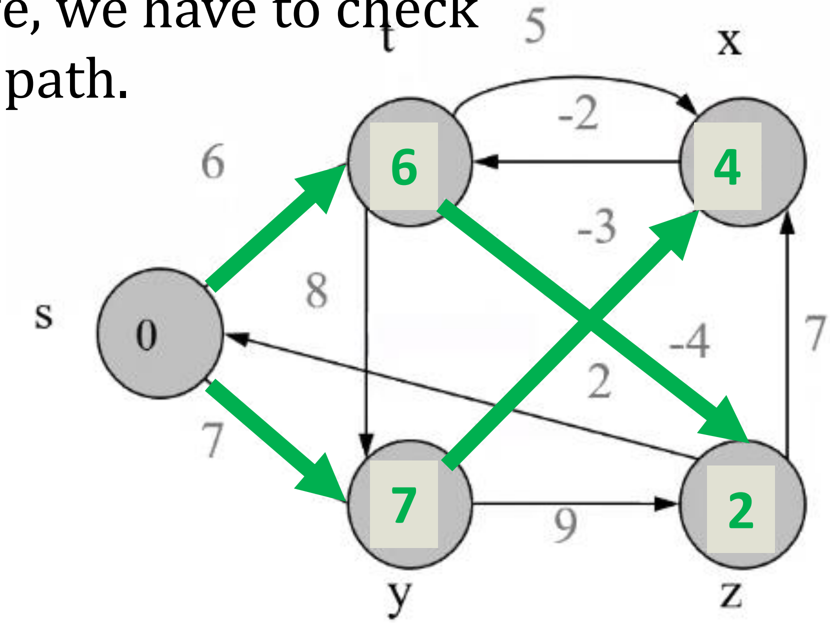
Graphs with Negative Edge Costs

- After the new update on the edge, we have to check for all edges if there is a shorter path.
- We can find x->t

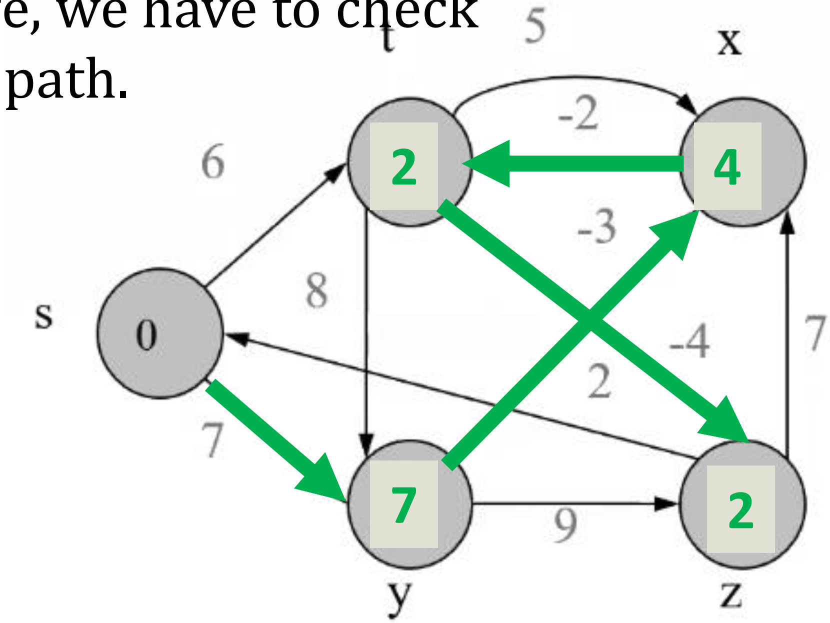
Node	Initial.	Iter. 1	Iter. 2	Iter. 2
s	0	0	0	
t	∞	6	6	
y	∞	7	7	
x	∞	∞	4	
z	∞	∞	2	



Graphs with Negative Edge Costs

- After the new update on the edge, we have to check for all edges if there is a shorter path.
- We can find x->t

