COMP 233 Discrete Mathematics

Sequences and Mathematical Induction

1

5.3 Mathematical Induction II

- Proving Divisibility Property
- Proving an Inequality
- Proving a Property of a Sequence

2

Proving Divisibility Property

For all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

proving that $3 \mid 2^{2n} - 1$

Is like proving that the output of this program for any input n will always be the same.

```
int n;
scanf("%d",&n);

if(n >= 0) {
    if( (pow(2,(2*n)) - 1) %3 == 0)
        printf("this property is true");
    else
        printf("this property isn't true");
}
```


Prove P(n): $3 \mid 2^{2n} - 1$, $n \geq 0$

For all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

$3 \mid 2^{2n} - 1$ ← $P(n)$

Basis Step: *Show that P(0) is true.*

P(0): $2^{2.0} - 1 = 2^0 - 1 = 1 - 1 = 0$. As $3 \mid 0$, thus $P(0)$ is true.

Inductive Step: *Show that for all integers $k \geq 0$, if P(k) is true then $P(k + 1)$ is also true*:

Suppose: $2^{2k} - 1$ is divisible by 3. ← $P(k)$ inductive hypothesis

$2^{2k} - 1 = 3r$ for some integer r.

To show: $2^{2(k+1)} - 1$ is divisible by 3. ← $P(k+1)$

$2^{2(k+1)} - 1 = 2^{2k+2} - 1$ by the laws of exponents

$= 2^{2k} \cdot 2^2 - 1 = 2^{2k} \cdot 4 - 1$

$= 2^{2k}(3 + 1) - 1 = 2^{2k} \cdot 3 + (2^{2k} - 1) = 2^{2k} \cdot 3 + 3r$

$= 3(2^{2k} + r)$ Which is integer

so, by definition of divisibility, $2^{2(k+1)} - 1$ is divisible by 3

Outline a proof by math induction for the statement:
For all integers $n \geq 0$, $5^n - 1$ is divisible by 4.

Proof by mathematical induction: Let the property P(n) be the sentence

$5^n - 1$ is divisible by 4. ← ***the property***

Show that the property is true for $n = 0$: We must show that $5^0 - 1$ is divisible by 4. But $5^0 - 1 = 1 - 1 = 0$, and 0 is divisible by 4 because $0 = 4 \cdot 0$.

Show that for all integers $k \geq 0$, if the property is true for $n = k$, then it is true for $n = k + 1$: Let k be an integer with $k \geq 0$, and **suppose** that *[the property is true for n = k. In other words, suppose that]*

$5^k - 1$ is divisible by 4. ← ***inductive hypothesis***

We must **show** that *[the property is true for n = k + 1.*

In other words, we must show that]

$5^{k+1} - 1$ is divisible by 4.

Ref: Sec. 4.3

Scratch Work for proving that
For all integers $n \geq 0$, $5^n - 1$ is divisible by 4.

Inductive hypothesis: $5^k - 1$ is divisible by 4.

Want to show: $5^{k+1} - 1$ is divisible by 4.

Working Backward: ***Want*** $5^{k+1} - 1 = 4 \cdot$(some integer)

Working Forward: ***Know*** $5^k - 1 = 4 \cdot r$ for some integer r

Experiment:

$$\begin{aligned} 5^{k+1} - 1 &= 5^k \cdot 5 - 1 \\ &= 5^k \cdot (4 + 1) - 1 \\ &= 5^k \cdot 4 + 5^k \cdot 1 - 1 \\ &= 5^k \cdot 4 + (5^k - 1) \end{aligned}$$

Note: *Each of these terms is divisible by 4.*

So:

$$\begin{aligned} 5^{k+1} - 1 &= 5^k \cdot 4 + 4 \cdot r \quad \textit{(where r is an integer)} \\ &= 4 \cdot (5^k + r) \end{aligned}$$

(For the complete proof, see the book.)

Proving Inequality

Proving that For all integers $n \geq 3$, $2n + 1 < 2^n$

Is like proving that the output of this program for any input n>=3 will always be the same

```
scanf("%d", &n);

If (n>=3)
{
    If (2n+1 < pow (2,n))       // is 2n+1 < 2^n ?? //
          printf ("true");
    Else
          printf ("false");
}
```


Prove P(n): $2n + 1 < 2^n$, $n \geq 3$

For all integers $n \geq 3$, $2n + 1 < 2^n$

Let P(n) be $2n+1<2^n$

Basis Step: *Show that P(3) is true.* P(3): $2.3+1 < 2^3$ which is true.

Inductive Step: *Show that for all integers $k \geq 3$, if P(k) is true then P(k + 1) is also true*:

Suppose: $2k+1<2^k$ is true ← $P(k)$ inductive hypothesis

To show $2(k+1) +1 < 2^{k+1}$ ← $P(k+1)$

$2k+3 = \mathbf{(2k+1)} +2$ by algebra

$< \mathbf{2^k} + 2^k$ as $2k + 1 < 2^k$ by the hypothesis and because $2 < 2^k$ $(k \geq 2)$

$\therefore 2k + 3 < 2 \cdot 2^k = 2^{k+1}$

[This is what we needed to show.]

