COMP 233 Discrete Mathematics

The Logic of Compound Statements (Propositional Logic)

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Propositional Logic

Propositional Logic is the logic of compound statements built from simpler statements using *Boolean connectives.*

Applications:

- Design of digital electronic circuits.
- Expressing conditions in programs.
- Queries to databases & search engines.

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Statements

Statement: A declarative sentence that is either true or false but not both.

Example: Which of the following are statements?

- 1 + 1 = 2   Is a statement - true sentence
- 1 + 1 = 5   Is a statement - false sentence
- 1 + x = 5   Not a statement; true for some x and false for others.

  (*We call this kind of sentence a "predicate." More about this later.*)

Definition of a Proposition

A *proposition* (*p*, *q*, *r*, …) is simply a *statement* (*i.e.*, a declarative sentence) *with a definite meaning*, having a *truth value* that's either *true* (T) or *false* (F) (**never** both, neither, or somewhere in between).

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Examples of Propositions

- "It is raining." (Given a situation.)
- "Beijing is the capital of China."
- "1 + 2 = 3"

The following are **NOT** propositions:

- "Who's there?" (interrogative, question)
- "La la la la la." (meaningless interjection)
- "Just do it!" (imperative, command)
- "Yeah, I sorta dunno, whatever..." (vague)
- "1 + 2" (expression with a non-true/false value)

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Logical Connectives

Used to put simple statements together to make compound statements

not	$\sim$	*negation*
and	$\wedge$	*conjunction*
or	$\vee$	*disjunction*
if-then	$\rightarrow$	*conditional*
if-and-only-if	$\leftrightarrow$	*biconditional*

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Truth Values for Compound Statement Forms

Not Statements (Negations): The negation of a statement is a statement that exactly expresses what it would mean for the given statement to be false.

The negation of a statement has opposite truth value from the statement.

p	$\sim p$
T	F
F	T

Truth Table for ∨ ("*inclusive or*")

In Logic (& Math, CS, etc.): The only time an **or** statement is false is when both components are false.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

So an **or** statement is false if, and only if, both components are false.

This is one of De Morgan's laws of logic – more about this later

Write a negation:

Sue left the door unlocked **or** she left a window open.

Negation:

Sue did **not** leave the door unlocked **and** she did **not** leave a window open.

Truth Table for ∧

The only time an **and** statement is true is when both components are true.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

So an **and** statement is false if, and only if, at least one component is false.

This is the other one of De Morgan's laws of logic

Write a negation:

Hal got an A on the midterm **and** Hal got an A on the final.

Negation:

Hal did **not** get an A on the midterm **or** Hal did **not** get an A on the final.

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If-then Statements (Conditionals)

Imagine that I promise you:

Hypothesis

"If you get an A on the final exam, then you will get an A in the course."

Conclusion

Jump forward to 1 week after the end of the quarter. Finding out your final exam grade and your course grade, you exclaim: "She lied."

What would have to be true to lead you to say that I lied?

You would have to have earned an A on the final exam and not received an A for the course. *In all other cases it would not be fair to say that I lied.*

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Truth Table for →

In Logic (& Math, CS, etc.): The only time a statement of the form ***if p then q*** is false is when the hypothesis (p) is true and the conclusion (q) is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Note: When the hypothesis of an if-then statement is false, we say that the if-then statement is "vacuously true" or "true by default." In other words, it is true because it is not false.

Write a negation:

If Jim got the right answer, **then** he solved the problem correctly.

Negation:

Jim got the right answer **and** he did **not** solve the problem correctly.

but

If-and-Only-If Statements (Biconditionals)

A statement of the form ***p* if and only if *q*** is true when both *p* and *q* have the same truth value; it is false when *p* and *q* have opposite truth values.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Write a negation:

This program is correct if, and only if, it produces correct output for all input data.

