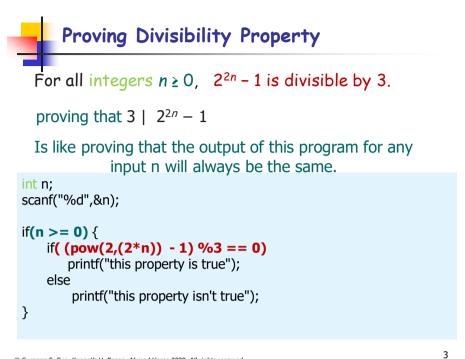


Sequences and Mathematical Induction

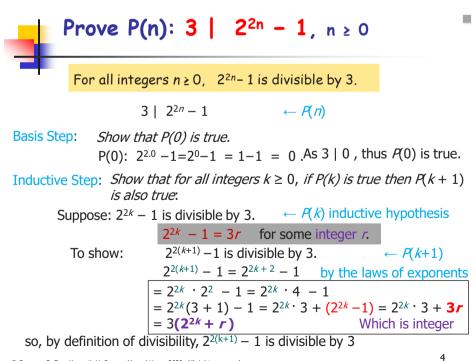
1

5.3 Mathematical Induction II

- Proving Divisibility Property
- Proving an Inequality
- Proving a Property of a Sequence



```
© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved
```



Outline a proof by math induction for the statement: For all integers $n \ge 0$, $5^n - 1$ is divisible by 4. Proof by mathematical induction: Let the property P(n) be the sentence $5^n - 1$ is divisible by 4. \leftarrow the property Show that the property is true for n = 0: We must show that $5^0 - 1$ is divisible by 4. But $5^0 - 1 = 1 - 1 = 0$, and 0 is divisible by 4 because $0 = 4 \cdot 0$. Show that for all integers $k \ge 0$, if the property is true for n = k, then it is true for n = k + 1: Let k be an integer with $k \ge 0$, and suppose that [the property is true for n = k. In other words, suppose that] $5^k - 1$ is divisible by 4. \leftarrow inductive hypothesis

We must **show** that [the property is true for n = k + 1.

In other words, we must show that]

 $5^{k+1} - 1$ is divisible by 4.

Ref: Sec. 4.3

5

© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved

Scratch Work for proving that For all integers $n \ge 0$, $5^n - 1$ is divisible by 4. Inductive hypothesis: $5^k - 1$ is divisible by 4. Want to show: $5^{k+1} - 1$ is divisible by 4. Working Backward: Want $5^{k+1} - 1 = 4 \cdot (\text{some integer})$

Working Forward: *Know* $5^k - 1 = 4 \cdot r$ for some integer *r*

Experiment: $5^{k+1}-1 = 5^{k} \cdot 5 - 1$ $= 5^{k} \cdot (4 + 1) - 1$ $= 5^{k} \cdot 4 + 5^{k} \cdot 1 - 1$ $= 5^{k} \cdot 4 + (5^{k} - 1)$ Note: Each of these terms is divisible by 4. So: $5^{k+1}-1 = 5^{k} \cdot 4 + 4 \cdot r$ (where r is an integer) $= 4 \cdot (5^{k} + r)$

(For the complete proof, see the book.)

Proving Inequality

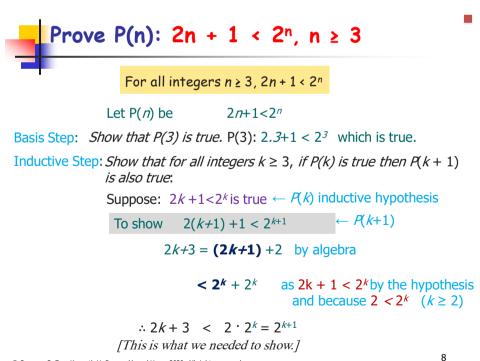
Proving that For all integers $n \ge 3$, $2n + 1 < 2^n$

Is like proving that the output of this program for any input n>=3 will always be the same

```
scanf(``%d", &n);

If (n>=3)
{
    If (2n+1 < pow (2,n)) // is 2n+1 < 2^n ?? //
        printf (``true");
    Else
        printf (``false");
}</pre>
```

© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved



© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved

From Appendix A

- T17. *Trichotomy Law* For arbitrary real numbers a and b, exactly one of the three relations a < b, b < a, or a = b holds.
- T18. *Transitive Law* If a < b and b < c, then a < c.
- T19. If a < b, then a + c < b + c.
- T20. If a < b and c > 0, then ac < bc.
- T21. If $a \neq 0$, then $a^2 > 0$.
- T22. 1 > 0.
- T23. If a < b and c < 0, then ac > bc.
- T24. If a < b, then -a > -b. In particular, if a < 0, then -a > 0.

