COMP 233 Discrete Mathematics

Functions

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Outline

- Introduction and Motivation
  - Domain/co-domain, image, inverse image, ordered pairs
  - Equality of functions
- Function Properties
  - One-to-One
  - Onto
  - One-to-One correspondence
- Proving/disproving Function properties P(x)
  - Direct proof method
  - Counter example

7.1: Functions Defined on General Sets

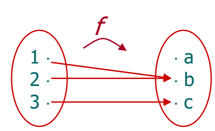
- Introduction and Motivation
  - Domain/co-domain, image, inverse image, ordered pairs
  - Equality of functions

Slightly Informal Definition of Function

Definition: A **function *f* from a set *X* to a set *Y*** is a relation between elements of *X*, called **inputs**, and elements of *Y*, called **outputs**, with the properties that:

a) **every** input has a related output

b) no input has more than one related output.



The notation ***f*: *X*→ *Y*** means that *f* is a function from *X* to *Y*.

X is called the **domain** of the function and *Y* is called its **co-domain**.

Given an input element *x* in *X*, there is a unique output element *y* that is related to *x* by *f*. We say that "*f* **sends** *x* to *y*."

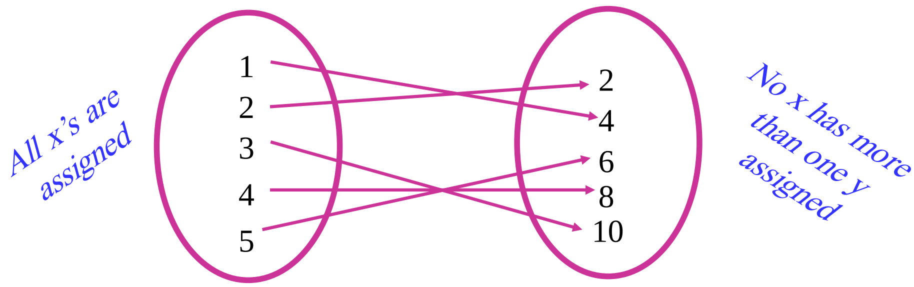
The unique element *y* to which *f* sends *x* is denoted ***f*(*x*)** and is called ***f* of *x***, or the **output** of *f* for the input *x*, or the **value** of *f* at *x*, or the **image** of *x* under *f*.

The **range** of *f* is $\{y \in Y \mid y = f(x) \text{ for some } x \text{ in } X\}$.

The **inverse image** of an element *y* in *Y* is $\{x \in X \mid y = f(x)\}$.

Function

A function *f* from set A to set B is a relation that assigns each element *x* in the set A to exactly one element *y* in the set B.



Set A is the domain

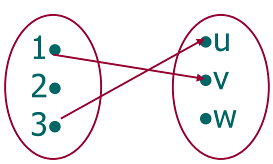
Set B is the Co-domain

1). Must use all the x's in A.

2). The x value can only be assigned to one y in B.

Definition of Function: Examples

Example: Which of the following arrow diagrams define functions? What are the ranges of those that are functions? For each function, what is the inverse image of v?



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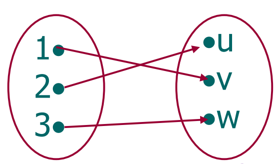
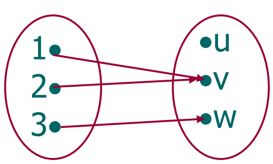
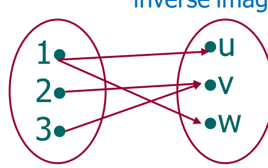


yes range = {u,v,w}
inverse image of v = {1}



yes range = {v,w}
inverse image of v = {1,2}



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Examples of Functions

1. $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 3n$. The domain and codomain are both the set of integers. However, the range is only the set of integer multiples of 3.
2. $g : \{1,2,3\} \to \{a,b,c\}$ defined by $g(1) = c$, $g(2) = a$ and $g(3) = a$. The domain is the set $\{1,2,3\}$, the codomain is the set $\{a,b,c\}$ and the range is the set $\{a,c\}$. Note that $g(2)$ and $g(3)$ are the same element of the codomain. This is okay since each element in the domain still has only one output.
3. $h : \{1,2,3,4\} \to \mathbb{N}$ defined by the table:

x	1	2	3	4
$h(x)$	3	6	9	12

Functions in programming

The domain and codomain of functions are often specified in programming language.

Example:

Java:

```
int f(float x){…}
```

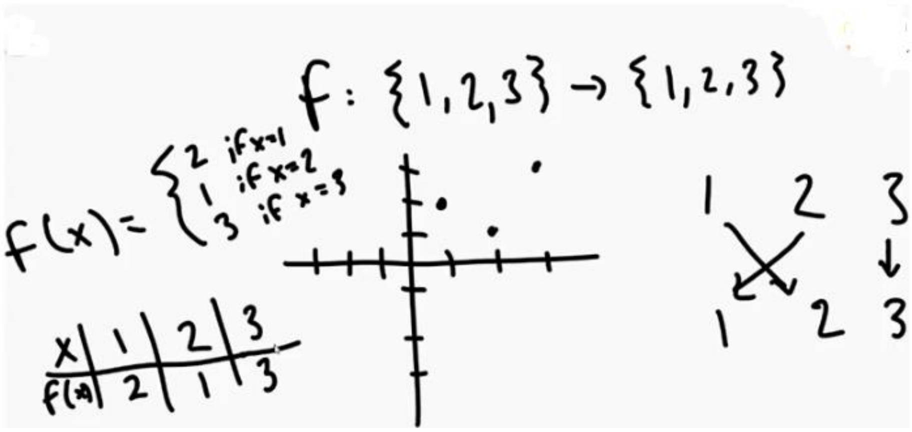
Pascal:

```
function f(x: real): integer
```

Domain of f: **R**

Codomain of f: **Z**

Examples: Representing Functions



Functions Defined on a Cartesian Product

Define functions *M*: **R** × **R** → **R** and *R*: **R** × **R** → **R** × **R** as follows:
For all ordered pairs *(a,b)* of integers,

M(a,b) = ab

Then *M* is the multiplication function that sends each pair of real numbers to the product of the two.

