COMP 233 Discrete Mathematics

Chapter 6

Set Theory

Outline

- A Glimpse into Set Theory
  - Set operations
- More later
  - Partition, Power set, Cartesian product
- Proving Set properties
  - Element argument method to prove
    - Subset property
    - Set Equality
    - Empty set properties
  - "Algebraic" method to prove set properties
    - Set Identities (Theorem 6.2.2)

A Glimpse into Set Theory

"Set" is an undefined term. We say that sets contain elements and are completely determined by the elements they contain.

So: Two sets are equal $\Leftrightarrow$ they have exactly the same elements.

Ex: Let $A = \{1, 3, 5\}$

$B = \{5, 1, 3\}$

the set of all

such that

$C = \{3, 1, 5\}$

$D = \{x \in \mathbb{Z} \mid x \text{ is an odd integer and } 0 < x < 6\}$

How are *A*, *B*, *C*, and *D* related?

Answer: They are all equal.

Notation: $x \in A$ is read "x is an element of A" (or "x is in A")
$x \notin A$ is read "x is not an element of A" (or "x is not in A").

A Glimpse into Set Theory cont.

The order of elements is irrelevant

{Ali, Adam, Sara} = {Adam, Sara, Ali}

Redundancy is not allowed

{Ali, Adam, Adam, Sara}

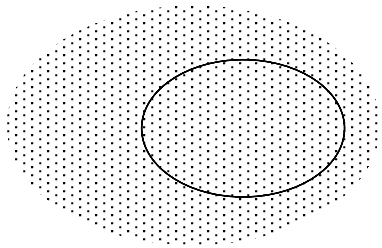
A set can be an element inside another set

{1, {1}} has two elements

Notation of elements

{Ali} ≠ Ali different elements

Defining Sets by a Property



$A = \{x \in S \mid P(x)\}$

The set of all
x is dummy

Property

Examples:

The set of all integers that are more than -2 and less than 5

$\{x \in \mathbf{Z} \mid -2 < x < 5\}$

The set of all persons who born in Palestine

$\{x \in \boldsymbol{Person} \mid BornIn(x, \text{Palestine})\}$

The set of all persons who born in Palestine and Love Homus

$\{x \in \boldsymbol{Person} \mid BornIn(x, \text{Palestine}) \wedge Love(x, \text{Homus})\}$

Subsets

Definition: Given sets A and B, $A \subseteq B$ (read "A is a subset of B")

$\Leftrightarrow \begin{cases} \text{every element in } A \text{ is also in } B. \\ \forall\, x, \text{ if } x \text{ is in } A \text{ then } x \text{ is in } B. \end{cases}$

Note 1: $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

Note 2: $A \nsubseteq B \quad \Leftrightarrow \exists\, x$ such that $x \in A$ and $x \notin B$.

Ex: Let $A = \{2,4,5\}$ and $B = \{1,2,3,4,6,7\}$. Is $A \subseteq B$?

Answer: No, because 5 is in A but 5 is not in B.

Ex: Let $C = \{2,4,7\}$ and $B = \{1,2,3,4,6,7\}$. Is $C \subseteq B$?

Answer: Yes, because every element in C is in B.

Distinction between $\in$ and $\subseteq$

Which of the following are true statements?

- ✔ $2 \in \{1, 2, 3\}$
- ✗ $\{2\} \in \{1, 2, 3\}$
- ✗ $2 \subseteq \{1, 2, 3\}$
- ✔ $\{2\} \subseteq \{1, 2, 3\}$
- ✗ $\{2\} \subseteq \{\{1\}, \{2\}\}$
- ✔ $\{2\} \in \{\{1\}, \{2\}\}$

Warm-up: proving set properties

Determining whether one set is a subset of another

Let $A = \{x \in \mathbb{Z} \mid x = 5a + 1 \text{ for some integer } a\}$

$B = \{y \in \mathbb{Z} \mid y = 10b - 9 \text{ for some integer } b\}$.

1. Is $A \subseteq B$? Justify your answer.

2. Is $B \subseteq A$? Justify your answer.

$A = \{x \in \mathbb{Z} \mid x = 5a + 1 \text{ for some integer } a\}$

$B = \{y \in \mathbb{Z} \mid y = 10b - 9 \text{ for some integer } b\}$

1. Is $A \subseteq B$? **Answer:** No

The reason is that $6 \in A$ because $6 = 5 \cdot 1 + 1$.