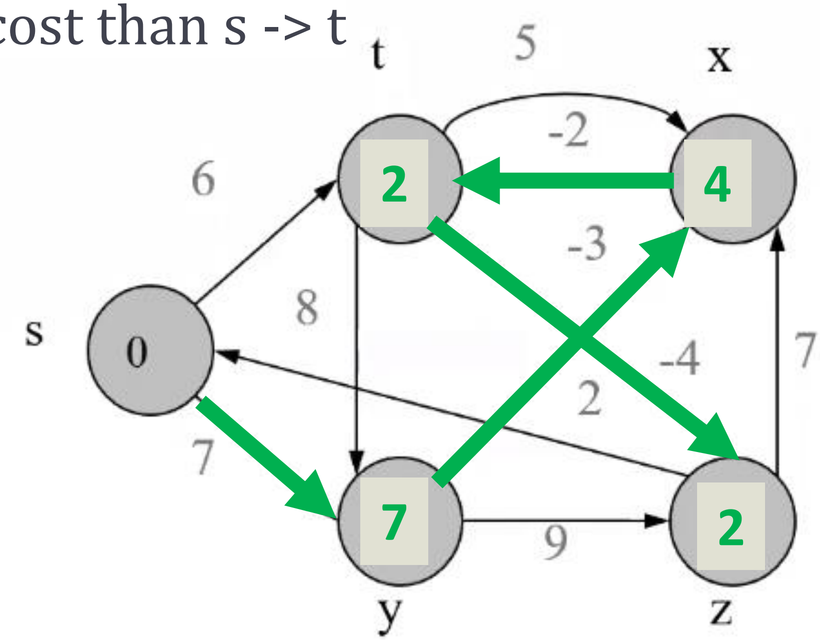
Node	Initial.	Iter. 1	Iter. 2	Iter. 3
s	0	0	0	0
t	∞	6	6	~~6~~ 2
y	∞	7	7	7
x	∞	∞	4	4
z	∞	∞	2	2



Graphs with Negative Edge Costs

from s -> y -> x -> t gives a shorter cost than s -> t

Node	Initial.	Iter. 1	Iter. 2	Iter. 3
s	0	0	0	0
t	∞	6	6	2
y	∞	7	7	7
x	∞	∞	4	4
z	∞	∞	2	2



