COMP 233 Discrete Mathematics

Chapter 3

The Logic of Quantified Statements
(First Order (Predicate) Logic)

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3.3: Statements with Multiple Quantifiers

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Statements with Multiple Quantifiers

consider a statement containing both ∀ and ∃, where the ∃ comes before the ∀ :

∃ an x in D such that ∀ y in E, x and y satisfy property P(x, y).

To show that a statement of this form is true:
You must find one single element (call it x) in D with the following property:
• After you have found your x, someone is allowed to choose any element whatsoever from E. The person challenges you by giving you that element. Call it y.
• Your job is to show that your x together with the person's y satisfy property P(x, y).

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Interpreting Statements with Two Different Quantifiers

If you want to establish the truth of a statement of the form

∀ x in D, ∃y in E such that P(x, y)

your challenge is to allow someone else to pick whatever element x in D they wish and then you must find an element y in E that "works" for that particular x.

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Interpreting Statements with Two Different Quantifiers

If you want to establish the truth of a statement of the form

$\exists x$ in D such that $\forall$ y in E, P(x, y)

your job is to find one particular x in D that will "work" no matter what y in E anyone might choose to challenge you with

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Multiply-Quantified Statements

Statements with more than one quantifier

Examples: What do the following statements mean?

1. $\forall$ integers x, $\exists$ an integer y such that $y < x$.

This means: No matter what integer you might pick, there is an integer that is less than the one you picked.

2. $\exists$ a positive integer x such that $\forall$ positive integers y, $x \leq y$.

This means: There is a positive integer that is less than or equal to every positive integer.

I.e., there is a smallest positive integer

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Nested Quantifiers

Two quantifiers are nested if one is within the scope of the other, such as $\forall x \, \exists y$ such that $(x + y = 0)$.

Note that everything within the scope of a quantifier can be thought of as a propositional function.

For example,

$\forall x \, \exists y$ such that $(x + y = 0)$, is the same thing as

$\forall x, Q(x)$, where $Q(x)$ is $\exists y \, P(x, y)$,

where $P(x, y)$ is $x + y = 0$.

Nested quantifiers commonly occur in mathematics and computer science.

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Example

Translate into English the statement
$\forall x \, \forall y \, \{(x > 0) \wedge (y < 0) \rightarrow (x \cdot y < 0)\}$,
where the domain for both variables consists of all real numbers.

Solution: This statement says that for every real number x and for every real number y, if $x > 0$ and $y < 0$, then $x \cdot y < 0$.

This can be stated more succinctly as
"The product of a positive real number and a negative real number is always a negative real number".

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Cont….

Write the following statements in English, using the predicate *S(x, y): "x shops in y", where* x represents people and *y represents stores:*

(a) $\forall y\, S(john, y)$.

(b) $\exists x \forall y\, S(x, y)$.

Sol: (a) The predicate states that if *y is a store, then john shops there. i.e.,*

"*john shops in every* store."

(b) The predicate states that there is a person *x with the property that x shops in every store y. That is,*

"There is a person who shops in every store."

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Table for the Quantification of two variables

Statement	***When True?***	***When False?***
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

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Example

Write the following statement in English, using the predicates
C(x): "x is a Computer Science major"
T (x, y): "x is taking y"
where ***x** represents students and **y** represents courses:*

$$\forall y\ \exists x\ (\sim C(x) \wedge T(x, y)).$$

Solution:
The statement $\forall y\ \exists x\ (\sim C(x) \wedge T(x, y))$ *says that for every course y there is a student x such that x is not a* Computer Science major and *x is taking y. That is, "In every course there is a student who is not a Computer* Science major."

