COMP 233 Discrete Mathematics

Relations

Outline

- Binary (n-ary) Relations
  - Representations of binary relations
    - set of ordered pairs, arrow diagram, directed graphs
  - Properties of binary relations
    - reflexivity, symmetry, transitivity
- Equivalence relations
  - equivalence classes
- Inverse relations
- Proving Properties of Relations on Sets
  - Equality relation, less then, divides, …

8.1 Relations on Sets

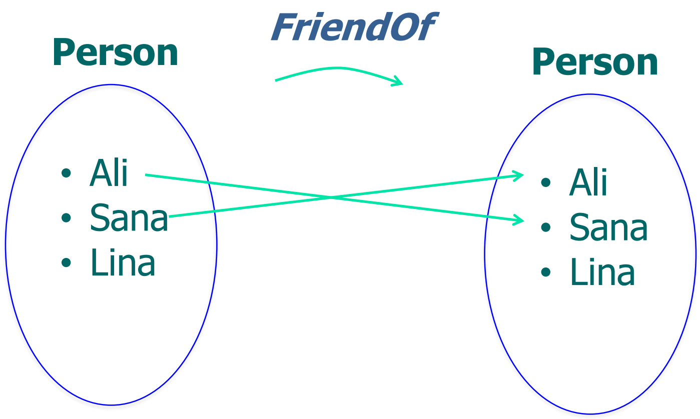
Relations on Sets

A more formal way to refer to the kind of relation defined on sets is to call it a **binary relation** because it is a subset of a **Cartesian product of two sets**.

At the end of this section we define an **n-ary relation** to be a subset of a Cartesian product of n sets, where n is any **integer greater than or equal to two**.

Such a relation is the fundamental structure used in **relational databases**. However, because we focus on binary relations in this text, when we use the term relation by itself, we will **mean binary relation**.

Relations: Motivation



Definition of Binary Relation

Definition: A binary relation R from a set A to a set B is a **subset** of $A \times B$. Given an ordered pair (a, b) in $A \times B$, we say that ***a* is related to *b***, written ***a R b***, if, and only if, $(a, b) \in R$. In symbols:

$$a\,R\,b \iff (a, b) \in R$$

Ex: Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$. Define a binary relation R from A to B as follows:

$$a\,R\,b \iff a > b.$$

a. Is 2 R 4? No Is 5 R 4? Yes Is $(7, 2) \in R$? Yes

b. Write R as a set of ordered pairs.

$R = \{(3,2), (5,2), (5,4), (7,2), (7,4), (7,6)\}$

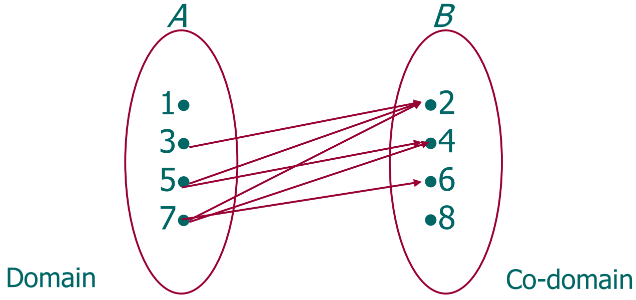
Representing Relations: ordered pairs

- $R = \{(3,2), (5,2), (5,4), (7,2), (7,4), (7,6)\}$.
- *EnrolledIn* ={(Ali, COMP233), (Sana,ENG231)}
- *FriendOf* = {(Ali, Sana), (Sana,Ali)}

Representing Relations: arrow diagram

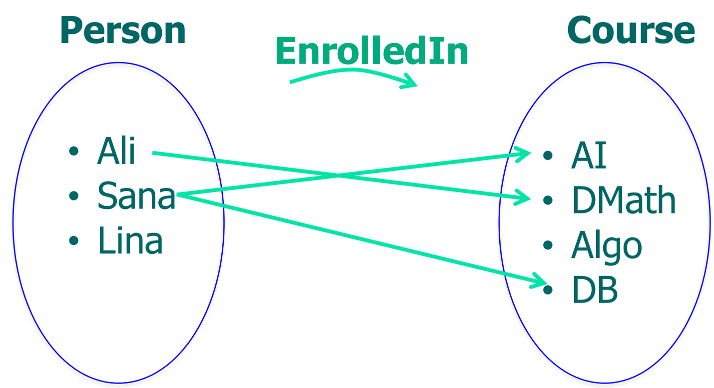
c. Draw an "arrow diagram" to represent *R,* where

$R = \{(3,2), (5,2), (5,4), (7,2), (7,4), (7,6)\}$.

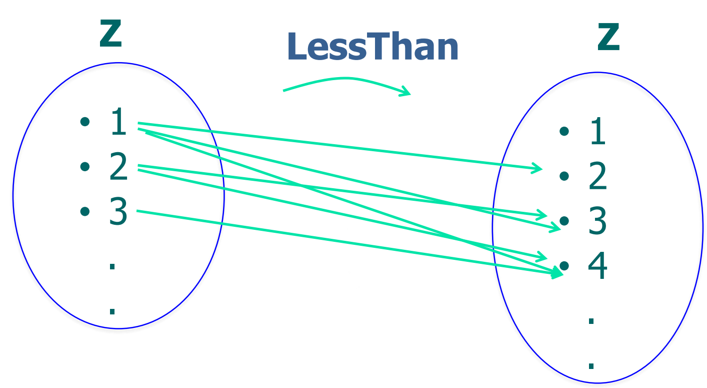


Note: An arrow diagram can be used to define a relation.

Representing Relations: Arrow diagram



Representing Relations: Arrow diagram



The Congruence Modulo 2 Relation

Let E be a relation from **Z** to **Z** as follows:

For all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$, $\quad m\,E\,n \Leftrightarrow m-n$ is even.

a. Is 4 E 0? Is 2 E 6? Is 3 E (–3)? Is 5 E 2?

Yes, 4 E 0 because 4 – 0 = 4 and 4 is even.

Yes, 2 E 6 because 2 – 6 = –4 and –4 is even.

Yes, 3 E (–3) because 3 – (–3) = 6 and 6 is even.

