Discrete Mathematics (COMP233)

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2020/2021

Previous Lecture Summery

- Introduction to the Course
- Propositions
- Compound propositions
- Translating English to logic and logic to English.
- Logical Equivalences.
- De Morgan’s laws.
- Tautologies and Contradictions.
- Laws of Logic.

Today's Lecture

- Conditional propositions.
- Contraposition
- Converse and inverse of the Conditional
- Biconditional (if and only if)
- Interpreting Necessary and sufficient conditions

2.2: Conditional Statements

Conditional Statements - Implication

Definition: Let *p* and *q* be propositions.

The *conditional statement* $p \rightarrow q$,

is the proposition "*If p, then q*".

where *p* is called **hypothesis**, antecedent or premise.

q is called **conclusion** or consequence

Example: If $\underbrace{\text{4,686 is divisible by 6}}_{\text{hypothesis}}$, then $\underbrace{\text{4,686 is divisible by 3}}_{\text{conclusion}}$

Such a sentence is called *conditional* because the *truth of statement q is conditioned on the truth of statement p*.

When you make a logical inference or deduction, you reason *from* a hypothesis *to* a conclusion. Your aim is to be able to say, "*If* such and such is known, *then* something or other must be the case."

Truth Table for →

In Logic (& Math, CS, etc.): The only time a statement of the form ***if p then q*** is false is when the hypothesis (p) is true and the conclusion (q) is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Note: When the hypothesis is **false**, we say that the if-then statement is "vacuously true" or "true by default." In other words, it is true because it is not false.

For example, the statement "*all cell phones in the room are turned off*" will be true whenever **there are no cell phones in the room**.

Interpreting Conditional Statements

Examples

"The online user is sent a notification of a link error if the network link is down".

The statement is equivalent to

"If the network link is down, then the online user is sent a notification of a link error."

Using

p : The network link is down,
q : the online user is sent a notification of a link error.

The statement becomes (q if p)

$$p \rightarrow q.$$

Examples

"When you study the theory, you understand the material".

The statement is equivalent to (using if for when)

"If you study the theory, then you understand the material."

Using

p : you study the theory,
q : you understand the material.

The statement becomes (when p, q)

$$p \rightarrow q.$$

Examples

"*Studying the theory is sufficient for solving the exercise*".

The statement is equivalent to

"If you study the theory, then you can solve the exercise."

Using

p : you study the theory,

q : you can solve the exercise.

The statement becomes (p is sufficient for q)

$$p \to q.$$

Activity

- Show that

$$p \rightarrow q \equiv \neg p \vee q$$

This shows that a conditional proposition is simple a proposition form that uses **a not and an or**.

- Show that

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

This means that negation of ‘**if p then q**’ is logically equivalent to ‘**p and not q**’.

Solution

p	q	$p \rightarrow q$	$\neg p \vee q$	$\neg(p \rightarrow q)$	$p \wedge \neg q$
T	T	T	T		
T	F	F	F		
F	T	T	T		
F	F	T	T		

From the above table it is obvious that conditional proposition is equivalent to a "not or proposition".

The Negation of a Conditional Statement

- We can also proof that

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

Using the laws of logic

- Start from the logical equivalence $p \rightarrow q \equiv \sim p \vee q$
- Take the negation of both sides to obtain:

$$
\begin{aligned}
\sim(p \rightarrow q) &\equiv \sim(\sim p \vee q) \\
&\equiv \sim(\sim p) \wedge (\sim q) && \text{by De Morgan's laws} \\
&\equiv p \wedge \sim q && \text{by the double negative law.}
\end{aligned}
$$

Examples: **Negations** of some Conditionals

Proposition: If my car is in the repair shop, then I cannot get the class.

Negation: My car is in the repair shop and I can get the class.

Proposition: If Sara lives in Athens, then she lives in Greece.

Negation: Sara lives in Athens and she does not live in Greece.

Order of Operations for Logical Operators - Precedence Rules

1. $\sim$
2. $\wedge$ and $\vee$ (need parentheses to avoid ambiguity)
3. $\rightarrow$ and $\leftrightarrow$ (need parentheses to avoid ambiguity)
4. Parentheses may be used to override rules 1-3

Ex: $p \wedge \sim q$ means the same as $p \wedge (\sim q)$

$p \vee q \wedge r$ is ambiguous. Need to add parentheses: $(p \vee q) \wedge r$ or $p \vee (q \wedge r)$

Example: Truth Table for p∨~q→~p

Construct a truth table for the statement form p∨~q→~p.

- Hypothesis?
- Conclusion?

		conclusion		hypothesis	
p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$p \vee \sim q \to \sim p$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

English Phrases Meaning $p \rightarrow q$

- "p implies q"
- "if p, then q"
- "if p, q"
- "when p, q"
- "whenever p, q"
- "q follows from p"
- "q is implied by p"
- "q when p"
- "q whenever p"
- "q if p"
- "p only if q"
- "p is sufficient for q"
- "q is necessary for p"

Example: Password validity as a compound proposition

A certain small college sends the following instructions to its users when they are required to change their password:

Your password is valid only if it is at least 8 characters long, you have not previously used it as your password, and it contains at least three different types of characters (lowercase letters, uppercase letters, digits, non-alphanumeric characters).

This compound proposition involves seven different atomic propositions:

- p: the password is valid
- q: the password is at least 8 characters long
- r: the password has been used previously by you
- s: the password contains lowercase letters
- t: the password contains uppercase letters
- u: the password contains digits
- v: the password contains non-alphanumeric characters

The form of the compound proposition is":

"p, only if q and not r and at-least-three of {s, t, u, v} are true.