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More Equivalence Laws

- $\neg\exists x\ P(x) \Leftrightarrow \forall x \neg P(x)$
  $\neg\forall x\ P(x) \Leftrightarrow \exists x \neg P(x)$
- $\forall x \forall y\ P(x,y) \Leftrightarrow \forall y \forall x\ P(x,y)$
  $\exists x \exists y\ P(x,y) \Leftrightarrow \exists y \exists x\ P(x,y)$
- $\forall x\ (P(x) \wedge Q(x)) \Leftrightarrow (\forall x\ P(x)) \wedge (\forall x\ Q(x))$
  $\exists x\ (P(x) \vee Q(x)) \Leftrightarrow (\exists x\ P(x)) \vee (\exists x\ Q(x))$

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Order of Quantifiers

Order matters

$\forall x\ \exists y$. Loves(x,y) Everyone loves someone	$\exists x\ \forall y$. Loves(x,y) Someone loves everyone
$\forall y\ \exists x$. Loves(x,y) Someone loves everyone	$\forall y\ \exists x$. Loves(y,x) Everyone loves someone

Order doesn't matters

$\forall x\ \forall y$. Loves(x,y) $\forall x,y$. Loves(x,y) Everyone loves everyone	$\exists x\ \exists y$. Loves(x,y) $\exists x,y$. Loves(x,y) someone loves someone

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Order of Quantifiers Is Important!!

P(x,y)="**x relies upon y**," verbalize the following:

$\forall x\ \exists y\ P(x,y)$: Everyone has *someone* to rely on.

$\exists y\ \forall x\ P(x,y)$: There's a poor overworked soul whom *everyone* relies upon (including himself)!

$\exists x\ \forall y\ P(x,y)$: There's some needy person who relies upon *everybody* (including himself).

$\forall y\ \exists x\ P(x,y)$: Everyone has *someone* who relies upon them.

$\forall x\ \forall y\ P(x,y)$: *Everyone* relies upon *everybody*, (including themselves)!

Previous example: there is somebody (a popular person) whom everyone likes? $\exists y\ \forall x\ Likes(x,y)$

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Negations of Multiply-Quantified Statements

$\sim(\forall\ x \text{ in } D, \exists\ y \text{ in } E \text{ such that } P(x,y))$ $\equiv \exists\ x \text{ in } D \text{ such that } \forall\ y \text{ in } E, \sim P(x,y)$
$\sim(\exists\ x \text{ in } D \text{ such that } \forall\ y \text{ in } E, P(x,y))$ $\equiv \forall\ x \text{ in } D, \exists\ y \text{ in } E \text{ such that } \sim P(x,y)$

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Negations of Multiple Statements

The negation of

$\forall\ x, \exists\ y$ *such that* $P(x, y)$

is logically equivalent to

$\exists\ x$ *such that* $\forall\ y, \sim P(x, y)$.

A similar sequence of reasoning can be used to derive the following:

The negation of

$\exists\ x$ *such that* $\forall\ y, Q(x, y)$.

is logically equivalent to

$\forall\ x, \exists\ y$ *such that* $\sim Q(x, y)$.

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Examples

a) ∀ integers *n*, ∃ an integer *k* such that *n* = 2*k*.

b) ∃ a person *x* such that ∀ people *y*, *x* loves *y*.

Sol: a. ∃ an integer *n* such that ∀ integers *k*, $n \neq 2k$

Or we can say

" there is a some integer that is not even"

b. ∀ people *x*, ∃ a person *y* such that *x* does not love *y*.

Or we can say

" Nobody Loves everybody"

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Interpreting Statements with Two Different Quantifiers example:3.3.3

A college cafeteria line has *four* stations:
salads, main courses, desserts, and beverages.

The **salad** station offers a choice of green salad or fruit salad; the **main course** station offers spaghetti or fish; the **dessert** station offers pie or cake; and the **beverage** station offers milk, soda, or coffee.