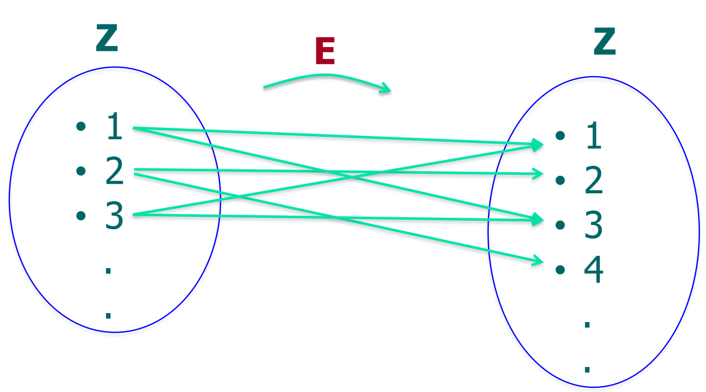
No, because 5 – 2 = 3 and 3 is not even.

b. List five integers that are related by E to 1.

1 because 1 – 1 = 0 is even,
3 because 3 – 1 = 2 is even,
5 because 5 – 1 = 4 is even,
–1 because –1 – 1 = –2 is even,
–3 because –3 – 1 = –4 is even.

c. Prove that if n is any odd integer, then n E 1

The Congruence Modulo 2 Relation



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The Congruence Modulo 2 Relation

Let E be a relation from **Z** to **Z** as follows:

For all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$, $m\ E\ n \Leftrightarrow m - n$ is even.

c. Prove that if m is any odd integer, then m E 1

Proof

Suppose m is any odd integer.

Then **m = 2k + 1** for some integer k.
Now by definition of E, m E 1 if, and only if, m – 1 is even.
But by substitution,
m – 1 = (2k + 1) – 1 = 2k,
and since k is an integer**, 2k is even.**

Hence m E 1 [as was to be shown]

Congruent Modulo 2

It can be shown that integers m and n are related by E if, and only if,
m mod 2 = n mod 2
(that is, both are even or both are odd).
When this occurs m and n are said to be **congruent modulo 2**.

N-ary Relations

Given sets $A_1, A_2, \ldots, A_n$, an ***n*-ary relation** R on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The special cases of 2-ary, 3-ary, and 4-ary relations are called **binary, ternary,** and **quaternary relations,** respectively.

"tuple or triple"

Example: Let A1 = {x, y}, A2 = {1,2,3}, and A3 = {a,b}.

Let R be the subset of
$A_1 \times A_2 \times A_3$ represented by
$\{(x,1,a),(x,2,a),(x,3,b),(y,1,a),(y,2,b),(y,3,b)\}$

Example: A Relation on a Power Set

Let $X = \{a, b, c\}$.
Then $P(X) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.
Define a relation S from $P(X)$ to **Z** as follows: For all sets A and B in $P(X)$ (i.e., for all subsets A and B of X),

$$A \text{ S } B \Leftrightarrow A \text{ has at least as many elements as } B.$$

Is $\{a,b\}$ S $\{b,c\}$? ✓ both sets have two elements.

Is $\{a\}$ S $\varnothing$? ✓ $\{a\}$ has one element and $\varnothing$ has zero elements, and $1 \geq 0$.

Is $\{b,c\}$ S $\{a,b,c\}$? ✗ $\{b, c\}$ has 2 elements and $\{a, b, c\}$ has 3 elements and $2 < 3$

Is $\{c\}$ S $\{a\}$? ✓ both sets have one element.

Relations and Functions

Definition

A **function *F* from a set *A* to a set *B*** is a relation from A to B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x,y) \in F$.
2. For all elements x in A and y and z in B,

$$\text{If } (x,y) \in F \text{ and } (x,z) \in F, \text{ then } y=z.$$

If F is a function from A to B, we write

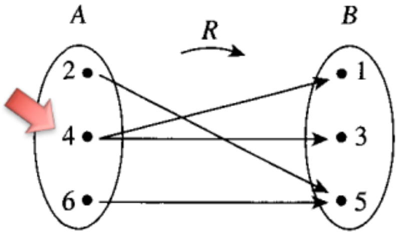
$$Y = F(x) \Leftrightarrow (x,y) \in F.$$

Example

Let A = {2 , 4, 6} and B = {1, 3 , 5}.

Is Relation R a Function from A to B?

R = {(2, 5), (4, 1), (4, 3), (6, 5)}.



The Inverse of a Relation

• Definition

Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

This definition can be written operationally as follows:

For all $x \in A$ and $y \in B$, $\quad (y, x) \in R^{-1} \quad \Leftrightarrow \quad (x, y) \in R$.

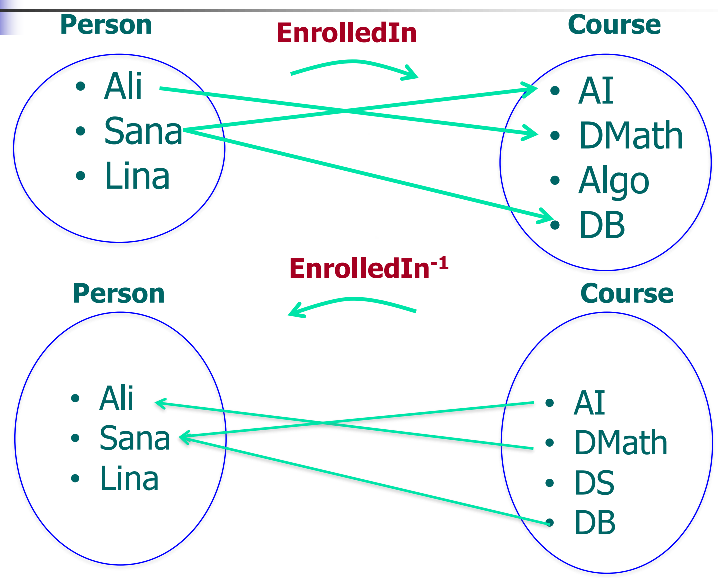
Inverse Relation R^{-1}

Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}.$$

For all $x \in A$ and $y \in B$, $(y,x) \in R^{-1} \Leftrightarrow (x,y) \in R$.

