COMP 233 Discrete Mathematics

Set Theory

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Outline

- A Glimpse into Set Theory
  - Set properties
  - Set operations
- Proving Set properties P(n)
  - Element argument method (defined/undefined sets)
    - Subset properties (Theorem 6.2.1)
    - Set Equality properties
    - Empty set properties
  - "Algebraic" method
    - Set Identities (Theorem 6.2.2)
  - Set Partition, Power set, Cartesian product

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6.1. Set Theory: Definitions and the Element Method of Proof

Set Versus Element

In Set theory → **Set** vs. **Element** ← Mathematical Set

In JAVA → Class vs. Object

In Logic/Philosophy → Concept vs. Instance

- The **extension** of a set is its elements.
- The **order** of elements is irrelevant
- In set theory: an element itself might be a set.
- In philosophy, an instance has no instances.

Basic Concepts and Notations

- Cantor suggested a set as a:

"collection into a whole M of definite and separate objects of our intuition or our thought".

M= {Ali, Hasan, Khalid }

- Each object is called an element (or member of) of M.

  Ali ∈ M (Ali belongs to M)

  Rami ∉ M (Rami does not belong to M)

- **Roster Notation:**

Roster notation is a complete listing of all the elements of the set.
A = {a, b, c, d} and
B = {2, 4, 6, 8, …, 20}
are examples of roster notation that define sets with 4 and 10 elements, respectively.

A Glimpse into Set Theory

"Set" is an undefined term. We say that sets contain elements and are completely determined by the elements they contain.

So: Two sets are equal ⇔ they have exactly the same elements.

Ex: Let $A = \{1,3,5\}$

$B = \{5,1,3\}$

$C = \{1,1,3,3,5\}$

$D = \{x \in \mathbb{Z} \mid x \text{ is an odd integer and } 0 < x < 6\}$

the set of all

such that

How do you read this out loud?

How are *A*, *B*, *C*, and *D* related?

Answer: They are all equal.

Notation: $x \in A$ is read "*x* is an element of *A*" (or "*x* is in *A*")
$x \notin A$ is read "*x* is not an element of *A*" (or "*x* is not in *A*").

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A Glimpse into Set Theory cont.

The order of elements is irrelevant

{Ali, Adam, Sara} = {Adam, Sara, Ali}

Redundancy is does not affect the set

{Ali, Adam, Adam, Sara}

A set can be an element inside another set

{1, {1}} has two elements

Notation of elements

{Ali} ≠ Ali different elements

Important Sets

- 1. The set of all natural numbers or positive integers {1, 2, 3, …} is denoted by N.
- 2. The set of integers {…, −3, −2, −1, 0, 1, 2, 3, …} is denoted by Z.
- 3. The set of rational numbers is denoted by Q.
- 4. The set of real numbers is denoted by R.
- 5. The set of complex numbers is denoted by C.
- 6. The set of positive real numbers is denoted by R^+

Defining Sets by a Property - **Set-Builder Notation**

$$A = \{x \in S \mid P(x)\}$$

The set of all
x is dummy

Property

Examples:

The set of all integers that are more than -2 and less than 5

$\{x \in \mathbf{Z} \mid -2 < x < 5\}$

The set of all persons who born in Palestine

$\{x \in \textit{Person} \mid \textit{BornIn}(x, \text{Palestine})\}$

The set of all persons who born in Palestine and Love Homus

$\{x \in \textit{Person} \mid \textit{BornIn}(x, \text{Palestine}) \wedge \textit{Love}(x, \text{Homus})\}$

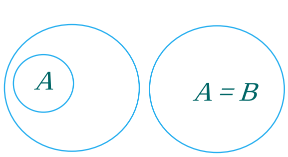
Subset definition

Definition: Given sets A and B, $A \subseteq B$ (read "A is a subset of B")

$\Leftrightarrow$
- every element in A is also in B.
- $\forall\ x$, if x is in A then x is in B.

Note 1: $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

Note 2: $A \not\subseteq B \Leftrightarrow \exists\ x$ such that $x \in A$ and $x \notin B$.



Note: $\subseteq$ denotes subset or equal
$A \subsetneq B$ denotes **proper subset**
(subset but not equal)

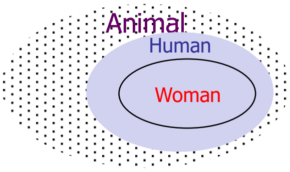
Subset examples

Ex: Let A = {2,4,5} and B = {1,2,3,4,6,7}.
Is $A \subseteq B$?
Answer: No, because 5 is in A but 5 is not in B.

