COMP 233 Discrete Mathematics

Chapter 5

Sequences and Mathematical Induction

Outline

- Sequences:
  - Explicit Formulas;
  - Summation Notation;
  - Sequences in Computer Programming;
- Proof by Mathematical Induction (I and II)
  - Proving sum of integers and geometric sequences
  - Proving a Divisibility Property and Inequality
  - Proving a Property of a Sequence

Sequences

Idea: Think of a sequence as a set of elements written in a row:

$a_1, a_2, a_3, \ldots, a_n$ *finite sequence*

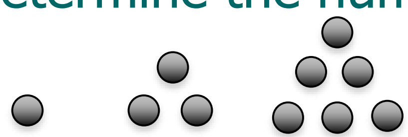
$a_1, a_2, a_3, \ldots, a_n, \ldots$ *infinite sequence*

Each individual element a_k is called a term.

The k in a_k is called a subscript or index

Observe patterns

Determine the number of points in the 4th and 5th figure



? ?

1 2 3 4 5

Determine the next 2 terms of the sequence

4, 8, 16, 32, 64

Induce the formula that could be used to determine any term in the sequence

Finding Terms of Sequences Given by Explicit Formulas

Define sequences $a_1, a_2, a_3, \ldots$ and $b_2, b_3, b_4, \ldots$ by the following explicit formulas:

$$a_k = \frac{k}{k+1} \quad \text{for some integers } k \geq 1$$

$$b_i = \frac{i-1}{i} \quad \text{for some integers } i \geq 2$$

Compute the first five terms of both sequences.

Compute the first six terms of the sequence $c_0, c_1, c_2, \ldots$ defined as follows: $c_j = (-1)^j$ for all integers $j \geq 0$.

Finding Explicit Formula to Fit Given Initial Terms

Find an explicit formula for a sequence that has the following initial terms:

$$1, \quad -\frac{1}{4}, \quad \frac{1}{9}, \quad -\frac{1}{16}, \quad \frac{1}{25}, \quad -\frac{1}{36}, \ldots\ldots$$

$a_k = \frac{-1^{k+1}}{k^2}$ for all integers $k \geq 1$.

OR

$a_k = \frac{-1^{k}}{(k+1)^2}$ for all integers $k \geq 0$.

Exercises

Example: Find an explicit formula for a sequence that has the following initial terms:

$$\frac{1}{3}, \ -\frac{2}{4}, \ \frac{3}{5}, \ -\frac{4}{6}, \ \frac{5}{7}, \ -\frac{6}{8}, \ldots$$

Solutions: The sequence satisfies the formulas

for all integers $n \geq 0$, $\quad a_n = (-1)^n \dfrac{n+1}{n+3}$

for all integers $n \geq 1$, $\quad a_n = (-1)^{n-1} \dfrac{n}{n+2}$

Summation Notation

Suppose $a_1, a_2, a_3, \ldots, a_n$ are real numbers. The "summation from i equals 1 to n of a-sub-i" is

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n.$$

• Definition

If m and n are integers and $m \le n$, the symbol $\sum_{k=m}^{n} a_k$, read the **summation from k equals m to n of a-sub-k**, is the sum of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We say that $a_m + a_{m+1} + a_{m+2} + \ldots + a_n$ is the **expanded form** of the sum, and we write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

We call k the **index** of the summation, m the **lower limit** of the summation, and n the **upper limit** of the summation.

Exercises

Ex: Use summation notation to write the following sum:

$$\frac{1}{3}-\frac{2}{4}+\frac{3}{5}-\frac{4}{6}+\frac{5}{7}-\frac{6}{8}.$$

Solution: By the example on the previous slide, we can write:

$$\frac{1}{3}-\frac{2}{4}+\frac{3}{5}-\frac{4}{6}+\frac{5}{7}-\frac{6}{8}=\sum_{n=0}^{5}(-1)^{n}\left(\frac{n+1}{n+3}\right).$$

or:

$$\frac{1}{3}-\frac{2}{4}+\frac{3}{5}-\frac{4}{6}+\frac{5}{7}-\frac{6}{8}=\sum_{n=1}^{6}(-1)^{n+1}\left(\frac{n}{n+2}\right).$$

