COMP 233 Discrete Mathematics

Set Theory

1

6.2 Properties of Sets and Element Argument

2

Theorem 6.2.1 Some Subset Relations

- *Inclusion of **Intersection**:* For all sets A and B,
  (a) $A \cap B \subseteq A$ and (b) $A \cap B \subseteq B$.

- *Inclusion in **Union**:* For all sets A and B,
  (a) $A \subseteq A \cup B$ and (b) $B \subseteq A \cup B$.

- ***Transitive** Property of **Subsets**:* For all sets A, B, and C,
  if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Warm-up: Proving set properties

- How do we read $\{x \in U \mid x \in A \text{ or } x \in B\}$ out loud?
  What is a shorthand notation for this set?
  Answers: The set of all x in U such that x is in A or x is in B.
  $A \cup B$
- If A and B are sets, what does it mean to say that $A \subseteq B$?
  Answer: Every element in A is also in B.
  $\forall\, x$, if x is in A then x is in B.
- If A and B are sets, what does it mean to say that A is not a subset of B?
  Answer: There is at least one element in A that is not in B.
  $\exists\, x$ such that x is in A but x is not in B.
- If A and B are sets, what does it mean to say that $A = B$?
  Answer: A is a subset of B and B is a subset of A.

Warm-up: Proving set properties

- Fill in the blanks: If A and B are sets,

$x \in A \cup B \Leftrightarrow$ $x \in A$ or $x \in B$

$x \in A \cap B \Leftrightarrow$ $x \in A$ and $x \in B$

$x \in A - B \Leftrightarrow$ $x \in A$ and $x \notin B$

$x \in A^c \Leftrightarrow$ $x \notin A$

- Is $\{3\} \in \{\{1\}, \{2\}, \{3\}\}$? Is $\{3\} \subseteq \{1, 2, 3\}$? Is $\{3\} \in \{1, 2, 3\}$?

  Answers: Yes, Yes, No

- When is an *if-then* statement false?

  Answer: when the hypothesis is true and the conclusion is false

- What is a negation for a statement of the form

  $\forall$ x in D, if P(x) then Q(x)?

  Answer: $\exists$ x in D such that P(x) and not-Q(x)

Set Identities

- An **identity** is an equation that is universally true for all elements in some set.

- **Example**

  a + b = b + a

  is an identity for real numbers, because it is true for all real numbers a and b.

- The collection of set properties in the next theorem consists entirely of set identities
- They are equations that are true for all sets in some universal set.

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

Theorem 6.2.2 Set Identities- Cont.

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

Proving Set Properties 1 & 2

Element Argument to Prove Subset Relations

To prove that a set X is a **subset** of a set Y,

suppose that x is a particular but arbitrarily chosen element of X, and

show that x is an element of Y.

Proving Set Equality

To prove that a set X **equals** a set Y,

prove that X is a subset of Y **and** Y is a subset of X.

Proving Set Properties 3

Element Proof That a Set Equals the Empty Set

To prove that a set X **equals the empty set** $\emptyset$,

suppose that this supposition leads to a **contradiction**.

show that X is not empty, i.e., suppose $\exists$ an element x in X

Element argument method

Use Element argument method to prove properties on undefined sets:

Given sets A, B, and C, what would you suppose and what would you show to prove that $(A \cap B) \cup C \subseteq A \cap (B \cup C)$?

In general: How do you show that one set is a subset of another set?

Answer: Show that every element in the one set is in the other. (Element method of proof)

Example: Proof of a Distributive Law

Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. That is:

Prove: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Suppose $x \in A \cup (B \cap C)$. [Show $x \in (A \cup B) \cap (A \cup C)$.]
…

Thus $x \in (A \cup B) \cap (A \cup C)$.

Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Prove: $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

Suppose $x \in (A \cup B) \cap (A \cup C)$. [Show $x \in A \cup (B \cap C)$.]
…

Thus $x \in A \cup (B \cap C)$.

Hence $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Thus $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.

