COMP 233 Discrete Mathematics

Elementary Number Theory and Methods of Proof

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4.2 Direct Proof and Counterexample II: Rational Numbers

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Rational Numbers

Definition: A real number is **rational** if, and only if, it can be written as a ratio of integers with a nonzero denominator.

In symbols:

r is **rational** $\Leftrightarrow \exists$ integers a and b such that $r = a/b$ and $b \neq 0$.

exercise Identify which of the following numbers are rational.

1. 43.205
2. $-\frac{6}{5}$
3. 0

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Class Exercise Solution

Identify which of the following numbers are rational.

1. **43.205** This number is rational: $43.205 = \frac{43205}{1000}$
2. $-\frac{6}{5}$ This number is rational: $-\frac{6}{5} = \frac{-6}{5} = \frac{6}{-5}$.
3. 0 This number is rational: $0 = \frac{0}{1}$.

Proof/disproof

- If m and n are nonzero integers, $\frac{m}{n}+\frac{n}{m}$ is a rational number
- A product of any two rational numbers is a rational number
- A quotient of any two rational numbers is a rational number.

Another Example

Example: Suppose m and n are nonzero integers. Is $\frac{m}{n}+\frac{n}{m}$ a rational number? Explain.

Solution: By algebra,
$$\frac{m}{n}+\frac{n}{m}=\frac{m^2}{mn}+\frac{n^2}{mn}=\frac{m^2+n^2}{mn}.$$

- Now both m^2+n^2 and mn are integers because products and sums of integers are integers.
- Also mn is nonzero by the zero product property

  Thus $\frac{m}{n}+\frac{n}{m}$ is a rational number.

Zero Product Property: If any two nonzero real numbers are multiplied, the product is nonzero.

Yet Another Example

True or false? A product of any two rational numbers is a rational number.

Write the first sentence (the "starting point") and the last sentence (the "conclusion to be shown") for a proof of the following statement:

∀ real numbers x and y, if x and y are rational then xy is rational.

Starting point:

Suppose x and y are any *[p.b.a.c.]* real numbers such that x and y are rational

Conclusion to be shown: xy is rational

Example, cont.

True or false? A product of any two rational numbers is a rational number.

(∀ real numbers x and y, if x and y are rational then xy is rational.)

Solution: This is true.

Proof: Suppose x and y are any rational numbers. *[We must show that xy is rational.]*

By definition of rational, $x = a/b$ and $y = c/d$ for some integers a, b, c, and d with $b \neq 0$ and $d \neq 0$. Then

$$xy = \frac{a}{b} \cdot \frac{c}{d} \quad \text{by substitution}$$

$$= \frac{ac}{bd} \quad \text{by algebra.}$$

But ac and bd are integers because they are products of integers, and $bd \neq 0$ by the zero product property.

Thus xy is a ratio of integers with a nonzero denominator, and hence xy is rational by definition of rational. QED

Final Example! (of this group)

True or false? A **quotient** of any two rational numbers is a rational number.

Solution: This is false.

Counterexample: Consider the numbers 1 and 0. Both are rational because $1 = \frac{1}{1}$ and $0 = \frac{0}{1}$. Then $\frac{1}{0}$ is a quotient of two rational numbers, but it is not even a real number. So it is not a rational number.

Final Example! (of this group)

True or false? A quotient of any two rational numbers is a rational number.

Solution: This is false.

Notice also that you can start the proof and conclude with a **general pattern** that cannot be true for all cases.

Suppose x and y are any rational numbers. *[We must show that x/y is rational.]*

By definition of rational, $x = a/b$ and $y = c/d$ for some integers a, b, c, and d with $b \neq 0$ and $d \neq 0$. Then x/y=ad/**bc**, but we cannot guarantee that $c \neq 0$. Let c=0, then bc=0 (Appendix A, T6) then, the resulting number x/y is not a real number, thus, x/y is not rational.

Corollary

- A **corollary** is a statement whose truth can be immediately deduced from a theorem that has already been proved.
- Example:

Theorem 4.2.2

The sum of any two rational numbers is rational.

Derive the following as a *corollary* of Theorem 4.2.2:

Corollary 4.2.3

The double of a rational number is rational.

∀ real numbers r, if R(r) then R(2r)

Proof:

Suppose r is any rational number. Then $2r = r + r$ is a sum of two rational numbers. So, by Theorem 4.2.2, $2r$ is rational. ■

Example: Deriving Additional Results about Even and Odd Integers

Suppose that you have already proved the following properties of even and odd integers:

1. The sum, product, and difference of any two even integers are even.
2. The sum and difference of any two odd integers are even.
3. The product of any two odd integers is odd.
4. The product of any even integer and any odd integer is even.
5. The sum of any odd integer and any even integer is odd.
6. The difference of any odd integer minus any even integer is odd.
7. The difference of any even integer minus any odd integer is odd.

Home Work!

Example Use the properties listed above to prove that if a is any even integer and b is any odd integer, then $\frac{a^2+b^2+1}{2}$ is an integer.

Solution Suppose a is any even integer and b is any odd integer. By property 3, b^2 is odd, and by property 1, a^2 is even. Then by property 5, $a^2 + b^2$ is odd, and because 1 is also odd, the sum $(a^2 + b^2) + 1 = a^2 + b^2 + 1$ is even by property 2. Hence, by definition of even, there exists an integer k such that $a^2 + b^2 + 1 = 2k$. Dividing both sides by 2 gives $\frac{a^2+b^2+1}{2} = k$, which is an integer. Thus $\frac{a^2+b^2+1}{2}$ is an integer *[as was to be shown]*. ■