COMP 233 Discrete Mathematics

Chapter 3
The Logic of Quantified Statements
(First Order (Predicate) Logic)

3.2: Predicates and Quantified Statements II

Today's Lecture

- Negations of Universal and Existential statements
- Negations of Universal Conditional Statements
- Variants of Universal Conditional Statements
- The Relation among ∀, ∃, ∧, and ∨
- Necessary and Sufficient Conditions, Only If
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Negations of Quantified Statements

The negation of a statement exactly expresses what it would mean for the statement to be false. Write negations for the following:

1. ∀ even integers x, x is positive.
2. ∃ an integer n such that n has an integer square root.

Answers

1. ∃ an even integer x such that x is not positive.
2. ∀ integers n, n does not have an integer square root.

In general:

~(∀ x in D, P(x)) ≡ ∃ x in D such that ~P(x)
~(∃ x in D such that P(x)) ≡ ∀ x in D, ~P(x)

So: The negation of a "for all" statement is a "there exists" statement, and the negation of a "there exists" statement is a "for all" statement.

Negation of Quantified Statements

The negation of the statement of the form

∃ x in D such that Q(x)

is logically equivalent to a statement of the form

∀ x in D, *~Q(x)*

Symbolically:

$$\sim(\exists x \in D \ni Q(x)) \equiv \forall x \in D, \sim Q(x).$$

Note: the negation of existential statement is logically equivalent to universal statement.

Examples

Negate "Some integer *x is positive and all integers y are negative.*"

Solution: Using all integers as the universe for *x and y, the statement is* **∃ x .(x > 0) ∧ ∀ y . (y < 0).** *The **negation** is*

~{∃x .(x > 0) ∧ ∀ y. (y < 0)}≡ ~∃x . (x > 0) ∨ ~∀y. (y < 0):

by De Morgan's law

≡ ∀ x . ~(x > 0) ∨ ∃ y . ~(y < 0) *properties of negation*

≡ ∀ x . (x ≤ 0) ∨ ∃ y . (y ≥ 0).

Therefore, the **negation** is "Every integer *x is non positive or there is an integer y that is nonnegative."*

Cont….

Negate "There is a student who came late to class and there is a student who is absent from class."

Solution: In symbols, if *L(x) : "x came late to class" and A(x) : "x is absent from class," this statement* can be written as **$\exists$ x st L(x) $\wedge$ $\exists$ y st A(y).**

Note that we must use a second variable *y. By one of De Morgan's laws the negation can be written as*

~($\exists$ x st L(x)) $\vee$ ~($\exists$ y st A(x)) $\equiv$ **$\forall$x, ~L(x)** $\vee$ $\forall$ **y, ~A(x).**

In English this is "No student came late to class or no student is absent from class."

Negations of Quantified Statements

All Palestinians like Zatar
  Some Palestinians do not like Zatar

Some Palestinians Like Zatar
  All Palestinians do not like Zatar

$\forall$ *p*$\in$ Prime . Odd(*p*)
  $\exists$*p*$\in$Prime . ~Odd(*p*)

Some computer hackers are over 40
  All computer hackers are not over 40

All computer programs are finite
  Some computer programs are not finite

No politicians are honest
  Some politicians are honest

Negation of a Universal Conditional Statement

$\sim(\forall\ x$ in D, if P(x) then Q(x)) $\equiv$?

$\equiv \exists\ x$ in D such that P(x) and ~Q(x)

Exercise: Write a negation for the following statements:

- $\forall$ real numbers x, if $x^2 > 4$ then $x > 2$.

$\exists\ x \in \mathbf{R}$, such that ($x^2 > 4$) and ~($x$> 2)

- $\forall$p$\in$Person . Blond(p) $\rightarrow$ BlueEyes(p)

$\exists$p$\in$Person . Blond(p) $\wedge$ ~BlueEyes(p)

- If a computer program has more than 10000 lines then it contains a bug

a computer program has more than 10000 and does not contains a bug

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Variants of Universal Conditional Statements

Consider a statement of the form: $\forall x \in D\ .\ P(x) \rightarrow Q(x)$.

1.Its **contrapositive** is the statement:

$\forall x \in D\ .\ \sim Q(x) \rightarrow \sim P(x)$

2. Its **converse** is the statement:

$\forall x \in D\ .\ Q(x) \rightarrow P(x)$.

3. Its **inverse** is the statement:

$\forall x \in D\ .\ \sim P(x) \rightarrow \sim Q(x)$.