- $r \leftrightarrow s \equiv (r \rightarrow s) \wedge (s \rightarrow r)$

Ex (Group Exercise): Let p be "It is winter," q be "It is cold," and r be "It is raining."

Write the following statements symbolically.

- It is winter but it is not cold. $p \wedge {\sim}q$
- Neither is it winter nor is it cold. ${\sim}p \wedge {\sim}q$
- It is not winter if it is not cold. ${\sim}q \rightarrow {\sim}p$
- It is not winter but it is raining or cold. ${\sim}p \wedge (r \vee q)$

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Boolean Operations Summary

- We have seen 1 unary operator (4 possible) and 5 binary operators (16 possible).

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
F	F	T	F	F	F	T	T
F	T	T	F	T	T	T	F
T	F	F	F	T	T	F	F
T	T	F	T	T	F	T	T

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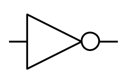
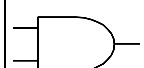
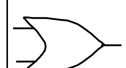
Some Alternative Notations

Name:	not	and	or	xor	implies	iff
Propositional logic:	$\neg$	$\wedge$	$\vee$	$\oplus$	$\rightarrow$	$\leftrightarrow$
Boolean algebra:	$\overline{p}$	pq	$+$	$\oplus$		
C/C++/Java (wordwise):	!	&&	||	!=		==
C/C++/Java (bitwise):	~	&	|	^		
Logic gates:						

Precedence Rules

1. $\sim$
2. $\wedge$ and $\vee$ (need parentheses to avoid ambiguity)
3. $\rightarrow$ and $\leftrightarrow$ (need parentheses to avoid ambiguity)
4. Parentheses may be used to override rules 1-3

Ex: $p \wedge \sim q$ means the same as $p \wedge (\sim q)$

$p \vee q \wedge r$ is ambiguous. Need to add parentheses:

$(p \vee q) \wedge r$ or $p \vee (q \wedge r)$

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Class Exercise (Exclusive Or)

1. Write a logical expression for a statement of the form *p* or *q* but not both.

$$(p \vee q) \wedge \sim(p \wedge q)$$

2. Construct a table showing how the truth values of $(p \vee q) \wedge \sim(p \wedge q)$ depend on the truth values of the components p and q.

References:

An *and* statement is true ⇔ both components are true.

An *or* statement is false ⇔ both components are false.

An *if-then* statement is false ⇔ its hypothesis is true and its conclusion is false.

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Truth Table for Exclusive Or

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

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Compound Statement Forms

Example

- p : Birzeit University is in Ramallah
- q : $1 + 1 = 5$
- r : Ramallah is next to Alquds

Knowing that p is T, q is F and r is T,
what is the truth value of the following compound statement?

$$(p \wedge \sim q) \vee \sim r$$

$$\underbrace{\underbrace{T \qquad T}_{T} \qquad\qquad F}_{T}$$

Answer: True

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Nested Propositional Expressions

- Use parentheses to *group sub-expressions*: "I just saw my old *f*riend, and either he's *g*rown or I've *s*hrunk." = $f \wedge (g \vee s)$
  - $(f \wedge g) \vee s$ would mean something different
  - $f \wedge g \vee s$ would be ambiguous
- By convention, "$\neg$" takes *precedence* over both "$\wedge$" and "$\vee$".
  - $\neg s \wedge f$ means $(\neg s) \wedge f$, **not** $\neg (s \wedge f)$

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Definition of Logical Equivalence

Definition: Two statement forms are **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables.

- Logical equivalence makes it convenient to express statements in more than one way.
- The symbol for logical equivalence is $\equiv$.

Testing Whether Two Statement Forms P and Q Are Logically Equivalent

- Construct the truth table
- Prove by laws

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Propositional Equivalence

Two *syntactically* (*i.e.,* textually) different compound propositions may be *semantically* identical (*i.e.,* have the same meaning).

