COMP 233 Discrete Mathematics

Sequences and Mathematical Induction

5.2 Mathematical Induction I

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Warm-up 1: Consider the expression $1 + 3 + 5 + 7 + \cdots + (2n - 1)$

1. What is $1 + 3 + 5 + 7 + \cdots + (2n - 1)$

when $n = 3$? $(2\cdot1 - 1) + (2\cdot2 - 1) + (2\cdot3 - 1) = 1 + 3 + 5$

when $n = 2$? $(2\cdot1 - 1) + (2\cdot2 - 1) = 1 + 3$

when $n = 1$? $(2\cdot1 - 1) = 1$ ← ***NOTE: The numbers 3, 5, and 7 don't appear!***

2. What is $1 + 3 + 5 + 7 + \cdots + (2n - 1)$

when $n = k$? $1 + 3 + 5 + 7 + \cdots + (2k - 1)$

when $n = k + 1$? $1 + 3 + 5 + \cdots + (2k - 1) + (2(k+1) - 1)$
$= 1 + 3 + 5 + \cdots + (2k + 1)$

What is the **next-to-last term** of

$$1 + 3 + 5 + 7 + \cdots + (2k + 1)?$$

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when $n = 1$? $(2\cdot 1 - 1) = 1$ ← ***NOTE: The numbers 3, 5, and 7 don't appear!***

2. What is $1 + 3 + 5 + 7 + \cdots + (2n - 1)$

when $n = k$? $1 + 3 + 5 + 7 + \cdots + (2k - 1)$

when $n = k + 1$? $1 + 3 + 5 + \cdots + (2k - 1) + (2(k+1) - 1)$ [$= 2k + 2 - 1 = 2k + 1$]

$= 1 + 3 + 5 + \cdots + (2k + 1)$

What is the **next-to-last term** of

$1 + 3 + 5 + 7 + \cdots + (2k + 1)$? $2k - 1$

Mathematical Induction Note

Do not mix between the index of a term (e.g, **k)** and the value of that term (e.g., **2k-1**)

the next term after **k** is **k+1** which makes the value 2(k+1)-1

When proving a **formula** by mathematical induction, it is virtually always desirable to make the next-to-last-term explicit, as was done above.

Doing so, makes it easier to see how the inductive hypothesis will apply.

Warm-up 2: Consider the property

$P(n)$: $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$

Why the colon?
Why not = ?

1. What is P(1)? $1 = 1^2$

2. What is P(k)? $1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$

3. What is P(k + 1)? $1 + 3 + 5 + 7 + \cdots + (2(k + 1) - 1) = (k + 1)^2$

Or: $1 + 3 + 5 + 7 + \cdots + (2k + 1) = (k + 1)^2$

Caution! P(1) is NOT $1 + 3 + 5 + 7 + \cdots + 1 = 1^2$.

Introduction

Any whole number of cents of at least 8c/ can be obtained using 3¢ and 5¢ coins.
More formally:

For all integers $n \geq 8$, n cents can be obtained using 3¢ and 5¢ coins.

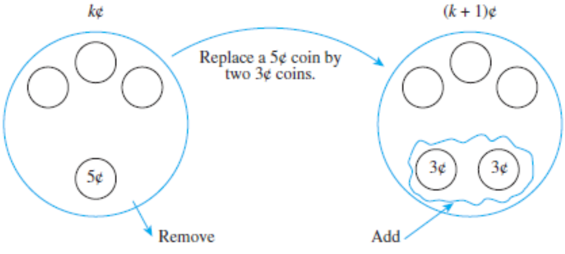
Even more formally:

For all integers $n \geq 8$, $P(n)$ is true, where $P(n)$ is the sentence
"n cents can be obtained using 3¢ and 5¢ coins."

Number of Cents	How to Obtain It
8¢	3¢ + 5¢
9¢	3¢ + 3¢ + 3¢
10¢	5¢ + 5¢
11¢	3¢ + 3¢ + 5¢
12¢	3¢ + 3¢ + 3¢ + 3¢
13¢	3¢ + 5¢ + 5¢
14¢	3¢ + 3¢ + 3¢ + 5¢
15¢	5¢ + 5¢ + 5¢
16¢	3¢ + 3¢ + 5¢ + 5¢
17¢	3¢ + 3¢ + 3¢ + 3¢ + 5¢

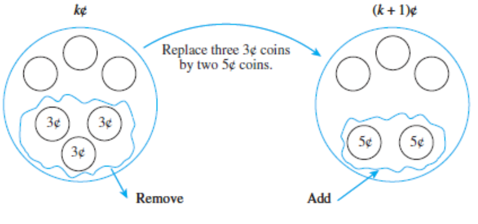
Cont….