R(a, b) = (-a,b)

R is the reflection function that sends each point in the plane that corresponds to a pair of real numbers to the mirror image of the point across the vertical axis.

Equality of Functions

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

Let $J = \{0, 1, 2\}$, and define functions f and g from J to J as follows: For all x *in* J

$$f(x) = (x^2 + x + 1) \text{ mod } 3 \quad \text{and } g(x) = (x + 2)^2 \text{ mod } 3.$$

Does $f = g$?

x	x^2+x+1	$f(x) = (x^2+x+1)$ *mod* 3	$(x+2)^2$	$g(x) = (x+2)^2$ *mod* 3
0	1	1 *mod* 3 = 1	4	4 *mod* 3 = 1
1	3	3 *mod* 3 = 0	9	9 *mod* 3 = 0
2	7	7 *mod* 3 = 1	16	16 *mod* 3 = 1

Equal functions

Sum/difference of Functions

Let **F**: **R** → **R** and **G**: **R** → **R** be functions.
Define new functions **F + G**: **R** → **R** and **G + F**: **R**→**R** as follows:

F and G must have same Domains and Codomains

For all $x \in \mathbf{R}$,

$$(F + G)(x) = F(x) + G(x) \quad \text{and}$$
$$(G + F)(x) = G(x) + F(x).$$

Does $F + G = G + F$?

$$(F + G)(x) = F(x) + G(x) \quad \text{by definition of } F + G$$
$$= G(x) + F(x) \quad \text{by the commutative law for addition of real numbers}$$
$$= (G + F)(x) \quad \text{by definition of } G + F$$

Hence $F + G = G + F$.

Example: f1+f2 and f1f2

Let f1 and f2 be functions from R to R such that:
f1(x)=x^2 and f2(x)=x−x^2.
What are the functions f1+f2 and f1f2 ?

Solution: From the definition of the sum and product of functions, it follows that:

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + (x - x^2) = x$$

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4.$$

Examples of Functions

The Identity Function on a Set

Given a set X, define a function I_X from X to X by

$I_X(x) = x$, for all x in X.

The function I_X is called the **identity function on *X*** because it sends each element of X to the element that is identical to it.

Thus the identity function can be pictured as a machine that sends each piece of input directly to the output chute without changing it in any way.

Examples: **Function defined on a power Set**

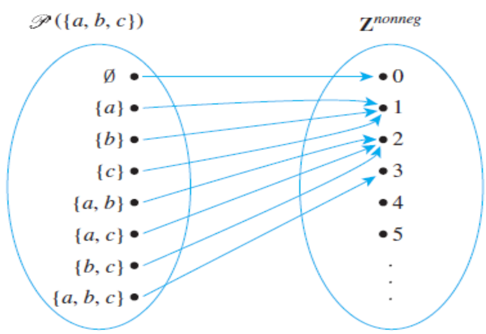
P(A) denotes the set of all subsets of the set *A*. Define a function

F: *P*({*a*, *b*, *c*}) → $\mathbf{Z}^{nonneg}$ as follows: For each $X \in P(\{a, b, c\})$,

F(*X*) = the number of elements in *X*.

Draw an arrow diagram for *F*.

$\mathscr{P}(\{a, b, c\}) \rightarrow \mathbf{Z}^{nonneg}$



Examples : **Boolean Function**

Consider the three-place Boolean function defined from the set of all 3-tuples of 0's and 1's to {0, 1} as follows:

For each triple (x_1, x_2, x_3) of 0's and 1's,

$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3) \mod 2.$

Describe *f* using an input/output table.

$f(1, 1, 1) = (1 + 1 + 1) \mod 2 = 3 \mod 2 = 1$

$f(1, 1, 0) = (1 + 1 + 0) \mod 2 = 2 \mod 2 = 0$

$f(0, 0, 1) = (0 + 0 + 1) \mod 2 = 1 \mod 2 = 1$

Examples of Functions

A Boolean Function

Input			Output
x_1	x_2	x_3	$(x_1 + x_2 + x_3)$ *mod* 2
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

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Well-defined functions

- It can sometimes happen that what appears to be a function defined by a rule is not really a function at all.
- A function is <u>not well defined</u> if it fails to satisfy at least one of the requirements of being a function
- **Example:**
- Define a function f : R → R by specifying that for all real numbers x, f(x) is the real number **y** such that $x^2+y^2 = 1$.