T25. If ab > 0, then both a and b are positive or both are negative.

T26. If a < c and b < d, then a + b < c + d.

T27. If 0 < a < c and 0 < b < d, then 0 < ab < cd. (2*k*+1)+2 < 2^{*k*} + 2^{*k*}

 $2k + 1 < 2^{k}$ and $2 < 2^{k}$ $(k \ge 2)$

a

© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved

Proving Inequalities

Example: Use mathematical induction to prove that $2^n < n!$ for every integer $n \ge 4$.

Solution: Let P(n) be the predicate that $2^n < n!$.

- **Basis:** P(4) is true since $2^4 = 16 < 4! = 24$.
- **Induction:** Assume P(k) holds, i.e., $2^k < k!$ for an arbitrary integer $k \ge 4$. To show that P(k + 1) holds: $2^{k+1} = 2 \times 2^k$

 $z^{n+1} = Z \times Z^n$

< $2 \times k!$ (by the induction hypothesis) < (k + 1)k! (since $k \ge 4$, hence k + 1 > 2) = (k + 1)!

Therefore, $2^n < n!$ holds, for every integer $n \ge 4$.

Note that here the basis step is P(4), since P(0), P(1), P(2), P(3) are all false.

Proving other properties of a Sequence

Define a sequence a_1 , a_2 , a_3 ... as follows: $a_1 = 2$ $a_k = 5a_{k-1}$ for all integers $k \ge 2$.

What are the terms?

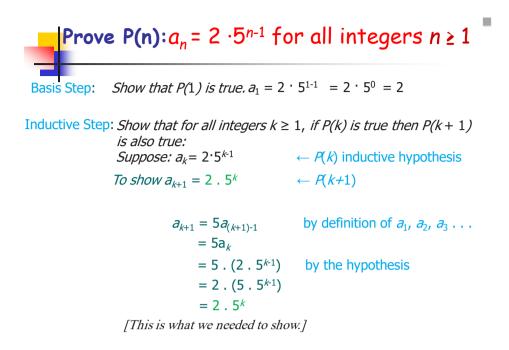
What is the explicit formula? Let it be P(n)

© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved

Proving other properties of a Sequence
Define a sequence a₁, a₂, a₃ ... as follows: a₁ = 2 a_k = 5a_{k-1} for all integers k ≥ 2.
a₁ = 2 a₂ = 5a₂₋₁ = 5a₁ = 5·2 = 10 a₃ = 5a₃₋₁ = 5a₂ = 5·10 = 50 a₄ = 5a₄₋₁ = 5a₃ = 5·50 = 250
P(n): The terms of the sequence satisfy the explicit formula a_n = 2 · 5ⁿ⁻¹

© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved

11



© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved

Induction Versus Deduction Thinking **Deduction Reasoning** Induction Reasoning If Every man is person and For all integers $n \ge 8$, n Sami is Man, cents can be obtained then Sami is Person using 3¢ and 5¢ coins. If my highest mark this We had a quiz each lecture semester is 82%, then my in the past months, so we average will not be more than will have a guiz next lecture 82%

Induction Versus Deduction Thinking

Deduction Reasoning

Based on facts, definitions, , theorems, laws

Moves from general observation to specific results

Provides proofs

Induction Reasoning

Based on observation, past experience, patterns

Moves from specific cases to create a general rule

حدس/Provides conjecture

© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved

15



Miscellaneous Examples



Prove P(n):

$$\prod_{i=0}^{n} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!}, \text{ for all integers } n \ge 0.$$

n!=n(n-1)(n-2)...1

© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved

Prove P(n):
$$\prod_{i=0}^{n} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!}$$
, for all integers $n \ge 0$.

Basis Step: Show that P(0) is true. P(0): (1/(0+1)).(1/(0+2)) = 1/(0+2)!1.1/2=1/2!1/2=1/2 which is true

Inductive Step: Show that for all integers $k \ge 0$, if P(k) is true then P(k + 1) is also true:

Suppose: P(k): $1/2 \dots 1/(2k+1)$.(1/(2k+2)) = 1/(2k+2)! is true

We need to show: P(k+1): $1/2 \dots 1/(2k+1)$.(1/(2k+2)).(1/(2(k+1)+1)).(1/(2(k+1)+2)) = 1/(2(k+1)+2)! =1/(2k+4)! is true But $1/2 \dots 1/(2k+1)$.(1/(2k+2)) = 1/(2k+2)! (by inductive hypothesis) 1/(2k+2)! (1/(2k+3)).(1/(2k+4)) = 1/[(2k+4)*(2k+3)*(2k+2)!] = 1/(2k+4)!Which is what we needed to show.