We call them *equivalent*. Learn:

- Various *equivalence rules* or *laws*.
- How to *prove* equivalences using *symbolic derivations*.

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Proving Equivalence via Truth Tables

Ex. Prove that $p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$.

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
F	F	F	T	T	T	F
F	T	T	T	F	F	T
T	F	T	F	T	F	T
T	T	T	F	F	F	T

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- Does the following statement are equivalent?

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$		$\sim p \wedge \sim q$
T	T	F	F	T	F		F
T	F	F	T	F	T	$\neq$	F
F	T	T	F	F	T	$\neq$	F
F	F	T	T	F	T		T

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$ have different truth values in rows 2 and 3, so they are not logically equivalent

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De Morgan's Laws: Negation of "If p Then q"

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$
$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$
} *De Morgan's Laws*

$$\sim(p \rightarrow q) \equiv p \wedge \sim q$$
← *Negation of $p \rightarrow q$*

Ex (Class Exercise): Write negations for each of the following statements:

1. If Tom is Ann's father, then Leo is her uncle.

   Ans: Tom is Ann's father **and** Leo is not her uncle.

2. $-4 < x \leq 7$ (This means $-4 < x$ **and** $x \leq 7$.)

   Ans: The negation is $-4 \geq x$ **or** $x > 7$.

Trichotomy law (see Appendix A): Given any two real numbers a and b, either $a < b$ or $a = b$ or $a > b$.

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- Why the negation of – ($p \rightarrow q$) = $p \wedge -q$
- Because $p \rightarrow q = -p \vee q$
- prove this

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Ex Division into Cases: Showing that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$p \vee q \to r$	$(p \to r) \wedge (q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

$p \vee q \to r$ and $(p \to r) \wedge (q \to r)$
always have the same truth values,
so they are logically equivalent

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Tautologies and Contradictions

A *tautology* is a compound proposition that is **true** *no matter what* the truth values of its atomic propositions are!

Ex. $p \vee \neg p$ [What is its truth table?]

A *contradiction* is a comp. prop. that is **false** no matter what!

Ex. $p \wedge \neg p$ [Truth table?]

Other comp. props. are *contingencies*.

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Equivalence Laws

- These are similar to the arithmetic identities you may have learned in algebra, but for propositional equivalences instead.
- They provide a pattern or template that can be used to match much more complicated propositions and to find equivalences for them.

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Equivalence Laws

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* **t** *and* **c**:	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

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Class Exercise: Simplifying Statement Forms

- Use equivalence laws to verify the logical equivalence $\sim(\sim p \wedge q) \wedge (p \vee q) \Leftrightarrow p$
- $\sim(\sim p \wedge q) \wedge (p \vee q)$ $\Leftrightarrow (\sim\sim p \vee \sim q) \wedge (p \vee q)$ (by demorgan's laws)
- $\Leftrightarrow (p \vee \sim q) \wedge (p \vee q)$ (by double negation)

$\Leftrightarrow p \vee (\sim q \wedge q)$ (by distributive law)

$\Leftrightarrow p \vee (q \wedge \sim q)$ (by commutative law for ^)

$\Leftrightarrow p \vee \mathbf{c}$ (by negation laws)

$\Leftrightarrow p$ (by identity law)

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Simplify

49. $(p \vee \sim q) \wedge (\sim p \vee \sim q)$

$$
\begin{aligned}
&\equiv (\sim q \vee p) \wedge (\sim q \vee \sim p) && \text{by } \underline{\text{(a)}} \\
&\equiv \sim q \vee (p \wedge \sim p) && \text{by } \underline{\text{(b)}} \\
&\equiv \sim q \vee \mathbf{c} && \text{by } \underline{\text{(c)}} \\
&\equiv \sim q && \text{by } \underline{\text{(d)}}
\end{aligned}
$$

Therefore, $(p \vee \sim q) \wedge (\sim p \vee \sim q) \equiv \sim q$.