$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$:

Suppose $x \in A \cup (B \cap C)$.
$x \in A$ or $x \in B \cap C$. (by def. of union)
Case 1 ($x \in A$): then
$x \in A \cup B$ (by def. of union) and
$x \in A \cup C$ (by def. of union)
$\therefore x \in (A \cup B) \cap (A \cup C)$ (def. of intersection)
Case 2 ($x \in B \cap C$): then
$x \in B$ and $x \in C$ (def. of intersection)
As $x \in B$, $x \in A \cup B$ (by def. of union)
As $x \in C$, $x \in A \cup C$, (by def. of union)
$\therefore x \in (A \cup B) \cap (A \cup C)$ (def. of intersection)

In both cases, $x \in (A \cup B) \cap (A \cup C)$.
Thus: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
by definition of subset

$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$:

Suppose $x \in (A \cup B) \cap (A \cup C)$.
$x \in A \cup B$ and $x \in A \cup C$. (def. of intersection)
Case 1 ($x \in A$): then
$x \in A \cup (B \cap C)$ (by def. of union)
Case 2 ($x \notin A$): then
$x \in B$ and $x \in C$, (def. of intersection)
Then, $x \in B \cap C$ (def. of intersection)
$\therefore x \in A \cup (B \cap C)$

In both cases $x \in A \cup (B \cap C)$.
Thus: $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$
by definition of subset,

Conclusion: Since both subset relations have been proved, it follows by definition of set equality that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof of a De Morgan's Law for Sets

Prove that for all sets A and B, $(A \cup B)^c = A^c \cap B^c$.

Starting Point: Suppose A and B are arbitrarily chosen sets.

To Show: $(A \cup B)^c = A^c \cap B^c$
To do this, you must show that $(A \cup B)^c \subseteq A^c \cap B^c$ and that $A^c \cap B^c \subseteq (A \cup B)^c$. To show the first containment means to show that

$$\forall x, \text{ if } x \in (A \cup B)^c \text{ then } x \in A^c \cap B^c.$$

And to show the second containment means to show that

$$\forall x, \text{ if } x \in A^c \cap B^c \text{ then } x \in (A \cup B)^c.$$

Since each of these statements is universal and conditional, for the first containment, you

suppose $x \in (A \cup B)^c$,

and then you **show** that $x \in A^c \cap B^c$.

And for the second containment, you

suppose $x \in A^c \cap B^c$,

and then you **show** that $x \in (A \cup B)^c$.

Proof that $(A \cup B)^c \subseteq A^c \cap B^c$:

Proof: Suppose $x \in (A \cup B)^c$. That is $x \notin A \cup B$.
But to say that $x \notin A \cup B$ means that it is false that:
(x is in A or x is in B).

By De Morgan's laws of logic, this implies that x is not in A and x is not in B, which can be written $x \notin A$ and $x \notin B$.

Hence $x \in A^c$ and $x \in B^c$ by definition of complement.

It follows, by definition of intersection, that $x \in A^c \cap B^c$.

So $(A \cup B)^c \subseteq A^c \cap B^c$ by definition of subset.

Cont… Proof: $A^c \cap B^c \subseteq (A \cup B)^c$

Suppose $x \in A^c \cap B^c$.

intersection, $x \in A^c$ and $x \in B^c$, and by definition of complement, $x \notin A$ and $x \notin B$.

In other words, x is not in A and x is not in B.

By De Morgan's laws of logic this implies that it is **false** that (x is in A or x is in B),

which can be written $x \notin A \cup B$ by definition of union.

Hence, by definition of complement, $x \in (A \cup B)^c$.

It follows that $A^c \cap B^c \subseteq (A \cup B)^c$ by definition of subset.

Proof of De Morgan's Law: $(A \cap B)^c = A^c \cup B^c$

To show equality, we must show
1) $(A \cap B)^c \subseteq A^c \cup B^c$
2) $(A \cap B)^c \supseteq A^c \cup B^c$ or $A^c \cup B^c \subseteq (A \cap B)^c$

Lemma 1: $(A \cap B)^c \subseteq A^c \cup B^c$

Proof using element argument

Suppose x is any element s.t. $x \in (A \cap B)^c$
We want to show that $x \in A^c \cup B^c$.
Since $x \in (A \cap B)^c$, we know that $x \notin (A \cap B)$ by the definition of complement.
This means x is not in BOTH A and B, by def of intersection.
Then $x \notin A$ or $x \notin B$. This means $x \in A^c$ or $x \in B^c$ by the def of compliment. By the definition of union, this means $x \in A^c \cup B^c$.
Which was to be shown. So $(A \cap B)^c \subseteq A^c \cup B^c$. ■