But $6 \notin B$ because

if $6 = 10b - 9$, then $15 = 10b$, which implies that $b = 1.5$, and 1.5 is not an integer.

So there is at least one element of A that is not in B, and hence A is not a subset of B.

2. Is $B \subseteq A$? **Answer:** Yes

Scratch Work: Suppose y is any *[pbac]* element of B. Then $y = 10b - 9$ for some integer b. Must y be in A??

Idea: Suppose y is in A. Then $y = 5a + 1$ for some integer a. Set the two values for y equal to each other. Deduce what a would have to be to make the two sides of the equation equal to each other. Then show that this value of a actually "works."

$$A = \{x \in \mathbb{Z} \mid x = 5a + 1 \text{ for some integer } a\}$$

$$B = \{y \in \mathbb{Z} \mid y = 10b - 9 \text{ for some integer } b\}$$

4 (continued). Is $B \subseteq A$? **Answer:** Yes

Proof: Suppose y is any *[pbac]* element in B.

Then $y = 10b - 9$ for some integer b.

$5a + 1 = 10b - 9$?

Let $a = 2b - 2$.

Note that a is an integer bkoz products and differences of integers are integers.

Moreover, $5a + 1 = 5(2b - 2) + 1 = 10b - 9 = y$.

So, by definition of A, y is an element in A.

[This argument shows that ***any*** *element in B is also in A.*

Hence B is a subset of A.]

Exercise

Define sets A and B as follows:

$$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$$
$$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$$

Is $A = B$?

Yes. To prove this, both subset relations $A \subseteq B$ and $B \subseteq A$ must be proved.

Part 1, Proof That $A \subseteq B$:

Part 2, Proof That $B \subseteq A$

$$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$$

$$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$$

***Part 1, Proof That A $\subseteq$ B*:**

Suppose x is a particular but arbitrarily chosen element of A.

[We must show that $x \in B$. By definition of B, this means we must show that $x = 2 \cdot$(some integer) $- 2$.]

By definition of A, there is an integer a such that $x = 2a$.

[Given that $x = 2a$, can x also be expressed as $2 \cdot$(some integer) $- 2$? *I.e., is there an integer, say b, such that $2a = 2b - 2$? Solve for b to obtain $b = (2a + 2)/2 = a + 1$. Check to see if this works.]*

Let $b = a + 1$.

[First check that b is an integer.]

Then b is an integer because it is a sum of integers.

[Then check that $x = 2b - 2$.]

Also $2b - 2 = 2(a + 1) - 2 = 2a + 2 - 2 = 2a = x$,

Thus, by definition of B, x is an element of B

[which is what was to be shown].

Definitions of Set Operations

Given sets A and B that are subsets of a "universal set" U, we define

$A \cup B = \{x \in U \mid x \in A$ **or** $x \in B\}$ "or" means "and/or"

$A \cap B = \{x \in U \mid x \in A$ **and** $x \in B\}$

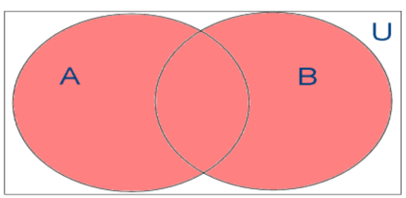
the set of all x in U such that x is in A and x is in B

$A - B = \{x \in U \mid x \in A$ and $x \notin B\}$

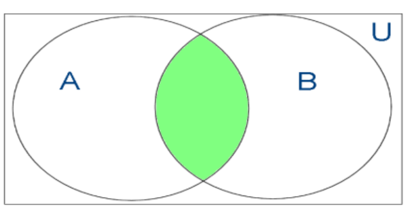
$A^c = \{x \in U \mid x \notin A\}$

Venn Diagrams

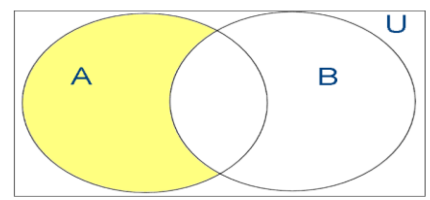
$A \cup B$



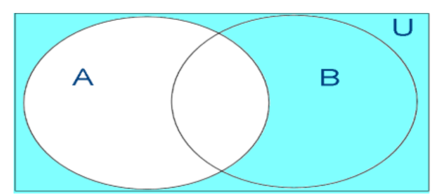
$A \cap B$



$A - B$



A^c



Set Difference and Subset

Definition: Given sets S and T, the **difference of T minus S**, denoted **$T - S$**, is the set consisting of all the elements that are in T but are not in S:

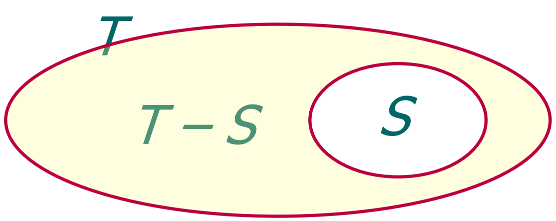
$$T - S = \{x \in T \mid x \text{ is not in } S\}$$

Read this: "The set of all x in T such that x is not in S."