Exercises

Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, and $a_5 = 2$.
Compute the following:

a. $\displaystyle\sum_{k=1}^{5} a_k$ b. $\displaystyle\sum_{k=2}^{2} a_k$ c. $\displaystyle\sum_{k=1}^{2} a_{2.k}$

Example 5.1.4 Computing Summations

Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, and $a_5 = 2$. Compute the following:

a. $\displaystyle\sum_{k=1}^{5} a_k$ b. $\displaystyle\sum_{k=2}^{2} a_k$ c. $\displaystyle\sum_{k=1}^{2} a_{2k}$

Solution

a. $\displaystyle\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = (-2) + (-1) + 0 + 1 + 2 = 0$

b. $\displaystyle\sum_{k=2}^{2} a_k = a_2 = -1$

c. $\displaystyle\sum_{k=1}^{2} a_{2k} = a_{2\cdot 1} + a_{2\cdot 2} = a_2 + a_4 = -1 + 1 = 0$

Summation to Expanded Form

Write the following summation in expanded form:

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$$

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1}$$

$$= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

Expanded Form to Summation

Express the following using summation notation:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \ldots + \frac{n+1}{2n}$$

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{k=0}^{n} \frac{k+1}{n+k}.$$

Separating Off a Final Term and Adding On a Final Term n

Rewrite $\sum_{i=1}^{n+1} \frac{1}{i^2}$ by separating off the final term.

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$$

Write $\sum_{k=0}^{n} 2^k + 2^{n+1}$ as a single summation.

$$\sum_{k=0}^{n} 2^k + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$$

Telescoping Sum

$$\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

$$= \left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cdots+\left(\cancel{\frac{1}{n-1}}-\cancel{\frac{1}{n}}\right)+\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$$

$$= 1-\frac{1}{n+1}. \quad \blacksquare$$

Product Notation

• Definition

If m and n are integers and $m \le n$, the symbol $\prod_{k=m}^{n} a_k$, read the **product from k equals m to n of a-sub-k**, is the product of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

A recursive definition for the product notation is the following: If m is any integer, then

$$\prod_{k=m}^{m} a_k = a_m \quad \text{and} \quad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n \quad \text{for all integers } n > m.$$

Computing Products

- Compute the following products:

a. $\prod_{k=1}^{5} k$

- = 1*2*3*4*5=120

b. $\prod_{k=1}^{1} \frac{k}{k+1}$

- = 1/2

Properties of Summations

Theorem 5.1.1

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$
2. $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ generalized distributive law
3. $\displaystyle\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k).$

Let $a_k = k+1$ and $b_k = k-1$ for all integers k. Write each of the following expressions as a single summation or product:

a. $\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k$ b. $\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right)$

Solution

a.
$$\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(k+1) + 2 \cdot \sum_{k=m}^{n}(k-1)$$ by substitution

$$= \sum_{k=m}^{n}(k+1) + \sum_{k=m}^{n} 2 \cdot (k-1)$$ by Theorem 5.1.1 (2)

$$= \sum_{k=m}^{n}((k+1) + 2 \cdot (k-1))$$ by Theorem 5.1.1 (1)

$$= \sum_{k=m}^{n}(3k-1)$$ by algebraic simplification

b.
$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \left(\prod_{k=m}^{n}(k+1)\right) \cdot \left(\prod_{k=m}^{n}(k-1)\right)$$ by substitution

$$= \prod_{k=m}^{n}(k+1) \cdot (k-1)$$ by Theorem 5.1.1 (3)

$$= \prod_{k=m}^{n}(k^2-1)$$ by algebraic simplification ■

Change of Variable

Example: Transform $\sum_{k=1}^{n} k^n$ by making the change of variable $j = k - 1$.