- There are two reasons why this function is <u>not well defined</u>:
- For almost all values of x either

(1) there is no y that satisfies the given equation or

(2) there are two different values of y that satisfy the equation

- Consider when x=2 : there is no real number *y* such that $x^2 + y^2 = 1$
- Consider when x=0 : both $y = -1$ and $y = 1$ satisfy the equation $x^2 + y^2 = 1$

Well-defined functions

- A function is not well defined if it fails to satisfy at least one of the requirements of being a function
- **Example:** $f: \mathbf{Q} \to \mathbf{Z}$ defines this formula:

$$f\left(\frac{m}{n}\right) = m \quad \text{for all integers } m \text{ and } n \text{ with } n \neq 0.$$

Is f a well defined function?

No, Example:

$$f\left(\frac{1}{2}\right) = 1 \quad \text{and} \quad f\left(\frac{3}{6}\right) = 3,$$

$$f\left(\frac{1}{2}\right) \neq f\left(\frac{3}{6}\right).$$

Sequences as Function

Can we define the following Sequence as a function? How? $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots, \frac{(-1)^n}{n+1}, \ldots$

This sequence is a function defined on set of integers that are greater than or equal to a particular integer.

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots, \frac{(-1)^n}{n+1}, \ldots$$

can be thought of as the function f from the nonnegative integers to the real numbers that associates $0 \to 1$, $1 \to -\frac{1}{2}$, $2 \to \frac{1}{3}$, $3 \to -\frac{1}{4}$, $4 \to \frac{1}{5}$, and, in general, $n \to \frac{(-1)^n}{n+1}$.

$g: \mathbf{Z}^+ \to \mathbf{R}$ by $g(n) = \frac{(-1)^{n+1}}{n}$, for each $n \in \mathbf{Z}^+$.

Logarithms and Logarithmic Functions

• Definition Logarithms and Logarithmic Functions

Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of x,** written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically,

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

The **logarithmic function with base b** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$.

Example: Logarithmic Function with Base *b*

Find the following:

a. $\log_3 9$ b. $\log_2\left(\frac{1}{2}\right)$ c. $\log_{10}(1)$ d. $\log_2(2^m)$ (m is any real number)

e. $2^{\log_2 m}$ $(m > 0)$

Solution

a. $\log_3 9 = 2$ because $3^2 = 9$.

b. $\log_2\left(\frac{1}{2}\right) = -1$ because $2^{-1} = \frac{1}{2}$.

c. $\log_{10}(1) = 0$ because $10^0 = 1$.

d. $\log_2(2^m) = m$

Because the exponent to which 2 must be raised to obtain 2^m is m.

e. $2^{\log_2 m} = m$

Because $\log_2 m$ is the exponent to which 2 must be raised to obtain m.

Examples: Strings

We have known that if **S** is a nonempty, finite set of characters, then a string over S is a finite sequence of elements of S.

The number of characters in a string is called the **length** of the string. The null string over S is the "string" with no characters.

It is usually denoted **ε** and is said to have length 0.

Functions Acting on Sets

• Definition

If $f: X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

and

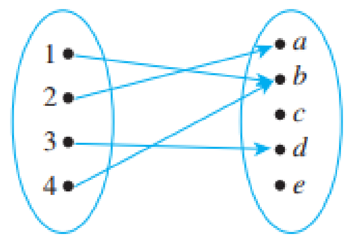
$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of A**, and $f^{-1}(C)$ is called the **inverse image of C**.

Example: The Action of a Function on Subsets of a Set

Let $X = \{1,2,3,4\}$ and $Y = \{a, b, c, d, e\}$, and define $F: X \to Y$ by the following arrow diagram:

Let $A = \{1,4\}$, $C = \{a,b\}$, and $D = \{c, e\}$.
Find $F(A)$, $F(X)$, $F^{-1}(C)$, and $F^{-1}(D)$.



Solution:

$F(A) = \{b\}$
$F(X) = \{a, b, d\}$
$F^{-1}(C) = \{1, 2, 4\}$
$F^{-1}(D) = \emptyset$

Example 7.1.14 Interaction of a Function with Union
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Let X and Y be sets, let F be a function from X to Y, and let A and B be any subsets of X. Prove that $F(A \cup B) \subseteq F(A) \cup F(B)$.

Thus to prove that $F(A \cup B) \subseteq F(A) \cup F(B)$, you only need **show** that if y is any element in $F(A \cup B)$, then y is an element of $F(A) \cup F(B)$.

Suppose $y \in F(A \cup B)$. *[We must show that $y \in F(A) \cup F(B)$.]* By definition of function, $y = F(x)$ for some $x \in A \cup B$. By definition of union, $x \in A$ or $x \in B$.

Case 1, $x \in A$: In this case, $y = F(x)$ for some x in A. Hence $y \in F(A)$, and so by definition of union, $y \in F(A) \cup F(B)$.

Case 2, $x \in B$: In this case, $y = F(x)$ for some x in B. Hence $y \in F(B)$, and so by definition of union, $y \in F(A) \cup F(B)$.

Thus in either case $y \in F(A) \cup F(B)$ *[as was to be shown].* ■

7.2 One-to-One and Onto, Inverse Functions

Must have properties of a function

- Give F: X → Y

1. $\forall x \in X, \exists y \in Y$ s.t. $(x,y) \in F$.
2. $\forall x \in X \wedge \forall y,z \in Y$ s.t.
   if $(x,y) \in F$ and $(x,z) \in F$ then $y = z$.

Properties a Function can have

Definition: Let $F: X \to Y$ be a function.

a. F is **onto** $\Leftrightarrow$ Co-domain = Range
$\forall y$ in Y, $\exists x$ in X such that $y = F(x)$.

b. F is **one-to-one** $\Leftrightarrow$ Each element has a unique output
$\forall x_1$ and x_2 in X, if $x_1 \neq x_2$, then $F(x_1) \neq F(x_2)$.