The cases shown in the **table** provide inductive evidence to support the claim that *P(n)* is true for general *n*. Indeed, *P(n) is true for all* $n \geq 8$ *if, and only if, it is possible to continue filling in the table for arbitrarily large values of n*. The k^{th} line of the table gives information about how to obtain *k*¢ using 3¢ and 5¢ coins. To continue the table to the next row, directions must be given for how to obtain *(k + 1)*¢ using 3¢ and 5¢ coins. **The secret** is to observe first that if *k*¢ can be obtained using at least one 5¢ coin, then *(k + 1)*¢ can be obtained by replacing the 5¢ coin by two 3¢ coins,



Cont…

If, **on the other hand**, *k*¢ is obtained without using a 5¢ coin, then 3¢ coins are used exclusively. And since the total is at least 8¢, three or more 3¢ coins must be included. Three of the 3¢ coins can be replaced by two 5¢ coins to obtain a total of *(k + 1)*¢. Any argument of this form is an argument by ***mathematical induction***. In general, mathematical induction is a method for proving that a property defined for integers *n* is true for all values of *n* that are greater than or equal to some initial integer.



Principal Of Mathematical Induction

Principle of Mathematical Induction

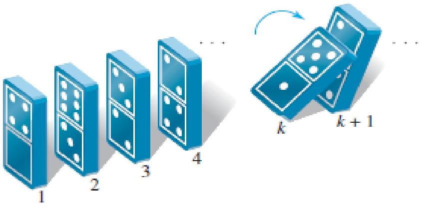
Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

$$\text{for all integers } n \geq a,\ P(n)$$

is true.



If the kth domino falls backward, it pushes the $(k+1)$st domino backward also.

Method Of Proof

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.
[This supposition is called the **inductive hypothesis.***]*

Then

show that $P(k+1)$ is true.

Induction is a technique for proving things on finite and infinite sequences.

Finding Terms of Sequences

Proposition: For all integers $n \geq 8$, n¢ can be obtained using 3¢ and 5¢ coins

Proof: Let the property $P(n)$ be the sentence "n¢ can be obtained using 3¢ and 5¢ coins". ← $P(n)$

Show that P(8) is true:

$P(8)$ is true because 8¢ can be obtained using one 3¢ coin and one 5¢ coin.

Show that for all integers $k \geq 8$, if P(k) is true then P(k+1) is also true:

Suppose that k is any integer with $k \geq 8$ such that k¢ can be obtained using 3¢ and 5¢ coins. ← $P(k)$ (inductive hypothesis)

Explicit formula

$(k + 1)$¢ can be obtained using 3¢ and 5¢ coins. ← $P(k + 1)$

Case 1 (There is a 5¢ coin among those used to make up the k¢

In this case replace the 5¢ coin by two 3¢ coins; the result will be $(k + 1)$¢

Case 2 (There is not a 5¢ coin among those used to make up the k¢

In this case, because $k \geq 8$, at least three 3¢ coins must have been used. So remove three 3¢ coins and replace them by two 5¢ coins; the result will be $(k + 1)$¢.

Thus in either case $(k + 1)$¢ can be obtained using 3¢ and 5¢ coins.

Prove P(n): 1+2+3+…+n=n(n+1)/2

Formula for the sum of the first *n* integers: For all integers $n \geq 1$,

Proving $$1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$ $\leftarrow P(n)$

Is like proving that these programs prints the same results for n ≥ *1*

```
for (i=1, i≤n; i++)
     S=S+i;
Print ("%d", S);
```

```
S=(n(n+1))/2;
Print ("%d",S);
```


Propositions

Proposition: For all integers $n \geq 1$,

$$1 + 2 + \cdots + n = n(n + 1)/2$$

Proof:

Show that P(1) is true:

To establish $P(1)$, we must show that

$$1 = \frac{1(1+1)}{2}$$ $\leftarrow P(1)$

But the left-hand side of this equation is 1 and the right-hand side is

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

also. Hence $P(1)$ is true.