When $k = 1$, then $j = 1 - 1 = 0$

When $k = n$, then $j = n - 1$

$j = k - 1 \Rightarrow k = j + 1$ Thus $k^n = (j + 1)^n$

So: $\sum_{k=1}^{n} k^n = \sum_{j=0}^{n-1} (j+1)^n$

Exercises

Transform the following summation by making the specified change of variable.

$$\sum_{k=0}^{6} \frac{1}{k+1}$$ Change variable $j = k+1$

For (k=0; k<=6; k++)
Sum = Sum + 1/(k+1)

$$\sum_{j=1}^{7} \frac{1}{j} = \sum_{k=1}^{7} \frac{1}{k}.$$

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{k=1}^{7} \frac{1}{k}$$

For (k=1; k<=7; k++)
Sum = Sum + 1/(k)

Exercises

Transform the following summation by making the specified change of variable.

$$\sum_{k=1}^{n+1} \frac{k}{n+k}$$

Change of variable: j = k - 1

For (k=1; k<=n+1; k++)
Sum = Sum + k/(n+k)

$$\sum_{j=0}^{n} \frac{j+1}{n+(j+1)} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}$$

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}$$

For (k=0; k<=n; k++)
Sum = Sum + (k+1)/(n+k+1)

Sequences in Computer Programming

- What is the difference

1. **for** $i := 1$ **to** n
   print $a[i]$
   next i

2. **for** $j := 0$ **to** $n - 1$
   print $a[j + 1]$
   next j

3. **for** $k := 2$ **to** $n + 1$
   print $a[k - 1]$
   next k

- Computing the sum

$s := a[1]$
for $k := 2$ **to** n
  $s := s + a[k]$
next k

$s := 0$
for $k := 1$ **to** n
  $s := s + a[k]$
next k

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}$$

```
Sum=0
For (k=0; k<=n; k++)
Sum = Sum + (k+1)/(n+k+1)
```

Factorial !

• Definition

For each positive integer n, the quantity ***n* factorial** denoted ***n*!,** is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1.$$

Zero factorial, denoted 0!, is defined to be 1:

$$0! = 1.$$

Example 5.1.16 Computing with Factorials

Simplify the following expressions:

a. $\dfrac{8!}{7!}$ b. $\dfrac{5!}{2!\cdot 3!}$ c. $\dfrac{1}{2!\cdot 4!}+\dfrac{1}{3!\cdot 3!}$ d. $\dfrac{(n+1)!}{n!}$ e. $\dfrac{n!}{(n-3)!}$

Solution

a. $\dfrac{8!}{7!}=\dfrac{8\cdot \cancel{7!}}{\cancel{7!}}=8$

b. $\dfrac{5!}{2!\cdot 3!}=\dfrac{5\cdot 4\cdot \cancel{3!}}{2!\cdot \cancel{3!}}=\dfrac{5\cdot 4}{2\cdot 1}=10$

c. $\dfrac{1}{2!\cdot 4!}+\dfrac{1}{3!\cdot 3!}=\dfrac{1}{2!\cdot 4!}\cdot\dfrac{3}{3}+\dfrac{1}{3!\cdot 3!}\cdot\dfrac{4}{4}$ by multiplying each numerator and denominator by just what is necessary to obtain a common denominator

$=\dfrac{3}{3\cdot 2!\cdot 4!}+\dfrac{4}{3!\cdot 4\cdot 3!}$ by rearranging factors

$=\dfrac{3}{3!\cdot 4!}+\dfrac{4}{3!\cdot 4!}$ because $3\cdot 2! = 3!$ and $4\cdot 3! = 4!$

$=\dfrac{7}{3!\cdot 4!}$ by the rule for adding fractions with a common denominator

$=\dfrac{7}{144}$

d. $\dfrac{(n+1)!}{n!}=\dfrac{(n+1)\cdot \cancel{n!}}{\cancel{n!}}=n+1$

e. $\dfrac{n!}{(n-3)!}=\dfrac{n\cdot(n-1)\cdot(n-2)\cdot\cancel{(n-3)!}}{\cancel{(n-3)!}}=n\cdot(n-1)\cdot(n-2)$

$=n^3-3n^2+2n$

n choose r

• Definition

Let n and r be integers with $0 \le r \le n$. The symbol

$$\binom{n}{r}$$

is read "***n* choose *r***" and represents the number of subsets of size r that can be chosen from a set with n elements.

