COMP 233 Discrete Mathematics

Chapter 3

The Logic of Quantified Statements (First Order (Predicate) Logic)

3.1: Predicates and Quantified Statements I

Today's Lecture

- Predicates
- Set Notation
- Universal and Existential Statement
- Translating between formal and informal language
- Universal conditional Statements
- Implicit Quantification

Predicate Logic (منطق الإسناد)

- *Predicate logic* is an extension of propositional logic that permits concisely reasoning about whole *classes* of entities.

  E.g., "$x>1$", "$x+y=10$"

- Such statements are neither true or false when the values of the variables are not specified.

What is First Order Logic?

Propositional Logic

P, Q *Propositions*

$\neg P$

$P \wedge Q$

$P \vee Q$

$P \rightarrow Q$

$P \leftrightarrow Q$

We regard the world as *Propositions*

First Order Logic

$P(x..y), Q(t,..s)$ *Predicates*

$\neg P$

$P \wedge Q$

$P \vee Q$

$P \rightarrow Q$

$P \leftrightarrow Q$

$\forall$ Universal quantification

$\exists$ *Existential* quantification

We regard the world as ***Quantified*** *Predicates*

Subjects and Predicates

- The proposition

  "The dog is sleeping"

  has two parts:

  - "the dog" denotes the ***subject*** the **entity** that the sentence is about.
  - "is sleeping" denotes the ***predicate*** – a **property** that the subject can have.

Predicates (aka Properties)

A **predicate** is a sentence that is not a statement but contains one or more variables and becomes a statement if specific values are substituted for the variables.

Example: $x^2 > 4$

- The **domain of a predicate variable** is the set of allowable values for the variable.
- **Ex:** Let P(x) be the sentence "$x^2 > 4$" where the domain of x is understood to be the set of all real numbers. Then P(x) is a predicate.
- **Question:** For what numbers x is P(x) true?

Ans: The set of all real numbers for which $x > 2$ or $x < -2$.

Example: Finding Truth Values of a Predicate

Let $P(x)$ be the predicate "$x^2 > x$" with domain the set **R** of all real numbers. Write $P(2)$, $P(\frac{1}{2})$, and $P(-\frac{1}{2})$, and indicate which of these statements are true and which are false.

Solution

$$P(2):\quad 2^2 > 2, \quad \text{or} \quad 4 > 2. \quad \text{True.}$$

$$P\left(\frac{1}{2}\right):\quad \left(\frac{1}{2}\right)^2 > \frac{1}{2}, \quad \text{or} \quad \frac{1}{4} > \frac{1}{2}. \quad \text{False.}$$

$$P\left(-\frac{1}{2}\right):\quad \left(-\frac{1}{2}\right)^2 > -\frac{1}{2}, \quad \text{or} \quad \frac{1}{4} > -\frac{1}{2}. \quad \text{True.}$$

Propositional Functions

- A *predicate* is modeled as a *function* $P(\cdot)$ from objects to propositions.
  - $P(x)$ = "x is sleeping" (where x is any object).
- The *result of applying* a predicate P to an object $x=a$ is the *proposition* $P(a)$.
  - *e.g.* if $P(x)$ = "$x > 1$", then $P(3)$ is the *proposition* "3 is greater than 1."
- Note: The predicate P **itself** (*e.g.* P="is sleeping") is **not** a proposition (not a complete sentence).

Propositional Functions

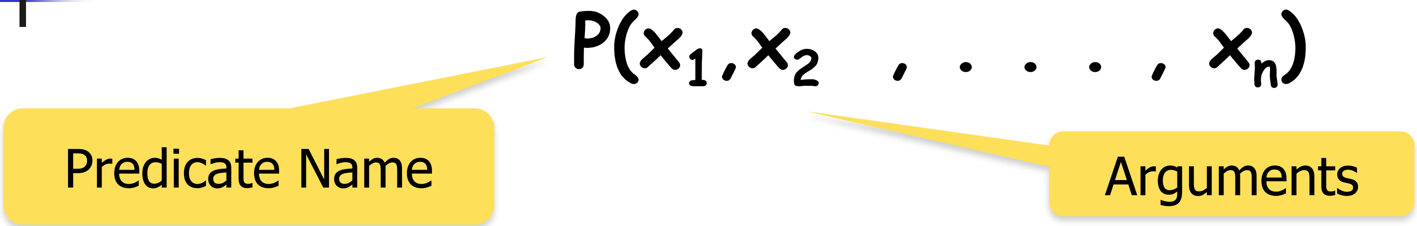
- Predicate logic includes propositional functions of **any** number of arguments.

  e.g. let $P(x,y,z)$ = "x gave y the grade z",

  x="Amal", y="Ali", z="A",

  $P(x,y,z)$ = "Amal gave Ali the grade A."