Proof of De Morgan's Law: $(A \cap B)^c = A^c \cup B^c$

Lemma 2: $A^c \cup B^c \subseteq (A \cap B)^c$

Proof using the element argument

Suppose x is any element s.t. $x \in A^c \cup B^c$.
We want to show that $x \in (A \cap B)^c$.
Since we know that $x \in A^c \cup B^c$, we know that $x \in A^c$ or $x \in B^c$ by the def of union. Then $x \notin A$ or $x \notin B$ by the def of compliment.
Since $x \notin A$ or $x \notin B$, we know $x \notin A \cap B$ by the def of intersection. So $x \in (A \cap B)^c$ by the def of compliment. Which was to be shown.
$\therefore A^c \cup B^c \subseteq (A \cap B)^c$. ■

Since $(A \cap B)^c \subseteq A^c \cup B^c$ and $A^c \cup B^c \subseteq (A \cap B)^c$, we can conclude that $(A \cap B)^c = A^c \cup B^c$ by the definition of set equality.

■

Example

Use Element argument method to prove properties on defined sets:

1. Let $A = \{x \in \mathbb{Z} \mid x = 5a + 1 \text{ for some integer } a\}$
   $B = \{y \in \mathbb{Z} \mid y = 10b - 9 \text{ for some integer } b\}$.

a. Is $A \subseteq B$? Justify your answer.

b. Is $B \subseteq A$? Justify your answer.

$A = \{x \in \mathbb{Z} \mid x = 5a + 1 \text{ for some integer } a\}$
$B = \{y \in \mathbb{Z} \mid y = 10b - 9 \text{ for some integer } b\}$

a. Is $A \subseteq B$? **Answer:** No

The reason is that $6 \in A$ because $6 = 5 \cdot 1 + 1$.

But $6 \notin B$ because

if $6 = 10b - 9$, then $15 = 10b$, which implies that $b = 1.5$, and 1.5 is not an integer.

So there is at least one element of A that is not in B, and hence A is not a subset of B.

$A = \{x \in \mathbb{Z} \mid x = 5a + 1 \text{ for some integer } a\}$
$B = \{y \in \mathbb{Z} \mid y = 10b - 9 \text{ for some integer } b\}$

b. Is $B \subseteq A$? **Answer:** Yes

Proof:

Suppose y is any *[pbac]* element in B.

Then $y = 10b - 9$ for some integer b.

But 10b-9=10b-10+1 = 5(2b-2)+1 (by algebra)

Note that $2b - 2$ is an integer b/c products and differences of integers are integers.

So, by definition of A, y is an element in A.

*[This argument shows that **any** element in B is also in A. Hence B is a subset of A.]*

21

Example

Use Element argument method to prove properties on defined sets:

Define sets A and B as follows:

$A = \{m \in \mathbf{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbf{Z}\}$
$B = \{n \in \mathbf{Z} \mid n = 3s \text{ for some } s \in \mathbf{Z}\}$.

Prove that $A \subseteq B$.

Suppose x is a p.b.a.c. element of A.

Show that $x \in B$, i.e., show that $x = 3\cdot$(some integer).

$x = 6r + 12$ *(Since $x \in A$)*
$= 3\cdot(2r + 4)$.

Let $s = 2r + 4$. *But s is integer ….*

Also, $3s = 3(2r + 4)$
$= 6r + 12$
$= x$

Therefore, x is an element of B.

Example

Use Element argument method to prove properties on defined sets:

Define sets A and B as follows:
$A = \{m \in Z \mid m = 2a$ for some integer $a\}$
$B = \{n \in Z \mid n = 2b - 2$ for some integer $b\}$
Prove that $A = B$

Yes. To prove this, both subset relations $A \subseteq B$ and $B \subseteq A$ must be proved.

Part 1, Proof That $A \subseteq B$:

…..

Part 2, Proof That $B \subseteq A$:

…..

23

Properties of the Empty set

- **Prove:** A set with no elements is a subset of every set **(Theorem 6.2.4).** I.e., **if** E is a set with no elements **and** A is any set, **then** $E \subseteq A$.

