COMP 233 Discrete Mathematics

Counting Theory

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Outline

- Counting elements of a list/sub-list
- Possibility Trees and the multiplication Rule
- Counting elements of sets (disjoint/overlapping)
  - Addition, Difference, and Inclusion/Exclusion Rules
- Counting the number of ways of choosing k elements from n (w/o repetition)
  - Order matters: Permutations (التباديل)
    - Proving a property of P(n,r)
  - Order does not matter: counting subsets of a set (Combinations) (التوافيق)

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Chapter Content

9.1 Basics of Probability and Counting

Part 1: **Probability and Sample Space**

Part 2: **Counting in Sub lists**

Assumptions

- This chapter assumes you are familiar with
- Probability theory
  - Events and probability
    - N(A): number of elements in set A
    - P(A): probability of a set A
- Possibility trees

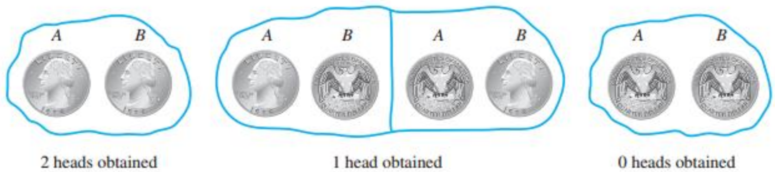
Motivation: Tossing Coins

Tossing two coins and observing whether 0, 1, or 2 heads are obtained.

What are the chances of having 0,1,2 heads?

Tossing Coins

- Tossing two coins 50 times and observing whether 0, 1, or 2 heads are obtained.
- What are the chances of having 0,1,2 heads?



Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	𝍸 𝍸 𝍷	11	22%
1 head obtained	𝍸 𝍸 𝍸 𝍸 𝍸 𝍷𝍷	27	54%
0 heads obtained	𝍸 𝍸 𝍷𝍷	12	24%

Motivation: 52-card deck

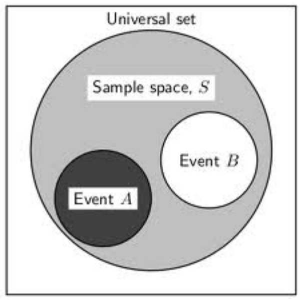
- **How many cards must you pick from a standard 52-card deck to be sure of getting at least 1 red card? Why?**
- There are only 26 black cards in a standard 52-card deck, so at most 26 black cards can be chosen.
- Hence if 27 are taken, at least one must be red.
- The answer is 27.

Sample Space

الفراغ العيني

• Definition

A **sample space** is the set of all possible outcomes of a random process or experiment. An **event** is a subset of a sample space.



Sample Space

In case an experiment has finitely many outcomes and all outcomes are **equally** likely to occur, the **probability** of an **event** (set of outcomes) is just the ratio of the number of outcomes in the event to the total number of outcomes

Equally Likely Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability of E**, denoted **$P(E)$**, is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S}.$$

• Notation

For any finite set A, $N(A)$ denotes the number of elements in A.

$$P(E) = \frac{N(E)}{N(S)}.$$

Example: Probabilities for a Deck of Cards

52 cards

diamonds (♦) hearts (♥)
clubs (♣) spades (♠)

a. What is the sample space of outcomes?

➔ the 52 cards in the deck.

b. What is the event that the chosen card is a black face card?

➔ E = {J♣, Q♣, K♣, J♠, Q♠, K♠}

c. What is the probability that the chosen card is a black face card?

$$P(E) = \frac{N(E)}{N(S)} = \frac{6}{52} \cong 11.5\%.$$

Rolling a Pair of Dice

a. Write the sample space S of possible outcomes (using compact notion).

S = {1 1, 1 2, 1 3, 1 4, 1 5, 1 6, 2 1, 2 2, 2 3, 2 4, 2 5, 2 6, 3 1, 3 2, 3 3, 3 4, 3 5, 3 6, 4 1, 4 2, 4 3, 4 4, 4 5, 4 6, 5 1, 5 2, 5 3, 5 4, 5 5, 5 6, 6 1, 6 2, 6 3, 6 4, 6 5, 6 6}.

b. write the event E that the numbers showing face up have a sum of 6 and find the **probability** of this event.

E = {1 5, 2 4, 3 3, 4 2, 5 1}. $P(E) = \frac{N(E)}{N(S)} = \frac{5}{36}$.

Counting the Elements of a List

Some counting problems are as simple as counting the elements of a list.
E.g., how many integers are there from 5 through 12?

list:	5	6	7	8	9	10	11	12
	↕	↕	↕	↕	↕	↕	↕	↕
count:	1	2	3	4	5	6	7	8

Theorem 9.1.1 The Number of Elements in a List

If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

12-5+1=8=#of elements

Counting the Elements of a Sublist

- How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

100	101	102	103	104	105	106	107	108	109	110	⋯	994	995	996	997	998	999
↕					↕					↕			↕				
$5 \cdot 20$					$5 \cdot 21$					$5 \cdot 22$			$5 \cdot 199$				

From the sketch it is clear that there are as many three-digit integers that are multiples of 5 as there are integers from 20 to 199 inclusive. By Theorem 9.1.1, **there are 199 − 20 + 1,** such integers. **Hence there are 180 three-digit integers that are divisible by 5.**

- What is the probability that a randomly chosen three-digit integer is divisible by 5?

Sample space: 999 − 100 + 1 = 900.

P(E)=180/900 = 1/5.

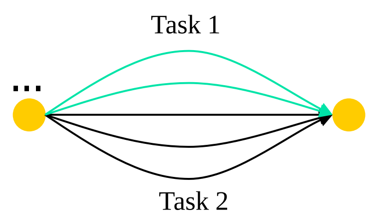
By Theorem 9.1.1 the total number of integers from 100 through 999 is 999 − 100 + 1 = 900. By part (a), 180 of these are divisible by 5. Hence the probability that a randomly chosen three-digit integer is divisible by 5 is 180/900 = 1/5.

Sum Rule

- Let us consider two tasks:
  - *m* is the number of ways to do **task 1**
  - *n* is the number of ways to do **task 2**
  - Tasks are <u>independent</u> of each other, i.e.,
    - Performing **task 1** does not accomplish **task 2** and vice versa.

- *Sum rule*: the number of ways that "**either** task 1 **or** task 2 can be done, but **not both**", is ***m* + *n***.

- Generalizes to multiple tasks ...



Example

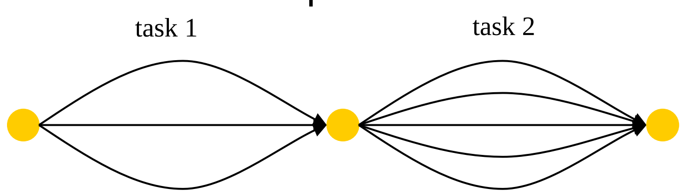
- A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects respectively. How many possible projects are there to choose from?

$$23+15+19$$

Product Rule

- Let us consider two tasks:
  - *m* is the number of ways to do **task 1**
  - *n* is the number of ways to do **task 2**
  - Tasks are <u>independent</u> of each other, i.e.,
    - Performing task 1does not accomplish task 2 and vice versa.
- *<u>Product rule</u>*: the number of ways that "**both** tasks 1 and 2 can be done" in *mn*.

- Generalizes to multiple tasks ...



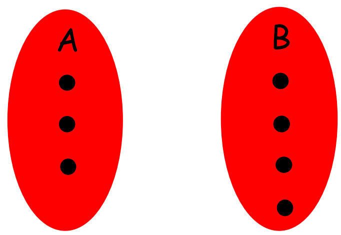
Product rule example

- There are 18 math majors and 325 CS majors
- How many ways are there to pick one math major and one CS major?

- Total is 18 * 325 = 5850

Product Rule

How many functions are there from set A to set B?



To define each function we have to make 3 choices, one for each element of A. Each has 4 options (to select an element from B).

4 4 4

How many ways can each choice be made?

$4^3 = 64 = |B|^{|A|}$

9.2 Possibility Trees and the Multiplication Rule

Part 1: Possibility Trees
Part 2: Multiplication Rule
Part3 : Permutations

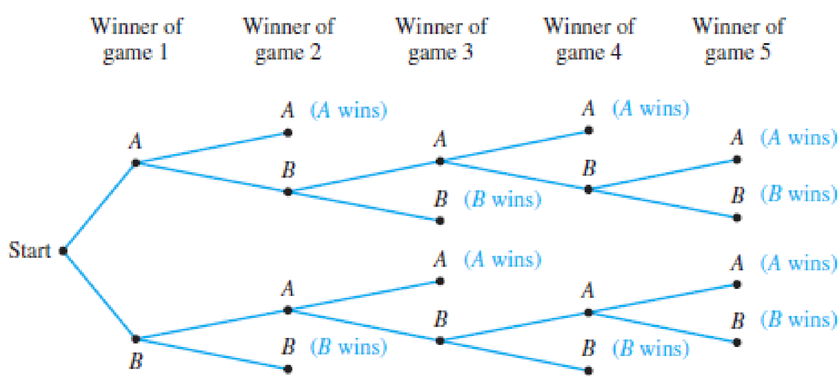
Example1: Possibility Trees

Teams *A* and *B* are to play each other repeatedly until one wins *two games in a row* or a total of three games. Example: *A* wins the first game, *B* wins the second, and *A* wins the third and fourth games. Denote this by writing *A*–*B*–*A*–*A*.