Definition: Given sets S and T, S is a **subset** of T if, and only if, every element in S is in T.

Example: Let $T = \{1,2,3,4,5\}$ and $S = \{1,3,5\}$. Then S is a subset of T and $T - S = \{2,4\}$.

Picture:



$T - S$ is shaded yellow

Class Exercise

Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ and suppose that the "universal set" $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Find

$A \cup B = \{1, 2, 3, 4, 5\}$

$A \cap B = \{3\}$

$A - B = \{1, 2\}$

$A^c = \{4, 5, 6, 7, 8\}$

- **Definition**

Unions and Intersections of an Indexed Collection of Sets

Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

Exercise

For each positive integer i, let $A_i = \left\{x \in \mathbf{R} \mid -\frac{1}{i} < x < \frac{1}{i}\right\} = A_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$.

a. Find $A_1 \cup A_2 \cup A_3$ and $A_1 \cap A_2 \cap A_3$.

$$A_1 \cup A_2 \cup A_3 \quad = \{x \in \mathbf{R} \mid -1 < x < 1\}$$
$$= (-1,\ 1)$$

$$A_1 \cap A_2 \cap A_3 \quad = \left\{x \in \mathbf{R} \mid -\frac{1}{3} < x < \frac{1}{3}\right\}$$
$$= \left(-\frac{1}{3}, \frac{1}{3}\right)$$

b. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.

$$\bigcup_{i=1}^{\infty} A_i = \quad = \{x \in \mathbf{R} \mid -1 < x < 1\}$$
$$= (-1,\ 1)$$
$$\bigcap_{i=1}^{\infty} A_i \quad = \{0\}$$

The Empty Set

Let ***A*** be *the set of all the people in the room who live in Ramallah* and ***B*** be the *set of all people in the room who live outside Ramallah*.

What is $A \cap B$?

Answer: This set contains no elements at all.

Notation: The symbol ∅ denotes a set with no elements. (One can prove that there is only one such set. We call it the *empty set* or the *null set*.)

The Empty Set

The empty set is not the same thing as nothing; rather, it is a set with nothing inside it and a set is always something. This issue can be overcome by **viewing a set as a bag—an empty bag undoubtedly still exists.**

Example: the set $D = \{x \in \mathbf{R} \mid 3 < x < 2\}$.

Axioms about the empty set:

$\emptyset \subseteq A$

$A \cup \emptyset \subseteq A$

$A \cap \emptyset \subseteq \emptyset$

$A \times \emptyset = \emptyset$

$A \times \emptyset \Rightarrow A = \emptyset$

Disjoint Sets

- **Definition**

Two sets are called **disjoint** if, and only if, they have no elements in common.
Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Are A and B disjoint?

Yes.

$$\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset.$$

Disjoint Sets

- **Definition**

Sets $A_1, A_2, A_3 \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all $i, j = 1, 2, 3, \ldots$

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

a. Let $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$. Are A_1, A_2, and A_3 mutually disjoint?

b. Let $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$. Are B_1, B_2, and B_3 mutually disjoint?

a. Yes. A_1 and A_2 have no elements in common, A_1 and A_3 have no elements in common, and A_2 and A_3 have no elements in common.

b. No. B_1 and B_3 both contain 4. ■

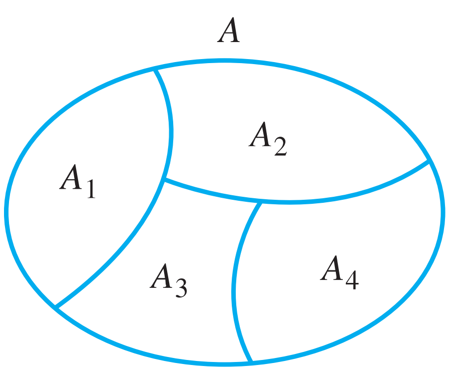
Partition of Set

- **Definition**

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i
2. The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

Man $\cap$ Woman $= \varnothing$
People = Man $\cup$ Woman



Let $A = \{1, 2, 3, 4, 5, 6\}$,

$A_1 = \{1, 2\}$,

$A_2 = \{3, 4\}$,

$A_3 = \{5, 6\}$

Is $\{A_1, A_2, A_3\}$ a partition of A?

