Discrete Mathematics (COMP233)

Ahmad Hamo

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Logic

- Accepted rules for making precise statements
- Logic for computer science: programming, artificial intelligence, logic circuits, database.
- Represents knowledge precisely
- Helps to extract information (inference)

Logical Thinking

- **formal logic**: symbolic manipulation of notation; logical not-thinking
- **propositional logic**: propositional calculus allows us to make logical deductions formally
- **predicate logic**: make a proposition to depend on a variable and we get a predicate; here the logical deductions include quantifiers (for all, there exists) in front of the predicates
- **methods of proof: direct, by contraposition, by contradiction** use what you learned in formal/symbolic logic, to guide your reasoning on mathematical proofs (written in paragraph form)
- **logic in programming**
  - imperative programming: conditional statements (if-then-else, do-while)
  - Logic programming languages (e.g. prolog): uses the rules of predicate logic
- **logic in circuits**

Propositional Logic

- Propositional Logic
- Logic of Compound Statements
- Propositional Equivalences

- Conditional Statements
- Logical Equivalences

- Valid and Invalid Arguments
- Applications: Digital Logic Circuits
- Predicates and Quantifiers
- Logic of Quantified Statements

Propositional Logic

Proposition: A proposition (or Statement) is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

Examples

1. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

Paris is the capital of France.

This makes a declarative statement, and hence is a proposition. The proposition is TRUE (T).

Examples (Propositions Cont.)

2. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

Can Ali come with you?.

This is a question not the declarative sentence and hence not a proposition.

Examples (Propositions Cont.)

3. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

Take two aspirins.

This is an imperative sentence not the declarative sentence and therefore not a proposition.

Examples (Propositions Cont.)

4. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

$x + 4 > 9$.

Because this is true for certain values of x (such as $x = 6$) and false for other values of x (such as $x = 5$), it is not a proposition.

Examples (Propositions Cont.)

5. Is the following sentence a proposition? If it is a proposition, determine whether it is true or false.

   He is a college student.

   Because truth or falsity of this proposition depend on the reference for the pronoun *he*. it is not a proposition.

Logical Form and Logical Equivalence

- The central concept of deductive logic is the concept of **argument form**. An argument is a *sequence of statements* aimed at demonstrating the truth of an assertion.
- The **assertion** at the end of the sequence is called the conclusion, and the preceding statements are called premises.
- To have <u>confidence in the conclusion</u> that you draw from an argument, you must be sure that the premises are acceptable on their own merits or follow from other statements that are known to be true.

Example 1 – *Identifying Logical Form*

- Fill in the blanks below so that argument (b) has the same form as argument (a). Then represent the common form of the arguments using letters to stand for component sentences.

a. **If** Jane is a math major **or** Jane is a computer science
- major, **then** Jane will take Math 150.
- Jane is a computer science major. **Therefore**, Jane will take Math 150.

- b. **If** logic is easy **or** (1?) , **then** (2?) .
- I will study hard.
- **Therefore**, I will get an A in this course.

Example 1: Solution

- 1. I (will) study hard.

- 2. I will get an A in this course.

- **Common form:**

  If p or q, then r .

  q.

  Therefore, r .

Notations

- The small letters are commonly used to denote the propositional variables, that is, variables that represent propositions, such as, p, q, r, s, ….

- The truth value of a proposition is
  - **true**, denoted by T or 1, if it is a true proposition and
  - **false**, denoted by F or 0, if it is a false proposition.

Compound Propositions

Producing new propositions from existing propositions.

Logical Operators or Connectives

1. Not $\neg$
2. And $\wedge$
3. Or $\vee$
4. Exclusive or $\oplus$
5. Implication $\rightarrow$
6. Biconditional $\leftrightarrow$

Logical Connectives

Used to put simple (atomic) statements together to make **compound statements**

not	$\sim$	negation
and	$\wedge$	conjunction
or	$\vee$	disjunction
if-then	$\rightarrow$	conditional
if-and-only-if	$\leftrightarrow$	biconditional

Ex. (Group Exercise)

Let **p** be "It is winter," **q** be "It is cold," and **r** be "It is raining." Write the following statements symbolically.