Three students, Uta, Tim, and Yuen, go through the line and make the following choices:

Uta: green salad, spaghetti, pie, milk
Tim: fruit salad, fish, pie, cake, milk, coffee
Yuen: spaghetti, fish, pie, soda

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Interpreting Statements with Two Different Quantifiers example:3.3.3

These choices are illustrated in Figure 3.3.2

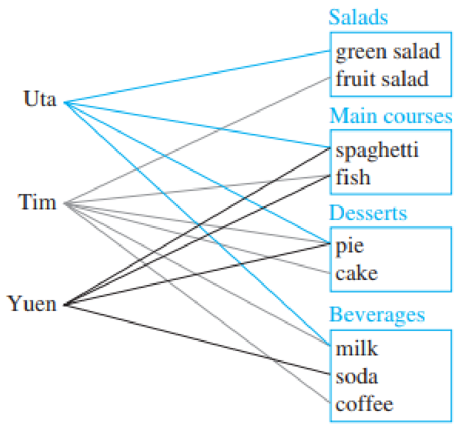

Figure 3.3.2

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Interpreting Statements with Two Different Quantifiers example:3.3.3

Write each of following statements informally and find its truth value.

a. ∃ an item I such that ∀ students S, S chose I.

There is an item that was chosen by every student

This is true; every student chose pie

b. ∃ a student S such that ∀ items I, S chose I.

There is a student who chose every available item

This is false; no student chose all nine items.

c. ∃ a student S such that ∀ stations Z,∃ an item I in Z such that S chose I.

There is a student who chose at least one item from every station

This is true; both Uta and Tim

d. ∀ students S and ∀ stations Z, ∃ an item I in Z such that S chose I.

Every student chose at least one item from every station

This is false; Yuen did not choose a salad.

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Exercises

Each of the following two statements makes a claim about positive integer numbers. For each statement,

a) rewrite the statement in a more formal language;
b) write a negation;
c) indicate whether the statement is true or false and give a reason for your answer.

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Exercise 1

1. For all positive integer *x*, there exists a positive integer *y* such that $y < x$.

1. a. $\forall$ positive integers *x*, $\exists$ a positive integer *y* such that $y < x$.

No matter what integer you might pick, there is an integer that is less than the one you picked.

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Exercise 2

2. There exists a positive integer x such that for all positive integer y, $x \leq y$.

2.a ∃ a positive integer x such that ∀ positive integers y , $x \leq y$.

There is a positive integer that is less than or equal to every positive integer.

I.e., there is a smallest positive integer

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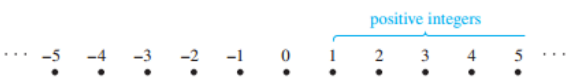
Exercise

Formalize these statements

There Is a Smallest Positive Integer

∃ a positive integer m such that ∀ positive integers n, m≤n.

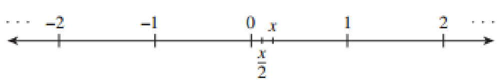
$\exists m \in Z^{+} \ \forall n \in Z^{+} \ .$ LessOrEqual(m,n)



There Is No Smallest Positive Real Number

∀ positive real numbers x,∃a positive real number y such that y<x.

$\forall x \in R^{+} \ \exists y \in R^{+} \ .$ Less(y,x)



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Exercise

The reciprocal (نظير ضربي) of a real number a is a real number b such that a*b = 1.
The following two statements are true.
Rewrite them formally using quantifiers and variables:

Every nonzero real number has a reciprocal.
$\forall$ u∈$\mathbf{R}_{nonZero}$, $\exists$ v∈R . uv = 1

There is a real number with no reciprocal.
$\exists$ u∈**R**, $\forall$ v∈R . uv ≠ 1

The number 0 has no reciprocal.

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Exercise

Not all people love someone.

~ (all people love someone)

~ ($\forall$x $\exists$y . Loves (x,y))

$\exists$*x* $\forall$*y* . ~**Love(*x,y*)**

Some people do not love everyone

Not all people love everyone.