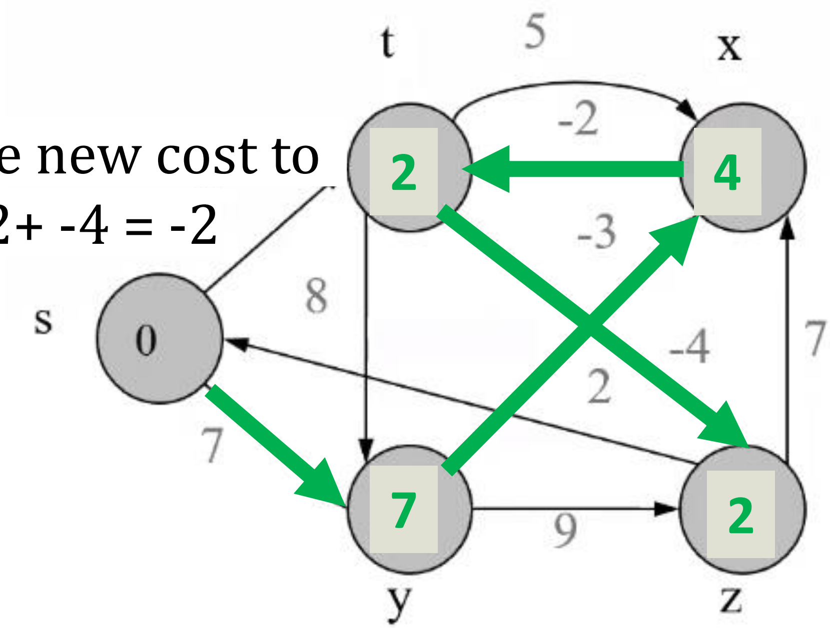
Graphs with Negative Edge Costs

Iteration 4

We check for all vertices

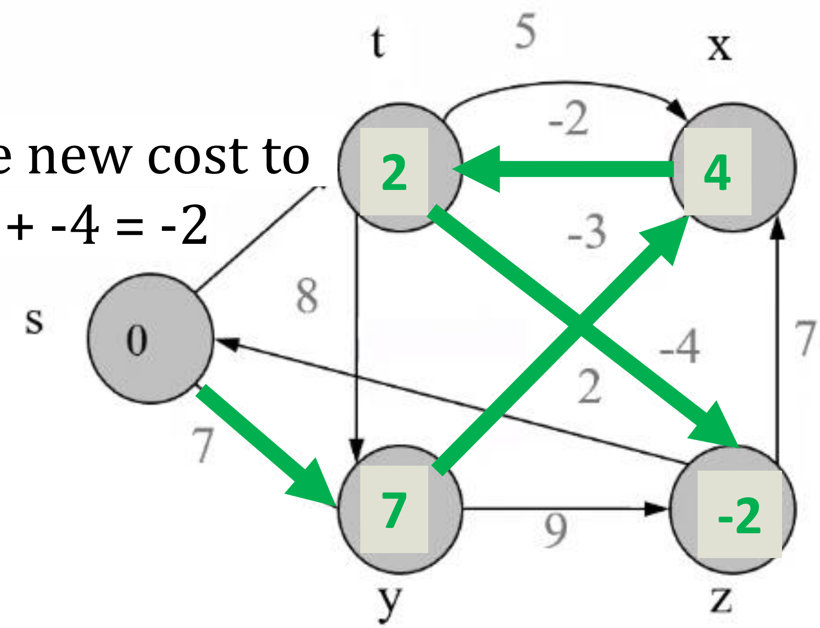
We can notice a change in t -> z (the new cost to reach t is 2, and from t -> z = -4) = 2+ -4 = -2

Node	Initial.	Iter. 1	Iter. 2	Iter. 3	Iter. 4
s	0	0	0	0	
t	∞	6	6	2	
y	∞	7	7	7	
x	∞	∞	4	4	
z	∞	∞	2	2	



Graphs with Negative Edge Costs

Iteration 4
We check for all vertices
We can notice a change in t -> z (the new cost to reach t is 2, and from t -> z = -4) = 2+ -4 = -2



Node	Initial.	Iter. 1	Iter. 2	Iter. 3	Iter. 4
s	0	0	0	0	0
t	∞	6	6	2	2
y	∞	7	7	7	7
x	∞	∞	4	4	4
z	∞	∞	2	2	-2