Recall

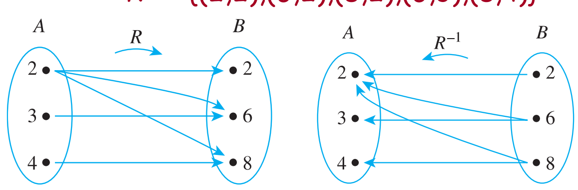


Example – The Inverse of a Finite Relation

Let $A = \{2,3,4\}$ and $B = \{2,6,8\}$ and let R be the "divides" relation from A to B: For all $(x, y) \in A \times B$, $x\,R\,y \Leftrightarrow x \mid y$ **x divides y.**

State explicitly which ordered pairs are in R and R^{-1}, and draw arrow diagrams for R and R^{-1}

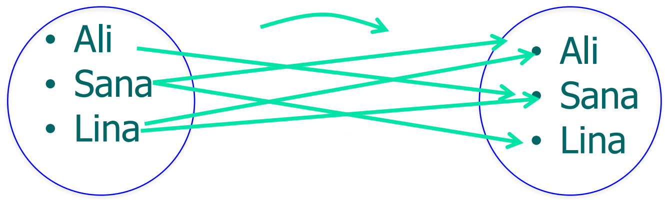
$R = \{(2,2),(2,6),(2,8),(3,6),(4,8)\}$
$R^{-1} = \{(2,2),(6,2),(8,2),(6,3),(8,4)\}$



Describe R^{-1} in words:

For all $(y, x) \in B \times A$, $y\,R^{-1}\,x \Leftrightarrow$ y is a multiple of x.

So far two representation styles



FriendOf = {(Ali, Sana), (Sana,Ali), (Sana,Lina),(Lina,Sana)}

How would you
Represent relations on
As a directed graph?

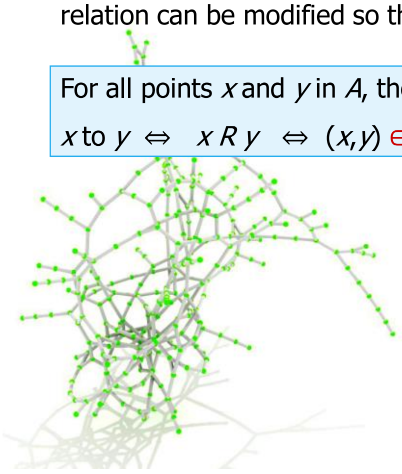
Directed Graphs of a relation from a set to itself

Definition. A relation on a set A is a relation from A to A.

When a relation *R* is defined *on* a set *A*, the arrow diagram of the relation can be modified so that it becomes a **directed graph.**

For all points x and y in A, there is an arrow from

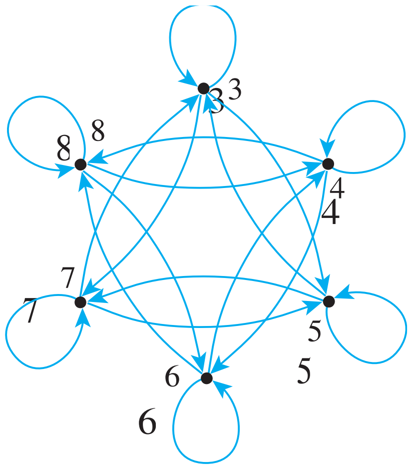
x to y $\Leftrightarrow$ $x \, R \, y$ $\Leftrightarrow$ $(x,y) \in R$.



It is important to distinguish clearly between a relation and the set on which it is defined.

Example: Directed Graph of a Relation

Let A = {3, 4, 5, 6, 7, 8} and define a relation R on A as follows: For all $x, y \in A$, $x\, R\, y \Leftrightarrow 2 \mid (x-y)$.



N-*ary Relations*

Example

Let R = {(a, b, c) | (a = 2b) ∧ (b = 2c) with a, b, c∈**N**}

What is the degree of R?

The degree of R is 3, so its elements are triples.

What are its domains?

Its domains are all equal to the set of integers.

Is (2, 4, 8) in R?

No.

Is (4, 2, 1) in R?

Yes.

N-ary Relations and Relational Databases

- N-ary relations form the mathematical foundation for relational database theory
- The following is a radically simplified version of a database that might be used in a hospital.
- Let A1 be a set of positive integers, A2 a set of alphabetic character strings, A3 a set of numeric character strings, and A 4 a set of alphabetic character strings.
- Define a quaternary relation R on A1 x A2 x A3 x A4 as follows:
  (a1, a2, a3, a4) ∈ R ⇔ a patient with patient ID number a1, named a2, was admitted on date a3,with primary diagnosis a4

Application: Relational Databases

Simplified version of a database that might be used in a hospital

Define R on $A1 \times A2 \times A3 \times A4$ as follows:

($a1$, $a2$, $a3$, $a4$) ∈ R ⇔ a patient with patient ID number $a1$, named $a2$, was admitted on date $a3$, with primary diagnosis $a4$.

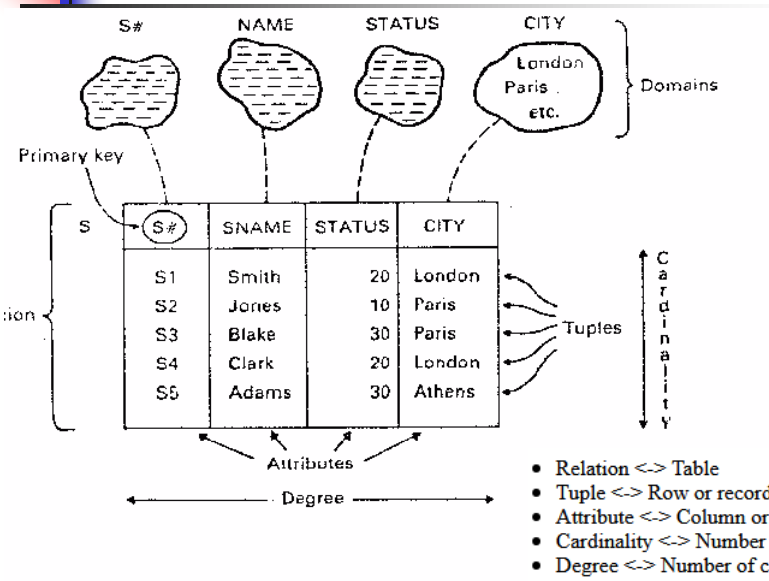
Each table in the database represents a Relation