- It is winter but it is not cold. $p \wedge \sim q$
- Neither is it winter nor is it cold. $\sim p \wedge \sim q$
- It is not winter if it is not cold. $\sim q \rightarrow \sim p$
- It is not winter but it is raining or cold. $\sim p \wedge (r \vee q)$

Compound Propositions

Negation of a proposition

Let p be a proposition. The negation of p, denoted by $\neg p$ *(also denoted by ~p)*, is the statement

"It is not the case that p".

The proposition $\neg p$ is read as "not p". The truth values of the negation of p, $\neg p$, is the opposite of the truth value of p.

Examples

1. Find the negation of the following proposition

p : Today is Friday.

The negation is

$\neg p$ *:* It is not the case that today is Friday.

This negation can be more simply expressed by

$\neg p$ *:* Today is not Friday.

Examples

2. Write the negation of

“6 is negative”.

The negation is

“It is not the case that 6 is negative”.

or “6 is nonnegative”.

Negation Truth Table (NOT)

- Unary Operator, Symbol: $\neg$

p	$\neg p$
true	false
false	true

Conjunction (AND)

Definition

Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$, is the proposition "*p and q*".

The conjunction $p \wedge q$ is true when *p and q* are both true and is false otherwise.

Examples

1. Find the conjunction of the propositions p and q, where

   p : Today is Friday.

   q : It is raining today.

   The conjunction is

   $p \wedge q$: Today is Friday and it is raining today.

Truth Table (AND)

- Binary Operator, Symbol: $\wedge$

p	q	$p \wedge q$
true	true	true
true	false	false
false	true	false
false	false	false

Disjunction (OR)

Definition

Let p and q be propositions. The disjunction of p and q, denoted by $p \vee q$, is the proposition "*p or q*".

The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Examples

1. Find the disjunction of the propositions p and q, where

p : Today is Friday.

q : It is raining today.

The disjunction is

$p \vee q$: Today is Friday or it is raining today.

Truth Table (OR)

- Binary Operator, Symbol: $\vee$

p	q	$p \vee q$
true	true	true
true	false	true
false	true	true
false	false	false

Exclusive OR (XOR)

Definition

Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is the proposition "$p \oplus q$".

The *exclusive or*, $p \oplus q$, is true when exactly one of p and q is true and is false otherwise.

Examples

1. Find the *exclusive or* of the propositions p and q, where

   p : Atif will pass the course CS102.

   q : Atif will fail the course CS102.

The *exclusive or* is

$p \oplus q$: Atif will pass or fail the course CSC102.

Truth Table (XOR)

- Binary Operator, Symbol: ⊕

p	q	$p \oplus q$
true	true	false
true	false	true
false	true	true
false	false	false

Examples (OR vs XOR)

The following proposition uses the (English) connective "or". Determine from the context whether "or" is intended to be used in the inclusive or exclusive sense.

1. "Nabeel has one or two brothers".

A person cannot have both one and two brothers.
Therefore, "or" is used in the exclusive sense.

Examples (OR vs XOR)

2. To register for BSC you must have passed the qualifying exam or be listed as an Math major.

Presumably, if you have passed the qualifying exam and are also listed as an Math major, you can still register for BCS. Therefore, “or” is inclusive.

Statement Form

- **Definition**

A **statement form** (or **propositional form**) is an expression made up of statement variables (such as p, q, and r) and logical connectives (such as $\sim$, $\wedge$, and $\vee$) that becomes a statement when actual statements are substituted for the component statement variables. The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

To compute the truth values for a statement form, follow *rules similar to those used to evaluate algebraic expressions*. **First** evaluate the expressions within the innermost parentheses, **then** evaluate the expressions within the next innermost set of parentheses, and so forth **until** you have the truth values for the complete expression.

Example: Truth Table for XOR as Statement Form

Construct the truth table for the statement form (p∨q) ∧ ~(p∧q)

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Exercise:

Construct the truth table for the statement form $(\neg p)\vee(\neg q)$

p	q	$\neg p$	$\neg q$	$(\neg p)\vee(\neg q)$
true	true	false	false	**false**
true	false	false	true	**true**
false	true	true	false	**true**
false	false	true	true	**true**

Translating English to Logic

I did not buy a lottery ticket this week or I bought a lottery ticket and won the million dollar on Friday.

Let p and q be two propositions

p: I bought a lottery ticket this week.

q: I won the million dollar on Friday.

In logic form

$$\neg p\vee(p\wedge q)$$

Logical Equivalence

• **Definition**

Two *statement forms* are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted by writing $P \equiv Q$.