Example: (Three (or more) of four, formalized)

Translate the following sentence into propositional logic:

"p, only if q and not r and at-least-three-of {s, t, u, v} are true."

Solution:
The *tricky part* will be translating "at least three of {s, t, u, v} are true."
One relatively simple way to do it is to explicitly write out four cases:

$$(s \wedge t \wedge u) \vee (s \wedge t \wedge v) \vee (s \wedge u \wedge v) \vee (t \wedge u \wedge v)$$

Finally:

$$(p \rightarrow q) \wedge \neg r \wedge ((s \wedge t \wedge u) \vee (s \wedge t \wedge v) \vee (s \wedge u \wedge v) \vee (t \wedge u \wedge v))$$

The Contrapositive of a Conditional Statement
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• One of the most fundamental laws of logic is the equivalence between a conditional statement and its contrapositive.

• Definition

The **contrapositive** of a conditional statement of the form "If p then q" is

$$\text{If } \sim q \text{ then } \sim p.$$

Symbolically,

$$\text{The contrapositive of } p \to q \text{ is } \sim q \to \sim p.$$

• The fact is that

A conditional statement is logically equivalent to its contrapositive.

Solution

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

From the above table it is obvious that conditional proposition is equivalent to its "contraposition".

Example – *Writing the Contrapositive*

- Write each of the following statements in its equivalent contrapositive form:

If today is Easter, then tomorrow is Monday.

- Solution:
- If tomorrow is not Monday, then today is not Easter.

Converse and **inverse** of the Conditional

Suppose a conditional proposition of the form

'If p then q' is given.

1. The <u>converse</u> is 'if q then p'. المقلوب
2. The <u>inverse</u> is 'if ⌐p then ⌐q'. المعكوس

Symbolically,

The <u>converse</u> of $p \rightarrow q$ is $q \rightarrow p$,

And

The <u>inverse</u> of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.

Example: Writing the Converse and the Inverse

Write the converse and inverse of each of the following statements:

a. If Howard can swim across the lake, then Howard can swim to the island.

b. If today is Easter, then tomorrow is Monday.

Solution:

a. *Converse:* If Howard can swim to the island, then Howard can swim across the lake.

Inverse: If Howard cannot swim across the lake, then Howard cannot swim to the island.

b. *Converse:* If tomorrow is Monday, then today is Easter.

Inverse: If today is not Easter, then tomorrow is not Monday. ■

Example: Writing the Contraposition, Converse and the Inverse

What are the contrapositive, the converse, and the inverse of the conditional statement:

"The home team wins whenever it is raining" ?

Solution: Because "q whenever p" is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as

"If it is raining, then the home team wins."

Consequently, the contrapositive of this conditional statement is

"If the home team does not win, then it is not raining."

The converse is

"If the home team wins, then it is raining."

The inverse is

"If it is not raining, then the home team does not win."

Truth Table for converse, inverse and Contrapositive

p	q	¬p	¬q	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$q \rightarrow p$	$\neg p \rightarrow \neg q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

1. A conditional statement and its converse are not logically equivalent.
2. A conditional statement and its inverse are not logically equivalent.
3. The converse and the inverse of a conditional statement are logically equivalent to each other.

The Biconditional

Definition Given proposition variables p and q, the biconditional of p and q is **p if and only if q** and is denoted $p \leftrightarrow q$.

It is **true** if *both p and q have the same truth values* and is **false** if p and q have opposite truth values.

The words if and only if are sometime abbreviated **iff.**

Example This computer program is correct **iff** it produces the correct answer for all possible sets of input data.

Biconditional (if and only if)

- Binary Operator, Symbol: $\leftrightarrow$

P	Q	$P \leftrightarrow Q$
true	true	true
true	false	false
false	true	false
false	false	true

Truth table

Truth Table Showing that $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ always have the same truth values, so they are logically equivalent

Necessary and Sufficient Conditions

Let r and s are two propositions

- **r** is a **<u>sufficient</u>** condition for **s** means 'if r then s'.

(the occurrence of r is sufficient to guarantee the occurrence of s).

- r is a **<u>necessary</u>** condition for **s** means 'if not r then not s'

(if r does not occur, then s cannot occur either).

- Applying equivalence between a <u>statement</u> and its <u>contrapositive</u>:

r is a necessary condition for s also means "if s then r."

Consequently**:**

- <u>**r** is **necessary and sufficient**</u> condition for **s** means

'r if and only if s'

Sufficient Condition

"*Studying the theory is sufficient for solving the exercise*".

The statement is equivalent to

"If you study the theory, then you can solve the exercise."

Using

p : you study the theory,

q : you can solve the exercise.

The statement becomes (p is sufficient for q)

$$p \rightarrow q.$$

Necessary Condition

"*Being above 16 is a necessary condition for getting ID Card*".

The statement is equivalent to

"If you are not above 16, then you can't get ID Card."

Using

p : you are above 16,

q : you can get ID Card.

The statement becomes (p is necessary for q)

$$\neg p \rightarrow \neg q.$$

Converting a Necessary Condition to If-Then Form

Use the ***contrapositive*** to rewrite the following statement in two ways:

"George's attaining age 35 is a necessary condition for his being president of the United States".

Solution:

- **Version 1**: If George has not attained the age of 35, then he cannot be president of the United States.

Inverse of version 1:

- **Version 2**: If George can be president of the United States, then he has attained the age of 35.