COMP 233 Discrete Mathematics

Sequences and Mathematical Induction

5.1 Sequences

Outline

- Finding terms from an explicit formula and vise verse;
- Separating off/adding on a Final Term and change of variable;
- Summation/Product notation to expanded form and vise verse and finding sum using closed form
- Sequences in Computer Programming;
- Proof by Mathematical Induction (I and II)
  - Proving any property P(n) of a sequence

Outline cont.

Given an integer variable n, we can consider a variety of properties $P(n)$ that might be true or false for various values of n. For instance, we could consider

$P(n)$: $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$
$P(n)$: divisibility properties: $4^n - 1$ is divisible by 3
$P(n)$: inequality properties: $2n + 1 < 2^n$
$P(n)$: formula for sum of integers
$P(n)$: formula for sum of geometric sequence

less trivial properties

$P(n)$: n cents can be obtained using 3¢ and 5¢ coins.
$P(n)$: two C programs give the same result

A **proof by mathematical induction** shows that a given property $P(n)$ is true for all integers greater than or equal to some <u>initial integer</u>.

Sequences

Idea: Think of a **sequence** as a set of **elements** written in a row:

$a_1, a_2, a_3, \ldots, a_n$ *finite sequence*

$a_1, a_2, a_3, \ldots, a_n, \ldots$ *infinite sequence*

Each individual element a_k is called a **term.**

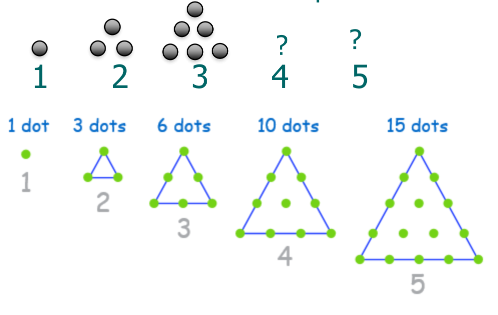
The k in a_k is called a **subscript** or **index**

• Definition

A **sequence** is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

Sequences Observe patterns

Determine the number of points in the 4th and 5th figure



Triangular Number

n(n+1)/2

Determine the next 2 terms of the sequence: 4, 8, 16, 32, 64, …

Determine the next 2 terms of the sequence:2, 8, 32, 128, …

Induce the formula that could be used to determine any term in the sequence

Finding Terms of Sequences Given by Explicit Formulas

Given sequences $a1, a2, a3, \ldots$ and $b2, b3, b4, \ldots$ defined by the following explicit formulas:

$$a_k = \frac{k}{k+1} \quad \text{for some integers } k \geq 1$$

$$b_i = \frac{i-1}{i} \quad \text{for some integers } i \geq 2$$

- Compute the first five terms of both sequences.

- Compute the first six terms of the sequence $c0, c1, c2, \ldots$ defined as follows:

$$c_j = (-1)^j \quad \text{for all integers } j \geq 0.$$

Finding Explicit Formula to Fit Given Initial Terms

Find an explicit formula for a sequence that has the following initial terms: (alternating sequence)

$$1, \; -\frac{1}{4}, \; \frac{1}{9}, \; -\frac{1}{16}, \; \frac{1}{25}, \; -\frac{1}{36}, \; \ldots\ldots$$

Solution:

$$\frac{1}{1^2}, \; \frac{(-1)}{2^2}, \; \frac{1}{3^2}, \; \frac{(-1)}{4^2}, \; \frac{1}{5^2}, \; \frac{(-1)}{6^2}.$$

$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6$$

$$a_k = \frac{(-1)^{k+1}}{k^2} \quad \text{for all integers } k \geq 1. \qquad \textbf{OR} \quad a_k = \frac{(-1)^k}{(k+1)^2} \quad \text{for all integers } k \geq 0.$$

Exercises

Example: Find an explicit formula for a sequence that has the following initial terms:

$$\frac{1}{3}, -\frac{2}{4}, \frac{3}{5}, -\frac{4}{6}, \frac{5}{7}, -\frac{6}{8}, \ldots$$

Solutions: The sequence satisfies the formulas

for all integers $n \geq 0$, $a_n = (-1)^n \dfrac{n+1}{n+3}$

for all integers $n \geq 1$, $a_n = (-1)^{n-1} \dfrac{n}{n+2}$

Summation Notation

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \ldots a_n$$

Upper limit; Lower limit; index

```
Sum =0;
for (k=m; k<=n; k++)
    Sum = Sum + a[k];
return Sum;
```

$$\sum_{k=1}^{7} \frac{1}{k}$$

```
Sum =0.0;
for (k=1; k<=7; k++)
    Sum = Sum + 1/k;
return Sum;
```