Patient

ID	Name	Date	Diagnosis
(011985,	John Schmidt,	020710,	asthma)
(574329,	Tak Kurosawa,	114910,	pneumonia)
(466581,	Mary Lazars,	103910,	appendicitis)
(008352,	Joan Kaplan,	112409,	gastritis)
(011985,	John Schmidt,	021710,	pneumonia)
(244388,	Sarah Wu,	010310,	broken leg)
(778400,	Jamal Baskers,	122709,	appendicitis)

Each row in the table is called tuple

Database Table as a Relation

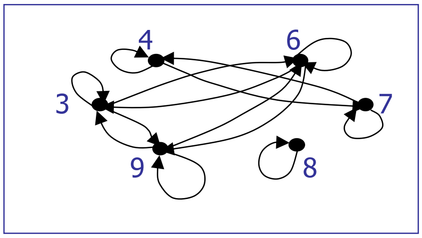


- Relation <-> Table
- Tuple <-> Row or record
- Attribute <-> Column or field
- Cardinality <-> Number of rows
- Degree <-> Number of columns
- Primary key <-> Unique identifier
- Domain <-> Pool of legal values

8.2 Reflexivity, Symmetry, and Transitivity

Properties of Relations

Definition: Let *A* be a set and let *R* be a binary relation "on" *A*.
(i.e., *R* is a binary relation from *A* to *A*).
R is **reflexive** متناظر ⇔ ∀ *x* in *A*, *x R x*.
R is **symmetric** متماثل ⇔ ∀ *x* and *y* in *A*, if *x R y* then *y R x*.
R is **transitive** متعدي ⇔ ∀ *x*, *y*, and *z* in *A*, if *x R y* and *y R z* then *x R z*.
R is an **equivalence relation** ⇔ *R* is reflexive, symmetric, and transitive.



Example: Consider the binary relation *S* defined on the set {3, 4, 6, 7, 8, 9} with directed graph shown at the left.
a. Is *S* reflexive?
b. Is *S* symmetric?
c. Is *S* transitive?
d. Is *S* an equivalence relation?

Properties of Relations- **reflexive**

We will now look at some useful ways to classify relations.

Definition: A relation R on a set A is called **reflexive,** if (a, a)∈R **for every** element **a∈A**.

Are the following relations on {1, 2, 3, 4} reflexive?

- **R = {(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)}** No
- **R = {(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)}** Yes
- **R = {(1, 1), (2, 2), (3, 3)}** No

A relation on a set A is **not reflexive** if there exist some element a∈A such that (a, a)∉R.

Properties of Relations - **symmetric**

Definitions

- A relation R on a set A is called **symmetric** if (b, a)∈R whenever (a, b)∈R for all a, b∈A.

***R* is not Symmetric:** there are elements *x* and *y* in *A* such that *x R y* but *y* $\not R$ *x [that is, such that (x , y)* ∈ *R but (y,x)* ∉ *R]*.

Properties of Relations - **transitive**

Definition: A relation R on a set A is called **transitive** if whenever (a, b)∈R and (b, c)∈R, then (a, c)∈R for a, b, c∈A.

Are the following relations on {1, 2, 3, 4} transitive?

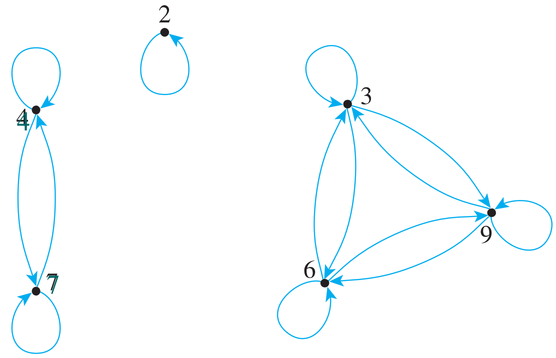
- R = {(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)} Yes
- R = {(1, 3), (3, 2), (2, 1)} No
- R = {(2, 4), (4, 3), (2, 3), (4, 1)} No

***R* is not transitive:** there are elements *x*,*y* and *z* in *A* such that *xRy* and *yRz* but *x* $\not R$ *z [that is, such that (x,y)* ∈ *R and (y,z)* ∈ *R but (x , z)* ∉ *R]* .

Exercise

Let $A = \{2,3,4,6,7,9\}$ and define a relation R on A as:
For all $x, y \in A$, $x\,R\,y \Leftrightarrow 3|(x-y)$.

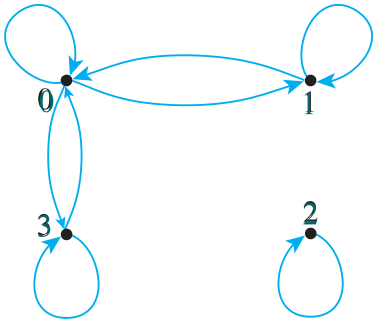
Is R Reflexive? ✓ Symmetric? ✓ Transitive? ✓



Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:
$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$

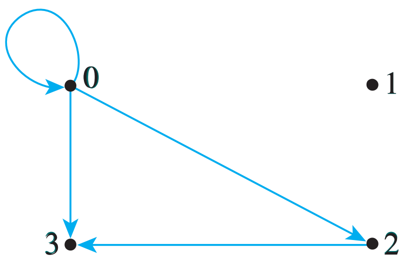
Is R Reflexive? ☑ Symmetric? ☑ Transitive? ☒



Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:
$R = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$

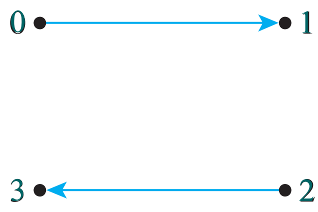
Is R Reflexive? ☒ *Symmetric?* ☒ *Transitive?* ☑



Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:
$R =\{(0,1),(2,3)\}$

Is R Reflexive? ☒ *Symmetric?* ☒ *Transitive?* ☑



The Transitive Closure of a Relation

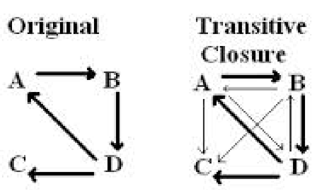
• Definition

Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the following three properties:

1. R^t is transitive.
2. $R \subseteq R^t$.
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$.

Informally:

- To obtain a transitive relation from one that is not transitive, it is necessary to add ordered pairs.
- Thus, R^t is the relation obtained by adding the least number of ordered pairs to ensure transitivity.