How many ways can the tournament be played?

Solution: The possible ways for the tournament to be played are represented by the distinct paths from "**root**" (the start) to "**leaf**" (a terminal point) in the **tree**, The **label** on each branching point indicates the winner of the game. The notations in parentheses indicate the winner of the tournament.

Outcome of the Tournament



Outcome of the Tournament

The fact that there are **ten** paths from the root of the tree to its **leafs** shows that there are ten possible ways for the tournament to be played. They are (moving from the top down):

A–A, A–B–A–A, A–B–A–B–A, *A–B–A–B–B, A–B–B, B–A–A*, *B–A–B–A–A*, *B–A–B–A–B, B–A–B–B*, and *B–B*.

In five cases *A* wins, and in the other five *B* wins. The **least** number of games that must be played to determine a winner is **two**, and the **most** that will need to be played is **five**.

Example2: Possibility Trees

Suppose a computer installation has four input/output units (*A*, *B*,*C*, and *D*) and three central processing units (*X*, *Y*, and *Z*). Any input/output unit can be **paired** with any central processing unit.

How many ways are there to pair an input/output unit with a central processing unit?

To answer this question, imagine the pairing of the two types of units as a two-step operation:

Step 1: Choose the input/output unit.
Step 2: Choose the central processing unit.

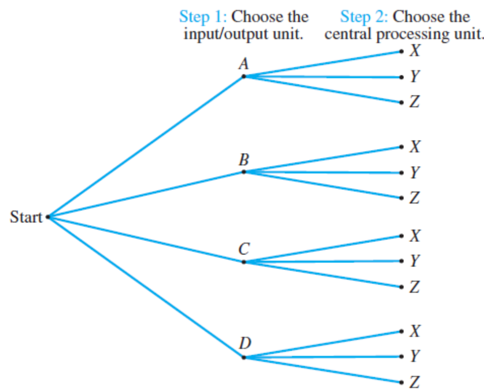
The possible outcomes of this operation are illustrated in the possibility tree

Example 2- cont.

The top most path from "root" to "leaf" indicates that input/output unit *A* is to be paired with central processing unit *X*. The next lower branch indicates that input/output unit *A* is to be paired with central processing unit *Y*. And so forth.

Thus the total number of ways to pair the two types of units is the same as the number of branches of the tree, which is

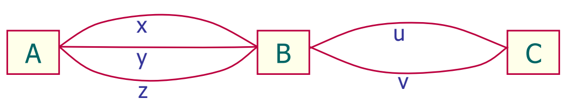
3 + 3 + 3 + 3 = 4·3 = 12.



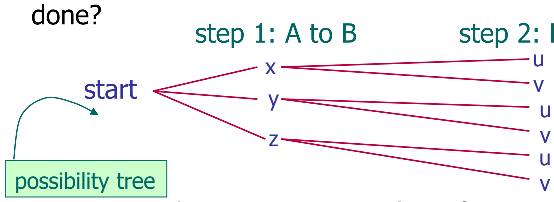
Pairing Objects Using a Possibility Tree

Example3: Possibility Trees

Example: There are 3 roads from A to B and 2 from B to C.



a. How many distinct routes are there from A to C if no backtracking is done?



6 routes:
xu, xv, yu, yv, zu, zv

b. How many distinct routes are there from A to C <u>and back</u> to A if <u>no road is reused</u>?

(xuvy, xuvz, xvuy, xvuz, yuvx, yuvz, yvux, yvuz, zuvx, zuvy, zvux, zvuy)

Answer: 12

The Multiplication Rule

Suppose an operation consists of k steps, and

step 1 can be performed in n_1 ways,
step 2 can be performed in n_2 ways,
step 3 can be performed in n_3 ways,
•
•
•
step k can be performed in n_k ways,

(where the *number* of ways to perform each step does not depend on how any of the previous steps were performed).

Then the number of ways to perform the operation as a whole is

$$n_1 \cdot n_2 \cdot n_3 \cdot \cdot \cdot n_k.$$

The Multiplication Rule: Example1

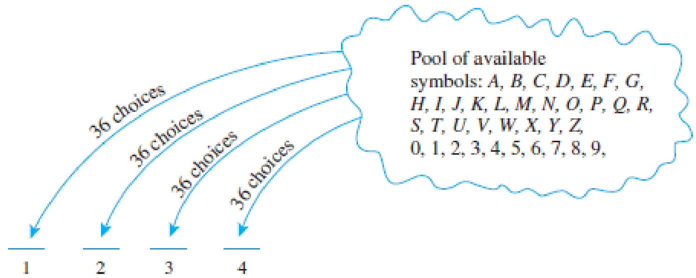
A typical **PIN** (personal identification number) is a sequence of any **four** symbols chosen from the **26** letters in the alphabet **and** the **ten** digits, with repetition allowed.

How many different PINs are possible?

Solution: Typical PINs are CARE, 3387, B32B, and so forth. You can think of forming a PIN as a four-step operation to fill in each of the four symbols in sequence.

Step 1: Choose the first symbol.
Step 2: Choose the second symbol.
Step 3: Choose the third symbol.
Step 4: Choose the fourth symbol.

Cont….



There is a fixed number of ways to perform each step, namely 36, regardless of how preceding steps were performed.
And so, by the multiplication rule, there are

$36 \cdot 36 \cdot 36 \cdot 36 = 36^4 = 1,679,616$

PINs in all.

Number of PINs **without Repetition**

we formed PINs using four symbols, either letters of the alphabet or digits, and supposing that letters could be repeated.
Now suppose that repetition is not allowed.

How many different PINs are there?

Solution: Again think of forming a PIN as a four-step operation: There are

36 ways to choose the first symbol,
35 ways to choose the second (the first symbol cannot be used again),
34 ways to choose the third
33 ways to choose the fourth.

Thus, the multiplication rule can be applied to conclude that there are

$36 \cdot 35 \cdot 34 \cdot 33 = 1,413,720$

different PINs with no repeated symbol.

The Multiplication Rule: more examples

1. A coin is tossed three times. Each time the result *H* for heads or *T* for tails is recorded. How many outcomes (THH, THT,…) can occur?

Solution: Imagine constructing an outcome as a 3-step operation:

Step 1: Choose *H* or *T* for the result of toss 1 ← *2 ways*

Step 2: Choose *H* or *T* for the result of toss 2 ← *2 ways*

Step 3: Choose *H* or *T* for the result of toss 3 ← *2 ways*

Since there are 2 ways to perform each of the three steps, by the multiplication rule there are $2 \cdot 2 \cdot 2 = 8$ ways to perform the entire operation.
So there are 8 possible outcomes.

continued

Definition: A **string** is a finite sequence of symbols written in a row.

Ex: 011 is a "string of 0's and 1's" (or "bit string") of length 3

2. How many bit strings are there of length 3?
Solution: Same as for example 1, but use 1 in place of *H* and 0 in place of *T*. So the answer is again 8.
(This is also the number of rows in a truth table with 3 variables.)

3. There are 16 "hexadecimal digits": 0,1,2,…,9,A,B,C,D,E,F
How many strings of hexadecimal digits are there that have length 3?
(Some typical strings are 4A8, DD3, 307, etc. Note that repetition of digits is allowed.)
Solution: Almost the same as for example 1, but instead of 2 choices for each step there are 16. So the answer is $16 \cdot 16 \cdot 16 = 4096$.

The Multiplication Rule: Programming

Consider the following algorithm segment:

```
for k := 1 to 2
        for j := 1 to 3
                [Statements in the body of the loop. None
                contain branching statements that lead
                outside the loop.]
        next j
next k
```

How many times will the **innermost loop be iterated** when this algorithm segment is implemented and run?

Solution

The number of iterations of the innermost loop equals the number of possible combinations of values for k and j.
This equals the number of columns (with numbers) in the table:

k	1	⟶	⟶	2	⟶	⟶
j	1	2	3	1	2	3

To count the number of columns:

Step 1: Choose a value for k. ← 2 choices

Step 2: Choose a value for j. ← 3 choices

So by the multiplication rule, the answer is $2 \cdot 3 = 6$.

Number of elements in a Cartesian product

Suppose A_1, A_2, A_3, and A_4 are sets with n_1, n_2, n_3, and n_4 elements, respectively.