Show that for all integers $k \geq 1$, if P(k) is true then P(k + 1) is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer $k \geq 1$. That is:] Suppose that k is any integer with $k \geq 1$ such that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$ $\leftarrow P(k)$ inductive hypothesis

Theorem: The sum of the first n integers is $\frac{n(n+1)}{2}$.

Theorem: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ property is that this relationship holds true.

Inductive step:

We want to show that for some integer $k \geq 1$, if $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$, then $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

suppose — show

IH: Suppose $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ for some $k \geq 1$.

We want to show that $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

Cont…

[We must show that $P(k+1)$ is true. That is:] We must show that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)[(k+1)+1]}{2},$$

or, equivalently, that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}. \quad \leftarrow P(k+1)$$

[We will show that the left-hand side and the right-hand side of $P(k+1)$ are equal to the same quantity and thus are equal to each other.]

The left-hand side of $P(k+1)$ is

$$1 + 2 + 3 + \cdots + (k+1)$$

$$= 1 + 2 + 3 + \cdots + k + (k+1) \quad \text{by making the next-to-last term explicit}$$

$$= \frac{k(k+1)}{2} + (k+1) \quad \text{by substitution from the inductive hypothesis}$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

Cont…

$$= \frac{k^2+k}{2} + \frac{2k+2}{2}$$

$$= \frac{k^2+3k+1}{2} \qquad \text{by algebra.}$$

And the right-hand side of $P(k+1)$ is

$$\frac{(k+1)(k+2)}{2} = \frac{k^2+3k+1}{2}.$$

Thus the two sides of $P(k+1)$ are equal to the same quantity and so they are equal to each other. Therefore the equation $P(k+1)$ is true *[as was to be shown]*.

[Since we have proved both the basis step and the inductive step, we conclude that the theorem is true.]

Sum of Geometric Series

For any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}.$$

Proof (by mathematical induction):

Suppose r is a particular but arbitrarily chosen real number that is not equal to 1, and let the property $P(n)$ be the equation

$$\sum_{i=0}^{n} r^i = \frac{r^{i+1}-1}{r-1} \qquad \leftarrow P(n)$$

We must show that $P(n)$ is true for all integers $n \geq 0$. We do this by mathematical induction on n.

Show that P(0) is true:

To establish $P(0)$, we must show that

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1}-1}{r-1} \qquad \leftarrow P(0)$$

Cont….

The left-hand side of this equation is $r^0 = 1$ and the right-hand side is

$$\frac{r^{0+1}-1}{r-1} = \frac{r-1}{r-1} = 1$$

also because $r^1 = r$ and $r \neq 1$. Hence $P(0)$ is true.

Show that for all integers $k \geq 0$, if P(k) is true then P(k + 1) is also true:
[Suppose that P(k) is true for a particular but arbitrarily chosen integer $k \geq 0$. That is:]
Let k be any integer with $k \geq 0$, and suppose that

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1}-1}{r-1} \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

[We must show that $P(k+1)$ is true. That is:] We must show that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1}-1}{r-1},$$

or, equivalently, that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2}-1}{r-1}. \qquad \leftarrow P(k+1)$$

Cont…

$$\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1} \qquad \text{by writing the } (k+1)\text{st term separately from the first } k \text{ terms}$$

$$= \frac{r^{k+1}-1}{r-1} + r^{k+1} \qquad \text{by substitution from the inductive hypothesis}$$

$$= \frac{r^{k+1}-1}{r-1} + \frac{r^{k+1}(r-1)}{r-1} \qquad \text{by multiplying the numerator and denominator of the second term by } (r-1) \text{ to obtain a common denominator}$$

$$= \frac{(r^{k+1}-1) + r^{k+1}(r-1)}{r-1} \qquad \text{by adding fractions}$$

$$= \frac{r^{k+1}-1+r^{k+2}-r^{k+1}}{r-1} \qquad \text{by multiplying out and using the fact that } r^{k+1}\cdot r = r^{k+1}\cdot r^1 = r^{k+2}$$

$$= \frac{r^{k+2}-1}{r-1} \qquad \text{by canceling the } r^{k+1}\text{'s.}$$

which is the right-hand side of $P(k+1)$ *[as was to be shown.]*

Prove P(n): $1+5+5^2+5^3+\cdots+5^n = \frac{5^{n+1}-1}{4}$ for all integers $n \geq 0$.