The **smallest** transitive relation that contains the relation.

Example1: The Transitive Closure of a Relation

Let $A = \{0, 1, 2, 3\}$ and consider the relation R defined on A as follows:

$$R = \{(0, 1), (1, 2), (2, 3)\}.$$

Find the transitive closure of R.

Solution: Every ordered pair in R is in R^t, so

$$\{(0, 1), (1, 2), (2, 3)\} \subseteq R^t.$$

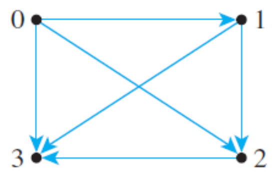
Thus the directed graph of R contains the arrows shown below.

Example1: The Transitive Closure of a Relation

- Since there are arrows going from 0 to 1 and from 1 to 2, R^t must have an arrow going from 0 to 2. Hence $(0, 2) \in R^t$.
- Then $(0, 2) \in R^t$ and $(2, 3) \in R^t$, so since R^t is transitive, $(0, 3) \in R^t$.
- Also, since $(1, 2) \in R^t$ and $(2, 3) \in R^t$, then $(1, 3) \in R^t$.
- Thus R^t contains at least the following ordered pairs:

$$\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}.$$

But this relation *is* transitive; hence it equals R^t. Note that the directed graph of R^t is as shown below.



8.3 Equivalence Class of Relations

Part 1: **Partitioned Sets**

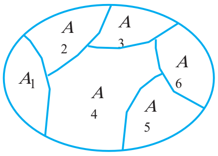
Part 2: **Equivalence Relation**

Part 3: **Equivalence Class**

Recall: Set Partition

Definition. A collection of nonempty **sets {A1, A2, A3, …} form a partition of a set A if, and only if,** A is the union of all the Ai The sets A1, A2, A3, … are mutually disjoint.

- Example:
- Let A = {0,1,2,3,4},
- one possible partition:{1,3}, {0,4}, {2}
- another: {0, 3, 4}, {1}, {2}



what is the relation R induced by each partition?

The Relation Induced by a Partition

Definition

Given a partition of a set *A*, the **relation induced by the partition,** *R*, is defined on *A* as follows:

For all *x*, *y* ∈ *A*, ***x R y*** ⇔ there is a subset A_i of the partition such that both *x* and *y* are in A_i .

Example: finding relation induced by Partition

Let A = {0, 1, 2, 3, 4} and consider the following **partition** of A:

{0, 3, 4}, {1}, {2}.

Find the relation R induced by this partition.

Solution Since {0, 3, 4} is a subset of the partition,

0 R 3 because both 0 and 3 are in {0, 3, 4},
3 R 0 because both 3 and 0 are in {0, 3, 4},
0 R 4 because both 0 and 4 are in {0, 3, 4},
4 R 0 because both 4 and 0 are in {0, 3, 4},
3 R 4 because both 3 and 4 are in {0, 3, 4},
4 R 3 because both 4 and 3 are in {0, 3, 4},

Also, 0 R 0 because both 0 and 0 are in {0, 3, 4},
3 R 3 because both 3 and 3 are in {0, 3, 4},
4 R 4 because both 4 and 4 are in {0, 3, 4}.

Example (Contd.)

Since {1} is a subset of the partition,

1 R 1 because both 1 and 1 are in {1},

and since {2} is a subset of the partition,

2 R 2 because both 2 and 2 are in {2}.

Hence

R = { (0, 0), (0, 3), (0, 4), (1, 1), (2, 2), (3, 0), (3, 3), (3, 4), (4, 0), (4, 3), (4, 4) }.

Relations induced by set partition

Theorem 8.3.1.

Let **A** be a set with a **partition** and
Let *R* be the relation induced by the partition.

Then ***R*** is **reflexive**, **symmetric**, and **transitive**

47

Equivalence Relation

Definition

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

➔ The relation induced by a partition is an equivalence relation

Example1

Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation R on X as follows: For all A and B in X,

$A\ R\ B \Leftrightarrow$ the least element of A equals the least element of B.

Prove that R is an equivalence relation on X.

An Equivalence Relation on a Set of Subsets

Let X be the set of all nonempty subsets of {1, 2, 3}. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation **R** on X as follows: For all A and B in X, A **R** $B \Leftrightarrow$ the least element of A equals the least element of B. Prove that **R** is an equivalence relation on X.

R *is reflexive*: Suppose A is a nonempty subset of X. *[We must show that A* **R** *A].* It is true to say that the least element of A equals the least element of A. Thus, by definition of R, A **R** A.

An Equivalence Relation on a Set of Subsets

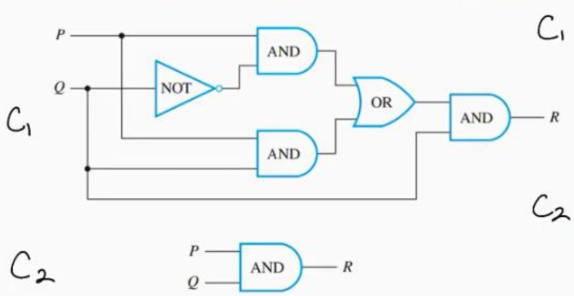
***R is symmetric*:**
Suppose *A* and *B* are nonempty subsets of X and *A* **R** *B*.
[We must show that B **R** *A.]*
Since *A* **R** *B*, the least element of *A* equals the least element of *B*. But this implies that the least element of *B* equals the least element of *A*, and so, by definition of **R**, *B* **R** *A*.

***R is transitive*:**
Suppose *A*, *B*, and *C* are nonempty subsets of X, *A* **R** *B*, and *B R C*.
[We must show that A **R** *C.]*
Since *A* **R** *B*, the least element of *A* equals the least element of *B* and since *B* **R** *C*, the least element of *B* equals the least element of *C*. Thus the least element of *A* equals the least element of *C*, and so, by definition of **R**, *A* **R** *C*.