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53. $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$

$\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q)$	
$\equiv \sim[\sim p \wedge (q \vee \sim q)] \vee (p \wedge q)$	by the distributive law
$\equiv \sim(\sim p \wedge \mathbf{t}) \vee (p \wedge q)$	by the negation law for $\vee$
$\equiv \sim(\sim p) \vee (p \wedge q)$	by the identity law for $\wedge$
$\equiv p \vee (p \wedge q)$	by the double negative law
$\equiv p$	by the absorption law

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Conditional statement and its variations

- If today is holiday, then Mos'ab will sleep until 10:00 o'clock.
- $P \rightarrow Q$
- $P \rightarrow Q \equiv -p \lor q$
- $-(P \rightarrow Q) \equiv p \wedge -q$

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Inverse, Converse, Contrapositive

Some terminology:

- The *inverse* of $p \to q$ is: $\neg p \to \neg q$
- The *converse* of $p \to q$ is: $q \to p$.
- The *contrapositive* of $p \to q$ is: $\neg q \to \neg p$.
- One of these has the *same meaning* (same truth table) as $p \to q$. Can you figure out which?

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Example

Conditional Statement: If today is Waqfat Arafa, then Tomorrow is Eid.

Inverse: If today is not Waqfat Arafa, then Tomorrow is not Eid.

Converse: If Tomorrow is Eid, then today is Waqfat Arafa.

Contrapositive:

If Tomorrow is not Eid, then today is not Waqfat Arafa.

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Which are Logically Equivalent?

Facts:

$$p \to q \equiv \sim q \to \sim p$$
$$p \to q \not\equiv q \to p$$
$$p \to q \not\equiv \sim p \to \sim q$$

Truth table verification:

p	q	$\sim q$	$\sim p$	$p \to q$	$\sim q \to \sim p$	$q \to p$	$\sim p \to \sim q$
T	T	F	F	T	*T*	*T*	*T*
T	F	T	F	F	*F*	*T*	*T*
F	T	F	T	T	*T*	*F*	*F*
F	F	T	T	T	*T*	*T*	*T*

Definition

A is a sufficient condition for *B*

means if *A* then *B*.

The occurrence of A guarantees the occurrence of B.

A is a necessary condition for *B*

means if ~*A* then ~B.

If A didn't occur, then B didn't occur either.

Or, equivalently, it means if *B* then *A*.

If B occurred then A also had to occur.

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Example 2.2.11 Converting a Sufficient Condition to If-Then Form

Rewrite the following statement in the form "If A then B":

Pia's birth on U.S soil is a sufficient condition
for her to be a U.S. citizen.

Solution If Pia was born on U.S. soil, then she is a U.S. citizen.

Example 2.2.12 Converting a Necessary Condition to If-Then Form

Use the contrapositive to rewrite the following statement in two ways:

George's attaining age 35 is a necessary condition
for his being president of the United States.

Solution *Version 1:* If George has not attained the age of 35, then he cannot be president of the United States.

Version 2: If George can be president of the United States, then he has attained the age of 35.

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Class Exercises

Write each of the following using if-then statements:

1. Wining the semi-final is a necessary condition for the مؤعclub to win the championship.

2. Getting a mark above 90% in high school is sufficient for entering BZU.

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Exercises, continued

Write each of the following using if-then statements:

1. Making it to the final four is a necessary condition for the Blue Demons to win the championship.

   If the B.D. don't make it to the final four, they won't win the championship.

   If the B.D. win the championship, then they made it to the final four.

2. Being appropriately dressed for a job interview is necessary (but not sufficient) for getting the job.

   If a person is not appropriately dressed for a job interview, then the person won't get the job, but it can happen that a person is appropriately dressed and still doesn't get the job.

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3. Getting all A's is sufficient (but not necessary) for graduating with honors.

If a person gets all A's then they will graduate with honors, but it's possible to graduate with honors even if a person doesn't get all A's.