Ex: Let C = {2,4,7} and B = {1,2,3,4,6,7}.
Is $C \subseteq B$?
Answer: Yes, because every element in C is in B.

Subset examples

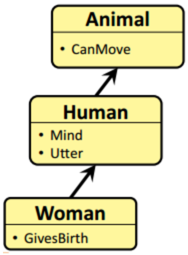
Woman ⊆ Human ⊆ Animal



$\textit{Animal} = \{x \in \text{LivingOrganism} \mid \textit{CanMove}(x)\}$

$\textit{Human} = \{x \in \text{Animal} \mid \text{Has}\textit{Mind}(x)\}$

$\textit{Woman} = \{x \in \text{Human} \mid \text{GivesBirth}(x)\}$



Every subclass inherits the properties of its super class, thus:

- Human is a living organism that can move, has mind and utter.
- Woman is a living organism that can move, has mind and utter, and able to give birth.

Distinction between ∈ and ⊆

Which of the following are true statements?

- ✔ $2 \in \{1, 2, 3\}$
- ✗ $\{2\} \in \{1, 2, 3\}$
- ✗ $2 \subseteq \{1, 2, 3\}$
- ✔ $\{2\} \subseteq \{1, 2, 3\}$
- ✗ $\{2\} \subseteq \{\{1\}, \{2\}\}$
- ✔ $\{2\} \in \{\{1\}, \{2\}\}$

Notations

Symbol	Meaning
Upper case	designates set name
Lower case	designates set elements
{ }	enclose elements in set
$\in$ (or $\notin$)	is (or is not) an element of
$\subseteq$	is a subset of (includes equal sets)
$\subset$	is a proper subset of
$\not\subset$	is not a subset of
$\supset$	is a superset of
| or :	such that (if a condition is true)
| |	the cardinality of a set

Subsets Notations

Not Subset:

$$A \not\subseteq B \Leftrightarrow \exists x \,.\, x \in A \text{ and } x \notin B$$

Notations:

$A = B$		A equals B
$A \subset B$	$B \supset A$	A is subset of B
$A \subseteq B$	$B \supseteq A$	A is subset or equal of B
$A \not\subset B$	$B \not\supset A$	A is not a subset of B
$A \not\subseteq B$	$B \not\supseteq A$	A is not a subset but not equal of B
$A \subsetneq B$	$B \supsetneq A$	A is a subset but not equal of B

Examples: Person ⊃ Man, $Z \supset Z+$, $R \supset Z$

Subsets: Proof and Disproof

Set *A* to be a subset of a set *B* as a formal universal conditional statement:

$$A \subseteq B \quad \Leftrightarrow \quad \forall x, \text{ if } x \in A \text{ then } x \in B.$$

The negation is, therefore, existential:

$$A \not\subseteq B \quad \Leftrightarrow \quad \exists x \text{ such that } x \in A \text{ and } x \notin B.$$

A *proper subset* of a set is a subset that is not equal to its containing set.

A is a **proper subset** of *B* $\Leftrightarrow$

(1) $A \subseteq B$, and
(2) there is at least one element in *B* that is not in *A*.

Proving and Disproving Subset Relations

Define sets A and B as follows:

$A = \{m \in \mathbf{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbf{Z}\}$
$B = \{n \in \mathbf{Z} \mid n = 3s \text{ for some } s \in \mathbf{Z}\}$.

Prove that $A \subseteq B$.

Please loot at details on page 338 example 6.1.2

Suppose x is a particular but arbitrarily chosen element of A.

Show that $x \in B$, means show that $x = 3\cdot(integer)$.

$$x = 6r + 12$$
$$= 3\cdot(2r + 4).$$

Let $s = 2r + 4$.
Also, $3s = 3(2r + 4)$

Therefore, x is an element of B.

Proving and Disproving Subset Relations

Define sets A and B as follows:

$A = \{m \in \mathbf{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbf{Z}\}$
$B = \{n \in \mathbf{Z} \mid n = 3s \text{ for some } s \in \mathbf{Z}\}$.

Disprove that B ⊆ A.