Example2: Equivalence of Digital Logic Circuits Is an Equivalence Relation

Let S be the set of all digital logic circuits with a fixed number n of inputs. Define the relation E as follows:

$$\forall C_1, C_2 \in S, C_1 \; E \; C_2 \iff C_1 \text{ has the same I/O table as } C_2$$



C_1

P	Q	R
1	1	
1	0	
0	1	
0	0	

C_2

P	Q	R
1	1	
1	0	
0	1	
0	0	

Example2: Equivalence of Digital Logic Circuits Is an Equivalence Relation

Let **S** be the set of all digital logic circuits with a fixed number *n* of inputs.

Define a relation **E** on *S* as follows:

For all circuits $C1$ and $C2$ in *S*, $C1$ **E** $C2$ ⇔ $C1$ has the same input/output table as $C2$.

If $C1$ **E** $C2$, then circuit $C1$ is said to be *equivalent* to circuit $C2$.

Prove that **E** is an equivalence relation on *S*.

Proof: **E *is reflexive*:** Suppose *C* is a digital logic circuit in *S*. *[We must show that C* **E** *C.]*
Certainly *C* has the same input/output table as itself.
Thus, by definition of **E**, *C* **E** *C [as was to be shown].*

Example2: Equivalence of Digital Logic Circuits Is an Equivalence Relation – cont.

E *is symmetric*:
Suppose $C1$ and $C2$ are digital logic circuits in *S* such that $C1$ **E** $C2$.
[We must show that $C2$ **E** $C1$.*]*
By definition of **E**, since $C1$ **E** $C2$, then $C1$ has the same input/output table as $C2$. It follows that $C2$ has the same input/output table as $C1$.
Hence, by definition of **E**, $C2$ **E** $C1$.

E *is transitive*:
Suppose $C1$, $C2$, and $C3$ are digital logic circuits in *S* such that $C1$ **E** $C2$ and $C2$ **E** $C3$. *[We must show that* $C1$ **E** $C3$.*]*
By definition of **E**, since $C1$ **E** $C2$ and $C2$ **E** $C3$, then $C1$ has the same input/output table as $C2$ and $C2$ has the same input/output table as $C3$. It follows that $C1$ has the same input/output table as $C3$. Hence, by definition of **E**, $C1$ **E** $C3$.

Since E is reflexive, symmetric, and transitive, E is an equivalence relation on *S*.

Equality relation over real numbers

$$\forall x, y \in \mathbb{R}, x\, R\, y \Leftrightarrow x = y$$

- Equality relation is an equivalence relation.
- What do you notice about the following set:

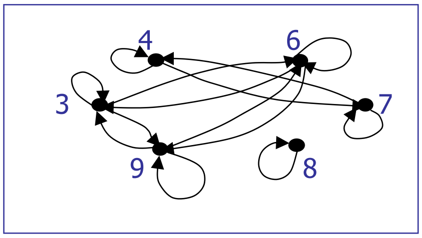
  {1/2, 2/4,3/6,-1/-2,30/60} ?
- They are equal to one another.
- In fact every real number has inf. number of numbers that are equal to it.
- Ex. [1/2] = {1/2, 2/4,3/6,-1/-2,30/60,…}

  [1/3] = {1/3,2/6,30/90,…}

Equivalence Class of an Element

Definition: Let *R* be an equivalence relation from *A* to *A*, and suppose *a* is any element of *A*. The **equivalence class of a**, denoted **[a]**, is defined as follows: **[a]** = {$x \in A$ | *x* is related to *a* by *R* }.

Example: Consider the binary relation *S* defined on the set {3, 4, 6, 7, 8, 9} with directed graph shown below. Find [3], [4], [6], [7], [8], and [9].



[3] = {3, 6, 9}
[4] = {4, 7}
[6] = {3, 6, 9}
[7] = {4, 7}
[8] = {8}
[9] = {3, 6, 9}

What are the *distinct* equivalence classes of this relation?

{3, 6, 9}, {4, 7}, {8}

Another example

Given a set A and an Equivalence Relation R, the equivalence class of a is: $[a] = \{x \in A \mid x\, R\, a\}$

Given the set $A = \left\{\frac{1}{2}, \frac{1}{3}, 1, \frac{2}{2}, \frac{2}{4}, \frac{2}{6}, 2, \frac{3}{9}\right\}$

How many Equivalence classes are there under the equality relation?

$$\left[\frac{1}{2}\right] = \left\{\frac{1}{2}, \frac{2}{4}\right\}$$
$$\left[\frac{1}{3}\right] = \left\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}\right\}$$
$$[1] = \left\{1, \frac{2}{2}\right\}$$
$$[2] = \{2\}$$

Equivalence Classes of an Element

Lemma 8.3.2

Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If $a\ R\ b$, then $[a] = [b]$.

The lemma says that if two elements of A are related by an equivalence relation R, **then their equivalence classes are the same.**

Lemma 8.3.3

If A is a set, R is an equivalence relation on A, and a and b are elements of A, then

$$\text{either} \quad [a] \cap [b] = \emptyset \quad \text{or} \quad [a] = [b].$$

The lemma says that any two equivalence classes of an equivalence relation **are either mutually disjoint or identical.**

- **Definition**

Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is any element a such that $[a] = S$.

Proof of Lemma 8.3.3

Suppose A is a set, R is an equivalence relation on A, a and b are elements of A, and

$$[a] \cap [b] \neq \emptyset.$$

[We must show that $[a] = [b]$.*]* Since $[a] \cap [b] \neq \emptyset$, there exists an element x in A such that $x \in [a] \cap [b]$. By definition of intersection,

$$x \in [a] \quad \text{and} \quad x \in [b]$$

and so

$$x\ R\ a \quad \text{and} \quad x\ R\ b$$

by definition of class. Since R is symmetric *[being an equivalence relation]* and $x\ R\ a$, then $a\ R\ x$. But R is also transitive *[since it is an equivalence relation]*, and so, since $a\ R\ x$ and $x\ R\ b$,

$$a\ R\ b.$$

Now a and b satisfy the hypothesis of Lemma 8.3.2. Hence, by that lemma,

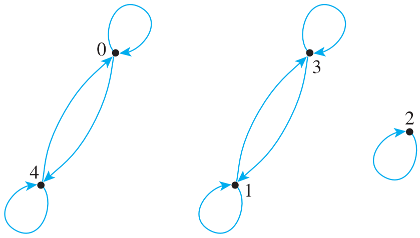
$$[a] = [b].$$

[This is what was to be shown.]