Formula for the sum of the terms of a geometric sequence: For all real numbers $r \neq 1$ and all integers $n \geq 0$, proving

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}.$$

Is like proving that these codes will return the same output.

```
int n, r, sum=0;
int i;
scanf("%d",&n);
scanf("%d",&r);

if(r != 1)  {
   for(i=0 ; i<=n ; i++)  {
         sum = sum + pow(r,i);
   }
   printf("%d\n", sum);
}
```

```
int n, r, sum=0;
scanf("%d",&n);
scanf("%d",&r);

if(r != 1) {
 sum=((pow(r,n+1))-1)/(r-1);
           printf("%d\n", sum);
 }
```


Outline a proof by math induction: $1+5+5^2+5^3+\cdots+5^n = \frac{5^{n+1}-1}{4}$ for all integers $n \geq 0$.

Proof by mathematical induction: Let the property P(n) be the equation

$$1+5+5^2+5^3+\cdots+5^n = \frac{5^{n+1}-1}{4}.$$

Show that the property is true for $n = 0$: We must show that

$$1 = \frac{5^{0+1}-1}{4}.$$

Show that for all integers $k \geq 0$, if the property is true for $n = k$, then it is true for $n = k + 1$: Let k be an integer with $k \geq 0$, and **suppose** that

$$1+5+5^2+5^3+\cdots+5^k = \frac{5^{k+1}-1}{4}.$$ *[This is the inductive hypothesis.]*

We must **show** that

$$1+5+5^2+5^3+\cdots+5^{k+1} = \frac{5^{(k+1)+1}-1}{4}.$$

Prove P(n): $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$

Example: Prove that for all integers $n \geq 1$,

$$1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2n - 1) = n^2.$$

Proof: Consider the equation

$$1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2n - 1) = n^2 \quad \leftarrow \textit{the property}$$

Show that the property is true for n = 1:

When $n = 1$, the property is the equation $1 = 1^2$.
But the left-hand side (LHS) of this equation is 1,
and the right-hand side (RHS) is 1^2, which equals 1 also.
So the property is true for $n = 1$.

It's really important to know what the property is.

Inductive Step for the proof that for all integers $n \geq 1$, $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$.

Show that $\forall$ integers $k \geq 1$, if the property is true for $n = k$ then it is true for $n = k + 1$:

Let k be any integer with $k \geq 1$, and suppose that the property is true for $n = k$. In other words, **suppose** that

$$1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k - 1) = k^2.$$

This supposition is called the **inductive hypothesis**.

We must show that the property is true for $n = k + 1$.
In other words, we must show that

$$1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2(k + 1) - 1) = (k + 1)^2,$$

or, equivalently, we must **show** that

$$1 + 3 + 5 + 7 + \cdot \cdot \cdot + (2k + 1) = (k + 1)^2.$$

Inductive hypothesis: $1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$.
Show: $1 + 3 + 5 + 7 + \cdots + (2k + 1) = (k + 1)^2$.

But the LHS of the equation to be shown is

$1 + 3 + 5 + 7 + \cdots + (2k + 1)$

$= 1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2(k + 1) - 1)$ *by making the next-to-last-term explicit*

$= k^2 + (2k + 1)$ *by substitution from the inductive hypothesis*

$= (k + 1)^2$ *by algebra,*

which equals the RHS of the equation to be shown.

This proves that ***if*** the property is true for $n = k$, then it is true for $n = k + 1$ and completes the proof by mathematical induction.

Example: Sum of Cubes

Proposition: For all integers $n \geq 1$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

Is there another way of writing this?

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$$

We know this is $= \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$

So, our theorem now is:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Base Case Step

Proof by Induction

Base Case: Let $n = 1$

LHS: $\sum_{i=1}^{1} i^3 = 1^3 = 1$

RHS: $\frac{1^2(1+1)^2}{4} = \frac{1 \cdot 2^2}{4} = \frac{1 \cdot 4}{4} = 1$

So our base case holds!

Inductive Step

Inductive Step:

We want to show that if $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$ for some $k \geq 1$, then $\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 \quad \text{by pulling off the last term}$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \text{by the IH}$$

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

} algebra

Cont.

$$\sum_{i=1}^{k+1} i^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2(k^2 + 4(k+1))}{4}$$

$$= \frac{(k+1)^2(k^2 + 4k + 4)}{4}$$

by algebra

$$= \frac{(k+1)^2(k+2)^2}{4}.$$

Which was to be shown. ∎

Example: Sum of Squares - HW

Proposition: For all integers $n \geq 1$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$