Two *statements* are called **logically equivalent** if, and only if, they have logically equivalent forms when identical component statement variables are used to replace identical component statements.

Example:
Show that the statement forms:

p ∧ q and q ∧ p

are logically equivalent.

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$p \wedge q$ and $q \wedge p$ always have the same truth values, so they are logically equivalent

Equivalence of Two Compound Propositions P and Q

1. Construct the truth table for P.
2. Construct the truth table for Q using the same proposition variables for identical component propositions.
3. Check each combination of truth values of the proposition variables to see whether the truth value of P is the same as the truth value of Q.

Equivalence Check

a. If in each row the truth value of P is the same as the truth value of Q, then P and Q are logically equivalent.

b. If in some row P has a different truth value from Q, then P and Q are not logically equivalent.

Example

- Prove that $\neg(\neg p) \equiv p$

Solution

p	¬p	¬ (¬p)
T	F	T
F	T	F

As you can see the corresponding truth values of p and ¬ (¬p) are same, hence equivalence is justified.

Example

Show that the proposition forms ¬(p∧q) and ¬p ∧ ¬q are NOT logically equivalent.

p	q	¬p	¬q	(p∧q)	¬(p∧q)	¬p∧¬q
T	T	F	F	T	F	F
T	F	F	T	F	**T**	**F**
F	T	T	F	F	**T**	**F**
F	F	T	T	F	T	T

Here the corresponding truth values differ and hence equivalence does not hold

De Morgan's laws

De Morgan's laws state that:
The negation of an **and** proposition is logically equivalent to the **or** proposition in which each component is negated.

The negation of an **or** proposition is logically equivalent to the **and** proposition in which each component is negated.

Symbolically (De Morgan's Laws)

1. $\neg(p \wedge q) \equiv \neg p \vee \neg q$

2. $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Example: Equivalent Statements

Show that the statements $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ are logically equivalent.

P	Q	$\neg(P \wedge Q)$	$(\neg P) \vee (\neg Q)$	$\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$
true	true	false	false	true
true	false	true	true	true
false	true	true	true	true
false	false	true	true	true

→ **De Morgan's Laws**

Applying De-Morgan's Law

Question: Negate the following compound Propositions

1. John is six feet tall and he weights at least 200 pounds.

2. The bus was late or Tom's watch was slow.

Solution

a) John is not six feet tall or he weights less than 200 pounds.

b) The bus was not late and Tom's watch was not slow.

Inequalities and De Morgan's Laws

Question Use De Morgan's laws to write the negation of

$$-1 < x \leq 4$$

Solution: The given proposition is equivalent to

$$-1 < x \quad \text{and} \quad x \leq 4,$$

By De Morgan's laws, the negation is

$$-1 \geq x \quad \text{or} \quad x > 4.$$

Tautology (حشو، تكرار) and Contradiction (تناقض)

Definition A **tautology (t)** is a proposition form that is always true regardless of the truth values of the individual propositions substituted for its proposition variables. A proposition whose form is a tautology is called a tautological proposition.

Definition A **contradiction (c)** is a proposition form that is always false regardless of the truth values of the individual propositions substituted for its proposition variables. A proposition whose form is a contradiction is called a contradictory proposition.

Logical Tautologies - Examples

- Either it will rain tomorrow, or it won't rain.
- Bill will win the election, or he will not win the election.
- She is brave, or she is not brave.
- I will get in trouble or not get in trouble.

Example: Tautologies and Contradictions

Show that the statement form **p∨∼p** is a tautology and that the statement form **p∧∼p** is a contradiction.

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

↑ all T's so $p \vee \sim p$ is a tautology

↑ all F's so $p \wedge \sim p$ is a contradiction

Summary of Logical Equivalences

- Knowledge of logically equivalent statements is very useful for constructing arguments.
- It often happens that it is difficult to see how a conclusion follows from one form of a statement, whereas it is easy to see how it follows from a logically equivalent form of the statement.
- A number of logical equivalences are summarized in Theorem 2.1.1 for future reference.

