Section 2.3

Valid and Invalid Arguments

Previous Lecture Summary

- Conditional Propositions.
- Negation, Inverse and Converse of the conditional statements.
- Contra positive .
- Bi conditional statements.
- Necessary and Sufficient Conditions.
- Conditional statements and their Logical equivalences.

Lecture`s outline

- Different forms of arguments
- Modus Ponens and Modus Tollens
- Additional Valid Arguments
- Valid Argument with False Conclusion
- Invalid argument with a true Conclusion
- Converse and Inverse error
- Contradictions and valid arguments

Arguments and Argument Forms

Argument: Sequence of statements. The final statement in the sequence is the **conclusion**; the preceding statements are **premises**.

If today is Weekend, then it is Friday.
It is not Friday. } premises
Therefore, today is not Weekend. ← conclusion

Argument Form: Obtained by replacing component statements in the argument by variables.

If p then q.
~ q.
Therefore, ~ p.

Example

1. What is the form of the following argument?

If Saja is guilty (p), then she was in Ramallah on Aug. 18 (q).

Saja was not in Ramallah on Aug. 18. (~q)

Therefore, Saja is not guilty. (~p)

What is the relationship between this argument and the previous one?

Validity of Arguments Forms

Valid form of argument: Every argument of that form that has true premises has a true conclusion. (A more formal version of the definition is in the book)

Claim: Modus tollens is a valid form of argument.

Proof: premises: $p \to q$, $\sim q$; conclusion: $\sim p$

p	q	p →q	~q	~p
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Modus Tollens
$p \to q$
$\sim q$
$\therefore \sim p$

(premises: $p \to q$, $\sim q$; conclusion: $\therefore \sim p$)

In the only case (represented by row 4) where all the premises are true, the conclusion is also true. So this form of argument is valid.

Invalid form of argument: There is at least one argument of that form that has true premises and a false conclusion.

Ex: Determine whether the following argument form is valid or invalid:

$\sim p \vee q$
$p \rightarrow r$
$q \rightarrow p$
$\therefore r$

Valid or Invalid? Class Exercise

p	q	r	~p	~p ∨ q	p → r	q → p	r
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

$\sim p \vee q$
$p \rightarrow r$
$q \rightarrow p$
$\therefore r$

Valid or Invalid? Class Exercise

premises (~p ∨ q, p → r, q → p); conclusion (r)

p	q	r	~p	~p ∨ q	p → r	q → p	r
T	T	T	F	T	T	T	T
T	T	F	F	T	F	T	F
T	F	T	F	F	T	T	T
T	F	F	F	F	F	T	F
F	T	T	T	T	T	F	T
F	T	F	T	T	T	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	F

$\sim p \vee q$
$p \rightarrow r$
$q \rightarrow p$
$\therefore r$

The 8th row of this truth table shows that it is possible for an argument of this form to have true premises and a false conclusion. So this form of argument is invalid.

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. A row of the truth table in which all the premises are true is called a **critical row.** *If there is a critical row in which the conclusion is false*, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is **invalid**. If the conclusion in *every* critical row is true, then the argument form is **valid**.

Example 1 – *Determining Validity or Invalidity*

Is this a valid argument form?

$$
\begin{aligned}
&p \rightarrow q \vee \sim r \\
&q \rightarrow p \wedge r \\
\therefore\ &p \rightarrow r
\end{aligned}
$$

Example 1 – *Solution*

						premises		conclusion
p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \rightarrow q \vee \sim r$	$q \rightarrow p \wedge r$	$p \rightarrow r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

This row shows that an argument of this form can have true premises and a false conclusion. Hence this form of argument is invalid.

Solution:

The truth table shows that even though there are several situations in which the premises and the conclusion are all true (rows 1, 7, and 8), there is one situation (row 4) where the premises are true and the conclusion is false.

Common Invalid Forms of Argument

Converse Error (Invalid – Avoid!):

If today is Thanksgiving, then it is Thursday.

It is Thursday.

Therefore, today is Thanksgiving.

Form: If p then q

q

$\therefore$ p

Common Invalid Forms of Argument

Inverse Error (Invalid – Avoid!):

If Ted is a math major, then Ted has to take MAT 152.

Ted is not a math major.

Therefore, Ted does not have to take MAT 152.

Form: If p then q

~p

Therefore, ~q.

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Rules of Interference

A **rule of inference** is a logical construct which takes premises, analyzes their syntax and returns a conclusion.

We already saw

$p \to q$; p; $\therefore q$	$p \to q$; $\neg q$; $\therefore \neg p$
Modus ponens (method of affirming)	Modus tollens (method of denying)

Rules of Inference

All valid arguments can be used as rules for inference.