To disprove a statement means to show that it is false, and to show it is false that B ⊆ A, you must find an element of B that is not an element of A.

let x = 3. Then x ∈ B because 3 = 3·1, but x ∉ A, because there is no integer r such that 3 = 6r + 12. For if there were such an integer, then

	$6r + 12 = 3$	by assumption
$\Rightarrow$	$2r + 4 = 1$	by dividing both sides by 3
$\Rightarrow$	$2r = 3$	by subtracting 4 from both sides
$\Rightarrow$	$r = 3/2$	by dividing both sides by 2,

but 3/2 is not an integer.
Thus 3 ∈ B but 3 ∉ A,
and so B ⊄ A

■

Element Argument

Element Argument: The Basic Method for Proving That One Set Is a Subset of Another

Let sets X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrarily chosen element of X,
2. **show** that x is an element of Y.

Set Equality

- **Definition**

Given sets A and B, A **equals** B, written $\boldsymbol{A = B}$, if, and only if, every element of A is in B and every element of B is in A.

Symbolically:

$$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

Let sets A, B, C, and D be defined as follows:

$$A = \{n \in \mathbf{Z} \mid n = 2p, \text{ for some integer } p\},$$
$$B = \text{the set of all even integers},$$
$$C = \{m \in \mathbf{Z} \mid m = 2q - 2, \text{ for some integer } q\},$$
$$D = \{k \in \mathbf{Z} \mid k = 3r + 1, \text{ for some integer } r\}.$$

a. Is $A = B$? b. Is $A = D$? c. Is $A = C$?

Example: Set Equality

Define sets A and B as follows:

$$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$$
$$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$$

Is $A = B$?

Yes. To prove this, both subset relations $A \subseteq B$ and $B \subseteq A$ must be proved.

Part 1, Proof That A $\subseteq$ ***B***:

.....

Part 2, Proof That B $\subseteq$ ***A***:

Part 1, Proof That A⊆ B

Suppose x is a particular but arbitrarily chosen element of A.

[We must show that $x \in B$. By definition of B, this means we must show that $x = 2 \cdot$ (some integer) $- 2$.]

By definition of A, there is an integer a such that $x = 2a$.

[Given that $x = 2a$, can x also be expressed as $2 \cdot$ (some integer) $- 2$? *I.e., is there an integer, say b, such that $2a = 2b - 2$? Solve for b to obtain $b = (2a + 2)/2 = a + 1$. Check to see if this works.]*

Let $b = a + 1$.

[First check that b is an integer.]

Then b is an integer because it is a sum of integers.

[Then check that $x = 2b - 2$.]

Also $2b - 2 = 2(a + 1) - 2 = 2a + 2 - 2 = 2a = x$,

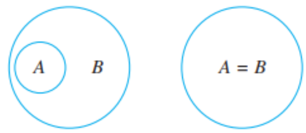
Thus, by definition of B, x is an element of B

[which is what was to be shown].

Part 2, Proof That B $\subseteq$ ***A***: This part of the proof is left as exercise 2 at the end of this section.

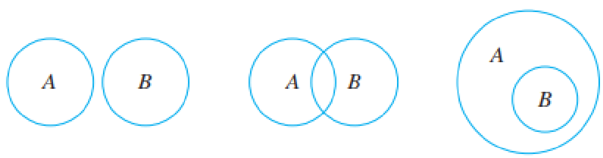
Venn Diagrams

- Relationship $A \subseteq B$ can be pictured in one of two ways



(a) (b)

- The relationship $A \not\subseteq B$ can be represented in 3 different ways with Venn diagrams



(a) (b) (c)

Operations on Sets: Union, Intersection, Difference, Complement

Given sets A and B that are subsets of a "universal set" U, we define

$A \cup B = \{x \in U \mid x \in A$ **or** $x \in B\}$ "or" means "and/or"

$A \cap B = \{x \in U \mid x \in A$ **and** $x \in B\}$

How do we read these symbols out loud?

the set of all

such that

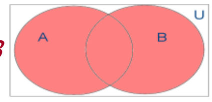
$A - B = \{x \in U \mid x \in A$ and $x \notin B\}$

$A^c = \{x \in U \mid x \notin A\}$

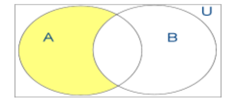
the set of all x in U such that x is in A and x is in B

Venn Diagrams

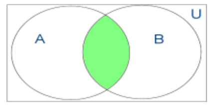
$A \cup B$



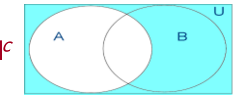
$A - B$



$A \cap B$



A^c



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Operations on Sets

• Definition

Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$, is the set of all elements that are in at least one of A or B.
2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$, is the set of all elements that are common to both A and B.
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $\boldsymbol{B - A}$, is the set of all elements that are in B and not A.
4. The **complement** of A, denoted $\boldsymbol{A^c}$, is the set of all elements in U that are not in A.