How many elements in $A_1 \times A_2 \times A_3 \times A_4$

Solution: Each element in $A_1 \times A_2 \times A_3 \times A_4$ is an ordered 4-tuple of the form (a_1,a_2,a_3,a_4)

By the multiplication rule, there are $n_1 n_2 n_3 n_4$ ways to perform the entire operation.
Therefore, there are $n_1 n_2 n_3 n_4$ distinct 4-tuples in $A_1 \times A_2 \times A_3 \times A_4$

Homework

1. Example 9.2.7 A More Subtle Use of the Multiplication Rule

2. We have 4 computers (A,B,C,D) and 3 printers (X,Y,Z). Each of these printers is connected with each of the computers. *Suppose you want to print something through one of the computers, How many possibilities for printing do you have?*

3. How many strings can be made using 2 of the letters "with repetition" from the word "DEPAUL"? How many strings using 1 letter from "DEPAUL"?

i.e., letters are allowed to be used more than once

When the Multiplication Rule Is Difficult or Impossible to Apply

Consider the following problem:

Three officers—a president, a treasurer, and a secretary—are to be chosen from among four people: Ann, Bob, Cyd, and Dan. Suppose that, for various reasons, Ann cannot be president and either Cyd or Dan must be secretary.

How many ways can the officers be chosen?

Try to solve this problem using the multiplication rule. A person might answer as follows:
There are three choices for president (all except Ann), three choices for treasurer (all except the one chosen as president), and two choices for secretary(Cyd or Dan).
Therefore, by the multiplication rule, there are $3 \cdot 3 \cdot 2 = 18$ choices in all.
Unfortunately, this analysis is incorrect. Please look at page 530.

Permutations

التباديل : عدد التشكيلات الممكنة لمجموعة جزئية من العناصر منتقاة من مجموعة كلية من العناصر مع مراعاة لأهمية تسلسل العناصر في تشكيلات المجموعة الجزئية

كم كلمة من خمس حروف ممكن ان نكون اذا كان لدينا عشرة حروف؟

كانت القاعدة التي تمكن من حساب عدد التبديلات لمجموعة ما، معروفة لدى الهنديين على الأقل في حوالي عام 1150م.

Permutations

A **permutation** of a **set** of objects is an ordering of the objects in a row.

Example: the set of elements $\{a, b, c\}$ has six permutations:

abc acb cba bac bca cab

Generally, given a set of *n* objects, how many permutations does the set have? Imagine forming a permutation as an *n*-step operation:

Step 1: Choose an element to write **first**.
Step 2: Choose an element to write **second**
. . .
Step *n*: Choose an element to write ***n*th**.

Permutations

by the multiplication rule, there are

$$n(n-1)(n-2)\cdot\cdot\cdot 2\cdot 1 = n!$$

ways to perform the entire operation.

Theorem 9.2.2

For any integer n with $n \geq 1$, the number of permutations of a set with n elements is $n!$.

Example 1

- How many ways can the letters in the word *COMPUTER* be arranged in a row?

  8! = 40,320

- How many ways can the letters in the word *COMPUTER* be arranged if the letters ***CO*** must remain next to each other (in order) as a unit?

  7! = 5,040

- If letters of the word *COMPUTER* are randomly arranged in a row, what is the **probability** that the letters ***CO*** remain next to each other (in order) as a unit?

$$\frac{5{,}040}{40{,}320} = \frac{1}{8} = 12.5\%.$$

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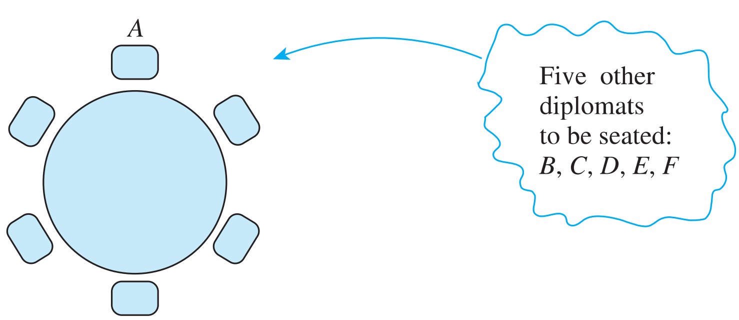
Example 2

كيف يمكن توزيع ستة دبلوماسيين حول طاولة مستديرة

لأنها مستديرة، ثبت واحدة، ويبقى خمسة يمكن تبديلها



5! = 120 ways

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Permutations of Selected Elements

Given the set $\{a, b, c\}$, there are six ways to select two letters from the set and write them in order.

$$ab \quad ac \quad ba \quad bc \quad ca \quad cb$$

Each such ordering of two elements of $\{a, b, c\}$ is called a 2-*permutation* of $\{a, b, c\}$.

أي مجموع التبديلات التي يمكن أن ننتقي بها أفراد المجموعة مع مراعاة الترتيب

• Definition

An ***r*-permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $\boldsymbol{P(n, r)}$.

How many permutations in $P(n, r)$?

Permutations of Selected Elements

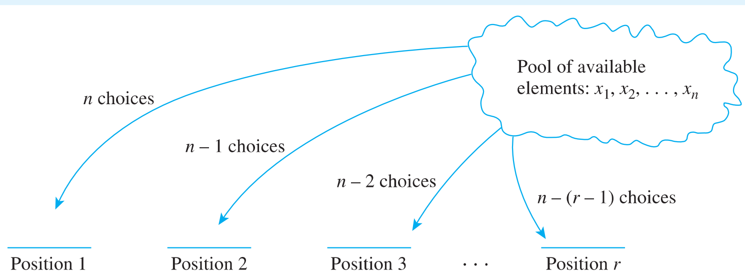
Theorem 9.2.3

If n and r are integers and $1 \leq r \leq n$, then the number of r-permutations of a set of n elements is given by the formula

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) \qquad \text{first version}$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!} \qquad \text{second version.}$$



Example 3

a. Evaluate $P(5, 2)$.

$$P(5,2) = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 20$$

b. How many 4-permutations are there of a set of 7 objects?

$$P(7,4) = \frac{7!}{(7-4)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 \cdot 5 \cdot 4 = 840.$$

c. How many 5-permutations are there of a set of 5 objects?

$$P(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5! = 120.$$

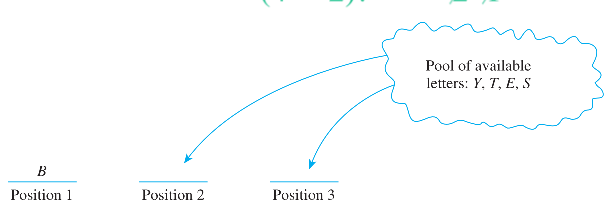
Example 4

How many different ways can 3 of the letters of the word *BYTES* be chosen and written in a row?

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 5 \cdot 4 \cdot 3 = 60.$$

How many different ways can this be done if the first letter must be *B*?

$$P(4,2) = \frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 4 \cdot 3 = 12.$$



Example 5

Prove that for all integers $n \geq 2$,

$$P(n, 2) + P(n, 1) = n^2.$$

$$P(n,2) = \frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!} = n(n-1)$$

and

$$P(n,1) = \frac{n!}{(n-1)!} = \frac{n \cdot (n-1)!}{(n-1)!} = n.$$

Hence

$$P(n,2) + P(n,1) = n \cdot (n-1) + n = n^2 - n + n = n^2,$$

Example: Permutation & Mult.

Mr. Jones has 10 books that he wants to put on his bookshelf:

4 mathematics books
3 chemistry books
2 history books
1 language book

4! ways to organize math
3! ways to organize chemistry
2! ways to organize history
1 way to organize language

He wants to arrange his books of the same subject are together on the shelf.

How many different arrangements are possible?

Also, have **4!** Ways of organizing the categories.

4! . 3! . 2! .1! . **4! = 6912** ways to organize the books**.**

Example: Permutation & Mult.

How many 7-digit numbers can be made from the digits 1, 2, 3, 4, 5, 6, 7 if there is no repetition and the odd digits must appear in an unbroken sequence. (So, 1357246 and 2753146 satisfy this condition, but 7654231 does not.) ?

Solution: This means that the first four digits must be odd, or the second four digits must be odd, or the third four or the fourth four. If the first four digits are odd, then there are 4! = 24 ways to do that and the remaining three digits have 3! = 6 possible entries. Therefore, there are (4!)(3!) = 144 ways to make 7-digit numbers where the first four digits are odd. What about if the second four digits are odd? Then, we have 4 options for the first digit, 3! ways of arranging the next three digits and 3! ways of arranging the final 3 digits, which is 4(3!)(3!) = (4!)(3!) = 144. A similar argument holds if the third three or fourth four digits are odd. Therefore, the total number of 7-digit numbers with the odd digits appearing in an unbroken sequence is given 4(3!)(4!) = 576

Example: Permutation & Mult.

This problem concerns lists made from the symbols A, B, C, D, E, F, G, H, I.

How many length-5 lists can be made if repetition is not allowed and the list must begin with a vowel?

Solution:
If the list must begin with a vowel, there are 3 possible choices (*A*, *E*, or *I*). Since no repetition is allowed, that leaves 8 choices for the second element, 7 choices for the third element, 6 choices for the fourth element, and 5 choices for the last element in the list. Therefore, there are $3 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 5,040$ length-5 lists starting with a vowel.

Example: Permutation & Mult.

This problem concerns lists made from the symbols A, B, C, D, E, F, G, H, I.

How many length-5 lists can be made if repetition is not allowed and the list must contain exactly one A?