Or, equivalently,
$\forall x_1$ and x_2 in X, if $F(x_1) = F(x_2)$ then $x_1 = x_2$.

Contrapositive!

Therefore:

a. F is **not onto** $\Leftrightarrow$
$\exists y$ in Y such that $\forall x$ in X, $y \neq F(x)$.

b. F is **not one-to-one** $\Leftrightarrow$
$\exists x_1$ and x_2 in X such that $F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

Called: Many-to-One

No two elements of the domain match the same element of the co-domain.

Can the size of Y be smaller than X?

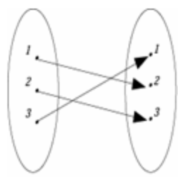
onto: surjective
1:1: injective

Injective vs. Surjective

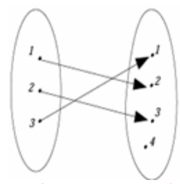
A function is **surjective (onto)** provides every element of the codomain is a the image of **at least one** element from the domain.

A function is **injective (one-to-one)** provides every element of the codomain is a the image of **at most one** element from the domain.

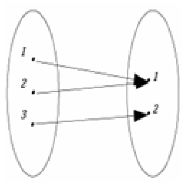
Function Properties: Examples



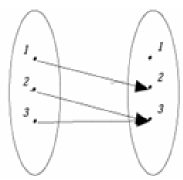
one-to-one onto



one-to-one not onto



not one-to-one onto



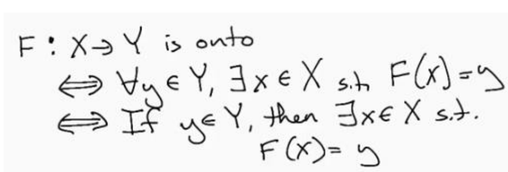
not one-to-one not onto

How do the functions in the top row differ from those in the bottom row?

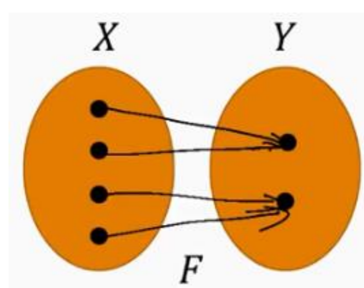
How do the functions in the left column differ from those in the right column?

Example: onto function

- A function is said to be onto (surjective):iff there is no element in the co-domain that is not matched to an element of the domain

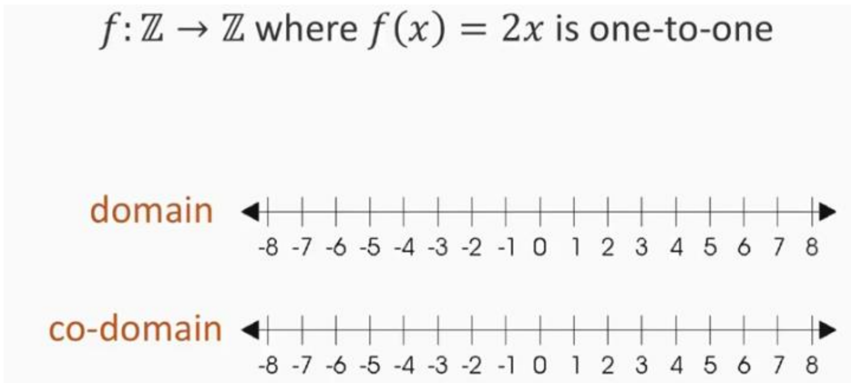


Size of X is bigger than the size of Y.



Many-to-One

Example: one-to-one function



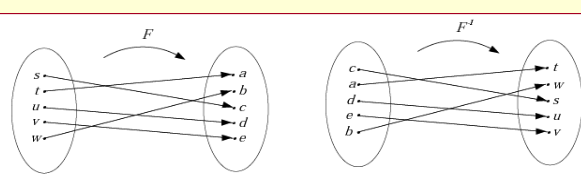
One-to-one Correspondences & Inverse Functions

Definition: A **one-to-one correspondence** from a set X to a set Y is a **function** from X to Y that is both one-to-one and onto.

Definition & Theorem: If $F: A \to B$ is a function that is 1-1 and onto, then for all y in B, there is a **unique** $\boldsymbol{x}$ in A that is sent to y by F.

Thus there is a 1-1, onto function from B to A, called the **inverse function for F**, and denoted F^{-1}.

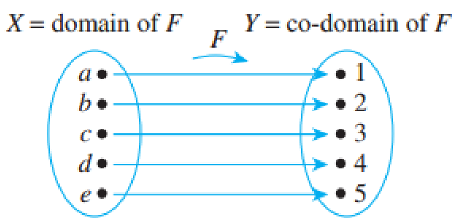
Picture:



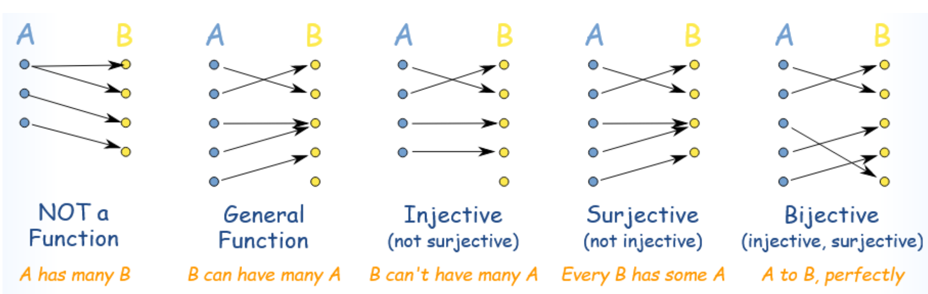
One-to-One Correspondence

• **Definition**

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.