$A = A_1 \cup A_2 \cup A_3$

A_1, A_2, and A_3 are mutually disjoint.

Let $\mathbf{Z}$ be the set of all integers and let

$$T_0 = \{n \in \mathbf{Z} \mid n = 3k, \text{ for some integer } k\},$$
$$T_1 = \{n \in \mathbf{Z} \mid n = 3k + 1, \text{ for some integer } k\}, \text{ and}$$
$$T_2 = \{n \in \mathbf{Z} \mid n = 3k + 2, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_2\}$ a partition of $\mathbf{Z}$?

b. Yes. By the quotient-remainder theorem, every integer n can be represented in exactly one of the three forms

$$n = 3k \quad \text{or} \quad n = 3k + 1 \quad \text{or} \quad n = 3k + 2,$$

for some integer k. This implies that no integer can be in any two of the sets T_0, T_1, or T_2. So T_0, T_1, and T_2 are mutually disjoint. It also implies that every integer is in one of the sets T_0, T_1, or T_2. So $\mathbf{Z} = T_0 \cup T_1 \cup T_2$. ■

Power Sets

- **Definition**

Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Find the power set of the set $\{x, y\}$. That is, find [redacted] $(\{x, y\})$

$$power\ (\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}.$$

Cartesian Products

- **Definition**

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\boldsymbol{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Example: Let $A_1 = \{x, y\}$, $A_2 = \{1,2,3\}$, and $A_3 = \{a, b\}$.

$$A_1 \times A_2 =$$

$$= \{(x,1),(x,2),(x,3),(y,1),(y,2),(y,3)\}$$

Example

Let $A = \{Ali, Ahmad\}$,
$B = \{AI, DM, DB\}$,
$C = \{P, F\}$

Find $A \times B =$

Find $A \times B \times C =$

Find $(A \times B) \times C$

Set Relations

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets A and B,

$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets A, B, and C,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

➔ We will prove some of these theories in the lecture, please prove others at home

How to prove?

- **Element Argument Method**
- **Algebraic Proof Method**

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in X \cup Y \quad \Leftrightarrow \quad x \in X$ or $x \in Y$
2. $x \in X \cap Y \quad \Leftrightarrow \quad x \in X$ and $x \in Y$
3. $x \in X - Y \quad \Leftrightarrow \quad x \in X$ and $x \notin Y$
4. $x \in X^c \quad \Leftrightarrow \quad x \notin X$
5. $(x, y) \in X \times Y \quad \Leftrightarrow \quad x \in X$ and $y \in Y$

The Element Argument Method

In details

Example: Prove that: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

That is:

Prove: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

That is, show $\forall x$, if $x \in A \cup (B \cap C)$ then $x \in (A \cup B) \cap (A \cup C)$

Suppose $x \in A \cup (B \cap C)$. [Show $x \in (A \cup B) \cap (A \cup C)$.]

...

Thus $x \in (A \cup B) \cap (A \cup C)$.

Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Prove: $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

That is, show $\forall x$, if $x \in (A \cup B) \cap (A \cup C)$ then $x \in A \cup (B \cap C)$.

Suppose $x \in (A \cup B) \cap (A \cup C)$. [Show $x \in A \cup (B \cap C)$.]

...

Thus $x \in A \cup (B \cap C)$.

Hence $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Thus $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.

Proving: **A Distributive Law for Sets**

Theorem 6.2.2(3)(a) A Distributive Law for Sets

For all sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$:

Suppose $x \in A \cup (B \cap C)$.
$x \in A$ or $x \in B \cap C$. (by def. of union)
Case 1 ($x \in A$): then
$x \in A \cup B$ (by def. of union) and
$x \in A \cup C$ (by def. of union)
$\therefore x \in (A \cup B) \cap (A \cup C)$ (def. of intersection)
Case 2 ($x \in B \cap C$): then
$x \in B$ and $x \in C$ (def. of intersection)
As $x \in B$, $x \in A \cup B$ (by def. of union)
As $x \in C$, $x \in A \cup C$, (by def. of union)
$\therefore x \in (A \cup B) \cap (A \cup C)$ (def. of intersection)