Solution:

If the list must contain an *A*, suppose that *A* is the first item in the list. Then, since no repetition is allowed, that leaves 8 choices for the second element, 7 choices for the third element, 6 choices for the fourth element, and 5 choices for the last element in the list. Of course, *A* could also be the second, third, fourth, or fifth element in the list. Therefore, there are 5 (1 · 8 · 7 · 6 · 5) = 8,040 length-5 lists containing exactly one *A*.

Loops examples:

Determine how many times the innermost loop will be iterated when the algorithm segment is implemented and run:

```
for i := 1 to m
    for j := 1 to n
        for k := 1 to p
            [Statements in body of inner loop.
            None contain branching statements that
            lead outside the loop.]
        next k
    next j
next i
```

The outer loop is iterated **m** times. During each iteration of the outer loop there are **n** iterations of the intermediate loop, and during each iteration of the intermediate loop, there are **p** iterations of the innermost loop. Hence, by the multiplication rule, the total number of iterations of the inner loop is **mnp**.

Loops examples:

Determine how many times the innermost loop will be iterated when the algorithm segment is implemented and run:

for i := 5 **to** 50
  for j := 10 **to** 20
    [Statements in body of inner loop.
    None contain branching statements that
    lead outside the loop.]
  next j
next i

The outer loop is iterated 50 - 5 + 1 = 46 times, and during each iteration of the outer loop there are 20 - 10 + 1 = 11 iterations of the inner loop.

Hence, by the multiplication rule, the total number of iterations of the inner loop is 46·11 = 506.

9.3: Counting Elements of Disjoint Sets

Part1: The Addition Rule
Part2: The Difference Rule
Part3:The Inclusion/Exclusion Rule

Counting elements in sets

So far.. counting problems that can be solved using *possibility trees*. **Multiplication rule**

Next.. we look at counting problems that can be solved by counting the number of elements in

- the union of two sets,
- the difference of two sets,
- or the intersection of two sets.

Underlying rule **Addition rule**

The Addition Rule

Notation: If A is a finite set, $N(A)$ = the number of elements in A.

The Addition Rule: Suppose a finite set A is a union of k subsets $A_1, A_2, A_3, \ldots, A_k$, where no two of the sets have any elements in common (**disjoint**). Then

$$N(A) = N(A_1) + N(A_2) + N(A_3) + \cdots + N(A_k).$$

Recall: Multiplication Rule Example

How many strings can be made using 3 of the letters "with repetition" in the word "DEPAUL"?

Solution:

D, E, P, A, U, L

___ 1   ___ 2   ___ 3

Step 1: Choose a letter to put in position 1. ← 6 ways

Step 2: Choose a letter to put in position 2. ← 6 ways

Step 3: Choose a letter to put in position 3. ← 6 ways

So the total number of strings that can be formed using 3 of the letters in "DEPAUL" is 6·6·6 = 216.

The Addition Rule: Example1

A computer access **password** consists of from one to three letters chosen from the **26** in the alphabet with repetitions allowed.

How many different passwords are possible?

Solution:

The set of all passwords can be partitioned into subsets consisting of those of length 1, those of length 2, and those of length 3 as shown in Figure

Cont.…

Set of All Passwords of Length ≤ 3

By the addition rule, the total number of passwords equals the number of passwords of length 1, plus the number of passwords of length 2, plus the number of passwords of length 3. Now the

number of passwords of length 1 $= 26$ — because there are 26 letters in the alphabet

number of passwords of length 2 $= 26^2$ — because forming such a word can be thought of as a two-step process in which there are 26 ways to perform each step

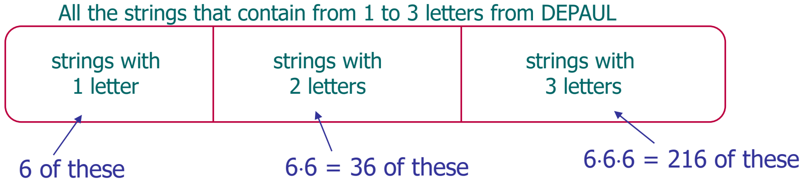
number of passwords of length 3 $= 26^3$ — because forming such a word can be thought of as a three-step process in which there are 26 ways to perform each step.

Hence the total number of passwords $= 26 + 26^2 + 26^3 = 18{,}278$.

The Addition Rule: Example2

How many strings of length from 1 to 3 letters "with repetition" can be created using the letters from the word DEPAUL?

All the strings that contain from 1 to 3 letters from DEPAUL



So the answer is $216 + 36 + 6 = 258$

The Addition Rule: Programming

Consider the following algorithm segment:

```
for k := 1 to 4
        for j := 1 to k
                [Statements in the body of the loop. None
                contain branching statements that lead
                outside the loop.]
        next j
next k
```

How many times will the innermost loop be iterated when this algorithm segment is implemented and run?

Solution

The number of iterations of the innermost loop equals the number of possible combinations of values for i and j.
This equals the number of columns (with numbers) in the table:

k	1	2		3			4			
j	1	1	2	1	2	3	1	2	3	4

1 2 3 4

So the answer is $1 + 2 + 3 + 4 = \frac{4 \cdot 5}{2} = 10.$

Another example

How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

Three-Digit Integers That Are Divisible by 5

3-digits integers That ends in 0	3-digits integers That ends in 5
↑ A_1	↑ A_2

$A_1 \cup A_2 =$ the set of all three-digit integers that are divisible by 5

$A_1 \cap A_2 = \emptyset$

[The number of 3-digit integers that are divisible by 5]

(1-9) (0-9) (0 or 5)
9 digit choices x 10 choices x 1 choice = 90

$= N(A_1) + N(A_2) = 90 + 90 = 180$

Another example

How many ways are there to create passwords of length 5,6 or 7 alpha num. chars. Repetition is allowed?

Solution:

Passwords of length 5: 35^5 = 60,466,176
" of length 6: 35^6 = 2,176,782,336
" of length 7: 35^7 = 78,364,164,099
+
80,601,412,608

Homework: How many ways are there to create passwords of length 5,6 or 7 alpha num. chars. Repetition is not allowed?

The Difference Rule

Number of students without boys =
number of **all** students – number of boys

Theorem 9.3.2 The Difference Rule

If A is a finite set and B is a **subset** of A, then

$$N(A - B) = N(A) - N(B).$$

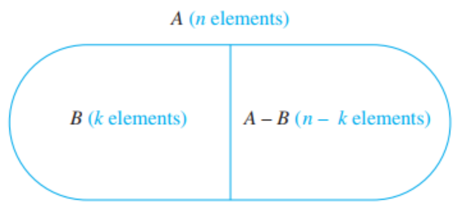


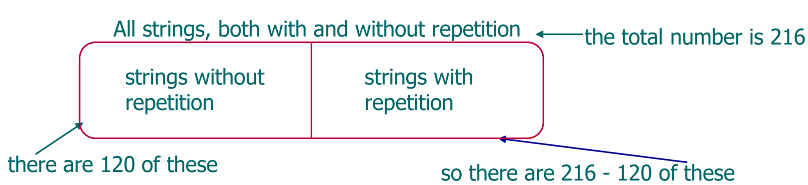
Figure 9.3.3 The Difference Rule

Strings example: With at least one repeated Symbol

How many strings of length 3 that are made from the letters of "DEPAUL" (with repetition allowed) have at least one repeated symbol? (Ex: DPD, EEE, UAA, etc.)

Hint: If you subtract the number of strings with no repeated symbol from the total number of strings, the result is the number of strings with at least one repeated symbol.

Solution: the total number of strings is 6*6*6 = 216 (repetition allowed), and there are 6*5*4 = 120 strings with no repeated symbol.

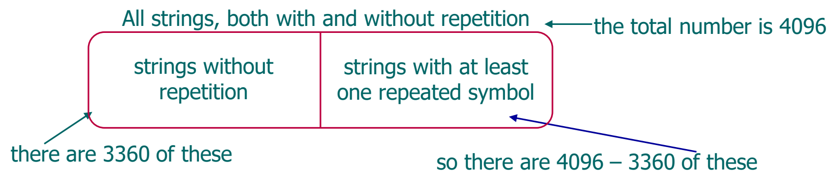


Thus the answer is 216 – 120 = 96

Another example

Example: How many strings of length 3 that consist of hexadecimal digits (with repetition allowed) have at least one repeated symbol? (Ex: D2D, EEE, 00A, etc.)

Solution: By example 3, there are 16.16.16=4096 strings of hexadecimal digits with repetition allowed, and by example 4 there are 16.15.14=3360 such strings if repetition is not allowed.



Thus the answer is 4096 – 3360 = 736

Probability of the Complement

Formula for the Probability of the Complement of an Event

If S is a finite sample space and A is an event in S, then

$$P(A^c) = 1 - P(A).$$

Exercise

The PIN codes are made from exactly four symbols chosen from the 26 letters and the 10 digits, with repetitions allowed.