Observe that the definition implies that $\binom{n}{r}$ will always be an integer because it is a number of subsets. In Section 9.5 we will explore many uses of n choose r for solving problems involving counting, and we will prove the following computational formula:

• Formula for Computing $\binom{n}{r}$

For all integers n and r with $0 \le r \le n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Example 5.1.17 Computing $\binom{n}{r}$ by Hand

Use the formula for computing $\binom{n}{r}$ to evaluate the following expressions:

a. $\binom{8}{5}$ b. $\binom{4}{0}$ c. $\binom{n+1}{n}$

Solution

a.
$$\binom{8}{5} = \frac{8!}{5!(8-5)!}$$
$$= \frac{8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3 \cdot 2 \cdot 1}}{(\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1) \cdot (\cancel{\cdot 3 \cdot 2 \cdot 1})}$$
$$= 56.$$

always cancel common factors before multiplying

b.
$$\binom{4}{4} = \frac{4!}{4!(4-4)!} = \frac{4!}{4!0!} = \frac{\cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{(4 \cdot 3 \cdot 2 \cdot 1)}(1)} = 1$$

The fact that $0! = 1$ makes this formula computable. It gives the correct value because a set of size 4 has exactly one subset of size 4, namely itself.

c.
$$\binom{n+1}{n} = \frac{(n+1)!}{n!((n+1)-n)!} = \frac{(n+1)!}{n!1!} = \frac{(n+1) \cdot \cancel{n!}}{\cancel{n!}} = n+1$$

5.2

Mathematical Induction

Mathematical Induction: A Way to Prove Such Formulas and Other Things

- Given an integer variable n, we can consider a variety of properties P(n) that might be true or false for various values of n. For instance, we could consider

P(n): $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$

P(n): $4^n - 1$ is divisible by 3

P(n): n cents can be obtained using 3¢ and 5¢ coins.

- A **proof by mathematical induction:**

shows that a given property P(n) is true for all integers greater than or equal to some initial integer.

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

$$\text{for all integers } n \geq a,\ P(n)$$

is true.

Outline of Proof by Mathematical Induction

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This supposition is called the **inductive hypothesis.***]*

Then

show that $P(k+1)$ is true.

Mathematical Induction: Example

Example: Prove that for all integers $n \geq 1$,

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2.$$

Proof: Let the property P(n) be the equation

$$\boxed{1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2}$$

← *The property P(n)*

Show that the property is true for n = 1:

Basis Step

When $n = 1$, the property is the equation $1 = 1^2$.
But the left-hand side (LHS) of this equation is 1,
and the right-hand side (RHS) is 1^2, which equals 1 also.
So the property is true for $n = 1$.

Inductive Step for the proof that for all integers $n \geq 1$, $1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2$.

- ***Show that $\forall$ integers $k \geq 1$, if $p(k)$ is true then it is true for $p(k+1)$:***
- Let k be any integer with $k \geq 1$, and suppose that the property is true for $n = k$. In other words, **suppose** that
- $1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$.

Inductive Hypothesis

- We must show that the property is true for $n = k + 1$.
- $P(k+1) = (k + 1)^2$,
- or, equivalently, we must **show** that
- $1 + 3 + 5 + 7 + \cdots + (2(k + 1) - 1) = (k + 1)^2$.

Inductive hypothesis: $1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2$.

Show: $1 + 3 + 5 + 7 + \cdots + (2k + 1) = (k + 1)^2$.

But the LHS of the equation to be shown is

$1 + 3 + 5 + 7 + \cdots + (2(k + 1)\text{-}1)$

$= 1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2(k + 1)\text{-}1)$

by making the next-to-last-term explicit

$= k^2 + (2k + 1)$ *by substitution from the inductive hypothesis*

$= (k + 1)^2$ *by algebra,*

which equals the RHS of the equation to be shown.

So, the property is true for $n = k+1$.

Therefore the property P(n) is true.