In both cases, $x \in (A \cup B) \cap (A \cup C)$.
Thus: $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
by definition of subset

$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$:

Suppose $x \in (A \cup B) \cap (A \cup C)$.
$x \in A \cup B$ and $x \in A \cup C$. (def. of intersection)
Case 1 ($x \in A$): then
$x \in A \cup (B \cap C)$ (by def. of union)
Case 2 ($x \notin A$): since $x \in A \cup B$, x is in at least one of A or B.
But x is not in A; hence x is in B.
Similarly, since $x \in A \cup C$, x is in at least one of A or C. But x is not in A; hence x is in C.
We have shown that both $x \in B$ and $x \in C$, and so by definition of intersection, $x \in B \cap C$. It follows by definition of union that $x \in A \cup (B \cap C)$.

Conclusion: Since both subset relations have been proved, it follows by definition of set equality that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proving: **A De Morgan's Law for Sets**

Theorem 6.2.2(9)(a) A De Morgan's Law for Sets

For all sets A and B, $(A \cup B)^c = A^c \cap B^c$

Same As: proving whether: the people who are not students or employees is the same as the people who are neither students nor employees.

$$\mathbf{(A \cup B)^c \subseteq A^c \cap B^c}$$

Suppose $x \in (A \cup B)^c$. *[We must show that $x \in A^c \cap B^c$.]* By definition of complement,

$$x \notin A \cup B.$$

But to say that $x \notin A \cup B$ means that

it is false that (x is in A or x is in B).

By De Morgan's laws of logic, this implies that

x is not in A and x is not in B,

which can be written $\quad x \notin A \quad$ and $\quad x \notin B$.

Hence $x \in A^c$ and $x \in B^c$ by definition of complement. It follows, by definition of intersection, that $x \in A^c \cap B^c$ *[as was to be shown]*. So $(A \cup B)^c \subseteq A^c \cap B^c$ by definition of subset.

$A^c \cap B^c \subseteq (A \cup B)^c$

Suppose $x \in A^c \cap B^c$. *[We must show that $x \in (A \cup B)^c$.]* By definition of intersection, $x \in A^c$ and $x \in B^c$, and by definition of complement,

$$x \notin A \quad \text{and} \quad x \notin B.$$

In other words, x is not in A and x is not in B.

By De Morgan's laws of logic this implies that

it is false that (x is in A or x is in B),

which can be written $x \notin A \cup B$

by definition of union. Hence, by definition of complement, $x \in (A \cup B)^c$ *[as was to be shown]*. It follows that $A^c \cap B^c \subseteq (A \cup B)^c$ by definition of subset.

Theorem 6.2.3 Intersection and Union with a Subset

For any sets A and B, if $A \subseteq B$, then

$$\text{(a) } A \cap B = A \quad \text{and} \quad \text{(b) } A \cup B = B.$$

Prove at home

Proof: If every person is a student, then the set of persons and students are students

Part (a): Suppose A and B are sets with $A \subseteq B$. To show part (a) we must show both that $A \cap B \subseteq A$ and that $A \subseteq A \cap B$. We already know that $A \cap B \subseteq A$ by the inclusion of intersection property. To show that $A \subseteq A \cap B$, let $x \in A$. *[We must show that $x \in A \cap B$.]* Since $A \subseteq B$, then $x \in B$ also. Hence

$$x \in A \quad \text{and} \quad x \in B,$$

and thus

$$x \in A \cap B$$

by definition of intersection *[as was to be shown].*

Theorem 6.2.4 A Set with No Elements Is a Subset of Every Set

If E is a set with no elements and A is any set, then $E \subseteq A$.

Proof by Contradiction

- Suppose not.
- We take the negation of the theorem and suppose it to be true.
- Suppose there exists a set **E** with no elements and a set **A** such that **E** $\nsubseteq$ **A**.
- Then, there would be an element of **E** that is not an element of **A** [by definition of subset].
- **But** there can not be such an element, since **E** has no elements. This is a contradiction.
- Hence the supposition that there are sets **E** and **A**, where **E** has no elements and **E** $\nsubseteq$ **A**, is **false**, and so the **theorem is true**.

Proving: **Uniqueness of the Empty Set**

Corollary 6.2.5 Uniqueness of the Empty Set

There is only one set with no elements.

Proof:

Suppose E_1 and E_2 are both sets with no elements.

By Theorem 6.2.4, $E_1 \subseteq E_2$ since E_1 has no elements.