- **How many PINs contain repeated symbols?**

- 36^4 = 1,679,616 PINs when repetition is allowed

- 36.35.34.33 =1,413,720 PINs when repetition is not allowed

$$1{,}679{,}616 - 1{,}413{,}720 = 265{,}896$$

PINs that contain at least one repeated symbol.

Exercise

The PIN codes are made from exactly four symbols chosen from the 26 letters and the 10 digits, with repetitions allowed.

- **If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?**

One way:

$$\frac{265{,}896}{1{,}679{,}616} \cong 0.158 = \mathbf{15.8\%}.$$

Another way:

$$P(S-A) = \frac{N(S-A)}{N(S)}$$ by definition of probability in the equally likely case

$$= \frac{N(S)-N(A)}{N(S)}$$ **by the difference rule**

$$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)}$$ by the laws of fractions

$$= 1 - P(A)$$ by definition of probability in the equally likely case

$$\cong 1 - 0.842$$ by Example 9.2.4

$$\cong 0.158 = \mathbf{15.8\%}$$

Example 9.3.4 Number of Python Identifiers of Eight or Fewer Characters

- Identifiers must start with one of 53 symbols: either one of the **52** letters of the upper- and lower-case Roman alphabet or an underscore (_)
- May be followed by additional characters chosen from a set of **63** symbols: the 53 symbols allowed as an initial character plus the ten digits
- 31 such reserved keywords, none of which has more than eight characters
- How many Python identifiers are there that are less than or equal to eight characters in length ?

Set of Python Identifiers with Eight or Fewer Characters

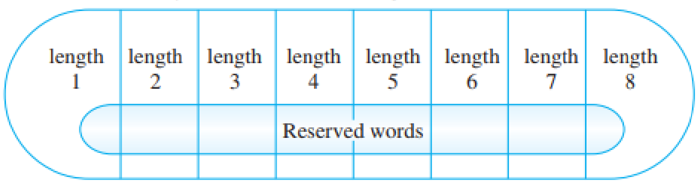


Figure 9.3.4

Example 9.3.4 Number of Python Identifiers of Eight or Fewer Characters (Cont.)

According to the rules for creating Python identifiers, there are

53 potential identifiers of length 1	because there are 53 choices for the first character
$53 \cdot 63$ potential identifiers of length 2	because the first character can be any one of 53 symbols, and the second character can be any one of 63 symbols
$53 \cdot 63^2$ potential identifiers of length 3	because the first character can be any one of 53 symbols, and each of the next two characters can be any one of 63 symbols
⋮	
$53 \cdot 63^7$ potential identifiers of length 8	because the first character can be any one of 53 symbols, and each of the next seven characters can be any one of 63 symbols.

Thus, by the addition rule, the number of potential Python identifiers with eight or fewer characters is

$$53 + 53 \cdot 63 + 53 \cdot 63^2 + 53 \cdot 63^3 + 53 \cdot 63^4 + 53 \cdot 63^5 + 53 \cdot 63^6 + 53 \cdot 63^7$$

$$= 53\left(\frac{63^8 - 1}{63 - 1}\right)$$

$$= 212{,}133{,}167{,}002{,}880.$$

Now 31 of these potential identifiers are reserved, so by the difference rule, the actual number of Python identifiers with eight or fewer characters is

$$212{,}133{,}167{,}002{,}880 - 31 = 212{,}133{,}167{,}002{,}849.$$ ■

Example 9.3.5 Internet Addresses

- Homework!

The Inclusion/Exclusion Rule

So far, we learned to count **union of sets** that they are **disjoint**. What if they **overlap**?

A={ 2,4,6,8,10,12,14} B={3,6,9,12,15}
$A \cap B$ = {6,12}
$A \cup B$ = {2,3,4,6,8,9,10,12,14,15}

Example: Let **A** be the set of numbers from 1 to 15 that are divisible by 2 and **B** be the set of numbers from 1 to 15 that are divisible by 3. What are

N(A)? 7 N(B)? 5 $N(A \cap B)$? 2 $N(A \cup B)$? 10

How are these related?

10 = 7+ 5 – 2

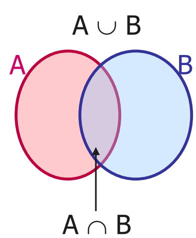
Inclusion/Exclusion Rule

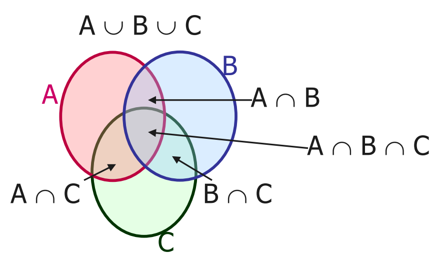
Theorem: If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$$





75

Example: Counting Elements of a General Union

a. How many integers from 1 through 1,000 are multiples of 3 **or** multiples of 5?

b. How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

Solution:

a. Let A = the set of all integers from 1 through 1,000 that are multiples of 3.

Let B = the set of all integers from 1 through 1,000 that are multiples of 5.

$A \cup B$ = the set of all integers from 1 through 1,000 that are multiples of 3 or multiples of 5

$A \cap B$ = the set of all integers from 1 through 1,000 that are multiples of both 3 and 5

Cont.…

1 2 3 4 5 6 … 996 997 998 999

$3 \cdot 1$ ↕ 3, $3 \cdot 2$ ↕ 6, $3 \cdot 332$ ↕ 996, $3 \cdot 333$ ↕ 999

There are 333 multiples of 3 from 1 through 1,000, and so
N(A) = **333**

1 2 3 4 5 6 7 8 9 10 … 995 996 997 998 999 1,000

$5 \cdot 1$ ↕ 5, $5 \cdot 2$ ↕ 10, $5 \cdot 199$ ↕ 995, $5 \cdot 200$ ↕ 1,000

Similarly, each multiple of 5 from 1 through 1,000 has the form 5*k*, for some integer *k* from 1 through **200** and so
N(B) = **200**

Cont.…

Finally, each multiple of 15 from 1 through 1,000 has the form 15*k*, for some integer *k* from 1 through **66** (since 990 = 66· 15).

1 2 … 15 … 30 … 975 … 990 … 999 1,000

$15 \cdot 1$ ↕ 15, $15 \cdot 2$ ↕ 30, $15 \cdot 65$ ↕ 975, $15 \cdot 66$ ↕ 990

It follows by the inclusion/exclusion rule that

$$
\begin{aligned}
N(A \cup B) &= N(A) + N(B) - N(A \cap B) \\
&= 333 + 200 - 66 \\
&= 467.
\end{aligned}
$$

b. There are 1,000 integers from 1 through 1,000, and by part (a), 467 of these are multiples of 3 or multiples of 5.
Thus, by the set difference rule, there are **1,000 − 467 = 533** that are neither multiples of 3 nor multiples of 5.

Example 9.3.7 Counting the Number of Elements in an Intersection

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

- How many students did not take any of the three courses?

$$50-47=3.$$

Exercise2

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

- How many students took all three courses?

$J =$ the set of students who took Java.
$P =$ the set of students who took precalculus
$C =$ the set of students who took calculus

$$N(P \cup C \cup J) =$$

$$N(P) + N(C) + N(J) - N(P \cap C) - N(P \cap J) - N(C \cap J) + N(P \cap C \cap J)$$

$$47 = 30 + 26 + 18 - 9 - 16 - 8 + N(P \cap C \cap J).$$

$$N(P \cap C \cap J) = 6.$$

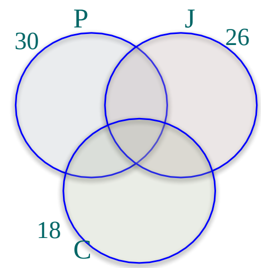
Exercise3

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➢ How many students took precalculus and calculus but not Java?

= (N(P∩C)) – (N(P∩C∩J) =?
9 – 6 = 3



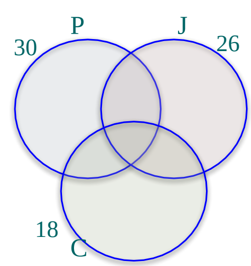
Exercise4

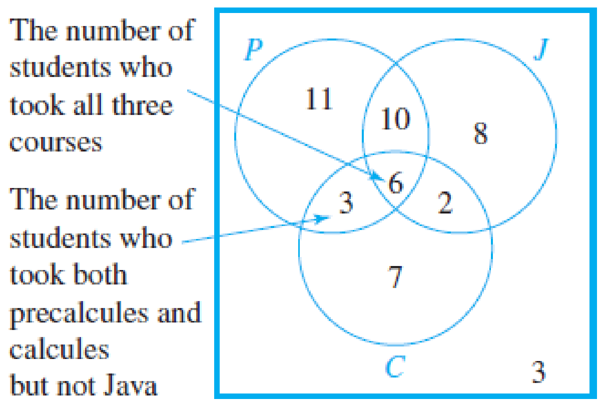
Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➢ How many students took precalculus but neither calculus nor Java?