Examples

Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, and $a_5 = 2$.
Compute the following:

a. $\sum_{k=1}^{5} a_k$ b. $\sum_{k=2}^{2} a_k$ c. $\sum_{k=1}^{2} a_{2.k}$

Solution

a. $\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = (-2) + (-1) + 0 + 1 + 2 = 0$

b. $\sum_{k=2}^{2} a_k = a_2 = -1$

c. $\sum_{k=1}^{2} a_{2k} = a_{2\cdot 1} + a_{2\cdot 2} = a_2 + a_4 = -1 + 1 = 0$

Example: When the Terms of a Summation Are Given by a Formula

Compute the following summation:

$$\sum_{k=1}^{5} k^2.$$

Solution

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

Useful Operations

- *Changing from Summation Notation to Expanded Form*
- *Changing from Expanded Form to Summation Notation*
- *Separating Off a Final Term*
- *Telescoping*

→**These concepts are very important to understand computer loops**

Changing from Summation Notation to Expanded Form

When the upper limit of a summation is a variable, an ellipsis is used to write the summation in **expanded form**.

Write the following summation in expanded form:

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$$

$$\begin{aligned}\sum_{i=0}^{n} \frac{(-1)^i}{i+1} &= \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1} \\ &= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1} \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}\end{aligned}$$

Changing from Expanded Form to Summation Notation

Express the following using summation notation:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \ldots + \frac{n+1}{2n}$$

Solution The general term of this summation can be expressed as $\frac{k+1}{n+k}$ for integers k from 0 to n. Hence

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{k=0}^{n} \frac{k+1}{n+k}.$$

■

Separating Off a Final Term and Adding On a Final Term

→recursive definition

a. Rewrite $\sum_{i=1}^{n+1} \frac{1}{i^2}$ by separating off the final term.

b. Write $\sum_{k=0}^{n} 2^k + 2^{n+1}$ as a single summation.

Solution

a. $\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$

b. $\sum_{k=0}^{n} 2^k + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$

When solving problems, it is often useful to rewrite a summation using the recursive form of the definition, either by separating off the final term of a summation or by adding a final term to a summation.

Telescoping

A **telescoping series** is a series whose partial sums eventually only have a fixed number of terms after cancellation [wiki].
A **telescoping sums** are sums that can be written as a simple expression.

Example: $\sum_{i=1}^{n} i-(i+1)$ $=(1-2)+(2-3)+\ldots+(n-(n+1))$

$= 1-(n+1)$

$=-n$

This is very useful in programing:

```
S=0
For (i=1;i<=n;i++)
    S= S+ i-(i+1);
```

→

```
S = -n;
```


Example: Telescoping

Some sums can be transformed into telescoping sums, which then can be rewritten as a simple expression. For instance, observe that

$$\frac{1}{k}-\frac{1}{k+1}=\frac{(k+1)-k}{k(k+1)}=\frac{1}{k(k+1)}.$$

```
S=0;
For (k=1;k<=n;k++)
    S=S+ 1/k*(k+1);
```

Use this identity to find a simple expression for $\sum_{k=1}^{n}\frac{1}{k(k+1)}$.

Solution

$$\sum_{k=1}^{n}\frac{1}{k(k+1)}=\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

$$=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}.$$

```
S = 1- (1/(n+1);
```

Important Formulas (closed form)

Formula for the sum of the first *n* integers: For all integers $n \geq 1$,

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}.$$

Formula for the sum of the terms of a geometric sequence: For all real numbers $r \neq 1$ and all integers $n \geq 0$,