Also $E_2 \subseteq E_1$ since E_2 has no elements.

Thus $E_1 = E_2$ by definition of set equality.

Proving: **a Conditional Statement**

Proposition 6.2.6

For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.

- Suppose A, B, and C are any sets such that $A \subseteq B$ and $B \subseteq C^c$.
- We must show that $A \cap C = \emptyset$.
- **Suppose not.**
- That is, suppose there is an element x in $A \cap C$.
- By definition of intersection, $x \in A$ and $x \in C$.
- Then, since $A \subseteq B$, $x \in B$ by definition of subset.
- Also, since $B \subseteq C^c$, then $x \in C^c$ by definition of subset again.
- It follows by definition of complement that $x \notin C$. Thus $x \in C$ and $x \notin C$, which is a contradiction.
- So the supposition that there is an element x in $A \cap C$ is **false**, and thus $A \cap C = \emptyset$ *[as was to be shown]*.

A Generalized Distributive Law

Prove that for all sets A and $B_1, B_2, B_3, \ldots, B_n$,

$$A \cup \left(\bigcap_{i=1}^{n} B_i\right) = \bigcap_{i=1}^{n} (A \cup B_i).$$

Solution Compare this proof to the one given in Example 6.2.2. Although the notation is more complex, the basic ideas are the same.

Proof:

Part 1, Proof that $\boldsymbol{A \cup \left(\bigcap_{i=1}^{n} B_i\right) \subseteq \bigcap_{i=1}^{n} (A \cup B_i)}$:

Suppose x is any element in $A \cup \left(\bigcap_{i=1}^{n} B_i\right)$. *[We must show that x is in $\bigcap_{i=1}^{n} (A \cup B_i)$.]*

By definition of union, $x \in A$ or $x \in \bigcap_{i=1}^{n} B_i$.

Case 1, $\boldsymbol{x \in A}$: In this case, it is true by definition of union that for all $i = 1, 2, \ldots, n$, $x \in A \cup B_i$. Hence $x \in \bigcap_{i=1}^{n} (A \cup B_i)$.

Case 2, $\boldsymbol{x \in \bigcap_{i=1}^{n} B_i}$: In this case, by definition of the general intersection, we have that for all integers $i = 1, 2, \ldots, n$, $x \in B_i$. Hence, by definition of union, for all integers $i = 1, 2, \ldots, n$, $x \in A \cup B_i$, and so, by definition of general intersection, $x \in \bigcap_{i=1}^{n} (A \cup B_i)$.

Thus, in either case, $x \in \bigcap_{i=1}^{n} (A \cup B_i)$ *[as was to be shown]*.

Part 2, Proof that $\bigcap_{i=1}^{n}(A \cup B_i) \subseteq A \cup \left(\bigcap_{i=1}^{n} B_i\right)$:

Suppose x is any element in $\bigcap_{i=1}^{n}(A \cup B_i)$. *[We must show that x is in $A \cup \left(\bigcap_{i=1}^{n} B_i\right)$.]*

By definition of intersection, $x \in A \cup B_i$ for all integers $i = 1, 2, \ldots, n$. Either $x \in A$ or $x \notin A$.

Case 1, $x \in A$: In this case, $x \in A \cup \left(\bigcap_{i=1}^{n} B_i\right)$ by definition of union.

Case 2, $x \notin A$: By definition of intersection, $x \in A \cup B_i$ for all integers $i = 1, 2, \ldots, n$. Since $x \notin$ A, x must be in each B_i for every integer $i = 1, 2, \ldots, n$. Hence, by definition of intersection, $x \in \bigcap_{i=1}^{n} B_i$, and so, by definition of union, $x \in A \cup \left(\bigcap_{i=1}^{n} B_i\right)$.

Conclusion: Since both set containments have been proved, it follows by definition of set equality that $A \cup \left(\bigcap_{i=1}^{n} B_i\right) = \bigcap_{i=1}^{n}(A \cup B_i)$. ■

Algebraic Proofs

Is the following set property true?
For all sets A, B, and C, $(A - B) \cup (B - C) = A - C$.

Counterexample 1:
Let $A = \{1, 2, 4, 5\}$, $B = \{2, 3, 5, 6\}$, and $C = \{4, 5, 6, 7\}$.
$A - B = \{1, 4\}$
$B - C = \{2, 3\}$
$A - C = \{1, 2\}$.
Hence
$(A - B) \cup (B - C) = \{1, 4\} \cup \{2, 3\} = \{1, 2, 3, 4\}$,
whereas $A - C = \{1, 2\}$.
Since $\{1, 2, 3, 4\} \neq \{1, 2\}$,
we have that $(A - B) \cup (B - C) \neq A - C$.