Modus Tollens
$p \to q$
$\sim q$
$\therefore \sim p$

Specialization
a. $p \wedge q$ $\therefore p$ b. $p \wedge q$ $\therefore q$

Conjunction
p
q
$\therefore p \wedge q$

Modus Ponens
$p \to q$
p
$\therefore q$

Generalization
a. P $\therefore p \vee q$ b. q $\therefore p \vee q$

Elimination
a. $p \vee q$, $\sim q$, $\therefore p$ b. $p \vee q$, $\sim p$, $\therefore q$

Rules of Inference

All valid arguments can be used as rules for inference.

Division into Cases

$p \vee q$

$p \to r$

$q \to r$

$\therefore r$

Transitivity

$p \to q$

$q \to r$

$\therefore p \to r$

Contradiction Rule

$\sim p \to c$

$\therefore p$

Modus Ponens

		premises		conclusion	
p	q	$p \to q$	p	q	
T	T	T	T	T	← critical row
T	F	F	T		
F	T	T	F		
F	F	T	F		

The first row is the only one in which both premises are true, and the conclusion in that row is also true. Hence the argument form is valid.

Example:

If it is raining, then there are clouds in the sky.
It is raining.
Therefore, there are clouds in the sky.

Modus Tollens

Now consider another valid argument form called **modus tollens**. It has the following form:

If p then q.
$\sim q$
$\therefore \sim p$

Activity:
Construct a truth table to prove that modus tollens is a valid form of argument.

Example:

If there is smoke then there is fire.
There is not fire
Therefore there is no smoke.

Additional Valid Argument Forms: Rules of Inference

A **rule of inference** is a form of argument that is valid. Thus modus ponens and modus tollens are both rules of inference.

The following are additional examples of rules of inference that are frequently used in deductive reasoning.

Generalization

The following argument forms are valid:

a. p
$\therefore p \vee q$

b. q
$\therefore p \vee q$

These argument forms are used for making generalizations. For instance, according to the first, if p is true, then, more generally, "p or q" is true for *any* other statement q.

Example:

Anton is a junior.
$\therefore$ (more generally) Anton is a junior or Anton holds a Phd..

Specialization

The following argument forms are valid:

a. $p \wedge q$
$\therefore p$

b. $p \wedge q$
$\therefore q$

These argument forms are used for specializing. When classifying objects according to some property, you often know much more about them than whether they do or do not have that property.

When this happens, you discard extraneous information as you concentrate on the particular property of interest.

Example 4: Specialization (intuition)

Suppose you are looking for a person who knows algorithms to work with you on a project. You discover that Nadine knows both numerical analysis and algorithms.

You reason as follows

Nadine knows numerical analysis and Nadine knows algorithms.

$\therefore$ **(in particular)**

Nadine knows algorithms.

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Case Elimination

The following argument forms are valid:

a.

$$p \vee q$$
$$\sim q$$
$$\therefore\ p$$

b.

$$p \vee q$$
$$\sim p$$
$$\therefore\ q$$

These argument forms say that when you have only two possibilities and you can rule one out, the other must be the case.

Transitivity

The following argument form is valid:

$$p \to q$$
$$q \to r$$
$$\therefore \quad p \to r$$

Many arguments in mathematics contain chains of if-then statements.

From the fact that one statement implies a second and the second implies a third, you can conclude that the first statement implies the third.

Proof of Transitivity Rule - Extra

$p \to q$
$q \to r$
$\therefore\ p \to r$

$(p \to q) \wedge (q \to r)$ (hypotheses; assumed true)

$\equiv (p \to q) \wedge (\neg q \vee r)$ (Conversion Theorem)

$\equiv [(p \to q) \wedge \neg q\,] \vee [(p \to q) \wedge r]$ (Distributive)

$\equiv [((p \to q) \wedge \neg q\,) \vee (p \to q)] \wedge [((p \to q) \wedge \neg q\,) \vee r]$ (Distributive)

$\equiv (p \to q) \wedge [((p \to q) \wedge \neg q\,) \vee r]$ (Recall absorption law: **$a \vee (a \wedge b) \equiv a$,**
hence $[((p \to q) \wedge \neg q\,) \vee (p \to q)] \equiv p \to q$)

$\equiv (p \to q) \wedge [((\neg p \vee q) \wedge \neg q\,) \vee r]$ (Conversion)

$\equiv (p \to q) \wedge [((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee r]$ (Distributive)

$\equiv (p \to q) \wedge [((\neg p \wedge \neg q) \vee F) \vee r]$ (Negation)

$\equiv (p \to q) \wedge [(\neg p \wedge \neg q) \vee r]$ (Unity)

$\equiv (p \to q) \wedge [(\neg p \vee r) \wedge (\neg q \vee r)]$ (Distributive)

$\equiv [(p \to q) \wedge (\neg q \vee r)] \wedge (\neg p \vee r)$ (Commutative; Associative)

$\therefore\ (\neg p \vee r) \equiv p \to r$ (Conjunctive simplification; conversion)

Proof by Division into Cases

The following argument form is valid:

$$\begin{array}{ll} & p \lor q \\ & p \to r \\ & q \to r \\ \therefore & r \end{array}$$

It often happens that you know one thing or another is true. If you can show that in either case a certain conclusion follows, then this conclusion must also be true.