From Appendix A

T17. *Trichotomy Law* For arbitrary real numbers a and b, exactly one of the three relations $a < b, b < a$, or $a = b$ holds.

T18. *Transitive Law* If $a < b$ and $b < c$, then $a < c$.

T19. If $a < b$, then $a + c < b + c$.

T20. If $a < b$ and $c > 0$, then $ac < bc$.

T21. If $a \neq 0$, then $a^2 > 0$.

T22. $1 > 0$.

T23. If $a < b$ and $c < 0$, then $ac > bc$.

T24. If $a < b$, then $-a > -b$. In particular, if $a < 0$, then $-a > 0$.

T25. If $ab > 0$, then both a and b are positive or both are negative.

T26. If $a < c$ and $b < d$, then $a + b < c + d$.

T27. If $0 < a < c$ and $0 < b < d$, then $0 < ab < cd$.

$2k + 1 < 2^k$ and $2 < 2^k$ $(k \geq 2)$

$(2k+1) + 2 < 2^k + 2^k$

Proving Inequalities

Example: Use mathematical induction to prove that $2^n < n!$ for every integer $n \geq 4$.

Solution: Let $P(n)$ be the predicate that $2^n < n!$.

- **Basis**: $P(4)$ is true since $2^4 = 16 < 4! = 24$.
- **Induction**: Assume $P(k)$ holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$. To show that $P(k + 1)$ holds:

$$\begin{aligned} 2^{k+1} &= 2 \times 2^k \\ &< 2 \times k! && \text{(by the induction hypothesis)} \\ &< (k+1)k! && \text{(since } k \geq 4\text{, hence } k+1 > 2) \\ &= (k+1)! \end{aligned}$$

Therefore, $2^n < n!$ holds, for every integer $n \geq 4$.

Note that here the basis step is $P(4)$, since $P(0)$, $P(1)$, $P(2)$, $P(3)$ are all false.

Proving other properties of a Sequence

Define a sequence a_1, a_2, a_3 . . . as follows:

a_1 = 2

a_k = $5a_{k-1}$ for all integers $k \geq 2$.

What are the terms?

What is the explicit formula? Let it be P(n)

Proving other properties of a Sequence

Define a sequence a_1, a_2, a_3 . . . as follows:

a_1 = 2

a_k = $5a_{k-1}$ for all integers $k \geq 2$.

$a_1 = 2$

$a_2 = 5a_{2-1} = 5a_1 = 5 \cdot 2 = \mathbf{10}$

$a_3 = 5a_{3-1} = 5a_2 = 5 \cdot 10 = 50$

$a_4 = 5a_{4-1} = 5a_3 = 5 \cdot 50 = \mathbf{250}$

P(n): The terms of the sequence satisfy the explicit formula

$a_n = 2 \cdot 5^{n-1}$

Prove P(n): $a_n = 2 \cdot 5^{n-1}$ for all integers $n \geq 1$

Basis Step: *Show that P(1) is true.* $a_1 = 2 \cdot 5^{1-1} = 2 \cdot 5^0 = 2$

Inductive Step: *Show that for all integers $k \geq 1$, if P(k) is true then P(k + 1) is also true:*

Suppose: $a_k = 2 \cdot 5^{k-1}$ ← $P(k)$ inductive hypothesis

To show $a_{k+1} = 2 \cdot 5^k$ ← $P(k+1)$

$a_{k+1} = 5a_{(k+1)-1}$ by definition of $a_1, a_2, a_3 \ldots$

$= 5a_k$

$= 5 \cdot (2 \cdot 5^{k-1})$ by the hypothesis

$= 2 \cdot (5 \cdot 5^{k-1})$

$= 2 \cdot 5^k$

[This is what we needed to show.]

Induction Versus Deduction Thinking

Deduction Reasoning	Induction Reasoning
If Every man is person and Sami is Man, then Sami is Person	For all integers $n \geq 8$, n cents can be obtained using 3¢ and 5¢ coins.
If my highest mark this semester is 82%, then my average will not be more than 82%	We had a quiz each lecture in the past months, so we will have a quiz next lecture

Induction Versus Deduction Thinking

Deduction Reasoning	Induction Reasoning
Based on facts, definitions, , theorems, laws	Based on observation, past experience, patterns
Moves from general observation to specific results	Moves from specific cases to create a general rule
Provides proofs	Provides conjecture/حدس

Miscellaneous Examples

Example 1

Prove P(n):

$$\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2n+2)!},\text{ for all integers } n\geq 0.$$

n!=n(n-1)(n-2)…1

Prove P(n): $\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2n+2)!}$, for all integers $n\geq 0$.