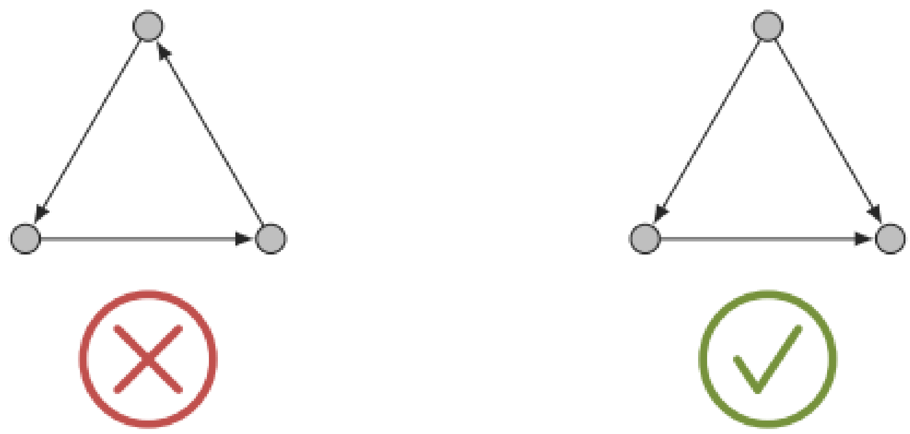
Bellman-Ford Algorithm

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
for i = 1 to |G.V| - 1
    for each edge (u, v) ∈ G.E
        RELAX(u, v, w)
for each edge (u, v) ∈ G.E
    if v.d > u.d + w(u, v)
        return FALSE
return TRUE
```

O(VE)

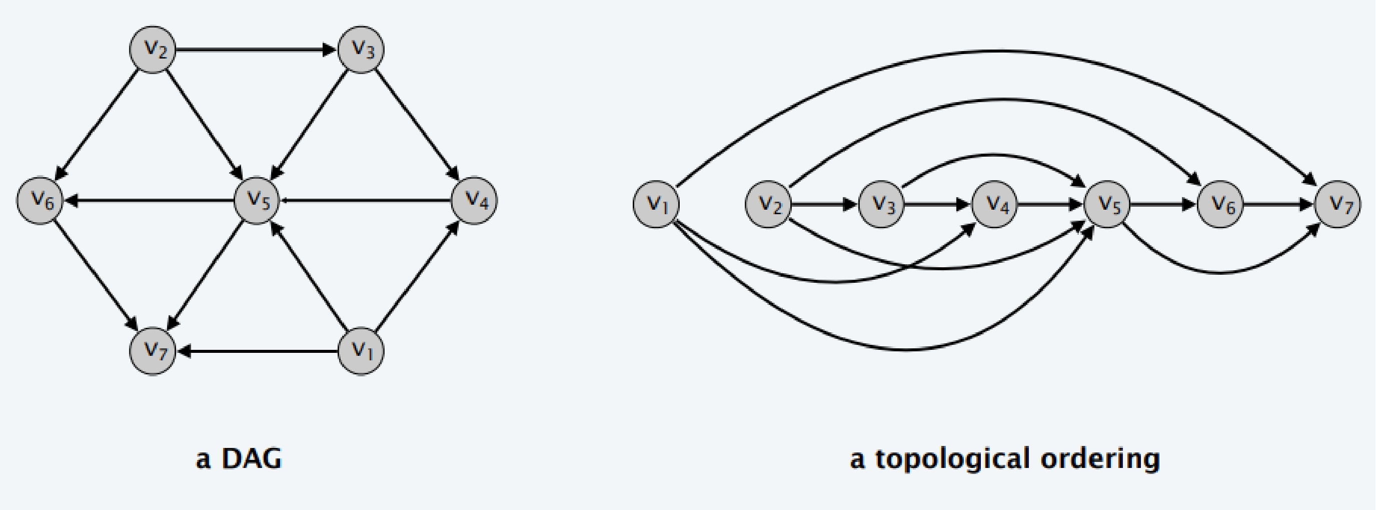
Acyclic Graphs

- We want to find the shortest path in acyclic graph (Directed Acyclic Graph)
- DAG contains no cycles



Acyclic Graphs

- If the graph is acyclic, we can use Bellman-Ford, but it takes O(VE)
- A better solution is to use Topological sort:
  - Initialize distances to all vertices as infinite and distance to source as 0
  - Then find a topological sorting of the graph

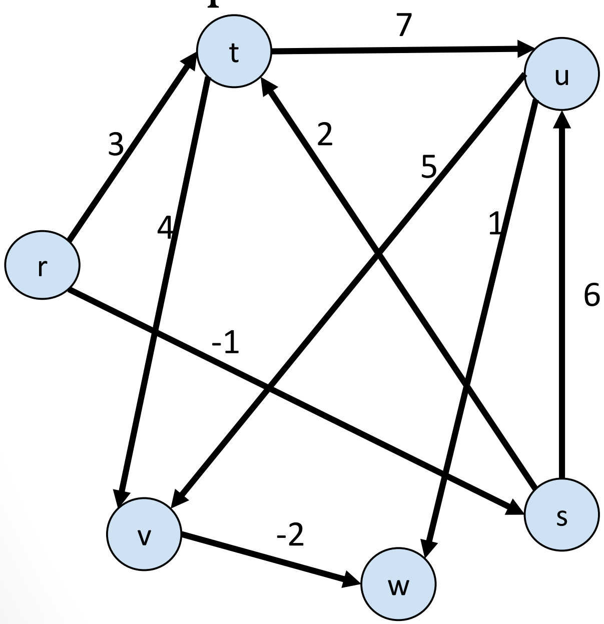