Example 7 – Proof by Division into Cases

Suppose you know that x is a particular nonzero real number. The trichotomy property of the real numbers says that any number is positive, negative, or zero.
Thus (by elimination) you know that x is positive or x is negative.

You can deduce that $x^2 > 0$ by arguing as follows:

x is positive or x is negative.
If x is positive, then $x^2 > 0$.
If x is negative, then $x^2 > 0$.
$\therefore x^2 > 0$.

Example 8 – *Application: A More Complex Deduction*

You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

b. If my glasses are on the kitchen table, then I saw them at breakfast.

c. I did not see my glasses at breakfast.

d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses?

Example 8 – *Application: A More Complex Deduction*

Solution:

Let *RK* = I was reading the newspaper in the kitchen.
GK = My glasses are on the kitchen table.
SB = I saw my glasses at breakfast.
RL = I was reading the newspaper in the living room.
GC = My glasses are on the coffee table.

(a) RK→GK
(b) GK→SB
(c) ~SB
(d) RL V RK
(e) RL→GC

(a),(b) ∴RK→SB by transitivity -- (1)
(1),(c) ∴~RK by modus tollens -- (2)
(d), (2) ∴RL by elimination -- (3)
(e),(3) ∴GC by modus ponens

Thus the glasses are on the coffee table.

Put it all together

a. Formalize the text in propositional logic

$RK \rightarrow GK$
$GK \rightarrow SB$
$\sim SB$
$RL \vee RK$
$RL \rightarrow GC$
$\therefore GC$

b.verify whether the **argument form** that you obtained in a is valid

use truth table

c. verify whether the **argument** you obtained in a is valid

use rules of inference to show that the premises lead to the conclusion

Fallacies

A **fallacy** is an error in reasoning that results in an invalid argument. Three common fallacies are …

1. **using ambiguous premises,** and treating them as if they were unambiguous,
2. **circular reasoning** (assuming what is to be proved without having derived it from the premises), and
3. **jumping to a conclusion** (without adequate grounds).

Two other fallacies

•*converse error* and
•*inverse error*
which give rise to arguments that superficially resemble those that are valid by modus ponens and modus tollens but are not, in fact, valid.

Example 9 – *Converse Error*

Show that the following argument is invalid:

If Jim is a cheater, then Jim sits in the back row.
Jim sits in the back row.
∴ Jim is a cheater.

Solution:
The first premise gives information about Jim *if* it is known he is a cheater. It doesn't give any information about him if it is not already known that he is a cheater.

Example 9 – *Solution*

The general form of the previous invalid argument is as follows:

$$p \to q$$
$$q$$
$$\therefore p$$

The fallacy underlying this invalid argument form is called the **converse error** because the conclusion of the argument would follow from the premises if the premise $p \to q$ were replaced by its converse.

Example 10 – *Inverse Error*

Consider the following invalid argument:

If Jim is a cheater, then Jim sits in the back row.
Jim is not a cheater.
∴ Jim does not sit in the back row.

Note that this invalid argument has the following form:

$$p \to q$$
$$\sim p$$
$$\therefore \sim q$$

The fallacy underlying this invalid argument form is called the **inverse error** because the conclusion of the argument would follow from the premises if the premise $p \to q$ were replaced by its inverse.

Soundness

An argument is *sound* **if and only if** it is valid and its premises are true.

An argument is sound = The conclusion is true

An argument is valid ≠ The conclusion is true

An argument is invalid ≠ The conclusion is false

Example 11 – *A Valid Argument with a False Premise and a False Conclusion*

The argument below is **valid** by modus ponens. But the first premise is false, and so is its conclusion.

If John Lennon was a rock star, then John Lennon had red hair.
John Lennon was a rock star.
∴ John Lennon had red hair.

Example 12 – *An Invalid Argument with True Premises and a True Conclusion*

The argument below is **invalid** by the converse error, but it has a true conclusion.

If New York is a big city, then New York has tall buildings.
New York has tall buildings.
∴ New York is a big city.

Contradictions and Valid Arguments

The concept of logical contradiction can be used to make inferences through a technique of reasoning called the *contradiction rule*. Suppose *p* is some statement whose truth you wish to deduce.

Contradiction Rule

If you can show that the supposition that statement *p* is false leads logically to a contradiction, then you can conclude that *p* is true.