Example: $\forall x \in$Person . Palestinian(x) $\rightarrow$ Smart(x)

Contrapostive: $\forall x \in$Person . ~ Smart(x) $\rightarrow$ ~Palestinian(x)

Converse: $\forall x \in$Person . Smart(x) $\rightarrow$ Palestinian(x)

Inverse: $\forall x \in$Person . ~Palestinian(x) $\rightarrow$ ~Smart(x)

Variants of Universal Conditional Statements

$\forall x \in R.$ MoreThan(x,2) $\rightarrow$ MoreThan(x^2,4)
$\forall x \in R \;.\; x > 2 \rightarrow x^2 > 4$

Contrapostive: $\forall x \in R \;.\; x^2 \leq 4 \rightarrow x \leq 2$

Converse: $\forall x \in R \;.\; x^2 > 4 \rightarrow x > 2$

Inverse: $\forall x \in R \;.\; x \leq 2 \rightarrow x^2 \leq 4$

The Relation among ∀,∃,∧,and ∨

If Q(x) is a predicate and the domain D of x is the set {x1,x2,...,xn}, then the statements:

∀x∈D,Q(x) and Q(x1)∧Q(x2)∧ ⋯ ∧Q(xn)
are logically equivalent.

Example: let Q(x) be "x·x=x" and D={0,1}.
Then ∀x∈D,Q(x) can be rewritten as
∀ binary digits x, x·x=x
This is equivalent to (0·0=0) and (1·1=1), which can be rewritten in symbols as:
Q(0)∧Q(1).

Necessary, Sufficient and Only If Conditions

• Definition

- "$\forall x, r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$."
- "$\forall x, r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$."
- "$\forall x, r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$."

Only If

"$\forall x . P(x)$ **only if** $Q(x)$" means "$\forall x, \sim Q(x) \rightarrow \sim P(x)$"
or, equivalently, "$\forall x$, if $P(x)$ then $Q(x)$"

Example:

You get the job only if you are the top.

If you are not a top you will not get a job

If you got the job then you are a top

$\forall x . \sim Top(x) \rightarrow \sim GotaJob(x)$

$\forall x . GotaJob(x) \rightarrow Top(x)$

Converse:

Inverse:

Contra Position:

Negation:

Necessary and Sufficient Conditions

"$\forall x \,.\, P(x)$ is a **sufficient condition** for $Q(x)$" means "$\forall x \,.\, P(x) \rightarrow Q(x)$"

"$\forall x \,.\, P(x)$ is a **necessary condition** for $Q(x)$" means "$\forall x, \sim P(x) \rightarrow \sim Q(x)$"
or, equivalently, "$\forall x, Q(x) \rightarrow P(x)$"

Examples:

Squareness is a sufficient condition for rectangularity.

If a shape is a square, then it is a rectangle.

$\forall x \,.\, \text{Square}(x) \rightarrow \text{Rectangle}(x)$

To get a job, it is sufficient to be loyal.

If one is loyal (s)he will get a job

$\forall x \,.\, \text{Loyal}(x) \rightarrow \text{GotaJob}(x)$

Necessary and Sufficient Conditions

"$\forall x \,.\, P(x)$ is a **sufficient condition** for $Q(x)$" means "$\forall x \,.\, P(x) \rightarrow Q(x)$"

"$\forall x \,.\, P(x)$ is a **necessary condition** for $Q(x)$" means "$\forall x, \sim P(x) \rightarrow \sim Q(x)$"
or, equivalently, "$\forall x, Q(x) \rightarrow P(x)$"

More examples:

Being smart is necessary to get a job.

If you are not smart you don't get a job

If you got a job then you are smart

$\forall x \,.\, \sim\text{Smart}(x) \rightarrow \sim\text{GotaJob}(x)$

$\forall x \,.\, \text{GotaJob}(x) \rightarrow \text{Smart}(x)$

Being above 40 years is necessary for being president of Palestine

$\forall x \,.\, \sim\text{Above}(x, 40) \rightarrow \sim\text{CanBePresidentOfPalestine}(x)$

$\forall x \,.\, \text{CanBePresidentOfPalestine}(x) \rightarrow \text{Above}(x, 40)$

Recall from Ch2: If and Only If

So: *P* only if *Q* means if *P* then *Q*
and *P* if *Q* means if *Q* then *P*

Thus:

P if and only if *Q*
means *P* only if *Q* and *P* if *Q*
which means if *P* then *Q* and if *Q* then *P*

Fact:

$$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$$

Recall from Ch2: Only If

P only if *Q* means if ~*Q* then ~*P*

If Q didn't occur, then P didn't occur either.

Or, equivalently, if *P* then *Q*

If P occurred then Q also had to occur.