Proving sum of integers and geometric sequence

Formula for the sum of the first *n* integers: For all integers $n \geq 1$,

$$\boxed{1+2+3+\cdots+n=\frac{n(n+1)}{2}.}$$

Formula for the sum of the terms of a geometric sequence: For all real numbers $r \neq 1$ and all integers $n \geq 0$,

$$\boxed{1+r+r^{2}+r^{3}+\cdots+r^{n}=\frac{r^{n+1}-1}{r-1}.}$$

Example

Theorem 5.2.2 Sum of the First n Integers

For all integers $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof (by mathematical induction):

Let the property $P(n)$ be the equation

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \qquad \leftarrow P(n)$$

Show that P(1) is true:

To establish $P(1)$, we must show that

$$1 = \frac{1(1+1)}{2} \qquad \leftarrow \quad P(1)$$

But the left-hand side of this equation is 1 and the right-hand side is

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

also. Hence $P(1)$ is true.

Show that for all integers k ≥ 1, if P(k) is true then P(k + 1) is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer $k \geq 1$. *That is:]* Suppose that k is any integer with $k \geq 1$ such that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \qquad \leftarrow P(k)$$

inductive hypothesis

[We must show that $P(k+1)$ is true. That is:] We must show that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)[(k+1)+1]}{2},$$

or, equivalently, that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}. \qquad \leftarrow P(k+1)$$

Exercises

The left-hand side of $P(k+1)$ is

$$
\begin{aligned}
&1+2+3+\cdots+(k+1) \\
&= 1+2+3+\cdots+k+(k+1) && \text{by making the next-to-last term explicit} \\
&= \frac{k(k+1)}{2}+(k+1) && \text{by substitution from the inductive hypothesis} \\
&= \frac{k(k+1)}{2}+\frac{2(k+1)}{2} \\
&= \frac{k^2+k}{2}+\frac{2k+2}{2} \\
&= \frac{k^2+3k+1}{2} && \text{by algebra.}
\end{aligned}
$$

And the right-hand side of $P(k+1)$ is

$$\frac{(k+1)(k+2)}{2}=\frac{k^2+3k+1}{2}.$$

Thus the two sides of $P(k+1)$ are equal to the same quantity and so they are equal to each other. Therefore the equation $P(k+1)$ is true *[as was to be shown]*.
[Since we have proved both the basis step and the inductive step, we conclude that the theorem is true.]

• Definition Closed Form

If a sum with a variable number of terms is shown to be equal to a formula that does not contain either an ellipsis or a summation symbol, we say that it is written **in closed form**.

a. Evaluate $2 + 4 + 6 + \cdots + 500$.

b. Evaluate $5 + 6 + 7 + 8 + \cdots + 50$.

c. For an integer $h \geq 2$, write $1 + 2 + 3 + \cdots + (h - 1)$ in closed form.

Solution

a. $$\begin{aligned} 2 + 4 + 6 + \cdots + 500 &= 2 \cdot (1 + 2 + 3 + \cdots + 250) \\ &= 2 \cdot \left(\frac{250 \cdot 251}{2}\right) && \text{by applying the formula for the sum of the first } n \text{ integers with } n = 250 \\ &= 62{,}750. \end{aligned}$$

b. $$\begin{aligned} 5 + 6 + 7 + 8 + \cdots + 50 &= (1 + 2 + 3 + \cdots + 50) - (1 + 2 + 3 + 4) \\ &= \frac{50 \cdot 51}{2} - 10 && \text{by applying the formula for the sum of the first } n \text{ integers with } n = 50 \\ &= 1{,}265 \end{aligned}$$

c. $$\begin{aligned} 1 + 2 + 3 + \cdots + (h - 1) &= \frac{(h - 1) \cdot [(h - 1) + 1]}{2} && \text{by applying the formula for the sum of the first } n \text{ integers with } n = h - 1 \\ &= \frac{(h - 1) \cdot h}{2} && \text{since } (h - 1) + 1 = h. \end{aligned}$$

Proving Sum of Geometric Sequences

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}.$$

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1}-1}{r-1} \quad \leftarrow P(0) \quad = \frac{r-1}{r-1} = 1$$