Injective, Surjective and Bijective



Inverse Functions

Theorem 7.2.2

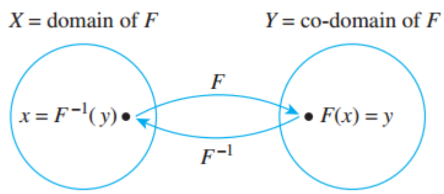
Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $F^{-1}: Y \to X$ that is defined as follows:

Given any element y in Y,

$F^{-1}(y)$ = that unique element x in X such that $F(x)$ equals y.

In other words,

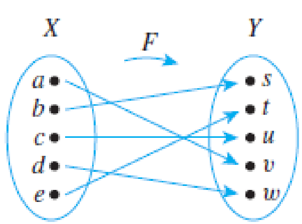
$$F^{-1}(y)=x \quad \Leftrightarrow \quad y=F(x).$$



➔ Is it always that the inverse of a function is a function?

Inverse Functions

Given an arrow diagram for a function. Draw the arrow diagram for the inverse of this function



Finding an Inverse Function

The function f: **R** → **R** defined by the formula

$f(x) = 4x - 1$, for all **real** numbers x

Solution For any *[particular but arbitrarily chosen]* y in **R**, by definition of f^{-1},

$$f^{-1}(y) = \text{that unique real number } x \text{ such that } f(x) = y.$$

But

$$f(x) = y$$

$$\Leftrightarrow \quad 4x - 1 = y \qquad \text{by definition of } f$$

$$\Leftrightarrow \quad x = \frac{y+1}{4} \qquad \text{by algebra.}$$

Hence $f^{-1}(y) = \frac{y+1}{4}$.

Direct proof or Counter example

- Whether a Function is One-to-one
- Whether a Function is Onto
- Whether a Function is One-to-Onto correspondence

Review: Formal Definitions of One-to-one and Onto

- Given a function f from a set X to a set Y, f is **onto** if, and only if,

  $\forall\ y$ in Y, $\exists\ x$ in X such that $y = f(x)$.
- Given a function f from a set X to a set Y, f is **not onto** if, and only if,

  $\exists\ y$ in Y such that $\forall x$ in X, $y \neq f(x)$.
- Given a function f from a set X to a set Y, f is **one-to-one** if, and only if,

  $\forall\ x_1$ and x_2 in X, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.
- Given a function f from a set X to a set Y, f is **not one-to-one** if, and only if,

  $\exists\ x_1$ and x_2 in X s. th. $f(x_1) = f(x_2)$ and $x_1 \neq x_2$.

How to prove or disprove?

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

How to prove: One-to-one Functions on Infinite Sets?

Now suppose f is a function defined on an infinite set X. By definition, f is one-to-one if, and only if, the following universal statement is true:

$$\forall x1, x2 \in X, \text{ if } f(x1) = f(x2) \text{ then } x1 = x2$$

Thus, to prove f is one-to-one, you will generally use the method of direct proof:

suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$

and **show** that $x_1 = x_2$.

To show that f is *not* one-to-one, you will ordinarily

find elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Writing Up the Proof That a Function is One-to-one

Define f: $\mathbf{Z} \to \mathbf{Z}$ by the formula

$f(n) = 2n + 1$ for all integers n.

Claim: f is one-to-one.

($\Leftrightarrow \forall$ x_1 and x_2 in X, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.)

Proof: Suppose n_1 and n_2 are any integers such that $f(n_1) = f(n_2)$.

[Show that $n_1 = n_2$.] ← To answer, must use the definition of f.

By definition of f, $2n_1 + 1 = 2n_2 + 1$

So $2n_1 = 2n_2$

and thus $n_1 = n_2$. QED

Proving That a Function is One-to-one

Define $f: \mathbb{Z} \to \mathbb{Z}$ by the formula

$f(n) = 3n - 2$ for all integers n.

Claim: f is one-to-one.
($\Leftrightarrow \forall n_1$ and n_2 in **Z**, if $f(n_1) = f(n_2)$ then $n_1 = n_2$.)

Proof: Suppose n_1 and n_2 are any integers such that $f(n_1) = f(n_2)$.

[Show that $n_1 = n_2$.]

By definition of f, $3n_1 - 2 = 3n_2 - 2$

So $3n_1 = 3n_2$

and thus $n_1 = n_2$. QED

Proving that a Function is Not One-to-one

Define f: **Z** → **Z** by the formula

$f(n) = n^2$ for all integers n.

Is f one-to-one? ($\Leftrightarrow \forall\ x_1$ and x_2 in X, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.)

Answer: No.

Counterexample:

Let $n_1 = 2$ and $n_2 = -2$. Then by definition of g,

$$g(n_1) = g(2) = 2^2 = 4 \quad \text{and also}$$
$$g(n_2) = g(-2) = (-2)^2 = 4.$$

Hence $g(n_1) = g(n_2)$ but $n_1 \neq n_2$,

and so g is not one-to-one.

Onto Functions on Infinite Sets

To prove that *f* is onto, you must prove

$\forall y \in Y, \exists x \in X$ such that $f(x) = y$.