Symbolically:

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\},$$
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\},$$
$$A^c = \{x \in U \mid x \notin A\}.$$

Class Exercise

Let A = {1, 2, 3} and B = {3, 4, 5} and suppose that the "universal set" U = {1, 2, 3, 4, 5, 6, 7, 8}. Find

$A \cup B$ = {1, 2, 3, 4, 5}

$A \cap B$ = {3}

$A - B$ = {1, 2}

A^c = {4, 5, 6, 7, 8}

The Empty Set

Let ***A*** be *the set of all the people in the room who live in Ramallah* and ***B*** be the *set of all people in the room who live outside Ramallah*.

What is $A \cap B$?

Answer: This set contains no elements at all.

Notation: The symbol ∅ denotes a set with no elements. (One can prove that there is only one such set. We call it the *empty set* or the *null set*.)

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The Empty Set

The empty set is not the same thing as nothing; rather, it is a set with nothing inside it and a set is always something. This issue can be overcome by **viewing a set as a bag—an empty bag undoubtedly still exists.**

Example: the set $D = \{x \in \mathbf{R} \mid 3 < x < 2\}$.

Axioms about the empty set:

$\forall A \,.\, \emptyset \subseteq A$

$\forall A \,.\, A \cup \emptyset \subseteq A$

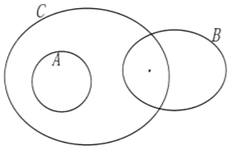
$\forall A \,.\, A \cap \emptyset \subseteq \emptyset$

$\forall A \,.\, A \times \emptyset = \emptyset$

Note: $\subseteq$ denotes subset or equal
$A \subsetneq B$ denotes **proper subset**
(subset but not equal)

Class Exercises

Draw Venn diagrams to represent sets A, B, and C so that $A \subseteq C$, $A \cap B = \varnothing$, and $B \cap C \neq \varnothing$:



Note: We allow Venn diagrams to contain empty spaces. For this reason we place a dot inside the region of overlap between sets B and C.

The diagram suggests a variety of discrete solutions. For instance:

Let $A = \{1\}$, $C = \{1, 2\}$, and $B = \{2, 3\}$.

Then $A \subseteq C$, $A \cap B = \varnothing$, and $B \cap C = \{2\} \neq \varnothing$.

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Interval Notation for subsets of real numbers

- **Notation**

Given real numbers a and b with $a \leq b$:

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\} \qquad [a, b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$$
$$(a, b] = \{x \in \mathbf{R} \mid a < x \leq b\} \qquad [a, b) = \{x \in \mathbf{R} \mid a \leq x < b\}.$$

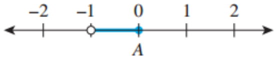
The symbols ∞ and $-\infty$ are used to indicate intervals that are unbounded either on the right or on the left:

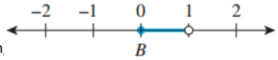
$$(a, \infty) = \{x \in \mathbf{R} \mid x > a\} \qquad [a, \infty) = \{x \in \mathbf{R} \mid x \geq a\}$$
$$(-\infty, b) = \{x \in \mathbf{R} \mid x < b\} \qquad [-\infty, b) = \{x \in \mathbf{R} \mid x \leq b\}.$$

Example:

Let the universal set be the set $\mathbf{R}$ of all real numbers and let

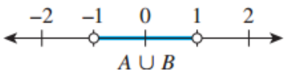
$$A = (-1, 0] = \{x \in \mathbf{R} \mid -1 < x \leq 0\} \text{ and } B = [0, 1) = \{x \in \mathbf{R} \mid 0 \leq x < 1\}.$$



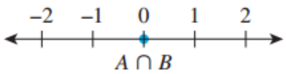


Example: Find $A \cup B$, $A \cap B$, $B - A$, and A^c.