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1}-1}{r-1} \quad \leftarrow P(k)$$

inductive hypothesis

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2}-1}{r-1}. \quad \leftarrow P(k+1)$$

$$= \sum_{i=0}^{k} r^i + r^{k+1}$$

$$= \frac{r^{k+1}-1}{r-1} + r^{k+1}$$

$$= \frac{r^{k+1}-1}{r-1} + \frac{r^{k+1}(r-1)}{r-1}$$

$$= \frac{(r^{k+1}-1) + r^{k+1}(r-1)}{r-1}$$

$$= \frac{r^{k+1}-1+r^{k+2}-r^{k+1}}{r-1}$$

$$= \frac{r^{k+2}-1}{r-1}$$

Theorem 5.2.3 Sum of a Geometric Sequence

For any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Proof (by mathematical induction):

Suppose r is a particular but arbitrarily chosen real number that is not equal to 1, and let the property $P(n)$ be the equation

$$\sum_{i=0}^{n} r^i = \frac{r^{i+1} - 1}{r - 1} \qquad \leftarrow P(n)$$

We must show that $P(n)$ is true for all integers $n \geq 0$. We do this by mathematical induction on n.

Show that P(0) is true:

To establish $P(0)$, we must show that

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1} - 1}{r - 1} \qquad \leftarrow P(0)$$

The left-hand side of this equation is $r^0 = 1$ and the right-hand side is

$$\frac{r^{0+1} - 1}{r - 1} = \frac{r - 1}{r - 1} = 1$$

also because $r^1 = r$ and $r \neq 1$. Hence $P(0)$ is true.

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer $k \geq 0$. That is:]

Let k be any integer with $k \geq 0$, and suppose that

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1} \quad \leftarrow P(k) \text{ inductive hypothesis}$$

[We must show that $P(k+1)$ is true. That is:] We must show that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1} - 1}{r - 1},$$

or, equivalently, that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2} - 1}{r - 1}. \qquad \leftarrow P(k+1)$$

[We will show that the left-hand side of $P(k+1)$ equals the right-hand side.]
The left-hand side of $P(k+1)$ is

$$\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1}$$ by writing the $(k+1)$st term separately from the first k terms

$$= \frac{r^{k+1} - 1}{r - 1} + r^{k+1}$$ by substitution from the inductive hypothesis

$$= \frac{r^{k+1} - 1}{r - 1} + \frac{r^{k+1}(r-1)}{r - 1}$$ by multiplying the numerator and denominator of the second term by $(r-1)$ to obtain a common denominator

$$= \frac{(r^{k+1} - 1) + r^{k+1}(r-1)}{r - 1}$$ by adding fractions

$$= \frac{r^{k+1} - 1 + r^{k+2} - r^{k+1}}{r - 1}$$ by multiplying out and using the fact that $r^{k+1} \cdot r = r^{k+1} \cdot r^1 = r^{k+2}$

$$= \frac{r^{k+2} - 1}{r - 1}$$ by canceling the r^{k+1}'s.

which is the right-hand side of $P(k+1)$ *[as was to be shown.]*
[Since we have proved the basis step and the inductive step, we conclude that the theorem is true.]

a. $1 + 3 + 3^2 + \cdots + 3^{m-2} = 3^{(m-2)+1} - 1/3 - 1$

by applying the formula for the sum of a geometric sequence with $r = 3$ and

$n = m - 2$

$$= \frac{3^{m-1} - 1}{2}$$

b.

$3^2 + 3^3 + 3^4 + \cdots + 3^m = 3^2 \cdot (1 + 3 + 3^2 + \cdots + 3^{m-2})$

by factoring out 3^2

$= 9 \times \dfrac{3^{m-1} - 1}{2}$ by part (a).