Proof by Contradiction:

Suppose not. [We take the negation of the theorem and suppose it to be true.]
That is, Suppose: E with no elements, and E $\nsubseteq$ A.
assuming (E $\nsubseteq$ A) means there $x \in$E and this x $\notin$ A [by definition of subset].

But there can be no such element since E has no elements. **This is a contradiction.**

Hence the supposition that there are sets E and A, where E has no elements and E $\nsubseteq$ A, is false, and so the theorem is true.

Properties of The "Empty Set"

Corollary 6.2.5 Uniqueness of the Empty Set

Prove: There is only one set with no elements.

Suppose $E1$ and $E2$ are both sets with no elements.

By Theorem 6.2.4, $E1 \subseteq E2$ since $E1$ has no elements.

Also $E2 \subseteq E1$ since $E2$ has no elements.

Thus $E1 = E2$ by definition of set equality.

25

Proving a Conditional Statement

- $\forall$ sets ST, SM, and FO, if $ST \subseteq SM$ and $SM \subseteq FO^c$, then $ST \cap FO = \varnothing$.
- "If every student is smart and every smart is not foolish, then there are no foolish students"
- Proof: **Suppose not**: i.e., there are sets ST,SM,FO s.t. $ST \subseteq SM$ and $SM \subseteq FO^c$, **but $ST \cap FO \neq \varnothing$.**
- This means that there is an element x in $ST \cap FO$.
- Then $x \in ST$ and $\mathbf{x \in FO}$ (By definition of intersection).
- since $ST \subseteq SM$ then $x \in SM$ (by definition of subset).
- Also, since $SM \subseteq FO^c$, then $x \in FO^c$ (by definition of subset).
- So, $\mathbf{x \notin FO}$ (by definition of complement)
- Thus, $x \in FO$ and $x \notin FO$, which is a **contradiction**.

- So the **supposition** that there is an element x in $ST \cap FO$ is **false**, and thus $ST \cap FO = \varnothing$ [as was to be shown].

Exercise (6.2 Q21 - Find the mistake)

"Theorem:" For all sets A and B, $A^c \cup B^c \subseteq (A \cup B)^c$.

"Proof: Suppose A and B are sets, and $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$ by definition of union. It follows that $x \notin A$ or $x \notin B$ by definition of complement, and so $x \notin A \cup B$ by definition of union. Thus $x \in (A \cup B)^c$ by definition of complement, and hence $A^c \cup B^c \subseteq (A \cup B)^c$."

The "proof" claims that because $x \notin A$ or $x \notin B$, it follows that $x \notin A \cup B$. But it is possible for "$x \notin A$ or $x \notin B$" to be true and "$x \notin A \cup B$" to be false. For example, let $A = \{1, 2\}$, $B = \{2, 3\}$, and $x = 3$. Then since $3 \notin \{1, 2\}$, the statement "$x \notin A$ or $x \notin B$" is true. But since $A \cup B = \{1, 2, 3\}$ and $3 \in \{1, 2, 3\}$, the statement "$x \notin A \cup B$" is false.

Exercise 6.2 Q24

Fill in the blanks in the following proof that for all sets A and B, $(A - B) \cap (B - A) = \emptyset$.

Proof: Let A and B be any sets and supppose $(A - B) \cap (B - A) \neq \emptyset$. That is, suppose there were an element x in __(a)__. By definition of __(b)__, $x \in A - B$ and $x \in$ __(c)__. Then by definition of set difference, $x \in A$ and $x \notin B$ and $x \in$ __(d)__ and $x \notin$ __(e)__. In particular $x \in A$ and $x \notin$ __(f)__, which is a contradiction. Hence *[the supposition that $(A - B) \cap (B - A) \neq \emptyset$ is false, and so]* __(g)__.

(a) $(A - B) \cap (B - A)$ **(b)** intersection **(c)** $B - A$
(d) B **(e)** A **(f)** A **(g)** $(A - B) \cap (B - A) = \emptyset$

Exercises

Use Element argument method to prove properties on undefined sets:

1. For all sets A, B, and C, $(A - B) \cup (C - B) \subseteq (A \cup C) - B$.

3. Given sets A and B, what would you suppose and what would you show to prove that $(A \cap B) \cap B^c = \varnothing$?

29

Prove: For all sets A, B, and C, $(A - B) \cup (C - B) \subseteq (A \cup C) - B$.