N(P) – (N(P∩C)) – N(P∩J)) + N(P∩C∩J) =?
30 – 9 – 16 + 6 = 11





Example - code

Let n be a positive integer, and consider the following algorithm segment:

```
for i := 1 to n
    for j := 1 to i
    [Statements in body of inner loop.
    None contain branching statements
    that lead outside the loop.]
    next j
next i
```

On the **i**th iteration of the outer loop, there are **i** iterations of the inner loop, and this is true for each $i = 1, 2, \ldots, n$.
Therefore, the total number of iterations of the inner loop is

$$1 + 2 + 3 + \ldots + n = n(n + 1)/2.$$

9.5 Counting Subsets of a Set: Combinations

- Part 1: **Permutation versus Combinations**
- Part 2: **How to Calculate Combinations**
- Part 3: **Permutations of a Set with Repeated Elements**

Apply these rules to count elements of union and disjoint sets

Counting Subsets of a Set Combinations (توافيق)

Suppose we have12 people,
How many distinct five-person teams can be selected?

Ordering is not important, as the result is a **set**.

- Recall that we cannot use the permutation rule here, because **permutation** produces **ordered sets without repetition**.
- Recall that we cannot use the r-permutation rule here, because **r-permutation** produces **ordered sets with repetition**.

Counting the Number of Integers

There are two distinct methods that can be used to select r objects from a set of n elements.

In an **ordered selection**, it is not only what **elements** are chosen but also the order in which they are chosen that matters. An ordered selection of r elements from a set of n elements is an r-permutation of the set.

In an **unordered selection**, on the other hand, it is only the identity of the chosen elements that matters. An unordered selection of r elements from a set of n elements is the same as a subset of size r or an r-combination of the set.

Permutations التباديل vs. Combinations التوافيق

An **ordered** selection of *r* elements from a set of *n* elements is **an *r-permutation*** *P(n, r)* of the set.

➔ How many 2-permutations we can produce from {a,b,c,d}

$= P(4,2)$

An **unordered** selection of *r* elements from a set of *n* elements is the same as a subset of size *r* or an ***r-combination*** of the set.

➔ How many 2-combanations (/subsets) can produce from {a,b,c,d}

$= \binom{4}{2}$

Combinations: "n choose r": $\binom{n}{r}$

• Definition

Let n and r be nonnegative integers with $r \le n$. An ***r*-combination** of a set of n elements is a subset of r of the n elements. As indicated in Section 5.1, the symbol

$$\binom{n}{r},$$

which is read "n choose r," denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

Theorem about $\binom{n}{r}$

Theorem: Let n and r be nonnegative integers and suppose r ≤ n. Then

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example: How many committees of 5 can be chosen from a club with 12 members?

$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot 7!} = 792$$

Properties of $\binom{n}{r}$

Examples: Use the definition of n choose r to find

a. $\binom{n}{1}$ = the number of subsets of size 1 that can be formed from a set with n elements
= n

b. $\binom{n}{n}$ = the number of subsets of size n that can be formed from a set with n elements
= 1

c. $\binom{n}{0}$ = the number of subsets of size 0 that can be formed from a set with n elements
= 1

More Properties of $\binom{n}{r}$

Example: What is $\binom{n}{n-r}$?

Solution:

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!\cdot(n-n+r)!}$$

$$= \frac{n!}{(n-r)!\cdot r!} = \frac{n!}{r!\cdot(n-r)!} = \binom{n}{r}$$

Example 1

Let S = {Ann, Bob, Cyd, Dan}, each **committee** consisting of three of the four people in S is a **3-combination** of S.

List all such 3-combinations of S.

{Bob, Cyd, Dan} leave out Ann
{Ann, Cyd, Dan} leave out Bob
{Ann, Bob, Dan} leave out Cyd
{Ann, Bob, Cyd} leave out Dan.

What is $\binom{4}{3}$ **?**

$= 4.$

Example 2

How many unordered selections of 2 elements can be made from the set {0, 1, 2, 3}?

{0, 1}, {0, 2}, {0, 3} subsets containing 0
{1, 2}, {1, 3} subsets containing 1 but not already listed
{2, 3} subsets containing 2 but not already listed.

Thus $\binom{4}{2} = 6$

Example: Relation between Permutations and Combinations

Write all 2-permutations of the set {0, 1, 2, 3}. Find an equation relating the number of 2-permutations, $P(4, 2)$, and the number of 2-combinations, $\binom{4}{2}$, and solve this equation for $\binom{4}{2}$.

Relation between P(n,r) and $\binom{n}{r}$

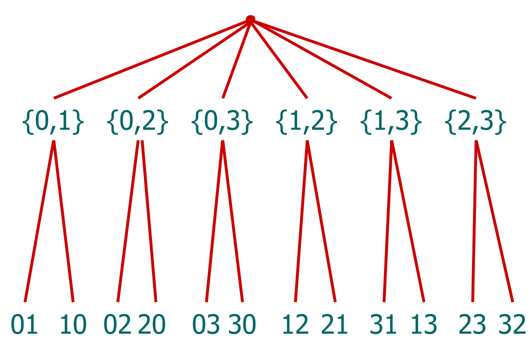
Given the following set: {0, 1, 2, 3}

Step 1: write the 2-combinations of {0,1,2,3}

$\binom{4}{2}$=6 ways

Step 2: order the 2-combinations To obtain the 2-permutations

2! ways



$$P(4, 2) = \binom{4}{2} \cdot 2!.$$

This is an equation that relates $P(4, 2)$ and $\binom{4}{2}$.

Example

Exercise: List all the ways to select two letters from the set $\{a, b, c\}$ and **write them in order**.

In other words: Deduce the value of P(3,2).

Solution: **ab** **ac** **ba** **bc** **ca** **cb**. Each such ordering of two elements of {a, b, c} is called a 2-permutation of {a, b, c}.

Exercise: List all the **sets** consisting of two elements from $\{a, b, c, d\}$.

In other words:

Deduce the value of $\binom{4}{2}$

Solution: $\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}$ So $\binom{4}{2} = 6$.

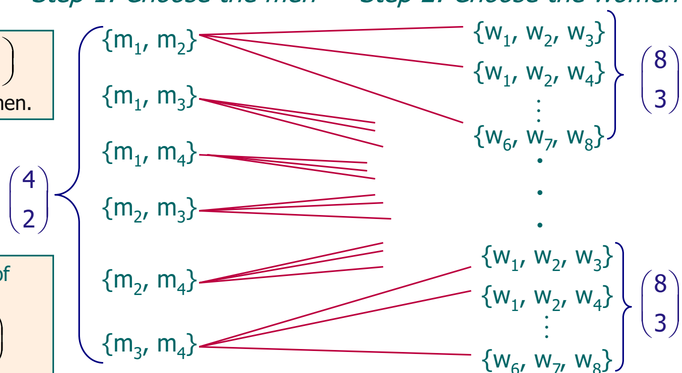
The Multiplication Rule & $\binom{n}{r}$

A club has 12 members. The club contains 4 men and 8 women. How many **committees** of **5** contain exactly 2 men?

Solution: *Step 1: Choose the men* *Step 2: Choose the women*

There are $\binom{4}{2}$ choices of **two** men.

For each choice of two men, there are $\binom{8}{3}$ choices of **three** women.

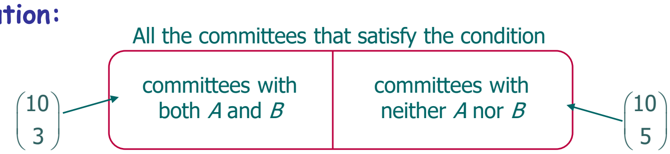


Answer: $\binom{4}{2} \cdot \binom{8}{3} = 6 \cdot 56 = 336$

The Addition Rule & $\binom{n}{r}$

Suppose two members of the group of twelve insist on working as a pair—any team must contain either both or neither.
How many five-person teams can be formed??

Solution:

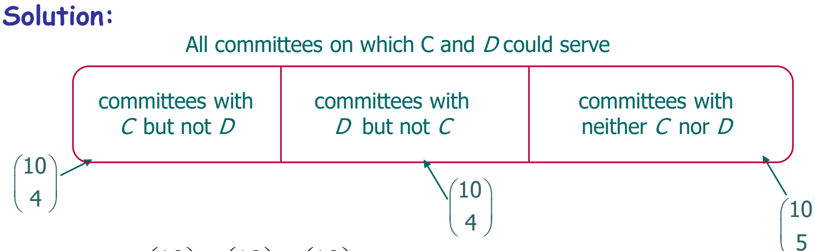


Answer: $\binom{10}{3} + \binom{10}{5} = 120 + 252 = 372$

HW: The Addition Rule & $\binom{n}{r}$ cont.

Two members of the 10 person club , say *C* and *D*, do not want to serve together on the same committee. How many ways can a committee of 5 be chosen that does not include these two together?

Solution:



Answer: $\binom{10}{4} + \binom{10}{4} + \binom{10}{5} = 210 + 210 + 252 = 672$

Do you see another solution?

The Multiplication Rule & $\binom{n}{r}$ cont.

The club contains 4 men and 8 women. How many committees of 5 contain no men?

Hint: If a committee contains no men, then it consists entirely of women.