Arty of Predicates



Unary Predicates: Person(Amjad), University(BZU)

Binary Predicates: StudyAt(Amjad, BZU)

Ternary Predicates StudyAt(Amjad,BZU, CS)

Quaternary Predicate: StudyAt(Amjad,BZU, CS, 2015)

n-ary Predicate: StudyAt(Amjad,BZU, CS, 2015, BA, ….)

Quantifier Expressions

- *Quantifiers* allow us to *quantify* (count) *how many* objects in u.d. satisfy a given predicate:
  - "∀" is the FOR ∀LL or *universal* quantifier.
    $\forall x\, P(x)$ means

    for all x in the u.d., *P* holds.

  - "∃" is the ∃XISTS or *existential* quantifier.
    $\exists x\, P(x)$ means

    there *exists* an x in the u.d.

    that is, one or more) such that $P(x)$ is true.

Universal statement

A <u>universal statement</u> is a statement saying that a certain property is **true** for all elements in a given set.

Ex: <u>All students in this room are registered for COMP 233</u>.

If a person in this room is a student, then that person is registered for COMP233.

> The symbol ∀ stands for the words "for all"
> – it is called the **universal quantifier**

∀ students *P* in this room, *P* is registered for COMP233.

∀ people *P* in this room, if *P* is a student, then *P* is registered for COMP233.

If *P* is a student in this room, then *P* is registered for COMP233.

> The universal quantification is "**implicit**" in this statement.

Universal Quantifier ∀: Example

- Let $P(x)$ be the *predicate* "x is full."
- Let the u.d. of x be parking spaces at BZU.
- The *universal quantification of* $P(x)$,

  $\forall x\, P(x)$, is the *proposition:*
  - "All parking spaces at BZU are full." or
  - "Every parking space at BZU is full." or
  - "For each parking space at BZU , that space is full."

Falsity of The Universal Quantifier $\forall$

- To prove that a statement of the form

  $\forall x\, P(x)$ is false,

  it suffices to find a **counterexample**

  (i.e., one value of x in the universe of discourse such that $P(x)$ is false)

  - e.g., $P(x)$ is the predicate "x>0"

Definition of Counterexample

Note: For a universal statement to be **false** means that there is **at least one** element of the set for which the property is false.

Definition: Given a universal statement of the form "∀ x in D, P(x)," a **counterexample** for the statement is a value of x for which P(x) is false.

Ex: True or false? ∀ COMP 233 students x, x has studied calculus.

Ans: The statement is false. _________ is a counterexample to the statement "∀ COMP 233 students x, x has studied calculus" because ____________ is a COMP 233 student and __________ has not studied calculus.

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Set Notation: Truth Set of a Predicate

The **truth set of a predicate** P(x) is the set of elements in the domain D of x for which P(x) is true.

We write

truth set of P(x) = $\{x \in D \mid P(x)\}$

How should we read this out loud?

"**the set of all** x in D **such that** P of x

Note: The vertical line denotes the words "such that" **for the set-bracket notation only**. In other contexts, the words "such that" are symbolized by "s.t." or "s. th."

Truth Set of a Predicate

• Definition

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}.$$

Note: Recall that we read these symbols as "the set of all x in D such that P(x).

Example: Finding the Truth Set of a Predicate

Let $Q(n)$ be the predicate "n is a factor of 8." Find the truth set of $Q(n)$ if

a. the domain of n is the set $\mathbf{Z}^+$ of all positive integers

b. the domain of n is the set $\mathbf{Z}$ of all integers.