1. There exists real number x such that y = f(x)?
2. Does f really send x to y?

Onto Functions on Infinite Sets

Now suppose F is a function from a set X to a set Y, and suppose Y is infinite. By definition, F is onto if, and only if, the following universal statement is true:

$$\forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Thus to prove F is onto, you will ordinarily use the method of generalizing from the generic particular:

suppose that y is any element of Y

and **show** that there is an element X of X with $F(x) = y$.

To prove F is *not* onto, you will usually

find an element y of Y such that $y \neq F(x)$ for *any* x in X.

Example: Onto Functions on Infinite Sets

If $f: \mathbf{R} \to \mathbf{R}$ is the function defined by the rule $f(x) = 4x - 1$ for all real numbers x, then f is onto.

Proof:

Let $y \in \mathbf{R}$. *[We must show that* $\exists x$ *in* $\mathbf{R}$ *such that* $f(x) = y$.*]* Let $x = (y+1)/4$. Then x is a real number since sums and quotients (other than by 0) of real numbers are real numbers. It follows that

$$\begin{aligned} f(x) &= f\left(\frac{y+1}{4}\right) && \text{by substitution} \\ &= 4 \cdot \left(\frac{y+1}{4}\right) - 1 && \text{by definition of } f \\ &= (y+1) - 1 = y && \text{by basic algebra.} \end{aligned}$$

[This is what was to be shown.]

Evaluating Whether a Function is Onto

Define $f: \mathbb{Z} \to \mathbb{Z}$ by the formula

$f(n) = 3n - 2$ for all integers n.

Is f onto? ($\Leftrightarrow \forall\ m$ in $\mathbb{Z}$ $\exists\ n$ in $\mathbb{Z}$ such that $m = f(n)$.)

Scratch work: *Start* as if to prove that it is: Suppose m is any element of the co-domain. I.e., m is any integer.

Then ask: Must there be an element n of the domain (i.e., an integer n) such that $f(n) = m$? ← To answer, must use the definition of f.

Def. of f $\Rightarrow$ $3n - 2 = m$ $\Rightarrow$ $3n = m + 2$ $\Rightarrow$ $n = \dfrac{m+2}{3}$

Will it always be true that n is an integer? No.

Example? $m = 0$

Therefore, the answer will be no, f is not onto.

Evaluating Whether a Function is Onto

Define f: $\mathbf{Z} \rightarrow \mathbf{Z}$ by the formula

$f(n) = 2n + 1$ for all integers n.

Is f onto? ($\Leftrightarrow \forall\ y$ in Y, $\exists\ x$ in X such that $y = f(x)$.)

Scratch work: *Start* as if to prove that it is: Suppose m is any element of the co-domain. I.e., m is any integer.
Then ask: Must there be an element n of the domain (i.e., an integer n) such that $f(n) = m$? To answer, need to use the definition of f.

Def. of f $\Rightarrow$ $2n + 1 = m$ $\Rightarrow$ $2n = m - 1$ $\Rightarrow$ $n = \dfrac{m-1}{2}$

Will it always be true that n is an integer? No.

Example? $m = 2$

Therefore, the answer will be no, f is not onto.

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Proving that a Function is Onto

Define a function $g: \mathbb{Z} \rightarrow \mathbb{Z}$ by the formula

$g(n) = 2 - n$ for all integers n.

Is g onto? ($\Leftrightarrow \forall\ m$ in $\mathbb{Z}$, $\exists\ n$ in $\mathbb{Z}$ such that $m = g(n)$.)

Proof: (Given an integer m, can we find an integer n such that $m = 2 - n$?)

1. Suppose m is any integer. Let $n = 2 - m$.

2. Then $g(n) = g(2 - m) = 2 - (2 - m) = 2 - 2 + m = m$. QED

Showing That a Function is Not Onto

Define f: $\mathbb{Z} \to \mathbb{Z}$ by the formula

$f(n) = 3n - 2$ for all integers n.

Is f onto? ($\Leftrightarrow \forall\ m\ \mathbb{Z}$, $\exists \mathbb{Z}$ in such that $m = f(n)$.)

Answer: No.

Counterexample:

Let $m = 0$, and note that $f(n) \neq 0$ for any integer n. To see why this is true, suppose it is not. That is, suppose that

$f(n) = 0$ for some integer n.

Then $3n - 2 = 0$

so $3n = 2$

and so $n = 2/3$, which is not an integer.

Thus n is an integer and n is not an integer, which is a contradiction. Hence the supposition is false, and, therefore, there is no integer n with $f(n) = 0$. Thus f is not onto.

Proving that a Function is Not Onto

Define f: $\mathbf{Z} \to \mathbf{Z}$ by the formula

$f(n) = 2n + 1$ for all integers n.

Is f onto?

Answer: No

Counterexample: Let $m = 2$. Suppose there is an integer n such that $f(n) = 2$. By definition of f,

$$2n + 1 = 2 \Rightarrow 2n = 1 \Rightarrow n = 1/2$$

But 1/2 is not an integer. So there is no integer n with $f(n) = 2$.

Proving One-to-One correspondence

Example: Define f : $\mathbf{Z} \rightarrow \mathbf{Z}$ by the formula

$f(n) = 2n + 1$ for all integers n.

a. Prove that f is one to one.
b. Prove that f is onto ($\Leftrightarrow \forall$ y in Y, $\exists$ x in X such that y = f (x).)
c. find the inverse function

Reducing Co-Domain

Given a function that is not onto, it is always possible to define a related, similar function that is onto by reducing the co-domain to be the range and keeping the rest of the definition the same.