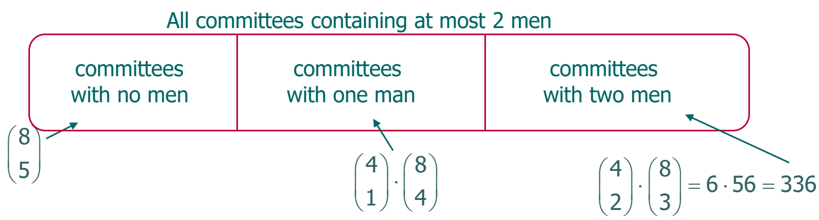
Solution: Because the committee consists entirely of women, the number of committees of 5 with no men is the same as the number of committees of 5 that can be chosen from the 8 women, namely

$$\binom{8}{5} = 56.$$

HW: Multiplication & Addition Rules & $\binom{n}{r}$

The club contains 4 men and 8 women. How many committees of 5 contain ***at most*** 2 men?

Solution:



Answer: $\binom{8}{5} + \binom{4}{1}\cdot\binom{8}{4} + \binom{4}{2}\cdot\binom{8}{3} = 56 + 280 + 336 = 672$

Example: Calculating the Number of Teams

Calculating the Number of Teams

Consider again the problem of choosing five members from a group of twelve to work as a team on a special project. How many distinct five-person teams can be chosen?

Solution The number of distinct five-person teams is the same as the number of subsets of size 5 (or 5-combinations) that can be chosen from the set of twelve. This number is $\binom{12}{5}$. By Theorem 9.5.1,

$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot 7!} = 11 \cdot 9 \cdot 8 = 792.$$

Thus there are 792 distinct five-person teams. ■

Cont…

Teams with Members of Two Types

Suppose the group of twelve consists of five men and seven women.

a. How many five-person teams can be chosen that consist of three men and two women?

b. How many five-person teams contain at least one man?

Solution

a. To answer this question, think of forming a team as a two-step process:

Step 1: Choose the men.

Step 2: Choose the women.

There are $\binom{5}{3}$ ways to choose the three men out of the five and $\binom{7}{2}$ ways to choose the two women out of the seven. Hence, by the product rule,

$$\begin{bmatrix}\text{number of teams of five that} \\ \text{contain three men and two women}\end{bmatrix} = \binom{5}{3}\binom{7}{2} = \frac{5!}{3!2!} \cdot \frac{7!}{2!5!}$$
$$= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 1}$$
$$= 210.$$

The Difference Rule & $\binom{n}{r}$

The club contains 4 men and 8 women. How many committees of 5 contain ***at least*** one man?

Hint: The number of committees with at least one man equals the total number of committees **minus** the number of committees with no men.

Answer: We can find that the total number of committees (12 choose 5) is 792 and we have seen that the number of committees with no men (8 choose 5) is 56. Thus the number of committees with at least one man is

$$792 - 56 = 736.$$

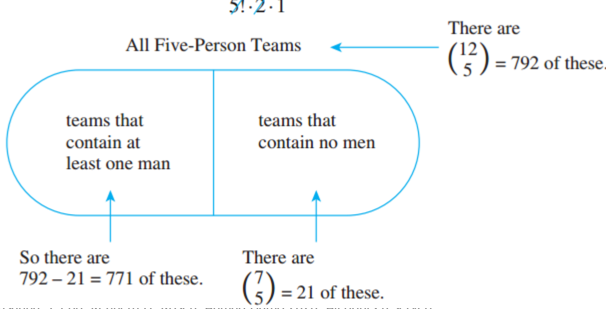
Cont.

b. This question can also be answered either by the addition rule or by the difference rule. The solution by the difference rule is shorter and is shown first.

$$\begin{bmatrix}\text{number of teams}\\ \text{with at least}\\ \text{one man}\end{bmatrix} = \begin{bmatrix}\text{total number}\\ \text{of teams}\\ \text{of five}\end{bmatrix} - \begin{bmatrix}\text{number of teams}\\ \text{of five that do not}\\ \text{contain any men}\end{bmatrix}$$

$$= \binom{12}{5} - \binom{7}{5} = 792 - \frac{7!}{5!\cdot 2!}$$

$$= 792 - \frac{7\cdot \overset{3}{\cancel{6}}\cdot \cancel{5!}}{\cancel{5!}\cdot \cancel{2}\cdot 1} = 792 - 21 = 771.$$



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Cont.

Alternatively, to use the addition rule, observe that the set of teams containing at least one man can be partitioned as:

$$= \binom{5}{1}\binom{7}{4} + \binom{5}{2}\binom{7}{3} + \binom{5}{3}\binom{7}{2} + \binom{5}{4}\binom{7}{1} + \binom{5}{5}\binom{7}{0}$$

$$= \frac{5!}{1!4!}\cdot\frac{7!}{4!3!} + \frac{5!}{2!3!}\cdot\frac{7!}{3!4!} + \frac{5!}{3!2!}\cdot\frac{7!}{2!5!} + \frac{5!}{4!1!}\cdot\frac{7!}{1!6!} + \frac{5!}{5!0!}\cdot\frac{7!}{0!7!}$$

$$= \frac{5\cdot 4!\cdot 7\cdot 6\cdot 5\cdot 4!}{4!\cdot 3\cdot 2\cdot 4!} + \frac{5\cdot 4\cdot 3!\cdot 7\cdot 6\cdot 5\cdot 4!}{3!\cdot 2\cdot 4!\cdot 3\cdot 2} + \frac{5\cdot 4\cdot 3!\cdot 7\cdot 6\cdot 5!}{2\cdot 3!\cdot 5!\cdot 2}$$

$$+ \frac{5\cdot 4!\cdot 7\cdot 6!}{4!\cdot 6!} + \frac{5!\cdot 7!}{5!\cdot 7!}$$

$$= 175 + 350 + 210 + 35 + 1 = 771.$$

Homework: c. How many five-person teams contain at most one man?

Number of Bit Strings with Fixed Number of 1's

How many eight-bit strings have exactly three 1's?

step 1, choose positions for the three 1's

step 2, put the 0's into place.

The number of ways to construct an eight-bit string with exactly three 1's is the same as the number of **subsets** of three positions that can be chosen from the eight into which to place the 1's. By Theorem 9.5.1, this equals:

$$\binom{8}{3} = \frac{8!}{3!\cdot 5!} = \frac{8\cdot 7\cdot 6\cdot 5!}{3\cdot 2\cdot 5!} = 56.$$

Warm-up: Permutations with sets of repeated elements

a. **How many distinguishable ways can the letters of the word MISSISSIPPI be arranged?**

M, I, I, I, I, S, S, S, S, P, P

_ _ _ _ _ _ _ _ _ _ _
1 2 3 4 5 6 7 8 9 10 11

Step 1: Choose 1 position for the M ← $\binom{11}{1}$ ways
(Ex: {9})

Step 2: Choose 4 positions for the I's ← $\binom{10}{4}$ ways
(Ex: {3, 4, 7, 11})

Step 3: Choose 4 positions for the S's ← $\binom{6}{4}$ ways
(Ex: {1, 2, 6, 10})

Step 4: Choose 2 positions for the P's ← $\binom{2}{2}$ ways
(Ex: {5, 8})

So the answer is

$$\binom{11}{1}\binom{10}{4}\binom{6}{4}\binom{2}{2} = \frac{11!}{(1!)(10!)}\frac{10!}{(4!)(6!)}\frac{6!}{(4!)(2!)}\frac{2!}{(2!)(0!)} = \frac{11!}{(1!)(4!)(4!)(2!)}.$$

Another example

b. **How many distinguishable ways can the letters of the word MISSISSIPPI be arranged if PPI stays together and the string begins with M?**

M, PPI, I, I, I, S, S, S, S

M _ _ _ _ _ _ _ _ _
1 2 3 4 5 6 7 8 9

Step 1: Choose 1 ("long") position for PPI ← $\binom{8}{1}$ ways
(Ex: {3})

Step 2: Choose 3 positions for the I's ← $\binom{7}{3}$ ways
(Ex: {1, 5, 7})

Step 3: Choose 4 positions for the S's ← $\binom{4}{4}$ ways
(Ex: {2, 4, 6, 8})

So

$$\binom{8}{1}\binom{7}{3}\binom{4}{4} = \frac{8!}{(1!)(7!)}\frac{7!}{(3!)(4!)}\frac{4!}{(4!)(0!)}$$

$$= \frac{8!}{(1!)(3!)(4!)} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4!}{(3\cdot 2\cdot 1)(4!)} = 280.$$

Permutations with sets of repeated elements

Theorem 9.5.2 Permutations with sets of Indistinguishable Objects

Suppose a collection consists of n objects of which

n_1 are of type 1 and are indistinguishable from each other

n_2 are of type 2 and are indistinguishable from each other

$\vdots$

n_k are of type k and are indistinguishable from each other,

and suppose that $n_1 + n_2 + \cdots + n_k = n$. Then the number of distinguishable permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}.$$

$$\binom{11}{1}\binom{10}{4}\binom{6}{4}\binom{2}{2} = \frac{11!}{(1!)(10!)}\frac{10!}{(4!)(6!)}\frac{6!}{(4!)(2!)}\frac{2!}{(2!)(0!)}$$

9.6: r-Combinations with Repetition Allowed

r-Combinations with Repetition Allowed

Definition: an r-combination with repetition allowed, or multiset of size r, chosen from a set X of n elements is an **unordered selection** of elements taken from X **with repetition allowed**.
If $X = \{x_1, x_2, \ldots, x_n\}$, we write an r-combination with repetition allowed, or multiset of size r, as

$[x_{i1}, x_{i2}, \ldots, x_{ir}]$ where each x_{ij} is in X and some of the x_{ij} may equal each other.