Solution

a. The truth set is $\{1, 2, 4, 8\}$ because these are exactly the positive integers that divide 8 evenly.

b. The truth set is $\{1, 2, 4, 8, -1, -2, -4, -8\}$ because the negative integers -1, -2, -4, and -8 also divide into 8 without leaving a remainder.

Example

- If D = {positive integers not exceeding 3} and P(x): $x^2 < 10$.
  What is the truth value of ∀x P(x)?
  T
  What is the truth set?
  {1,2,3}

- If D = {positive integers not exceeding 4} and P(x): $x^2 < 10$.
  What is the truth value of ∃x P(x)?

Example: Truth and Falsity of Universal Statements

a. Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Show that this statement is true.

b. Consider the statement

$$\forall x \in \mathbf{R}, x^2 \geq x.$$

Find a counterexample to show that this statement is false.

Solution

a. Check that "$x^2 \geq x$" is true for each individual x in D.

$$1^2 \geq 1, \qquad 2^2 \geq 2, \qquad 3^2 \geq 3, \qquad 4^2 \geq 4, \qquad 5^2 \geq 5.$$

Hence "$\forall x \in D, x^2 \geq x$" is true.

b. *Counterexample:* Take $x = \frac{1}{2}$. Then x is in $\mathbf{R}$ (since $\frac{1}{2}$ is a real number) and

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \not\geq \frac{1}{2}.$$

Hence "$\forall x \in \mathbf{R}, x^2 \geq x$" is false. ■

Existential statement ∃

∃ stands for the words "there exists" or "there exists at least one" – it is called the **existential quantifier**

Ex: Rephrase the following statement in <u>less formal</u> language, and determine whether it is true or false.

∃ a COMP 233 student x such that x has studied calculus.

(What is the Domain? What is/are the Predicates?)

Ans:

There is a COMP 233 student who has studied calculus.

Some COMP 233 student has studied calculus.

Some COMP 233 students have studied calculus.

At least one COMP 233 student has studied calculus.

Etc.

(<u>Alternative formal version</u>:

∃ x such that x is a COMP 233 student and x has studied calculus.)

(What is the Domain? What is/are the Predicates?)

Existential Quantifier ∃ :Example

- Let $P(x)$ be the *predicate* "x is full."
- Let the u.d. of x be parking spaces at BZU .
- The *Existential quantification of* $P(x)$,

  $\exists x\, P(x)$, is the *proposition*:
  - "Some parking space at BZU is full." or
  - "There is a parking space at BZU that is full." or
  - "At least one parking space at BZU is full."

Example: Truth and Falsity of Existential Statements

a. Consider the statement

$$\exists m \in \mathbf{Z}^+ \text{ such that } m^2 = m.$$

Show that this statement is true.

b. Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

Solution

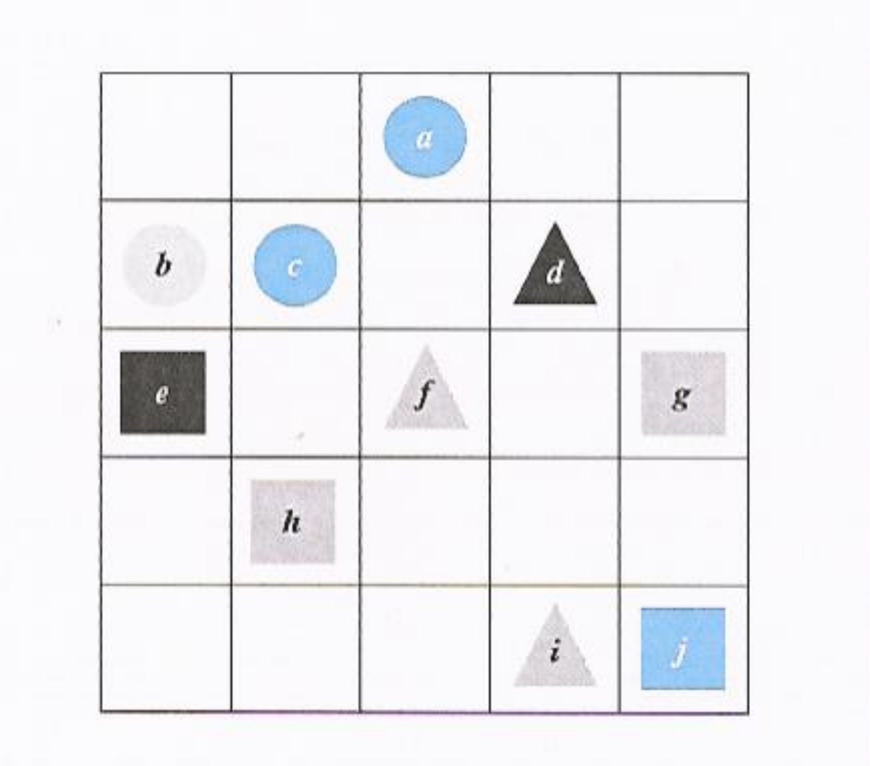
a. Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m. Hence "$\exists m \in \mathbf{Z}$ such that $m^2 = m$" is true.