$$1+r+r^2+r^3+\cdots+r^n=\frac{r^{n+1}-1}{r-1}.$$

Exercises: find the closed form of the following – Homework

a. $1+2+3+\cdots+100=\frac{100(100+1)}{2}=50(101)=5050$

b. $1+2+3+\cdots+k=\frac{k(k+1)}{2}$

c. $1+2+3+\cdots+(k-1)=\frac{(k-1)((k-1)+1)}{2}=\frac{(k-1)k}{2}$

d. $4+5+6+\cdots+(k-1)=(1+2+3+\cdots+(k-1))-(1+2+3)$

$$=\frac{k(k-1)}{2}-(1+2+3)=\frac{k(k-1)}{2}-6$$

e. $3+3^2+3^3+\cdots+3^k=(1+3+3^2+3^3+\cdots+3^k)-1=\frac{3^{k+1}-1}{3-1}-1$

$$=\frac{3^{k+1}-1}{2}-1=\frac{3^{k+1}-1}{2}-\frac{2}{2}=\frac{3^{k+1}-3}{2}$$

f. $3+3^2+3^3+\cdots+3^k=3(1+3+3^2+\cdots+3^{k-1})$

$$=3\left(\frac{3^{(k-1)+1}-1}{3-1}\right)=\frac{3(3^k-1)}{2}$$

Product notation

• Definition

If m and n are integers and $m \leq n$, the symbol $\prod_{k=m}^{n} a_k$, read the **product from k equals m to n of a-sub-k**, is the product of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

Recursive definition for the product notation:

$$\prod_{k=m}^{m} a_k = a_m \quad \text{and} \quad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n \quad \text{for all integers } n > m.$$

Example: Computing Products

Compute the following products:

a. $\prod_{k=1}^{5} k$

b. $\prod_{k=1}^{1} \frac{k}{k+1}$

Solution

a. $\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

b. $\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$

Exercises - Homework

Write the following using product notation

$$(2^2 - 1) \cdot (3^2 - 1) \cdot (4^2 - 1)$$

$$(1 - t) \cdot (1 - t^2) \cdot (1 - t^3) \cdot (1 - t^4)$$

Factorial Notation

- **Definition**

For each positive integer n, the quantity ***n* factorial** denoted ***n*!**, is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1.$$

Zero factorial, denoted 0!, is defined to be 1:

$$0! = 1.$$

$0! = 1$

$1! = 1$

$2! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$

$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$= 40{,}320$

$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$= 362{,}880$

Recursive definition for factorial

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1. \end{cases}$$

Example using this definition:

a. $\frac{8!}{7!} = \frac{8 \cdot \cancel{7!}}{\cancel{7!}} = 8$ b. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{2! \cdot \cancel{3!}} = \frac{5 \cdot 4}{2 \cdot 1} = 10$

c. $\frac{(n+1)!}{n!} = \frac{(n+1) \cdot \cancel{n!}}{\cancel{n!}} = n + 1$

More Exercises - Homework

d. $\frac{1}{2! \cdot 4!} + \frac{1}{3! \cdot 3!} = \frac{1}{2! \cdot 4!} \cdot \frac{3}{3} + \frac{1}{3! \cdot 3!} \cdot \frac{4}{4}$ by multiplying each numerator and denominator by just what is necessary to obtain a common denominator

$= \frac{3}{3 \cdot 2! \cdot 4!} + \frac{4}{3! \cdot 4 \cdot 3!}$ by rearranging factors

$= \frac{3}{3! \cdot 4!} + \frac{4}{3! \cdot 4!}$ because $3 \cdot 2! = 3!$ and $4 \cdot 3! = 4!$

$= \frac{7}{3! \cdot 4!}$ by the rule for adding fractions with a common denominator

$= \frac{7}{144}$

e. $\frac{n!}{(n-3)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \cancel{(n-3)!}}{\cancel{(n-3)!}} = n \cdot (n-1) \cdot (n-2)$

$= n^3 - 3n^2 + 2n$

Use for the factorial notation

• Definition

Let n and r be integers with $0 \le r \le n$. The symbol

$$\binom{n}{r}$$

is read "***n* choose *r***" and represents the number of subsets of size r that can be chosen from a set with n elements.

In Section 9.5 we will explore many uses of n choose r for solving problems involving counting, and we will prove the following computational formula:

• Formula for Computing $\binom{n}{r}$

For all integers n and r with $0 \le r \le n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Use for the factorial notation

n choose r is always an integer (because it is a number of subsets), you can be sure that all the factors in the denominator of the formula will be canceled out by factors in the numerator.

Examples:

Use the formula for computing $\binom{n}{r}$ to evaluate the following expressions:

a. $\binom{8}{5}$ b. $\binom{4}{0}$ c. $\binom{n+1}{n}$

a. $\binom{8}{5} = \frac{8!}{5!(8-5)!}$

$= \frac{8\cdot 7\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{(\cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1)\cdot(\cancel{3}\cdot \cancel{2}\cdot \cancel{1})}$ always cancel common factors before multiplying

$= 56.$

b. $\binom{4}{4} = \frac{4!}{4!(4-4)!} = \frac{4!}{4!0!} = \frac{\cancel{4\cdot 3\cdot 2\cdot 1}}{(\cancel{4\cdot 3\cdot 2\cdot 1})(1)} = 1$

c. Look at page 239 textbook.