Example

Let A = {0,1,2,3,4} and define a relation R on A as :
R = {(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)}.

Find the distinct equivalence classes of R.



$$[0] = \{x \in A \mid x\ R\ 0\} = \{0, 4\}$$
$$[1] = \{x \in A \mid x\ R\ 1\} = \{1, 3\}$$
$$[2] = \{x \in A \mid x\ R\ 2\} = \{2\}$$
$$[3] = \{x \in A \mid x\ R\ 3\} = \{1, 3\}$$
$$[4] = \{x \in A \mid x\ R\ 4\} = \{0, 4\}$$

$[0] = [4]$ and $[1] = [3]$.

So, the distinct equivalence classes of R:

{0, 4}, {1,3}, and {2}

Relations induced by set partition

Theorem 8.3.4 The Partition Induced by an Equivalence Relation

If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A, and the intersection of any two distinct classes is empty.

Let A be a set and let R be an equivalence relation on A. Then the **distinct** equivalence classes of R form a **partition** of A.

1. The **union** of the equivalence classes is all of ***A***, and
2. The **intersection** of any two distinct classes is **empty**.

Proof of Theorem 8.3.4 is on page 469 and page 470

61

Congruence to n Modulo d

• **Definition**

Let m and n be integers and let d be a positive integer. We say that **m is congruent to n modulo d** and write

$$m \equiv n \pmod{d}$$

if, and only if, $d \mid (m - n)$.

Symbolically: $m \equiv n \pmod{d} \Leftrightarrow d \mid (m - n)$

Examples: Determine which of the following congruences are true and which are false:

a. $12 \equiv 7 \pmod{5}$

a. True. $12 - 7 = 5 = 5 \cdot 1$. Hence $5 \mid (12 - 7)$, and so $12 \equiv 7 \pmod{5}$.

b. $6 \equiv -8 \pmod{4}$

b. False. $6 - (-8) = 14$, and $4 \nmid 14$ because $14 \neq 4 \cdot k$ for any integer k. Consequently, $6 \not\equiv -8 \pmod{4}$.

c. $3 \equiv 3 \pmod{7}$

c. True. $3 - 3 = 0 = 7 \cdot 0$. Hence $7 \mid (3 - 3)$, and so $3 \equiv 3 \pmod{7}$

Equivalence Classes of Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set **Z** of all integers. That is, for all integers m and n,

$$m\ R\ n \Leftrightarrow 3 \mid (m - n)$$

Describe the distinct **equivalence classes** of R.

For each integer a,

$$[a] = \{x \in \mathbf{Z} \mid x\ R\ a\}$$
$$= \{x \in \mathbf{Z} \mid 3 \mid (x - a)\}$$
$$= \{x \in \mathbf{Z} \mid x - a = 3k, \text{ for some integer } k\}.$$

Therefore,

[a] = {x ∈ Z | x = 3k + a, for some integer k}.

In particular, $[0] = \{x \in \mathbf{Z} \mid x = 3k + 0, \text{ for some integer } k\}$
$= \{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\}$
$= \{. . . -9, -6, -3, 0, 3, 6, 9, . . .\}$,

Equivalence Classes of Congruence Modulo 3

$[1] = \{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\}$
$= \{. . . -8, -5, -2, 1, 4, 7, 10, . . .\}$,

$[2] = \{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}$
$= \{. . . -7, -4, -1, 2, 5, 8, 11, . . .\}$.

Now since $3\ R\ 0$,

$$[3] = [0].$$

More generally, by the same reasoning,

$[0] = [3] = [-3] = [6] = [-6] = . . . ,$ and so on.

Similarly,

$[1] = [4] = [-2] = [7] = [-5] = . . . ,$ and so on.

And

$[2] = [5] = -1 = [8] = [-4] = . . . ,$ and so on.

Equivalence Classes of Congruence Modulo 3

Notice that every integer is in class [0], [1], or [2]. Hence the distinct equivalence classes are:

$\{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\}$,

$\{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\}$, and

$\{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}$.

In words, the three classes of congruence modulo 3 are

(1) The set of all integers that are divisible by 3,

(2) The set of all integers that leave a remainder of 1 when divided by 3, and

(3) The set of all integers that leave a remainder of 2 when divided by 3.

Equivalence Classes of Congruence Modulo 3

$$[0] = \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\},$$

$$[1] = \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\},$$

$$[2] = \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}.$$

Partition of Z :

$$Z = [0] \cup [1] \cup [2]$$

And

$$[0] \cap [1] \cap [2] = \emptyset$$

Rational Numbers are really equivalence classes

Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero. Symbolically,

$A = Z \times (Z - \{0\})$.

Define a relation R on A as follows: For all $(a, b), (c, d) \in A$,

$(a, b)\ R\ (c, d) \Leftrightarrow ad = bc$.

<u>The fact is that R is an **equivalence relation**.</u>

a. Prove that R is transitive. (Proofs that R is reflexive and symmetric are left to exercise 42 at the end of the section.)
b. Describe the distinct equivalence classes of R.

Solution (a)

a. *[We must show that for all $(a, b), (c, d), (e, f) \in A$, if $(a, b)\ R\ (c, d)$ and $(c, d) R\ (e, f)$, then $(a, b)\ R\ (e, f)$.]* Suppose $(a, b), (c, d)$, and (e, f) are particular but arbitrarily chosen elements of A such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. *[We must show that $(a, b)\ R\ (e, f)$.]* By definition of R,

$$(1)\ ad = bc \quad \text{and} \quad (2)\ cf = de.$$

Since the second elements of all ordered pairs in A are nonzero, $b \neq 0$, $d \neq 0$, and $f \neq 0$. Multiply both sides of equation (1) by f and both sides of equation (2) by b to obtain

$$(1')\ adf = bcf \quad \text{and} \quad (2')\ bcf = bde.$$

Thus

$$adf = bde$$

and, since $d \neq 0$, it follows from the cancellation law for multiplication (T7 in Appendix A) that

$$af = be.$$

It follows, by definition of R, that $(a, b)\ R\ (e, f)$ *[as was to be shown]*.