b. Note that $m^2 = m$ is not true for any integers m from 5 through 8:

$$5^2 = 25 \neq 5, \qquad 6^2 = 36 \neq 6, \qquad 7^2 = 49 \neq 7, \qquad 8^2 = 64 \neq 8.$$

Thus "$\exists m \in E$ such that $m^2 = m$" is false. ■

Tarski's World Example



True or False?

1. ∀ objects *x*, if *x* is a circle, then *x* is blue.

False, there is a circle, **b**, that is not blue.

2. ∃ an object *x* such that *x* is a square and *x* is to the left of c.

True, **e** is a square and e is to the left of **c**.

True & False Quantified Statements

Statement	When Is It True?	When Is It False?
$\exists x\ p(x)$	For some (at least one) a in the universe, $p(a)$ is true.	For every a in the universe, $p(a)$ is false.
$\forall x\ p(x)$	For every replacement a from the universe, $p(a)$ is true.	There is at least one replacement a from the universe for which $p(a)$ is false.
$\exists x\ \neg p(x)$	For at least one choice a in the universe, $p(a)$ is false, so its negation $\neg p(a)$ is true.	For every replacement a in the universe, $p(a)$ is true.
$\forall x\ \neg p(x)$	For every replacement a from the universe, $p(a)$ is false and its negation $\neg p(a)$ is true.	There is at least one replacement a from the universe for which $\neg p(a)$ is false and $p(a)$ is true.

Verbalizing Formal Statements

Write the following formal statements in an informal language:

$\forall x \in \mathbf{R} \cdot \; x^2 \geq 0$

The square of every real number is greater than or equal to zero

$\forall x \in \mathbf{R} \cdot \; x^2 \neq -1$

The square of any real number does not equal -1

$\exists m \in \mathbf{Z}^+ \cdot \; m^2 = m$

There is a positive integer that is equal to its square

$\forall x \in \mathbf{R} \cdot \; x > 2 \;\; \rightarrow x^2 > 4$

If a real number is greater than 2 then is square is greater than 4

Formalize Statements

Write the following informal statements in a formal language:

All triangles have three sides

$\forall t \in Triangle \cdot ThreeSided(t)$

No dogs have wings

$\forall d \in Dog \cdot \neg HasWings(d)$

Some programs are structured

$\exists p \in Program \cdot structured(p)$

All bytes have eight bits

$\forall b \in Byte \cdot EightBits(b)$

No fire trucks are green

$\forall t \in FireTruck \cdot \neg Green(t)$

Different Writings

$\forall x \in Square \,.\, Rectangle(x)$

$\forall x$. If x is a square then x is a rectangle
$\forall$ *Squares* x . x *is a* a rectangle

Although the book uses this notation but it's not recommended as predicates are not clear.

$\forall p \in Palestinian \,.\, Likes(p, Zatar)$
$\forall p \,.\, Palestinian(p) \wedge Likes(p, Zatar)$

$\exists p \in Person \,.\, Likes(p, Zatar)$
$\exists p.\ Person(p) \wedge Likes(p, Zatar)$

Quantifications might be Implicit

Formalize the following:

If a real number is an integer, then it is a rational number.