4. Suppose a teacher says: Getting 100% correct on all the exams is both necessary and sufficient for earning an A in the course. What does this mean?

If a person earns an A in the course, then the person got 100% correct on all the exams, and if a person got 100% correct on all the exams, then the person got an A in the course.

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Definition: Only If

r **only if** *s* means **if** ~*s* **then** ~*r*

If s didn't occur, then r didn't occur either.

Or, equivalently, **if** *r* **then** *s*

If r occurred then s also had to occur.

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Interpretation of If and Only If

So: *r* only if *s* means if *r* then *s*

and *r* if *s* means if *s* then *r*

Thus:

> *r* if <u>and</u> only if *s*
>
> means *r* only if *s* <u>and</u> *r* if *s*
>
> which means if *r* then *s* <u>and</u> if *s* then *r*

Fact:

$$\boxed{r \leftrightarrow s \equiv (r \rightarrow s) \wedge (s \rightarrow r)}$$

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Arguments and Argument Forms

Argument: Sequence of statements. The final statement in the sequence is the **conclusion**; the preceding statements are **premises**.

Argument Form: Obtained by replacing component statements in the argument by variables.

Ex (modus tollens): If p then q. $\left.\begin{matrix}\\ \\ \end{matrix}\right\}$ *premises*

Not q.

Therefore, not p. $\leftarrow$ *conclusion*

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Arguments in Logic

premises

1. How do you know today isn't a holiday?

   If today is holiday, then the university doors should be closed.

   University doors are open.

   Therefore, today is not holiday. ← *conclusion*

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Valid and Invalid Arguments

Valid form of argument: Every argument of that form that has true premises, its conclusion is true.

(A more formal version of the definition is in the book)

Claim: Modus tollens is a valid form of argument.

Proof:

p	q	$p \to q$	$\sim q$	$\sim p$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

premises: $p \to q$, $\sim q$ — *conclusion*: $\sim p$

In the only case (represented by row 4) where all the premises are true, the conclusion is also true. So this form of argument is valid.

Testing an Argument for Validity

- Identify the premises and conclusion of the argument form.
- Construct a truth table of the argument form.
- Identify the critical row (s).
- **critical row:** A row of the truth table in which all the premises are true.
- If the conclusion in *every* critical row is true, then the argument form is **valid**. Otherwise it is **invalid.**

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Invalid form of argument: There is at least one argument of that form that has true premises and a false conclusion.

Ex: Determine whether the following argument form is valid or invalid:

$$\begin{array}{l} \sim p \vee q \\ p \rightarrow r \\ q \rightarrow p \\ \therefore\ r \end{array}$$

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Valid or Invalid? Class Exercise

p	q	r	$\sim p$	$\sim p \vee q$	$p \to r$	$q \to p$	r
				premises			conclusion
T	T	T	F	T	T	T	T
T	T	F	F	T	F	T	F
T	F	T	F	F	T	T	T
T	F	F	F	F	F	T	F
F	T	T	T	T	T	F	T
F	T	F	T	T	T	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	F

$$\sim p \vee q$$
$$p \to r$$
$$q \to p$$
$$\therefore r$$

The 8th row of this truth table shows that it is possible for an argument of this form to have true premises and a false conclusion. So this form of argument is invalid.

Modus tollens :method of denying

Common form:

If p then q.
~ q.
Therefore, ~ p.

modus tollens

p	q	$p \to q$	$\sim q$	$\sim p$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

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Modus tollens

Modus tollens is a valid form of argument.

Proof:

1. What is the form of the following argument?

If $\overbrace{\text{Ahmad is a tall person}}^{p}$, then $\overbrace{\text{he can touch the celling}}^{q}$.

$\overbrace{\text{Ahmad can not touch the celling}}^{\sim q}$.

Therefore, $\overbrace{\text{Ahmad is not a tall person}}^{\sim p}$.