Mathematics in Programming

Example : Finding the sum of a integers

Same Question: Prove that these programs prints the same results in case $n \geq 1$

```
For (i=1, i≤n; i++)
   S=S+i;
Print ("%d", S);
```

```
S=(n(n+1))/2
Print ("%d",S);
```

Mathematics in Programming

Example : Finding the sum of a geometric series

Prove that these codes will return the same output.

```
int n, r, sum=0;
int i;
scanf("%d",&n);
scanf("%d",&r);

if(r != 1)  {
   for(i=0 ; i<=n ; i++)  {
         sum = sum + pow(r,i);
   }
   printf("%d\n“, sum);
}
```

```
int n, r, sum=0;
scanf("%d",&n);
scanf("%d",&r);

if(r != 1) {
          sum=((pow(r,n+1))-1)/(r-1);
          printf("%d\n", sum);
 }
```

5.3

Mathematical Induction II

Proving Divisibility

Mathematics in Programming
Proving Divisibility Property

What will the output of this program be for any input n?

```
int n;
scanf("%d",&n);

if(n >= 0) {
   if( (pow(2,(2*n))  - 1) %3 == 0)                \\ does 2^2n -1 | 3?? \\
     printf("this property is true");
   else
     printf("this property isn't true");
}
```

Proving a Divisibility Property

For all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

$$3 \mid 2^{2n} - 1 \qquad \leftarrow P(n)$$

Basis Step: *Show that P(0) is true.*

P(0): $2^{2.0} - 1 = 2^0 - 1 = 1 - 1 = 0$ as $3 \mid 0$, thus $P(0)$ is true.

Inductive Step: *Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is also true*:

Suppose: $2^{2k} - 1$ is divisible by 3. $\leftarrow P(k)$ inductive hypothesis

$$2^{2k} - 1 = 3r \text{ for some integer } r.$$

We want to prove $2^{2(k+1)} - 1$ is divisible by 3. $\leftarrow P(k+1)$

$2^{2(k+1)} - 1 = 2^{2k+2} - 1$ by the laws of exponents

$= 2^{2k} \cdot 2^2 - 1 = 2^{2k} \cdot 4 - 1$

$= 2^{2k}(3 + 1) - 1 = 2^{2k} \cdot 3 + (2^{2k} - 1) = 2^{2k} \cdot 3 + 3r$

$= 3(2^{2k} + r)$ Which is integer

so, by definition of divisibility, $2^{2(k+1)} - 1$ is divisible by 3

Outline a proof by math induction for the statement:

For all integers $n \geq 0$, $5^n - 1$ is divisible by 4.

<u>**Proof by mathematical induction**</u>:

Let the property P(*n*) be the sentence

$5^n - 1$ is divisible by 4. ← ***the property P(n)***

<u>Show that the property is true for $n = 0$</u>:

We must show that $5^0 - 1$ is divisible by 4.

But $5^0 - 1 = 1 - 1 = 0$, and 0 is divisible by 4 because $0 = 4 \cdot 0$.

<u>Show that for all integers $k \geq 0$, if the property is true for $n = k$, then it is true for $n = k + 1$</u>:

Let *k* be an integer with $k \geq 0$, and **suppose** that

[the property is true for n = k.

$5^k - 1$ is divisible by 4. ← ***inductive hypothesis***

We must **show** that *P(k + 1) is true.*

$5^{k+1} - 1$ is divisible by 4.

Scratch Work for proving that
For all integers $n \geq 0$, $5^n - 1$ is divisible by 4.

$$\begin{aligned} 5^{k+1} - 1 &= 5^k \cdot 5 - 1 \\ &= 5^k \cdot (4 + 1) - 1 \\ &= 5^k \cdot 4 + 5^k \cdot 1 - 1 \\ &= 5^k \cdot 4 + (5^k - 1) \end{aligned}$$

Note: *Each of these terms is divisible by 4.*

So: $5^{k+1} - 1 = 5^k \cdot 4 + 4 \cdot r$ *(where r is an integer)*

$$= 4 \cdot (5^k + r)$$

$(5^k + r)$ is an integer because it is a sum of products of integers, and so, by definition of divisibility $5^{k+1} - 1$ is divisible by 4.

Proving Inequality

For all integers $n \geq 3$, $2n + 1 < 2^n$

Let P(n) be $2n+1<2^n$

Basis Step: *Show that P(3) is true.* P(3): $2.3+1 < 2^3$ which is true.