> n!=n(n-1)(n-2)...1, so (2k+4)!=(2k+4)*(2k+3)*(2k+2)!

© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved



Prove that $7^m - 1$ is divisible by 6 for all positive integers m.

Base case: $m = 1, 7^1 - 1 = 6$ which is obviously divisible by 6.

Inductive step: Assume $7^m - 1$ is divisible by 6 for some $m \ge 1$ (inductive hypothesis). Then $7^{m+1} - 1 =$ $7^{m+1} - 7 + 6 = 7(7^m - 1) + 6$. But $7^m - 1$ is divisible by 6 (by the inductive hypothesis) and so is 6, so $7^{m+1} - 1$ is also divisible by 6. Hence proved by induction.

© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved

Consider the sequence of real numbers defined by the relations mple 3

$$x_1 = 1 \text{ and } x_{n+1} = \sqrt{1 + 2x_n} \text{ for } n \ge 1.$$

19

Use the Principle of Mathematical Induction to show that $x_n < 4$ for all $n \ge 1$.

For any $n \ge 1$, let P_n be the statement that $x_n < 4$.

<u>Base Case.</u> The statement P_1 says that $x_1 = 1 < 4$, which is true.

Inductive Step. Fix $k \ge 1$, and suppose that P_k holds, that is, $x_k < 4$.

It remains to show that P_{k+1} holds, that is, that $x_{k+1} < 4$. $x_{k+1} = \sqrt{1 + 2x_k}$ $<\sqrt{1+2(4)}$ Therefore P_{k+1} holds. $=\sqrt{9}$ = 3< 4.

Thus by the principle of mathematical induction, for all $n \ge 1$, P_n holds.

Example 4 - Homework

Prove that for any integer number n>1, n³ + 2 n is divisible by 3

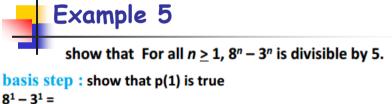
Basis Step: show that p (1) is true. Let n = 1 and calculate $n^3 + 2n$ $1^3 + 2(1) = 3$ 3 is divisible by 3 ,hence p (1) is true.

Inductive Step: Show that for all integers k>0, if p(k) is true then p(pk+1) is true: suppose that p(k) is true $\leftarrow P(k)$ inductive hypothesis $(k + 1)^3 + 2(k + 1)$ = $k^3 + 3k^2 + 5k + 3$ = $[k^3 + 2k] + [3k^2 + 3k + 3]$ = $3[k^3 + 2k] + 3[k^2 + k + 1]$ = $3[[k^3 + 2k] + k^2 + k + 1]$ Hence $(k + 1)^3 + 2(k + 1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

[This is what we needed to show.]

© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved

۲٦



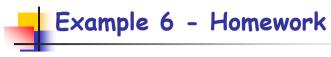
= 8 - 3

= 5 which is clearly divisible by 5.

inductive step : Show that for all integers k>0, if p(k) is true then p(pk+1) is true: Suppose p(k) is true ($8^k - 3^k$ is divisible by 5) \leftarrow P(k) inductive hypothesis

 $\begin{aligned} 8^{k+1}-3^{k+1} &= \\ &= 8^{k+1}-3\times8^k+3\times8^k-3^{k+1} \\ &= 8^k(8-3)+3(8^k-3^k) \\ &= 8^k(5)+3(8^k-3^k) \\ &\text{The first term in } 8^k(5)+3(8^k-3^k) \text{ has 5 as a factor (explicitly), and the second term is divisible } \\ &\text{by 5 (by assumption). Since we can factor a 5 out of both terms, then the entire expression, } \\ &8^k(5)+3(8^k-3^k)=8^{k+1}-3^{k+1}, \text{ must be divisible by 5.} \end{aligned}$

[This is what we needed to show.]



Show that For any integer $n \ge 5$, $4n < 2^n$.

basis step : show P(n = 5) is true. $4n = 4 \times 5 = 20$, and $2^n = 2^5 = 32$. Since 20 < 32, thus p(n=5) is true

inductive step : Show that for all integers k>0, if p(k) is true then p(pk+1) is true: suppose p(k) is true for $k\geq 5 \leftarrow P(k)$ inductive hypothesis

p(k+1): 4(k + 1) = 4k + 4, and, by assumption $[4k] + 4 < [2^k] + 4$ Since $k \ge 5$, then $4 < 32 \le 2^k$. Then we get $2^k + 4 < 2^k + 2^{k=}$ $= 2 \times 2^k$ $= 2^1 \times 2^k$

Then $4(k+1) < 2^{k+1}$, hence p(k+1) is true

© Susanna S. Epp, Kenneth H. Rosen, Ahmad Hamo 2020, All rights reserved