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More Valid and Invalid Forms of Argument

Modus ponens (valid): method of affirming

If Ted is a CS student, then Ted has to take COMP233.

Ted is a CS student.

Therefore, Ted has to take COMP233.

Form: If p then q

p

$\therefore\ q$

Is it possible for an argument of this form to have true premises and a false conclusion? No

Therefore, this form of argument is valid.

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More Examples

Converse Error (Invalid – Avoid!):

If today is Thanksgiving, then it is Thursday.

It is Thursday.

Therefore, today is Thanksgiving.

Form: If p then q

q

$\therefore\ p$

Is it possible for the premises of an argument of this form to be true and its conclusion false? Yes

Therefore, this form of argument is invalid.

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Inverse Error (Invalid – Avoid!):

If Ted is a math major, then Ted has to take MAT 152.

Ted is not a math major.

Therefore, Ted does not have to take MAT 152.

Form: If p then q

~p

Therefore, ~q.

Is it possible for the premises of an argument of this form to be true and its conclusion false? Yes

Therefore, this form of argument is invalid.

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Note

Crucial fact about valid argument is that the truth of its conclusion follows necessarily from its premises. It is impossible to have a valid argument with true premises and false conclusion. When an argument is valid and its premises are true, the truth of the conclusion is said to be **inferred**

from the truth of the premises

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Rules of Inference

All valid arguments can be used as rules for inference.

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \lor q$ $\sim q$ $\therefore p$ **b.** $p \lor q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \lor q$ **b.** q $\therefore p \lor q$	**Proof by Division into Cases**	$p \lor q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \land q$ $\therefore p$ **b.** $p \land q$ $\therefore q$		
Conjunction	p q $\therefore p \land q$	**Contradiction Rule**	$\sim p \to c$ $\therefore p$

Inference Example

Formalize the following text in propositional logic and use the inference rules find the glasses.

If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table. $RK \rightarrow GK$

If my glasses are on the kitchen table, then I saw them at breakfast. $GK \rightarrow SB$

I did not see my glasses at breakfast. $\sim SB$

I was reading the newspaper in the living room or I was reading the newspaper in the kitchen. $RL \vee RK$

If I was reading the newspaper in the living room then my glasses are on the coffee table. $RL \rightarrow GC$

Where are the glasses?

Inferencing Example

Let

RK = I was reading the newspaper in the kitchen.
GK = My glasses are on the kitchen table.
SB = I saw my glasses at breakfast.
RL = I was reading the newspaper in the living room.
GC = My glasses are on the coffee table.

$RK \rightarrow GK$
$GK \rightarrow SB$
$\sim SB$
$RL \vee RK$
$RL \rightarrow GC$

$RK \rightarrow GK$
$GK \rightarrow SB$
$\therefore RK \rightarrow SB$ by transitivity

$RL \vee RK$
$\sim RK$
$\therefore RL$ by elimination

$RK \rightarrow SB$
$\sim SB$
$\therefore \sim RK$ by modus tollens

$RL \rightarrow GC$
RL
$\therefore GC$ by modus ponenes

Thus the glasses are on the coffee table.

Given the following information about a computer program, find the mistake in the program.

a. There is an undeclared variable or there is a syntax error in the first five lines.

b. If there is a syntax error in the first five lines, then there is a missing semicolon or a variable name is misspelled.

c. There is not a missing semicolon.

d. There is not a misspelled variable name.

36. The program contains an undeclared variable.

One explanation:

1. There is not a missing semicolon and there is not a misspelled variable name. *(by (c) and (d) and definition of $\wedge$)*
2. It is not the case that there is a missing semicolon or a misspelled variable name. *(by (1) and De Morgan's laws)*
3. There is not a syntax error in the first five lines. *(by (b) and (2) and modus tollens)*
4. There is an undeclared variable. *(by (a) and (3) and elimination)*