Precedence Rules

1. $\sim$
2. $\wedge$ and $\vee$ (need parentheses to avoid ambiguity)
3. $\rightarrow$ and $\leftrightarrow$ (need parentheses to avoid ambiguity)
4. Parentheses may be used to override rules 1-3

Ex: $p \wedge \sim q$ means the same as $p \wedge (\sim q)$

$p \vee q \wedge r$ is ambiguous. Need to add parentheses: $(p \vee q) \wedge r$ or $p \vee (q \wedge r)$

Laws of Logic

1. Commutative laws

   $p \wedge q \equiv q \wedge p$; $p \vee q \equiv q \vee p$

2. Associative laws

   $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$; $p \vee (q \vee r) \equiv (p \vee q) \vee r$

3. Distributive laws

   $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Laws of Logic

4. Identity laws
   $p \wedge t \equiv p$; $p \vee c \equiv p$

5. Negation laws
   $p \vee \neg p \equiv t$; $p \wedge \neg p \equiv c$

6. Double negation law
   $\neg(\neg p) \equiv p$

7. Idempotent laws
   $p \wedge p \equiv p$; $p \vee p \equiv p$

Laws of Logic

8. Universal bound laws

$p \vee t \equiv t$; $p \wedge c \equiv c$

9. De Morgan's Laws

$\neg(p \wedge q) \equiv \neg p \vee \neg q$; $\neg(p \vee q) \equiv \neg p \wedge \neg q$

10. Absorption laws

$p \wedge (p \vee q) \equiv p$; $p \vee (p \wedge q) \equiv p$

11. Negation of *t* and *c*

$\neg t \equiv c$; $\neg c \equiv t$

Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology **t** and a contradiction **c**, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* **t** *and* **c**:	$\sim\mathbf{t} \equiv \mathbf{c}$	$\sim\mathbf{c} \equiv \mathbf{t}$

Example: Simplifying Statement Forms

Use Theorem 2.1.1 to verify the logical equivalence:

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$$

$$
\begin{aligned}
\sim(\sim p \wedge q) \wedge (p \vee q) &\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q) && \text{by De Morgan's laws} \\
&\equiv (p \vee \sim q) \wedge (p \vee q) && \text{by the double negative law} \\
&\equiv p \vee (\sim q \wedge q) && \text{by the distributive law} \\
&\equiv p \vee (q \wedge \sim q) && \text{by the commutative law for } \wedge \\
&\equiv p \vee \mathbf{c} && \text{by the negation law} \\
&\equiv p && \text{by the identity law.} \quad \blacksquare
\end{aligned}
$$

Another Example

Prove that $\neg[r \vee (q \wedge (\neg r \rightarrow \neg p))] \equiv \neg r \wedge (p \vee \neg q)$

$$
\begin{aligned}
&\neg[r \vee (q \wedge (\neg r \rightarrow \neg p))] \\
&\equiv \neg r \wedge \neg(q \wedge (\neg r \rightarrow \neg p)), && \text{De Morgan's law} \\
&\equiv \neg r \wedge \neg(q \wedge (\neg\neg r \vee \neg p)), && \text{Conditional rewritten as disjunction} \\
&\equiv \neg r \wedge \neg(q \wedge (r \vee \neg p)), && \text{Double negation law} \\
&\equiv \neg r \wedge (\neg q \vee \neg(r \vee \neg p)), && \text{De Morgan's law} \\
&\equiv \neg r \wedge (\neg q \vee (\neg r \wedge p)), && \text{De Morgan's law, double negation} \\
&\equiv (\neg r \wedge \neg q) \vee (\neg r \wedge (\neg r \wedge p)), && \text{Distributive law} \\
&\equiv (\neg r \wedge \neg q) \vee ((\neg r \wedge \neg r) \wedge p), && \text{Associative law} \\
&\equiv (\neg r \wedge \neg q) \vee (\neg r \wedge p), && \text{Idempotent law} \\
&\equiv \neg r \wedge (\neg q \vee p), && \text{Distributive law} \\
&\equiv \neg r \wedge (p \vee \neg q), && \text{Commutative law}
\end{aligned}
$$

Example: Password validity as a compound proposition

A certain small college sends the following instructions to its users when they are required to change their password:

Your password is valid only if it is at least 8 characters long, you have not previously used it as your password, and it contains at least three different types of characters (lowercase letters, uppercase letters, digits, non-alphanumeric characters).

This compound proposition involves seven different atomic propositions:

- p: the password is valid
- q: the password is at least 8 characters long
- r: the password has been used previously by you
- s: the password contains lowercase letters
- t: the password contains uppercase letters
- u: the password contains digits
- v: the password contains non-alphanumeric characters

The form of the compound proposition is":

"p, only if q and not r and at-least-three of {s, t, u, v} are true.