Example 13 – *Contradiction Rule*

Show that the following argument form is valid:

$\sim p \to c$, where c is a contradiction
$\therefore p$

Solution:
Construct a truth table for the premise and the conclusion of this argument.

			premises	conclusion
p	$\sim p$	c	$\sim p \to c$	p
T	F	F	T	T
F	T	F	F	

There is only one critical row in which the premise is true, and in this row the conclusion is also true. Hence this form of argument is valid.

Contradictions and Valid Arguments

The contradiction rule is the logical heart of the method of proof by contradiction.

A <u>slight variation</u> also provides the basis for solving many logical puzzles by eliminating contradictory answers: <u>*If an assumption leads to a contradiction, then that assumption must be false.*</u>

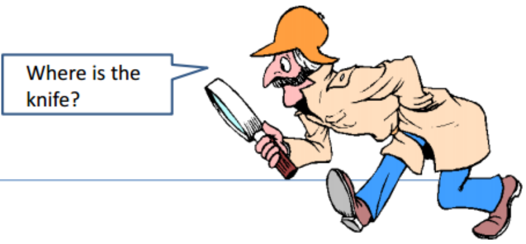
Summary of Rules of Inference

Table 2.3.1 summarizes some of the most important rules of inference.

Modus Ponens	$p \to q$ p • q	**Elimination**	**a.** $p \lor q$ $\sim q$ • p **b.** $p \lor q$ $\sim p$ • q
Modus Tollens	∴ $\to q$ $\sim q$ • $\sim p$	**Transitivity**	∴ $p \to q$ ∴ $q \to r$ • $p \to r$
Generalization	**a.** ∴ p • $p \lor q$ **b.** q • $p \lor q$	**Proof by Division into Cases**	∴ $p \lor q$ $p \to r$ $q \to r$ ∴ • r
Specialization	**a.** ∴ $p \land q$ • ∴ **b.** ∴ $p \land q$ ∴ q		
Conjunction	p q • ∴ $\land q$	**Contradiction Rule**	$\sim p \to c$ ∴ • p

Example: Where is the knife?

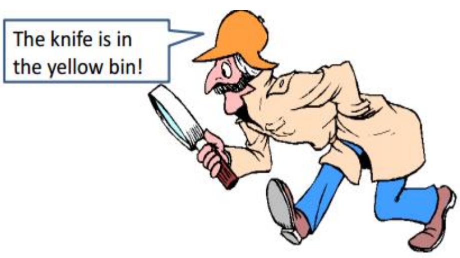
1. if the knife is in the **s**tore room, then we saw it when we **c**leared the store room;
2. the murder was committed at the **b**asement or inside the **a**partment;
3. if the murder was committed at the **b**asement, then the knife is in the **y**ellow dust bin;
4. we did not see a knife when we **c**leared the store room;
5. if the murder was committed **o**utside the building, then we are **u**nable to find the knife;
6. if the murder was committed inside the **a**partment, then the knife is in the **s**tore room.



Solution: Applying Inference Rules

1. if **s**, then **c**;
2. **b** or **a**;
3. if **b**, then **y**;
4. not **c**;
5. if **o**, then **u**;
6. if **a**, then **s**

1. $s \rightarrow c$
2. $b \vee a$
3. $b \rightarrow y$
4. $\neg c$
5. $o \rightarrow u$
6. $a \rightarrow s$



7. $\neg s$ 1, 4; modus tollens
8. $\neg a$ 6, 7; modus tollens
9. b 2, 8; case elimination

$\therefore y$ 3, 9; modus ponens

Sample Exam Question

Use a truth table to determine whether the following <u>argument form is valid</u>. Indicate which columns represent the **premises** and which represent the **conclusion**, and include a sentence explaining how the truth table supports your answer.

Your explanation should show that you understand what it means for a **form of argument to be valid**.

(q→~r) → p

~p →~q

~r →~p ∨ q

∴ r

Sample Exam Question - Solution

p	q	r	$\sim p$	$\sim q$	$\sim r$	$q \to \sim r$	$\sim p \vee q$	$(q \to \sim r) \to p$	$\sim p \to \sim q$	$\sim r \to \sim p \vee q$	r
T	T	T	F	F	F	F	T	T	T	T	T
T	T	F	F	F	T	T	T	T	T	T	F
T	F	T	F	T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	F	T	T	F	F
F	T	T	T	F	F	F	T	T	F	T	T
F	T	F	T	F	T	T	T	F	F	T	F
F	F	T	T	T	F	T	T	F	T	T	T
F	F	F	T	T	T	T	T	F	T	T	F

The 9th, 10th and 11th columns are the **premises**, the 12th column is the **conclusion**.

The 2nd **row is a critical** row with a false conclusion, hence the **argument form is invalid**.