Properties of Summations and products

Theorem 5.1.1

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(a_k + b_k)$

2. $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ generalized distributive law

3. $\displaystyle\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n}(a_k \cdot b_k).$

The proof of the theorem is discussed in Section 5.6

Example: Using Properties of Summation

Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k.

Write the following expression as a single summation

$$\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k$$

$$= \sum_{k=m}^{n}(k+1) + 2 \cdot \sum_{k=m}^{n}(k-1)$$ by substitution

$$= \sum_{k=m}^{n}(k+1) + \sum_{k=m}^{n} 2 \cdot (k-1)$$ by Theorem 5.1.1 (2)

$$= \sum_{k=m}^{n}((k+1) + 2 \cdot (k-1))$$ by Theorem 5.1.1 (1)

$$= \sum_{k=m}^{n}(3k-1)$$ by algebraic simplification

Example: Using Properties of Product

Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k.
Write the following expression as a single product

$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right)$$

$$= \left(\prod_{k=m}^{n} (k+1)\right) \cdot \left(\prod_{k=m}^{n} (k-1)\right)$$ by substitution

$$= \prod_{k=m}^{n} (k+1) \cdot (k-1)$$ by Theorem 5.1.1 (3)

$$= \prod_{k=m}^{n} (k^2 - 1)$$ by algebraic simplification

Change of Variable

Observe
$$\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$$
and
$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2.$$

Hence,
$$\sum_{k=1}^{3} k^2 = \sum_{i=1}^{3} i^2.$$

Replaced Index by any other symbol (called a dummy variable)

Also Observe

$$\sum_{j=2}^{4} (j-1)^2 = (2-1)^2 + (3-1)^2 + (4-1)^2$$
$$= 1^2 + 2^2 + 3^2$$
$$= \sum_{k=1}^{3} k^2.$$

Programing Loops

Any difference between these loops?

1. **for** $i := 1$ **to** n
   print $a[i]$
   next i

2. **for** $j := 0$ **to** $n - 1$
   print $a[j + 1]$
   next j

3. **for** $k := 2$ **to** $n + 1$
   print $a[k - 1]$
   next k

What about these loops?

$s := a[1]$
for $k := 2$ **to** n
  $s := s + a[k]$
next k

$s := 0$
for $k := 1$ **to** n
  $s := s + a[k]$
next k

Change of Variable

Example: Transform $\sum_{k=1}^{n} k^n$ by making the change of variable $j = k - 1$.

1. When $k = 1$, then $j = 1 - 1 = 0$

2. When $k = n$, then $j = n - 1$

3. $j = k - 1 \Rightarrow k = j + 1$ Thus $k^n = (j + 1)^n$

So: $\sum_{k=1}^{n} k^n = \sum_{j=0}^{n-1} (j+1)^n$

Change of Variable - Homework

Example: Transform $\prod_{k=1}^{n} \frac{k}{k^2+4}$ by making the change of Variable $i = k + 1$.

Example: Transform $\prod_{i=n}^{2n} \frac{n-i+1}{n+i}$ by making the change of Variable $j = i - 1$.

Exercises

Transform the following summation by making the specified change of variable. Change of variable $j = k+1$

$$\sum_{k=0}^{6} \frac{1}{k+1}$$

```
For (k=0; k<=6; k++)
    Sum = Sum + 1/(k+1)
```

$$\sum_{j=1}^{7} \frac{1}{j} = \sum_{k=1}^{7} \frac{1}{k}.$$

```
For (k=1; k<=7; k++)
    Sum = Sum + 1/(k)
```

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{k=1}^{7} \frac{1}{k}$$

Exercises

Transform the following summation by making the specified change of variable.

$$\sum_{k=1}^{n+1} \frac{k}{n+k}$$

For (k=1; k<=n+1; k++)
Sum = Sum + k/(n+k)

Change of variable: j = k - 1

$$\sum_{j=0}^{n} \frac{j+1}{n+(j+1)} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}$$

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}$$

For (k=0; k<=n; k++)
Sum = Sum + (k+1)/(n+k+1)

Homework

Application: Algorithm to Convert from Base 10 to Base 2 Using Repeated Division by 2

Page 240 textbook

Programing Loops

All questions in the exams will be loops

Thus, I suggest:
Convert all previous examples into loops and play with them