Example: Let $\mathbf{Z}^{odd}$ be the set of **all odd integers**.

Define f : $\mathbf{Z} \rightarrow \mathbf{Z}^{odd}$ by the formula

$f(n) = 2n + 1$ for all integers n.

Proof: Suppose m is any odd integer s.t. $2n + 1 = m$.

$\Rightarrow 2n = m - 1 \Rightarrow n = \frac{m-1}{2}$.

Is $\frac{m-1}{2} \in Z$? Yes! Basically, because m is odd.

(I.e., $m = 2k + 1$ for some integer k, and so $k = \frac{m-1}{2}$.) QED

or simply (By definition of odd, $m = 2n + 1$ for some integer n. But then by definition of f, for all m, there is n s.t. $m = f(n)$).

Thus, we proved that *f* is a one-to-one correspondence

$f^{-1}(m) = \frac{m-1}{2}$ for all $m \in \mathbf{Z}^{odd}$.

Examples

- f is a function from {a, b, c} to {1, 2, 3} with f(a)=2, f(b)=3, f(c)=1. Is it invertible? What is it its inverse?
- Let f: Z→Z such that f(x)=x+1, Is f invertible? If so, what is its inverse?

  $y=x+1$, $x=y-1$, $f^{-1}(y)=y-1$
- Let f: R→R with $f(x)=x^2$, Is it invertible?
  - Since f(2)=f(-2)=4, f is not one-to-one, and so not invertible

Theorem – Homework !

If X and Y are sets and $F: X \to Y$ is one-to-one and onto, then $F^{-1}: Y \to X$ is also one-to-one and onto.

Proof:

F^{-1} is one-to-one: Suppose y_1 and y_2 are elements of Y such that $F^{-1}(y_1) = F^{-1}(y_2)$. *[We must show that $y_1 = y_2$.]* Let $x = F^{-1}(y_1) = F^{-1}(y_2)$. Then $x \in X$, and by definition of F^{-1},

$$F(x) = y_1 \quad \text{since } x = F^{-1}(y_1)$$

and

$$F(x) = y_2 \quad \text{since } x = F^{-1}(y_2).$$

Consequently, $y_1 = y_2$ since each is equal to $F(x)$. This is what was to be shown.

F^{-1} is onto: Suppose $x \in X$. *[We must show that there exists an element y in Y such that $F^{-1}(y) = x$.]* Let $y = F(x)$. Then $y \in Y$, and by definition of F^{-1}, $F^{-1}(y) = x$. This is what was to be shown.

Examples of functions

- Hash functions
- String Functions
- Cartesian Products Functions (2 variables)
- Logarithmic functions

Example: Hash Functions

Hash functions are functions that when given an input, map it to a certain value.

Define a function *Hash* from the set of all Palestinian ids I to the set $M=\{0, 1, 2, 3, 4, 5, 6\}$ as follows:

$Hash(n) = n \bmod 7$ for all Palestinian ids n

Is Hash one-to-one?

no, 14 and 7 both give mod of 0 when divided by 7.

0	356-63-3102
1	
2	513-40-8716
3	223-79-9061
4	
5	328-34-3419
6	

Example: String Functions

String functions take a sequence of characters as input (e.g., 0's and 1's or a's and b's etc.).

Let S be the set of all strings of a's and b's, and define N: $S \to Z$ by

$N(s)$ = the number of a's in s, for all $s \in S$.

- Is N one-to-one? Prove or give a counterexample.
- Is N onto? Prove or give a counterexample.

Example: Cartesian Products Functions

Define a function $F: \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ as follows: For all $(x, y) \in \mathbf{R} \times \mathbf{R}$,

$$F(x, y) = (x + y, x - y).$$

Is F a one-to-one correspondence from $\mathbf{R} \times \mathbf{R}$ to itself?

Proof that F is one-to-one: Suppose that (x_1, y_1) and (x_2, y_2) are any ordered pairs in $\mathbf{R} \times \mathbf{R}$ such that

$$F(x_1, y_1) = F(x_2, y_2).$$

[We must show that $(x_1, y_1) = (x_2, y_2)$.] By definition of F,

$$(x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2).$$

For two ordered pairs to be equal, both the first and second components must be equal. Thus x_1, y_1, x_2, and y_2 satisfy the following system of equations:

$$x_1 + y_1 = x_2 + y_2 \quad (1)$$
$$x_1 - y_1 = x_2 - y_2 \quad (2)$$

Adding equations (1) and (2) gives that

$$2x_1 = 2x_2, \quad \text{and so} \quad x_1 = x_2.$$

Substituting $x_1 = x_2$ into equation (1) yields

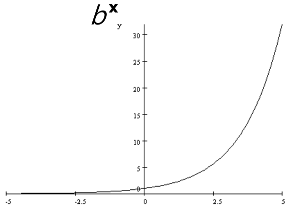
$$x_1 + y_1 = x_1 + y_2, \quad \text{and so} \quad y_1 = y_2.$$

Thus, by definition of equality of ordered pairs, $(x_1, y_1) = (x_2, y_2)$ *[as was to be shown]*.

Full Proof on Page 409

The Exponential and Logarithmic Functions

The **exponential function with positive real number base $b \neq 1$** is the function that sends each positive real number x to b^x, where $b^0 = 1$ and $b^{-x} = \frac{1}{b^x}$.



