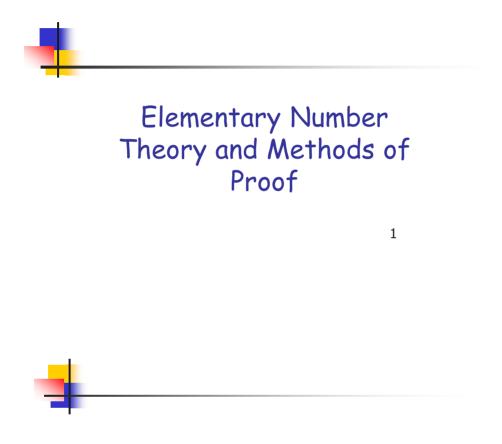
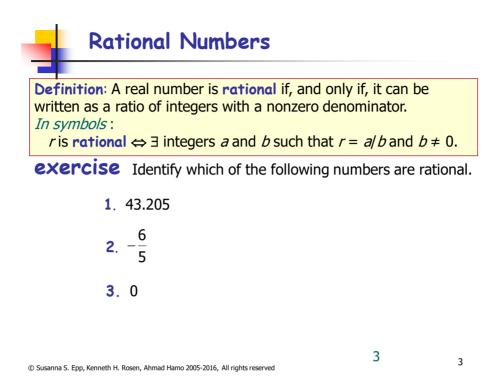
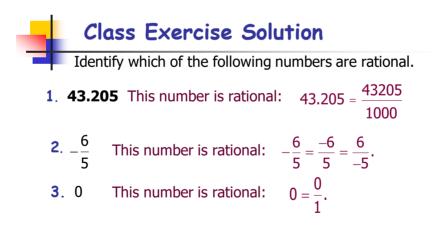
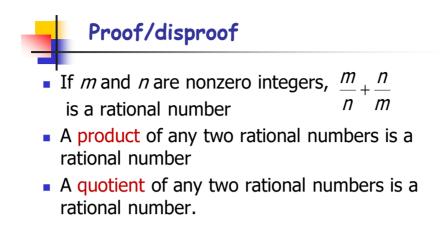
## COMP 233 Discrete Mathematics



4.2 Direct Proof and Counterexample II: Rational Numbers







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# Another Example

**Example:** Suppose *m* and *n* are nonzero integers. Is  $\frac{m}{n} + \frac{n}{m}$  a rational number? Explain.

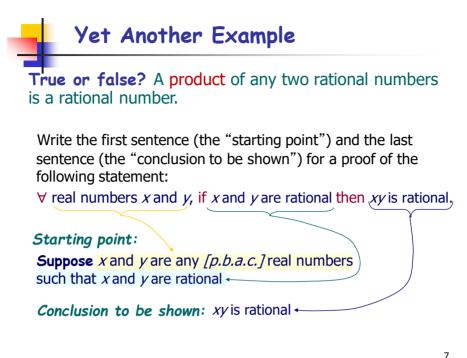
**Solution:** By algebra,  $\frac{m}{n} + \frac{n}{m} = \frac{m^2}{mn} + \frac{n^2}{mn} = \frac{m^2 + n^2}{mn}$ .

- Now both m<sup>2</sup>+ n<sup>2</sup> and mn are integers because products and sums of integers are integers.
- Also *mn* is nonzero by the zero product property

Thus  $\frac{m}{n} + \frac{n}{m}$  is a rational number.

**Zero Product Property:** If any two nonzero real numbers are multiplied, the product is nonzero.

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Example, cont.

**True or false?** A product of any two rational numbers is a rational number.

( $\forall$  real numbers x and y, if x and y are rational then xy is rational.)

Solution: This is true.

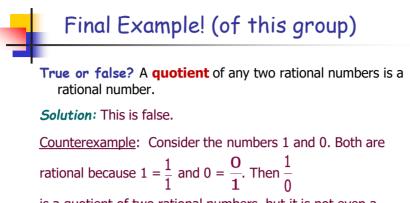
<u>Proof</u>: Suppose *x* and *y* are any rational numbers. *[We must show that xy is rational.]* 

By definition of rational, x = a/b and y = c/d for some integers a, b, c, and d with  $b \neq 0$  and  $d \neq 0$ . Then  $xy = \frac{a}{d} \cdot \frac{c}{d}$  by substitution

 $\frac{ac}{bd}$  by algebra.

But *ac* and *bd* are integers because they are products of integers, and  $bd \neq 0$  by the zero product property.

<u>Thus xy is a ratio of integers with a nonzero denominator</u>, and hence xy is rational by definition of rational. QED

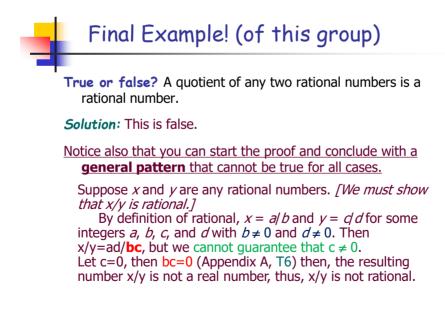


is a quotient of two rational numbers, but it is not even a

real number. So it is not a rational number.

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A corollary is a statement whose truth can be immediately deduced from a theorem that has already been proved. Theorem 4.2.2 Example:

The sum of any two rational numbers is rational.

Derive the following as a *corollary* of Theorem 4.2.2:

**Corollary 4.2.3** 

Corollary

The double of a rational number is rational.

### $\forall$ real numbers r, if R(r) then R(2r)

**Proof:** 

Suppose r is any rational number. Then 2r = r + r is a sum of two rational numbers. So, by Theorem 4.2.2, 2r is rational.

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#### Example: Deriving Additional Results about Even and Odd Integers

Suppose that you have already proved the following properties of even and odd integers:

- 1. The sum, product, and difference of any two even integers are even.
- 2. The sum and difference of any two odd integers are even.
- 3. The product of any two odd integers is odd.
- 4. The product of any even integer and any odd integer is even.
- 5. The sum of any odd integer and any even integer is odd.
- 6. The difference of any odd integer minus any even integer is odd.
- 7. The difference of any even integer minus any odd integer is odd.
- Use the properties listed above to prove that if a is any even integer and b is any odd Example integer, then  $\frac{a^2+b^2+1}{2}$  is an integer.
- Solution Suppose a is any even integer and b is any odd integer. By property 3,  $b^2$  is odd, and by property 1,  $a^2$  is even. Then by property 5,  $a^2 + b^2$  is odd, and because 1 is also odd, the sum  $(a^2 + b^2) + 1 = a^2 + b^2 + 1$  is even by property 2. Hence, by definition of even, there exists an integer k such that  $a^2 + b^2 + 1 = 2k$ . Dividing both sides by 2 gives  $\frac{a^2+b^2+1}{2} = k$ , which is an integer. Thus  $\frac{a^2+b^2+1}{2}$  is an integer [as was to be shown1.

# Home Work!