Acyclic Graphs

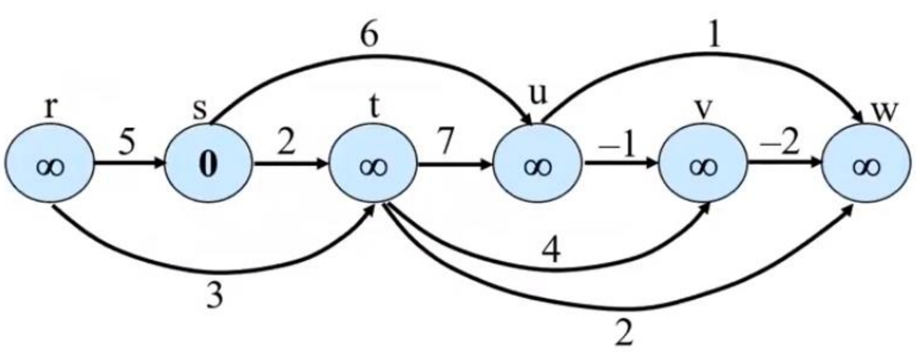
- Precedence constraints: Edge (v_i , v_j) means task v_i must occur before v_j
- Examples of DAG
- Course prerequisite graph: course v_i must be taken before v_j
- Compilation: module v_i must be compiled before v_j
- Pipeline of computing jobs: output of job v_i needed to determine input of job v_j

Acyclic Graphs

- Topological sort represents a linear ordering of a graph
- Example



DAG



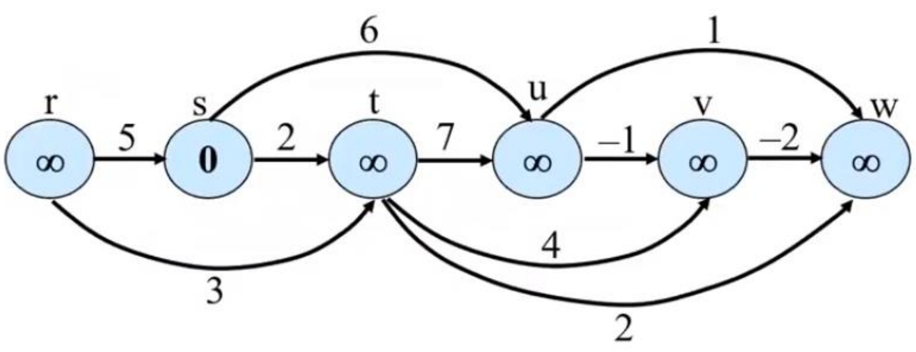
Topologically sorted

Acyclic Graphs

- The idea: process vertices on each shortest path from left to right
- Every path in DAG is a subsequence of topologically sorted vertex order. So processing vertices in that order will do each path in forward order