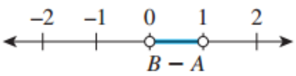
$A \cup B = \{x \in \mathbf{R} \mid x \in (-1, 0] \text{ or } x \in [0, 1)\} = \{x \in \mathbf{R} \mid x \in (-1, 1)\} = (-1, 1).$



$A \cap B = \{x \in \mathbf{R} \mid x \in (-1, 0] \text{ and } x \in [0, 1)\} = \{0\}.$



$B - A = \{x \in \mathbf{R} \mid x \in [0, 1) \text{ and } x \notin (-1, 0]\} = \{x \in \mathbf{R} \mid 0 < x < 1\} = (0, 1)$



A^c Homework!

Unions and Intersections of an Indexed Collection of Sets

• **Definition**

Unions and Intersections of an Indexed Collection of Sets

Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

Note $\bigcup_{i=0}^{n} A_i$ is read "the union of the A-sub-i from i equals zero to n."

An alternative notation for $\bigcup_{i=0}^{n} A_i$ is $A_0 \cup A_1 \cup \ldots \cup A_n$

Example: Finding Unions and Intersections of More than Two Sets

For each positive integer *i*, let A_i= $\left\{x \in \mathbf{R} \mid -\frac{1}{i} < x < \frac{1}{i}\right\} = A_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$

A_1: set of all real numbers between -1 and 1
A_2: set of all real numbers between -1/2 and 1/2
A_3: set of all real numbers between - 1/3 and 1/3

Find $A_1 \cup A_2 \cup A_3$ $= (-1,1)$, because $\left(-\frac{1}{2}, \frac{1}{2}\right)\left(-\frac{1}{3}, \frac{1}{3}\right)$ included

Find $A_1 \cap A_2 \cap A_3$ $= \left(-\frac{1}{3}, \frac{1}{3}\right)$, because (-1,1)$\left(-\frac{1}{2}, \frac{1}{2}\right)$ are included

Find $\bigcup_{i=1}^{\infty} A_i = (-1,1)$ Find $\bigcap_{i=1}^{\infty} A_i = \{0\}$

Partitions of Sets (Disjoint)

• **Definition**

Two sets are called **disjoint** if, and only if, they have no elements in common. Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

Example: Disjoint Sets

Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Are A and B disjoint?

Solution Yes. By inspection A and B have no elements in common, or, in other words, $\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$. ■

Mutually Disjoint Sets

• Definition

Sets $A_1, A_2, A_3 \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all $i, j = 1, 2, 3, \ldots$

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

Example: Mutually Disjoint Sets

a. Let $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$. Are A_1, A_2, and A_3 mutually disjoint?

a. Yes. A_1 and A_2 have no elements in common, A_1 and A_3 have no elements in common, and A_2 and A_3 have no elements in common.

b. Let $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$. Are B_1, B_2, and B_3 mutually disjoint?

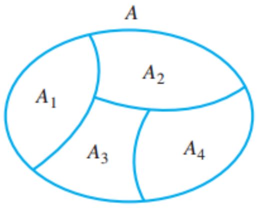
b. No. B_1 and B_3 both contain 4.

Partitions of Sets

• Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i
2. The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.



Example: Partition of Set

Let **Z** be the set of all integers and let

$$T_0 = \{n \in \mathbf{Z} \mid n = 3k, \text{ for some integer } k\},$$
$$T_1 = \{n \in \mathbf{Z} \mid n = 3k + 1, \text{ for some integer } k\}, \text{ and}$$
$$T_2 = \{n \in \mathbf{Z} \mid n = 3k + 2, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_2\}$ a partition of **Z**?

a. Yes. By inspection, $A = A_1 \cup A_2 \cup A_3$ and the sets A_1, A_2, and A_3 are mutually disjoint.

b. Yes. By the quotient-remainder theorem, every integer n can be represented in exactly one of the three forms

$$n = 3k \quad \text{or} \quad n = 3k + 1 \quad \text{or} \quad n = 3k + 2,$$

for some integer k. This implies that no integer can be in any two of the sets T_0, T_1, or T_2. So T_0, T_1, and T_2 are mutually disjoint. It also implies that every integer is in one of the sets T_0, T_1, or T_2. So $\mathbf{Z} = T_0 \cup T_1 \cup T_2$. ■

Power Sets

• Definition

Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Example: Power Set of a Set

Find the power set of the set $\{x, y\}$. That is, find $\mathscr{P}(\{x, y\})$.