~ (All people love everyone)

~ $\forall$x$\forall$y Like(x, y)

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Exercises (↔)

- A number *x* is *even* if and only if it is equal to 2 times some other number

$$\forall x\ \exists y\ Even(x) \leftrightarrow x=2y$$

- A number is *prime*, iff it isn't the product of two non-unity numbers (no-unity means they are not equal 1)

$$\forall x\ \exists y,z\ Prime(x) \leftrightarrow x \neq yz \wedge y \neq 1 \wedge z \neq 1)$$

- One's mother is one's female parent

$\forall$m, $\forall$c (Mother(c, m) ↔ (Female(m) ∧ Parent(m,c))

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Exercise

Using the predicates student(x), study(x), play_soccer(x), and appropriate quantifiers, represent in predicate logic the following sentences:

- All students study.

∀ x, student(x) → study(x)

- Some students play soccer

∃x, student(x) Λ play_soccer(x)

- Some soccer players are not students

∃x, play_soccer(x) Λ ~ student(x)

- Students that play soccer are healthy.

∀x, students(x) Λ soccer_player(x), → healthy(x).

- Some healthy students play soccer

∃x, students(x) Λ soccer_player(x) Λ healthy(x).

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Exercise – cont.

- Some healthy soccer players are students

∃x, healthy(x) Λ soccer_player(x) Λ students(x)

- All healthy soccer players are students

∀x, healthy(x) Λ soccer_player(x), → students(x).

- All soccer players are healthy students

∀x, soccer_player(x), → healthy(x) Λ students(x)

- All students are healthy soccer players

∀x, students(x) → healthy(x) Λ soccer_player(x)

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Exercises

- Every farmer who owns a donkey buys hay
  ∀x, ∃y ((Farmer(x) ∧ (Donkey(y) ∧ Owns(x, y)) →BuysHay(x)))

- Given any real number, you can find a real number so that the sum of the two is zero. Alternatively: Every real number has an additive inverse.
  $\forall x \in R, \exists y \in R, x + y = 0$

- There is a real number, which added to any other real number results in the other number. Alternatively: Every real number has an additive identity.
  $\exists x \in R, \forall y \in R, x + y = y$

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More Exercises

Express the statement "there is a number x such that when it is added to any number, the result is that number, **and** if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let P (x, y) be the expression "x + y = y".
- Let Q(x, y) be the expression "xy = x".
- Then the expression is

$$\exists x\ \forall y\ (P(x, y) \land Q(x, y))$$

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Group Exercise - Nono

The law says that it is a crime for an American to sell weapons to hostile nations. The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

Rewrite the paragraph in a more formal language:

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Solution1

... it is a crime for an American to sell weapons to hostile nations:

Nono … has some missiles, i.e.,

… all of its missiles were sold to it by Colonel West

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Solution 2- Nono

... it is a crime for an American to sell weapons to hostile nations:

$\forall x \in People, y \in Missiles, z \in Countries . American(x) \wedge Weapon(y) \wedge Sells(x,y,z) \wedge Hostile(z) \Rightarrow Criminal(x)$

Nono … has some missiles, i.e.,

$\exists x \in Missiles . Owns(Nono,x)$

… all of its missiles were sold to it by Colonel West

$\forall x \in Missiles \ \ Owns(Nono,x) \Rightarrow Sells(West,x,Nono)$

it is necessary to indicate that it is not all missles but the ones **owned by Nono**.

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Solution1

Missiles are weapons:

..An enemy of America counts as "hostile":

West, who is American …

The country Nono, an enemy of America …

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Solution 2- Nono

Missiles are weapons:
∀x ∈ Missiles. Weapon(x)

..An enemy of America counts as "hostile":
∀x ∈ Countries . Enemy(x,America) ⇒ Hostile(x)

West, who is American …
∃x ∈ West . American(x)
OR American(West)

What can you infer about West? 😈

The country Nono, an enemy of America …
Enemy(Nono,America)

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Domains of numbers

Important sets:

R, the set of all real numbers (on paper: ℝ)

Q, the set of all rational numbers (on paper: ℚ)

Z, the set of all integers (on paper: ℤ)

$\mathbf{R}^+$, the set of all positive real numbers

$\mathbf{Z}^{nonneg}$, the set of all nonnegative integers

Etc.

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