- Just one pass.
- Time complexity O(V + E)

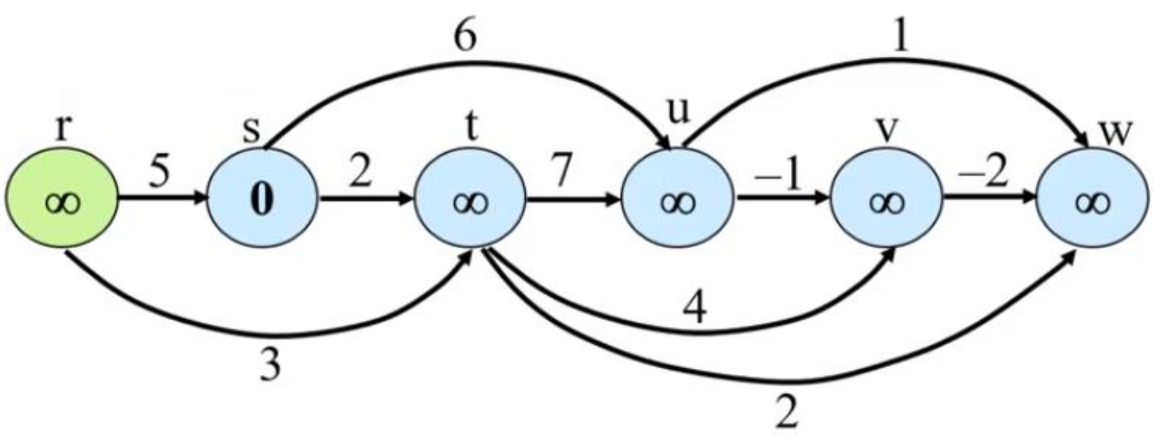
Acyclic Graphs

- Topologically sorted graph
- Now we have vertex s as the source
- We want to find the shortest path from s to all vertices
- Start with r, what is the path from s to r?
  - There is no path (infinity)



Acyclic Grap

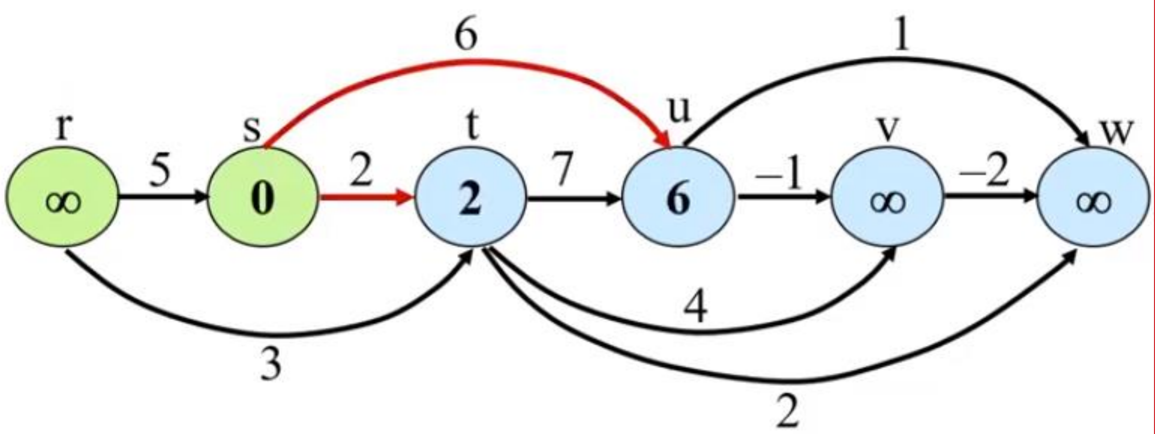
- So the first iteration, r = ∞



- Now the second pass
- Take the adjacent of s. From s to t =2, which is less than ∞, so update t and the predecessor is s
- From s to u is the same, 6 is less than ∞, so update u and the predecessor is s

Acyclic Graph

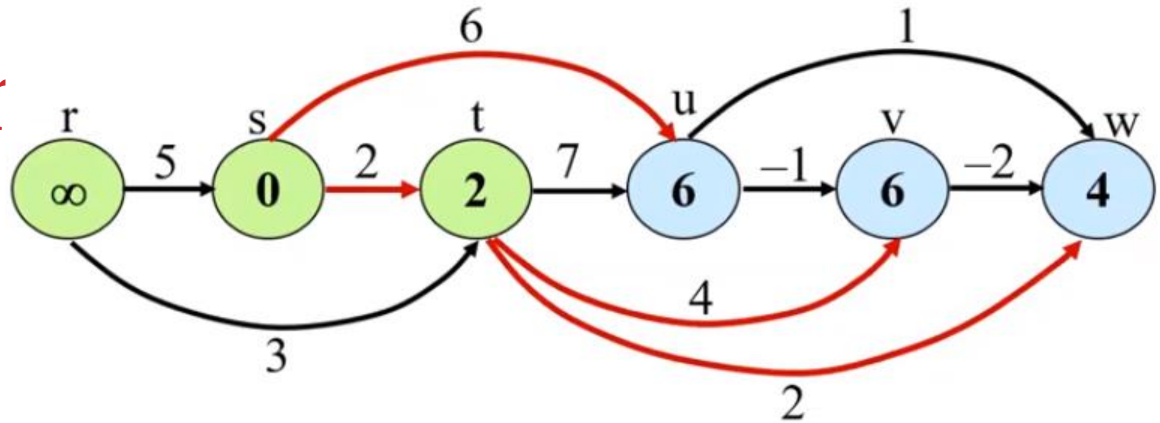
- Next iteration, check the adjacents of t