Basis Step: *Show that P(0) is true.* P(0): (1/(0+1)).(1/(0+2)) = 1/(0+2)!
1.1/2=1/2!
½=1/2 which is true

Inductive Step: *Show that for all integers k* ≥ 0, *if P(k) is true then P(k* + 1) *is also true*:

Suppose: P(k): 1/2 … 1/(2k+1)).(1/(2k+2)) = 1/(2k+2)! is true

We need to show: P(k+1):
1/2 … 1/(2k+1)).(1/(2k+2)).(1/(2(k+1)+1))).(1/(2(k+1)+2)) = 1/(2(k+1)+2)!
=1/(2k+4)! is true
But 1/2 … 1/(2k+1)).(1/(2k+2)) =1/(2k+2)! (by inductive hypothesis)
1/(2k+2)! (1/(2k+3)).(1/(2k+4)) = 1/[(2k+4)*(2k+3)*(2k+2)!]=1/(2k+4)!
Which is what we needed to show.

n!=n(n-1)(n-2)…1, so
(2k+4)!=(2k+4)*(2k+3)*(2k+2)!

Example 2

Prove that $7^m - 1$ is divisible by 6 for all positive integers m.

Base case: $m = 1$, $7^1 - 1 = 6$ which is obviously divisible by 6.

Inductive step: Assume $7^m - 1$ is divisible by 6 for some $m \geq 1$ (inductive hypothesis). Then $7^{m+1} - 1 = 7^{m+1} - 7 + 6 = 7(7^m - 1) + 6$. But $7^m - 1$ is divisible by 6 (by the inductive hypothesis) and so is 6, so $7^{m+1} - 1$ is also divisible by 6. Hence proved by induction.

Example 3

Consider the sequence of real numbers defined by the relations

$$x_1 = 1 \text{ and } x_{n+1} = \sqrt{1 + 2x_n} \text{ for } n \geq 1.$$

Use the Principle of Mathematical Induction to show that $x_n < 4$ for all $n \geq 1$.

For any $n \geq 1$, let P_n be the statement that $x_n < 4$.

Base Case. The statement P_1 says that $x_1 = 1 < 4$, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is, $x_k < 4$.

It remains to show that P_{k+1} holds, that is, that $x_{k+1} < 4$.

$$\begin{aligned} x_{k+1} &= \sqrt{1 + 2x_k} \\ &< \sqrt{1 + 2(4)} \\ &= \sqrt{9} \\ &= 3 \\ &< 4. \end{aligned}$$

Therefore P_{k+1} holds.

Thus by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

Example 4 - Homework

Prove that for any integer number $n \geq 1$, $n^3 + 2n$ is divisible by 3

Basis Step: show that p (1) is true.
Let n = 1 and calculate $n^3 + 2n$
$1^3 + 2(1) = 3$
3 is divisible by 3 ,hence p (1) is true.

Inductive Step: Show that for all integers k>0 , if p(k) is true then p(pk+1) is true:
suppose that p (k) is true ← P(k) inductive hypothesis
$(k + 1)^3 + 2(k + 1)$
$= k^3 + 3k^2 + 5k + 3$
$= [k^3 + 2k] + [3k^2 + 3k + 3]$
$= 3[k^3 + 2k] + 3[k^2 + k + 1]$
$= 3[[k^3 + 2k] + k^2 + k + 1]$
Hence $(k + 1)^3 + 2(k + 1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

[This is what we needed to show.]

Example 5

show that For all $n \geq 1$, $8^n - 3^n$ is divisible by 5.

basis step : show that p(1) is true
$8^1 - 3^1 =$
$= 8 - 3$
$= 5$ which is clearly divisible by 5.

inductive step : Show that for all integers k>0 , if p(k) is true then p(pk+1) is true:
Suppose p(k) is true ($8^k - 3^k$ is divisible by 5) ← P(k) inductive hypothesis

$8^{k+1} - 3^{k+1} =$
$= 8^{k+1} - 3\times 8^k + 3\times 8^k - 3^{k+1}$
$= 8^k(8 - 3) + 3(8^k - 3^k)$
$= 8^k(5) + 3(8^k - 3^k)$
The first term in $8^k(5) + 3(8^k - 3^k)$ has 5 as a factor (explicitly), and the second term is divisible by 5 (by assumption). Since we can factor a 5 out of both terms, then the entire expression, $8^k(5) + 3(8^k - 3^k) = 8^{k+1} - 3^{k+1}$, must be divisible by 5.

[This is what we needed to show.]

Example 6 - Homework

Show that For any integer $n \geq 5$, $4n < 2^n$.

basis step : show $P(n = 5)$ is true.
$4n = 4\times5 = 20$, and $2^n = 2^5 = 32$.
Since $20 < 32$, thus $p(n=5)$ is true

inductive step : Show that for all integers $k>0$, if $p(k)$ is true then $p(pk+1)$ is true:
suppose $p(k)$ is true for $k \geq 5$ ← P(k) inductive hypothesis

$p(k+1)$: $4(k + 1) = 4k + 4$, and, by assumption $[4k] + 4 < [2^k] + 4$
Since $k \geq 5$, then $4 < 32 \leq 2^k$. Then we get
$2^k + 4 < 2^k + 2^{k=}$
$= 2\times2^k$
$= 2^1\times2^k$

Then $4(k+1) < 2^{k+1}$,hence $p(k+1)$ is true [This is what we n