Inductive Step: *Show that for all integers $k \geq 3$, if P(k) is true then $P(k + 1)$ is also true*:

Suppose: $2k+1<2^k$ is true $\leftarrow$ $P(k)$ inductive hypothesis

$2(k+1)+1 < 2^{k+1}$ $\leftarrow$ $P(k+1)$

$2k+3 = (2k+1) +2$ by algebra

$< 2^k + 2^k$ as $2k + 1 < 2^k$ by the hypothesis and because $2 < 2^k$ $(k \geq 2)$

$\therefore 2k + 3 < 2 \cdot 2^k = 2^{k+1}$

[This is what we needed to show.]

Exercise

For each positive integer n, let $P(n)$ be the property

$$2^n < (n+1)!.$$

Proving a Property of a Sequence

Define a sequence $a_1, a_2, a_3 \ldots$ as follows:

$$a_1 = 2$$
$$a_k = 5a_{k-1} \qquad \text{for all integers } k \geq 2.$$

$$a_1 = 2$$
$$a_2 = 5a_{2-1} = 5a_1 = 5 \cdot 2 = \mathbf{10}$$
$$a_3 = 5a_{3-1} = 5a_2 = 5 \cdot 10 = 50$$
$$a_4 = 5a_{4-1} = 5a_3 = 5 \cdot 50 = \mathbf{250}$$

Property ➔ The terms of the sequence satisfy the equation $a_n = 2 \cdot 5^{n-1}$

Proving a Property of a Sequence

Prove this property:

$a_n = 2 \cdot 5^{n-1}$ for all integers $n \geq 1$

Basis Step: *Show that P(1) is true.* $a_1 = 2 \cdot 5^{1-1} = 2 \cdot 5^0 = 2$

Inductive Step: *Show that for all integers k ≥ 1, if P(k) is true then P(k + 1) is also true:*

Suppose: $a_k = 2 \cdot 5^{k-1}$ ← $P(k)$ inductive hypothesis

$a_{k+1} = 2 \,.\, 5^k$ ← $P(k+1)$

$= 5a_{(k+1)-1}$ by definition of $a_1, a_2, a_3 \ldots$

$= 5a_k$

$= 5 \,.\, (2 \,.\, 5^{k-1})$ by the hypothesis

$= 2 \,.\, (5 \,.\, 5^{k-1})$

$= 2 \,.\, 5^k$

[This is what we needed to show.]

Important Formulas

Formula for the sum of the first *n* integers: For all integers $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Formula for the sum of the terms of a geometric sequence: For all real numbers $r \neq 1$ and all integers $n \geq 0$,

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

Exercises

a. $1 + 2 + 3 + \cdots + 100 = \dfrac{100(100+1)}{2} = 50(101) = 5050$

b. $1 + 2 + 3 + \cdots + k = \dfrac{k(k+1)}{2}$

c. $1 + 2 + 3 + \cdots + (k-1) = \dfrac{(k-1)((k-1)+1)}{2} = \dfrac{(k-1)k}{2}$

d. $4 + 5 + 6 + \cdots + (k-1) = (1 + 2 + 3 + \cdots + (k-1)) - (1 + 2 + 3)$

$$= \frac{k(k-1)}{2} - (1+2+3) = \frac{k(k-1)}{2} - 6$$

e. $3 + 3^2 + 3^3 + \cdots + 3^k = (1 + 3 + 3^2 + 3^3 + \cdots + 3^k) - 1 = \dfrac{3^{k+1}-1}{3-1} - 1$

$$= \frac{3^{k+1}-1}{2} - 1 = \frac{3^{k+1}-1}{2} - \frac{2}{2} = \frac{3^{k+1}-3}{2}$$

f. $3 + 3^2 + 3^3 + \cdots + 3^k = 3(1 + 3 + 3^2 + \cdots + 3^{k-1})$

$$= 3\left(\frac{3^{(k-1)+1}-1}{3-1}\right) = \frac{3(3^k-1)}{2}$$