Solution $\mathscr{P}(\{x, y\})$ is the set of all subsets of $\{x, y\}$. In Section 6.2 we will show that $\emptyset$ is a subset of every set, and so $\emptyset \in \mathscr{P}(\{x, y\})$. Also any set is a subset of itself, so $\{x, y\} \in \mathscr{P}(\{x, y\})$. The only other subsets of $\{x, y\}$ are $\{x\}$ and $\{y\}$, so

$$\mathscr{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}.$$ ■

Ordered n-tuple

- **Definition**

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered *n*-tuple, $(x_1, x_2, \ldots, x_n)$,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

Example: Ordered n-tuples

a. Is $(1, 2, 3, 4) = (1, 2, 4, 3)$?

No. By definition of equality of ordered 4-tuples,

$$(1,\ 2,\ 3,\ 4) = (1,\ 2,\ 4,\ 3) \Leftrightarrow 1 = 1,\ 2 = 2,\ 3 = 4,\ \text{and } 4 = 3$$

b. Is $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$?

Yes. By definition of equality of ordered triples,

$$\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right) \quad \Leftrightarrow \quad 3 = \sqrt{9} \text{ and } (-2)^2 = 4 \text{ and } \frac{1}{2} = \frac{3}{6}.$$

Cartesian product

- **Definition**

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\mathbf{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Example: Cartesian Products

Let $A_1 = \{x, y\}$, $A_2 = \{1, 2, 3\}$, and $A_3 = \{a, b\}$.

a. Find $A_1 \times A_2$. b. Find $(A_1 \times A_2) \times A_3$. c. Find $A_1 \times A_2 \times A_3$.

a. $A_1 \times A_2 = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$

b. The Cartesian product of A_1 and A_2 is a set, so it may be used as one of the sets making up another Cartesian product. This is the case for $(A_1 \times A_2) \times A_3$.

$$\begin{aligned}(A_1 \times A_2) \times A_3 &= \{(u, v) \mid u \in A_1 \times A_2 \text{ and } v \in A_3\} \quad \text{by definition of Cartesian product} \\ &= \{((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a), \\ &\quad ((y, 2), a), ((y, 3), a), ((x, 1), b), ((x, 2), b), ((x, 3), b), \\ &\quad ((y, 1), b), ((y, 2), b), ((y, 3), b)\}\end{aligned}$$

c. The Cartesian product $A_1 \times A_2 \times A_3$ is superficially similar

Formalizing Sets in Set Theory

Let **S** represent the set of **students**, **SM** represent the set of **smart**, **P** represent the set of **Palestinians**, **A** represent the set of **Americans**, and **W** represent the set of **women**. Formalize the following in Set notation:

1.Set of smart students

$S \cap SM$

Between sets → $\cap$ and $\wedge$ ← Between predicate and propositions

Between sets → $\cup$ and $\vee$ ← Between predicate and propositions

2.Set of Students who are not Smart

$S \cap SM^c$

or $S - SM$

3.Set of Palestinian Americans except Women

$(A \cap P) - W$ or $A \cap P \cap W^c$

Formalizing Statements in Set Theory

Let **ST** represent the set of **students**, **SM** represent the set of **smart**, **P** represent the set of **Palestinians**, **A** represent the set of **Americans**, and **W** represent the set of **women**. Let **WI** represent the set of **Winners.** Formalize the following in Set notation:

1. There are no smart students from Palestine

$\forall$ sets P, ST, SM, $P \cap ST \cap SM = \emptyset$

2. There are no smart students from Palestine among the winners

$\forall$ sets P, ST, SM, WI, $WI \cap P \cap ST \cap SM = \emptyset$

Formalizing Statements in Set Theory

Let **ST** represent the set of **students**, **SM** represent the set of **smart**, and **FO** represent the set of **foolish**. Formalize the following in Set notation:

1. Every student is smart
   $\forall$ sets ST, SM, $ST \subseteq SM$

2. Every smart is not-foolish
   $\forall$ sets FO, SM, $SM \subseteq FO^c$

3. There are no foolish students
   $\forall$ sets ST, FO, $ST \cap FO = \emptyset$

4. "If every student is smart and every smart is not-foolish, then there are no foolish students"

$\forall$ sets ST, SM, FO, if $ST \subseteq SM$ and $SM \subseteq FO^c$, then $ST \cap FO = \emptyset$