Solution(b)

b. There is one equivalence class for each distinct rational number. Each equivalence class consists of all ordered pairs (a, b) that, if written as fractions a/b, would equal each other. The reason for this is that the condition for two rational numbers to be equal is the same as the condition for two ordered pairs to be related. For instance, the class of $(1, 2)$ is

$$[(1,2)] = \{(1,2), (-1,-2), (2,4), (-2,-4), (3,6), (-3,-6), \ldots\}$$

since $\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6}$ and so forth. ■

Proving Properties on Relations on Infinite Sets

Proving Properties on Relations on Infinite Sets

Outline of proof.
To prove a relation is reflexive, symmetric, or transitive, first write down what is to be proved, in **First Order Logic**.

For instance, for symmetry

$\forall x, y \in A$, if $x\ R\ y$ then $y\ R\ x$.

Then use **direct proof method**

Recall: Definition of Relation Properties and their consequences

Let A be a set and let ***R* be a binary relation** on A. Complete the following sentences.

R is not reflexive $\Leftrightarrow$ there is an element x in A such that $x \not R\ x$ *[that is, such that* $(x, x) \notin R$*].*

R is not symmetric $\Leftrightarrow$ there are elements x and y in A such that $x\ R\ y$ but $y \not R\ x$ *[that is, such that* $(x, y) \in R$ *but* $(y, x) \notin R$*].*

R is not transitive $\Leftrightarrow$ there are elements x, y and z in A such that $x\ R\ y$ and $y\ R\ z$ but $x \not R\ z$ *[that is, such that* $(x, y) \in R$ *and* $(y, z) \in R$ *but* $(x, z) \notin R$*].*

Properties of "**Less Than**" relation

Define a relation R on **R** (the set of all real numbers) as follows:

For all $x, y \in R$, $x\ R\ y \Leftrightarrow x < y$.

a. Is R reflexive? b. Is R symmetric? c. Is R transitive?

***R is not reflexive*:**

R is reflexive if, and only if, $\forall x \in \mathbf{R}, x\ R\ x$. By definition of R, this means that $\forall x \in \mathbf{R}, x < x$.

But this is false: $\exists x \in \mathbf{R}$ such that $x \not< x$.

As a counterexample, let $x = 0$ and note that $0 \not< 0$.

Hence R is not reflexive.

Properties of "Less Than" relation

Define a relation R on **R** (the set of all real numbers) as follows:

For all $x, y \in R$, $x\ R\ y \Leftrightarrow x < y$.

a. Is R reflexive? b. Is R symmetric? c. Is R transitive?

***R is not symmetric*:**

R is symmetric if, and only if, $\forall x, y \in \mathbf{R}$, if $x\ R\ y$ then $y\ R\ x$.

By definition of R, this means that $\forall x, y \in \mathbf{R}$, if $x < y$ then $y < x$.

But this is false: $\exists x, y \in \mathbf{R}$ such that $x < y$ and $y \not< x$.

As a counterexample, let $x = 0$ and $y = 1$ and note that $0 < 1$ but $1 \not< 0$.

Hence R is not symmetric.

Properties of "Less Than" relation

Define a relation R on **R** (the set of all real numbers) as follows:

For all $x, y \in R$, $x \, R \, y \Leftrightarrow x < y$.

a. Is R reflexive? b. Is R symmetric? c. Is R transitive?

***R is transitive*:**

R is transitive if, and only if, for all $x, y, z \in$ **R**, if $x \, R \, y$ and $y \, R \, z$ then $x \, R \, z$.

By definition of R, this means that for all $x, y, z \in$ **R**, if $x < y$ and $y < z$, then $x < z$.

But this statement is true by the transitive law of order for real numbers.

Hence R is transitive.

Properties of **Congruence Modulo 3**

Define a relation T on **Z** (the set of all integers) as follows:

For all integers m and n, $m\ T\ n \Leftrightarrow 3 \mid (m - n)$.

This relation is called **congruence modulo 3.**

a. Is T reflexive? b. Is T symmetric? c. Is T transitive?

T is Reflexive

Suppose m is a particular but arbitrarily chosen integer.

[We must show that m T m.]

Now $m - m = 0$. But $3 \mid 0$ since $0 = 3 \cdot 0$. Hence $3 \mid (m - m)$.

Thus, by definition of T, **m T m**.

Hence T is reflexive.

Properties of Congruence Modulo 3

Define a relation T on **Z** (the set of all integers) as follows:

For all integers m and n, $m\ T\ n \Leftrightarrow 3 \mid (m - n)$.

Is T reflexive? b. Is T symmetric? c. Is T transitive?

T is Symmetric

Suppose m and n are particular but arbitrarily chosen integers that satisfy the condition **m T n**. *[We must show that n T m.]*

By definition of T, since $m\ T\ n$ then $3 \mid (m - n)$.

By definition of "divides," this means that $m - n = 3k$, for some integer k.

Multiplying both sides by -1 gives $n - m = 3(-k)$.

Since $-k$ is an integer, this equation shows that $3 \mid (n - m)$.

Thus, by definition of T, **n T m**.

Hence T is Symmetric

Properties of Congruence Modulo 3

Define a relation T on **Z** (the set of all integers) as follows:

For all integers m and n, $m\ T\ n \Leftrightarrow 3 \mid (m - n)$.

Is T reflexive? b. Is T symmetric? c. Is T transitive?

T is Transitive

Suppose m, n, and p are particular but arbitrarily chosen integers that satisfy the condition $m\ T\ n$ and $n\ T\ p$. *[We must show that* **m T p**.*]*
By definition of T, since $m\ T\ n$ and $n\ T\ p$, then $3 \mid (m - n)$ and $3 \mid (n - p)$.
By definition of "divides," this means that $m - n = 3r$ and $n - p = 3s$, for some integers r and s.
Adding the two equations gives $(m - n) + (n - p) = 3r + 3s$, and simplifying gives that $m - p = 3(r + s)$.
Since $r + s$ is an integer, this equation shows that $3 \mid (m - p)$.
Thus, by definition of T, **m T p**. *Hence, T is transitive*

Exercises

1. A = {BZU students}. Define R on A by:

$$x\ R\ y \Leftrightarrow x \text{ lives within 1 mile of } y.$$

Is R reflexive? Is R symmetric? Is R transitive?
Is R an equivalence relation?

2. $A = \{0, 1, 2, 3\}$. Define R on A by:

$$R = \{(1,3), (2,3)\}$$

Is R reflexive? Is R symmetric? Is R transitive?
Is R an equivalence relation?