$y = 2^x$

> For any positive real number $b \neq 1$,
>
> if $b^u = b^v$ then $u = v$.
>
> The exponential function with base b is one-to-one.

The **logarithmic function with positive real number base $b \neq 1$** is the inverse function for the exponential function with base b.

Logarithms

Definition: Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of x**, denoted $\log_b x$, is defined as follows:

$\log_b x$ = the exponent to which b must be raised to obtain x

That is, $\log_b x = y \Leftrightarrow b^y = x$.

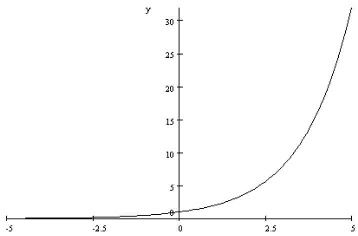
Exercises: 1. $\log_2 8$ 2. $\log_2 2$ 3. $\log_2\left(\frac{1}{4}\right)$ 4. $\log_2 1$ 5. $\log_2(2^k)$

3 1 -2 0 k

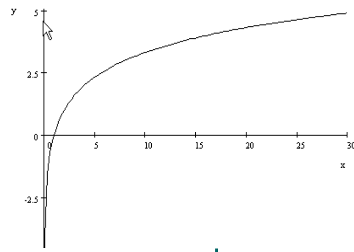
The **logarithmic function with base $b \neq 1$** is the function that sends each positive real number x to $\log_b(x)$.

the exponent to which 2 must be raised to obtain 2^k

Graphs of Exponential and Logarithmic Functions



$y = 2^x$



$y = \log_2 x$

Note: $(u,v) \in$ graph of $y = 2^x \Leftrightarrow (v,u) \in$ graph of $y = \log_2 x$

Laws of Exponents

If b and c are any positive real numbers with $b \neq 1$ and $c \neq 1$, and if u and v are any real numbers, then

$$b^u b^v = b^{u+v}$$

$$(b^u)^v = b^{uv}$$

$$\frac{b^u}{b^v} = b^{u-v}$$

$$(bc)^u = b^u c^u$$

Ex: $2^2 2^3 = (2\cdot2)(2\cdot2\cdot2) = 2^5$

Ex: $(2^2)^3 = (2\cdot2)(2\cdot2)(2\cdot2) = 2^6$

Ex: $\frac{2^2}{2^3} = \frac{2\cdot2}{2\cdot2\cdot2} = \frac{1}{2} = 2^{-1}$

Ex: $2^3 5^3 = (2\cdot2\cdot2)(5\cdot5\cdot5) = (2\cdot5)^3$

Fact: For any positive real number $b \neq 1$,

if $b^u = b^v$ then $u = v$.

Properties of Logarithms

Theorem 7.2.1 Properties of Logarithms

For any positive real numbers b, c and x with $b \neq 1$ and $c \neq 1$:

a. $\log_b(xy) = \log_b x + \log_b y$

b. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

c. $\log_b(x^a) = a \log_b x$

d. $\log_c x = \dfrac{\log_b x}{\log_b c}$

We want proof d!

Using the One-to-Oneness of the Exponential Function

Use the definition of logarithm, the laws of exponents, and the one-to-oneness of the exponential function (property 7.2.5) to prove part (d) of Theorem 7.2.1: For any positive real numbers b, c, and x, with $b \neq 1$ and $c \neq 1$,

$$\log_c x = \frac{\log_b x}{\log_b c}.$$

Solution Suppose positive real numbers b, c, and x are given. Let

$$(1)\ \ u = \log_b c \qquad (2)\ \ v = \log_c x \qquad (3)\ \ w = \log_b x.$$

Then, by definition of logarithm,

$$(1')\ \ c = b^u \qquad (2')\ \ x = c^v \qquad (3')\ \ x = b^w.$$

Substituting (1′) into (2′) and using one of the laws of exponents gives

$$x = c^v = (b^u)^v = b^{uv} \qquad \text{by 7.2.2}$$

But by (3), $x = b^w$ also. Hence

$$b^{uv} = b^w,$$

and so by the one-to-oneness of the exponential function (property 7.2.5),

$$uv = w.$$

Substituting from (1), (2), and (3) gives that

$$(\log_b c)(\log_c x) = \log_b x.$$

And dividing both sides by $\log_b c$ (which is nonzero because $c \neq 1$) results in

$$\log_c x = \frac{\log_b x}{\log_b c}.$$

Cartesian product

Define functions $M: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $R: \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ as follows: For all ordered pairs (a, b) of integers,

$$M(a, b) = ab \quad \text{and} \quad R(a, b) = (-a, b).$$

M is the multiplication function that sends each pair of real numbers to the product of the two. *R* is the reflection function that sends each point in the plane that corresponds to a pair of real numbers to the mirror image of the point across the vertical axis.

Find the following:

a. $M(-1, -1)$ b. $M\left(\frac{1}{2}, \frac{1}{2}\right)$ c. $M(\sqrt{2}, \sqrt{2})$
d. $R(2, 5)$ e. $R(-2, 5)$ f. $R(3, -4)$

Functions in programming

Given the C functions below, for each function:

- Use the notation *F: S → S where F(x)=y* to define the function.
- Is F one-to-one correspondence? Prove or give a counterexample.
- If you answered "yes" for b above, what is formula for F^{-1}?

```
float f(float x) {
return (3*x - 4);
}
```

```
float g(float x){
      if (x!=0)
       return ((x+1)/x);
}
```