$\forall n \in RealNumber \cdot Integer(n) \rightarrow Rational(n)$

If a person was born in Hebron then s/he is Khalili

$\forall x \in Person \cdot BornInHebron(x) \rightarrow Khalili(x)$

People who like Homos are smart

$\forall x \in Person \cdot Like(x, Homos) \rightarrow Smart(x)$

Translating from informal Language to Formal language

Write the following statement in English, using the predicates

F(x): "x is a Freshman"

T (x, y): "x is taking y"

where **x** represents *students* and **y** represents *courses*:

$$\exists x\,(F(x) \land T(x, \text{Discrete Math}))$$

Solution

The statement $\exists x\,(F(x) \land T(x, \text{Discrete}))$ *says that there is a student x with two properties: x is a freshman and x is taking Discrete.*

In English,

"Some Freshman is taking Discrete Math."

Translating from informal Language to Formal language

"Every freshman at the College is taking CSC 102."

Solution: There are various ways to answer this question, depending on the domain.

- If we take as our domain all freshmen at the College and use the predicate *T (x) : "x is taking CSC* 102", then the statement can be written as $\forall x, T(x)$.

- We can take the students as domain and we are making a conditional statement:

  "If the student is a freshman, then the student is taking CSC 101;"

$$\forall x, (F(x) \rightarrow T(x)).$$

Note that we cannot say $\forall x (F(x) \wedge T(x))$, *because this says that every student is a freshman, which is not* something we can assume here.

Universal Conditional Statements

Definition: Any statement of the following form is called a **universal conditional statement:**

$$\forall\ x \text{ in } D, \text{ if } P(x) \text{ then } Q(x).$$

For all

a collection of objects

properties that x might satisfy

Universal conditional statements are the most important form of statement in mathematics!

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Example: A Universal Conditional Statement

$\forall$ real numbers x, if $x > 2$ then $x^2 > 4$.

Less formal versions:

1. All real numbers that are greater than 2 have squares that are greater than 4.
2. If a real number is greater than 2, then its square is greater than 4.

"Implicit" Quantification

More formal versions:

1. $\forall\, x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4$.
2. $x > 2 \Rightarrow x^2 > 4$.

Notation: The symbol $\Rightarrow$ denotes a "universalized if-then."
So $x > 2 \Rightarrow x^2 > 4$ means $\forall\, x \in \mathbf{R}$, $x > 2 \rightarrow x^2 > 4$

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Example2: A Universal Conditional Statement

Everyone who visited France stayed in Paris.”

- if we take all people as the universe , then we need to introduce the predicates

  F(x) for "x visited France.” and

  P(x) is "x stayed in Paris.”

In this case, the proposition can be written as

$$\forall x, (F(x) \rightarrow P(x)).$$

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- We can write the following statements in a variety of informal ways.

$$\forall x \in R, \text{ if } x > 2 \text{ then } x^2 > 4$$

Sol:

- if a real number is greater **then** 2, then the square is greater than 4.
- Whenever a real number is greater then 2, its square is greater than 4.
- The squares of real number, greater than 2, are greater than 4.

Exercise

Rewrite the following statements in the form

$\forall$ __________, *if* __________ *then* _________.

a) If a real number is an integer, then it is a rational number.

b) All bytes have eight bits.

c) No fire trucks are green.

Sol:

a). $\forall x \in R, if\, x \in Z, then\, x \in Q.$

b). *∀ x, if x is a byte, then x has eight bits.*

c). ∀ x, if x is a fire truck, then x is not green.

Observe that the two statements

"*∀ real numbers x, if x is an integer then x is rational*" *and*
"∀ integers x, x is rational mean the same thing.

In fact, a statement of the form

$\forall x \in U,$ if P(x) then Q(x).

Can always be rewritten in the form

$$\forall x \in D, Q(x)$$

by narrowing U to be the domain D consisting of all values of the variable x that make P(x) true.

And Can be rewritten as

∀x, if x is in D then Q(x).

Example: Equivalent Forms for Universal Statements

Rewrite the following statement in the two forms:
"∀x, if then " and
"∀ x, ":

All squares are rectangles.

Solution:

∀x, if x is a square then x is a rectangle.

∀ squares x, x is a rectangle.

Equivalent Forms of Existential statements

.

The existential statements

∃ x belongs to U such that P(x) and Q(x).

And

∃ x belongs to D such that Q(x)

Are also equivalent provided D is taken to consist of all elements in U that make P(x) true.