$\mathbf{(A - B) \cup (C - B) \subseteq (A \cup C) - B}$**:** Suppose that x is any element in $(A - B) \cup (C - B)$. *[We must show that $x \in (A \cup C) - B$.]* By definition of union, $x \in A - B$ or $x \in C - B$.

Case 1 **($x \in A - B$):** Then, by definition of set difference, $x \in A$ and $x \notin B$. But because $x \in A$, we have that $x \in A \cup C$ by definition of union. Hence $x \in A \cup C$ and $x \notin B$,

Case 2 **($x \in C - B$):** Then, by definition of set difference, $x \in C$ and $x \notin B$. But because $x \in C$, we have that $x \in A \cup C$ by definition of union. Hence $x \in A \cup C$ and $x \notin B$, and so, by definition of set difference, $x \in (A \cup C) - B$.

Thus, in both cases, $x \in (A \cup C) - B$ *[as was to be shown]*. So $(A - B) \cup (C - B) \subseteq (A \cup C) - B$.

Proof that $(A \cup C) - B \subseteq (A - B) \cup (C - B)$:

Suppose that x is any element in $(A \cup C) - B$. *[We must show that* $x \in (A - B) \cup (C - B)$*.]*

By definition of set difference, $x \in (A \cup C)$ and $x \notin B$.

And, by definition of union, $x \in A$ or $x \in C$, and in both cases, $x \notin B$.

Case 1 $(x \in A$ ***and*** $x \notin B)$: Then, by definition of set difference, $x \in A - B$, and so by definition of union, $x \in (A - B) \cup (C - B)$.

Case 2 $(x \in C$ ***and*** $x \notin B)$: Then, by definition of set difference, $x \in C - B$, and so by definition of union, $x \in (A - B) \cup (C - B)$.

In both cases, $x \in (A - B) \cup (C - B)$ *[as was to be shown]*.

So $(A \cup C) - B \subseteq (A - B) \cup (C - B)$.

Exercise

Given sets A and B, what would you suppose and what would you show to prove that:

$(A \cap B) \cap B^c = \varnothing$?

Exercise

Given sets A and B, what would you suppose and what would you show to prove that $(A \cap B) \cap B^c = \varnothing$?

In general: How do you show that a set equals the empty set?

Answer: Show that the set has no elements. Go by contradiction. Suppose the set has an element. Show that this supposition leads to a contradiction.

Home Exercises

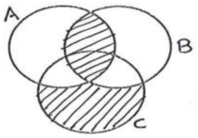
Prove:

1. Given sets A, B, and C, prove/disprove that for all sets A, B, and C,
$(A \cap B) \cup C = A \cap (B \cup C)$. We proved the forward direction previously

2. For all sets A and B, if $A \subseteq B$ then $A - B = \varnothing$.

Ex: The description of the shaded region in the following figure using the operations on set is,

(a) $(C - (A \cap C) \cup (C \cap B)) \cup (A \cap B)$
(b) $A \cup B \cup C - (C \cup (A \cap B))$
(c) $(C - ((A \cap C) \cup (C \cap B))) \cup (A \cap B)$



Proof & Cartesian product

Recall that the **Cartesian product** (or simply the **product**) $A \times B$ of two sets A and B is defined as

$$A \times B = \{(a, b) \ : \ a \in A \text{ and } b \in B\}.$$

If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$.

Before looking at several examples of proofs concerning Cartesian products of sets, it is important to keep in mind that an arbitrary element of the Cartesian product $A \times B$ of two sets A and B is of the form (a, b), where $a \in A$ and $b \in B$.

Example:

Prove for sets A, B, C and D that

If $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

Proof:

Let $(x, y) \in A \times B$. Then $x \in A$ and $y \in B$ *by definition of Cartesian product.*
Since $A \subseteq C$ and $B \subseteq D$, it follows that $x \in C$ and $y \in D$ *by* definition of subset.
Hence, $(x, y) \in C \times D$ *by definition of Cartesian product.*