- From t to v is 2+4 = 6 which is less than ∞, so update v and the predecessor is t
- From t to w is 2+2 = 4 which is less than ∞, so update w and the predecessor is t

Acyclic Grapl

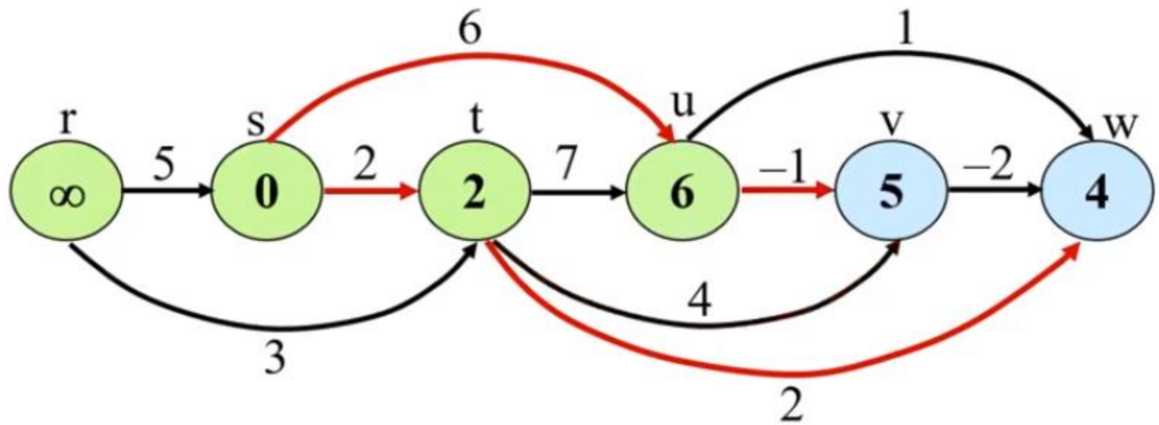
- Next iteration, check the adjacents of u



- From u to v is 6 + -1 = 5 which is less than 6, so update v and the predecessor is u
- From u to w is 6 + 1 = 7 which is more than 4, so no updates

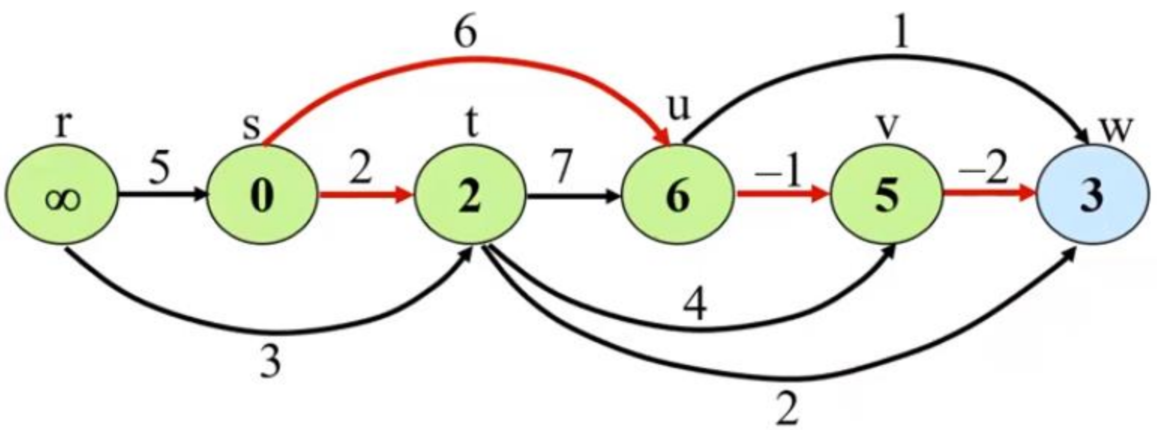
Acyclic Grapl

- Next iteration, check the adjacents of v



- From v to w is 5 + -2 = 3 which is less than 4, so update w and the predecessor is v instead of t

Acyclic Graph



- We are left with 1 iteration for w
- Notice that w has no adjacents
- Thus we reached the shortest path from the source s