Examples:

- buy 20 drinks of cola, 7up, or fanta. How many ways?
- select a committee of 3 people, from 10 persons, but one person may play one or more roles.
- Given a set on n elements $\{x_1, x_2, \ldots, x_n\}$ {cola, fanta, 7up}

Choose r element multisets $[x_{i1}, x_{i2}, \ldots x_{ik}]$ [cola,cola,7up,7up,...,fanta]
With repetition allowed, and unordered.

Example: r-Combinations with Repetition Allowed

Write a complete list to find the number of 3-combinations with repetition allowed, or multisets of size 3, that can be selected from {1,2,3,4}.

Solution

[1, 1, 1] ; [1, 1, 2]; [1, 1, 3]; [1, 1, 4]	all combinations with 1, 1
[1, 2, 2] ; [1, 2, 3]; [1, 2, 4];	all additional combinations with 1, 2
[1, 3, 3] ; [1, 3, 4]; [1, 4, 4];	all additional combinations with 1, 3 or 1, 4
[2, 2, 2] ; [2, 2, 3]; [2, 2, 4];	all additional combinations with 2, 2
[2, 3, 3] ; [2, 3, 4]; [2, 4, 4];	all additional combinations with 2, 3 or 2, 4
[3, 3, 3] ; [3, 3, 4]; [3, 4, 4];	all additional combinations with 3, 3 or 3, 4
[4, 4, 4]	the only additional combination with 4, 4

Thus there are twenty 3-combinations with repetition allowed. ■

How could the number **twenty** have been predicted other than by making a complete list?

Calculating r-Combinations with Repetition Allowed

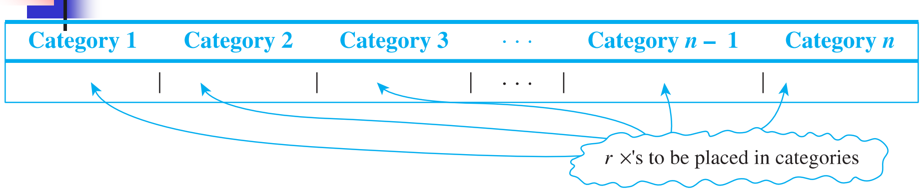
Consider the numbers 1, 2, 3, and 4 as categories and imagine choosing a total of three numbers from the categories with multiple selections from any category allowed.

Category 1	Category 2	Category 3	Category 4	Result of the Selection
|	× |	|	× ×	1 from category 2 2 from category 4
× |	|	× |	×	1 each from categories 1, 3, and 4
× × × |	|	|		3 from category 1

× × | | × | means [1,1,3]

The problem now became like selecting 3 positions out of 6, because once 3 positions have been chosen for the × 's, the |'s are placed in the remaining 3 positions, which is: $\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{\not{6}\cdot 5\cdot 4\cdot \not{3!}}{\not{3}\cdot\not{2}\cdot 1\cdot\not{3!}} = 20,$

Viewing it as a string problem



n – 1 vertical bars (to separate the n categories)
r crosses (to represent the r elements to be chosen).

➔ **r-combanations of x and |**

Given (r+n-1) select r $\binom{r+n-1}{r}$

r-Combinations with Repetition Allowed

Theorem 9.6.1

The number of r-combinations with repetition allowed (multisets of size r) that can be selected from a set of n elements is

$$\binom{r+n-1}{r}.$$

This equals the number of ways r objects can be selected from n categories of objects with repetition allowed.

Example

A person wants to buy 15 cans of drinks. He shops at a store that sells 5 different types of soft drinks.

How many different selections of cans of 15 soft drinks can he make?

We can represent the problem by a string of 5 – 1 = 4 vertical bars (to separate the categories of soft drinks) and 15 crosses (to represent the cans selected). For instance,

$$\times \times \times \mid \times \times \times \times \times \times \times \mid \quad \mid \times \times \times \mid \times \times$$

$$\binom{15+5-1}{15} = \binom{19}{15} = \frac{19 \cdot \overset{6}{\cancel{18}} \cdot 17 \cdot \overset{2}{\cancel{16}} \cdot \cancel{15!}}{\cancel{15!} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 3{,}876.$$

Example cont.

If FANTA is one of the types of soft drink, how many different selections include at least 6 cans of FANTA?

Thus we need to select 9 cans from the 5 types.
The nine additional cans can be represented as 9 × 's and 4 |'s.

$$\binom{9+4}{9} = \binom{13}{9} = \frac{13 \cdot \cancel{12} \cdot 11 \cdot \cancel{10}^{5} \cdot \cancel{9!}}{\cancel{9!} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 715.$$

Counting Triples (*i*, *j*, *k*) with 1 ≤ *i* ≤ *j* ≤ *k* ≤ *n*

If n is a positive integer, how many triples of integers from 1 through n can be formed in which the elements of the triple are written in increasing order but are not necessarily distinct (repetition allowed)?

In other words, how many triples of integers (i, j, k) are there with 1 ≤ i ≤ j ≤ k ≤ n?
e.g., n=5 (3,3,5), (1,2,4),…

Category 1		2		3		4		5	Result of the Selection
	|		|	× ×	|		|	×	(3, 3, 5)
×	|	×	|		|	×	|		(1, 2, 4)

Observation. There are exactly as many triples of integers (i, j, k) with 1 ≤ i ≤ j ≤ k ≤ n as there are 3-combinations of integers from 1 through n with repetition allowed: $\binom{3+n-1}{3} = \frac{n(n+1)(n+2)}{6}.$

Counting Iterations of a Loop

How many times will the innermost loop be iterated when the algorithm segment below is implemented and run?

for $k := 1$ **to** n

  for $j := 1$ **to** k

    for $i := 1$ **to** j

      [*Statements in the body of the inner loop, none containing branching statements that lead outside the loop*]

    next i

  next j

next k

$$\binom{3 + (n-1)}{3} = \frac{n(n+1)(n+2)}{6}$$

Illustration to prev. example

k	1	2 →			3 →						. . .
j	1	1	2 →		1	2 →		3 →			. . .
i	1	1	1	2	1	1	2	1	2	3	. . .

. . .	n →						
. . .	1	2 →		. . .	n →		
. . .	1	1	2	. . .	1	. . .	n

Homework: Counting Iterations of a Loop

```
For   I:=1 to n
   for j:= 1 to i
      for k:= 1 to j
          x = x  + 1
        next k
      next j
      y=y * x
   next i
```

Suppose initially n=5, x=0, y=1

What will be the values of x and y when this algorithm segment is implemented and run?

That is, how many times will the innermost loop be iterated?

Solution

To know how many times the statements in the k^{th} loop will be executed, we might observe that there are exactly as many triples of integers (*k*, *l*, *m*) with $1 \le i \le j \le k \le 5$ as there are 3-combinations of integers from 1 through 5 with repetition allowed.

Hence,

$$v = \binom{3+5-1}{3}$$

$= 35$

Y = 1*2*…*35

Number of Integral Solutions of an Equation

How many solutions are there to the equation **x1+x2+x3+x4=10** if x1, x2 , x3 ,and x4 are nonnegative integers?

Each solution can be represented by a string of three vertical bars (to separate the four categories) and ten crosses (to represent the ten individual units).

Categories x_1 | x_2 | x_3 | x_4	Solution to the equation $x_1 + x_2 + x_3 + x_4 = 10$
× × | × × × × × | | × × ×	$x_1 = 2,\ x_2 = 5,\ x_3 = 0,$ and $x_4 = 3$
× × × × | × × × × × × | |	$x_1 = 4,\ x_2 = 6,\ x_3 = 0,$ and $x_4 = 0$

$$\binom{10+3}{10} = \binom{13}{10} = \frac{13!}{10!(13-10)!} = \frac{13\cdot 12\cdot 11\cdot \cancel{10!}}{\cancel{10!}\cdot 3\cdot 2\cdot 1} = 286.$$

Counting the number of ways of choosing k elements from n

Which Formula to Use?

	Order Matters	Order Doesn't Matter
Repetition Allowed	n^k **Select 4-digites PIN, from 0-9** =10.10.10.10 = 10^4 = 10000	$\binom{k+n-1}{k}$ **Select 10 cans, from 4 types of drinks =?**
Repetition not Allowed	$P(n,k)$ **Select 4-digites PIN, from 0-9** 10!/6!=10.9.8.7 = 5040	$\binom{n}{k}$ **Select 4-position team, from 10 people** =10!/4!.6! = 5040/24 = 210