Equivalence form for existential statement

The following statements are equivalent:

∃ a number n such that n is prime and n is even

And

∃ a prime number n such that n is even.

Implicit Quantifications

- Consider " If a number is an integer, then it is a rational number"

The clue to indicate its universal quantifications comes from the presence of the indefinite article "a".

Existential quantification can also be implicit.

for instance, " the number 24 can be written as a sum of sum of two even integers"

"∃ even integers m and n such that 24=m + n."

Implicit Quantifications

- **Notation**

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- The notation $\boldsymbol{P(x) \Rightarrow Q(x)}$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x, P(x) \to Q(x)$.
- The notation $\boldsymbol{P(x) \Leftrightarrow Q(x)}$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x, P(x) \leftrightarrow Q(x)$.

Please look at example 3.1.12 page 104

Example 3.1.12 Using $\Rightarrow$ and $\Leftrightarrow$

Let

$Q(n)$ be "n is a factor of 8,"

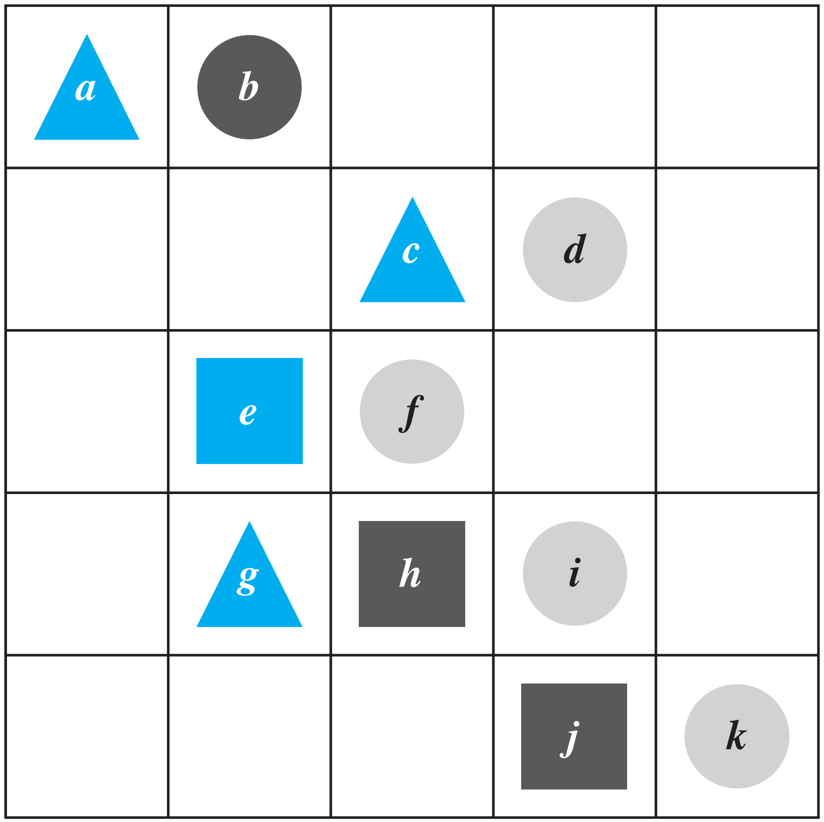
$R(n)$ be "n is a factor of 4,"

$S(n)$ be "$n < 5$ and $n \neq 3$,"

and suppose the domain of n is $\mathbf{Z}^+$, the set of positive integers. Use the $\Rightarrow$ and $\Leftrightarrow$ symbols to indicate true relationships among $Q(n)$, $R(n)$, and $S(n)$.

Tarski's World Example

Verbalize and test the following statements



$\forall t\,.\ \mathrm{Triangle}(t) \rightarrow \mathrm{Blue}(t)$ ✓

All the triangles are blue

$\forall x\,.\ \mathrm{Blue}(x) \rightarrow \mathrm{Triangle}(x)$ ✗

As a counterexample, note that *e* is blue and it is not a triangle

$\exists y\,.\ \mathrm{Square}(y) \wedge \mathrm{RightOf}(d, y)$ ✓

RightOf: even if not in same row!

$\exists z\,.\ \mathrm{Square}(z) \wedge \mathrm{Gray}(z)$ ✗