Problem-Solving Strategy

- How can you discover whether a given universal statement about sets is true or false?
- There are two basic approaches: the optimistic and the pessimistic.
- In the optimistic approach,
  "What do I need to show?" and "How do I show it?"
- In the pessimistic approach, you start by searching your mind for a set of conditions that must be fulfilled to construct a counterexample.

- The trick is to be ready to switch to the other approach if the one you are trying does not look promising.

Algebraic Proofs
Deriving a Set Difference Property

Construct an algebraic proof that for all sets A, B, and C,

$$(A \cup B) - C = (A - C) \cup (B - C).$$

$$\begin{aligned}
(A \cup B) - C &= (A \cup B) \cap C^c && \text{by the set difference law} \\
&= C^c \cap (A \cup B) && \text{by the commutative law for } \cap \\
&= (C^c \cap A) \cup (C^c \cap B) && \text{by the distributive law} \\
&= (A \cap C^c) \cup (B \cap C^c) && \text{by the commutative law for } \cap \\
&= (A - C) \cup (B - C) && \text{by the set difference law.}
\end{aligned}$$

Cite a property from Theorem 6.2.2 for every step of the proof.

Algebraic Proofs

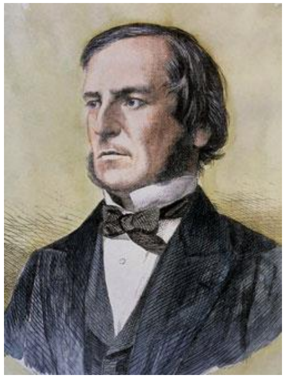
Deriving a Set Identity Using Properties of ∅

Construct an algebraic proof that for all sets A and B,

$$A - (A \cap B) = A - B.$$

$$\begin{aligned}
A - (A \cap B) &= A \cap (A \cap B)^c && \text{by the set difference law} \\
&= A \cap (A^c \cup B^c) && \text{by De Morgan's laws} \\
&= (A \cap A^c) \cup (A \cap B^c) && \text{by the distributive law} \\
&= \emptyset \cup (A \cap B^c) && \text{by the complement law} \\
&= (A \cap B^c) \cup \emptyset && \text{by the commutative law for } \cup \\
&= A \cap B^c && \text{by the identity law for } \cup \\
&= A - B && \text{by the set difference law.}
\end{aligned}$$

Boolean Algebra



George Boole
1815-1864,
England

Introduced by George Boole in his first book The Mathematical Analysis of Logic (1847),

A structure abstracting the computation with the truth values false and true.

Instead of elementary algebra where the values of the variables are numbers, and the main operations are addition and multiplication, the main operations of Boolean algebra are the conjunction (∧) the disjunction (∨) and the negation not (¬).

Used extensively in the simplification of logic Circuits

Compare

Logical Equivalences	Set Properties
For all statement variables p, q, and r:	For all sets A, B, and C:
a. $p \vee q \equiv q \vee p$ b. $p \wedge q \equiv q \wedge p$	a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$ b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$	a. $A \cup (B \cup C) \equiv A \cup (B \cup C)$ b. $A \cap (B \cap C) \equiv A \cap (B \cap C)$
a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	a. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$
a. $p \vee \mathbf{c} \equiv p$ b. $p \wedge \mathbf{t} \equiv p$	a. $A \cup \emptyset = A$ b. $A \cap U = A$
a. $p \vee \sim p \equiv \mathbf{t}$ b. $p \wedge \sim p \equiv \mathbf{c}$	a. $A \cup A^c = U$ b. $A \cap A^c = \emptyset$
$\sim(\sim p) \equiv p$	$(A^c)^c = A$
a. $p \vee \mathbf{t} \equiv \mathbf{t}$ b. $p \wedge \mathbf{c} \equiv \mathbf{c}$	a. $A \cup U = U$ b. $A \cap \emptyset = \emptyset$

Both are special cases of the same general structure, known as a *Boolean Algebra.*

Boolean Algebra

- **Definition: Boolean Algebra**

A **Boolean algebra** is a set B together with two operations, generally denoted $+$ and $\cdot$, such that for all a and b in B both $a + b$ and $a \cdot b$ are in B and the following properties hold:

1. *Commutative Laws:* For all a and b in B,

$$\text{(a) } a + b = b + a \quad \text{and} \quad \text{(b) } a \cdot b = b \cdot a.$$

2. *Associative Laws:* For all a, b, and c in B,

$$\text{(a) } (a + b) + c = a + (b + c) \quad \text{and} \quad \text{(b) } (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

3. *Distributive Laws:* For all a, b, and c in B,

$$\text{(a) } a + (b \cdot c) = (a + b) \cdot (a + c) \quad \text{and} \quad \text{(b) } a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

4. *Identity Laws:* There exist distinct elements 0 and 1 in B such that for all a in B,

$$\text{(a) } a + 0 = a \quad \text{and} \quad \text{(b) } a \cdot 1 = a.$$

5. *Complement Laws:* For each a in B, there exists an element in B, denoted $\overline{a}$ and called the **complement** or **negation** of a, such that

$$\text{(a) } a + \overline{a} = 1 \quad \text{and} \quad \text{(b) } a \cdot \overline{a} = 0.$$

Properties of a Boolean Algebra

Theorem 6.4.1 Properties of a Boolean Algebra

Let B be any Boolean algebra.

1. *Uniqueness of the Complement Law:* For all a and x in B, if $a + x = 1$ and $a \cdot x = 0$ then $x = \overline{a}$.
2. *Uniqueness of* 0 *and* 1: If there exists x in B such that $a + x = a$ for all a in B, then $x = 0$, and if there exists y in B such that $a \cdot y = a$ for all a in B, then $y = 1$.
3. *Double Complement Law:* For all $a \in B$, $\overline{(\overline{a})} = a$.
4. *Idempotent Law:* For all $a \in B$,

$$\text{(a) } a + a = a \quad \text{and} \quad \text{(b) } a \cdot a = a.$$

5. *Universal Bound Law:* For all $a \in B$,

$$\text{(a) } a + 1 = 1 \quad \text{and} \quad \text{(b) } a \cdot 0 = 0.$$

6. *De Morgan's Laws:* For all a and $b \in B$,

$$\text{(a) } \overline{a + b} = \overline{a} \cdot \overline{b} \quad \text{and} \quad \text{(b) } \overline{a \cdot b} = \overline{a} + \overline{b}.$$

7. *Absorption Laws:* For all a and $b \in B$,

$$\text{(a) } (a + b) \cdot a = a \quad \text{and} \quad \text{(b) } (a \cdot b) + a = a.$$

8. *Complements of* 0 *and* 1:

$$\text{(a) } \overline{0} = 1 \quad \text{and} \quad \text{(b) } \overline{1} = 0.$$

Set Theory

6.4 Boolean Algebra

In this lecture:

- ❑ Part 1: History of Algebra
- ❑ Part 2: What is Boolean Algebra
- ❑ Part 3: **Proving Boolean Algebra Properties**

Proving of Boolean Algebra Properties

Uniqueness of the Complement Law: For all a and x in B, if $a + x = 1$ and $a \cdot x = 0$ then $x = \overline{a}$.

Proof:

Suppose a and x are particular, but arbitrarily chosen, elements of B that satisfy the following hypothesis: $a + x = 1$ and $a \cdot x = 0$. Then

$x = x \cdot 1$	because 1 is an identity for $\cdot$
$= x \cdot (a + \overline{a})$	by the complement law for $+$
$= x \cdot a + x \cdot \overline{a}$	by the distributive law for $\cdot$ over $+$
$= a \cdot x + x \cdot \overline{a}$	by the commutative law for $\cdot$
$= 0 + x \cdot \overline{a}$	by hypothesis
$= a \cdot \overline{a} + x \cdot \overline{a}$	by the complement law for $\cdot$
$= (\overline{a} \cdot a) + (\overline{a} \cdot x)$	by the commutative law for $\cdot$
$= \overline{a} \cdot (a + x)$	by the distributive law for $\cdot$ over $+$
$= \overline{a} \cdot 1$	by hypothesis
$= \overline{a}$	because 1 is an identity for $\cdot$.

Proof of an Idempotent Law

Fill in the blanks in the following proof that for all elements a in a B
$a + a = a$.

Proof:

Suppose B is a Boolean algebra and a is any element of B. Then

$$
\begin{aligned}
a &= a + 0 && \underline{\quad(a)\quad} \\
&= a + (a \cdot \overline{a}) && \underline{\quad(b)\quad} \\
&= (a + a) \cdot (a + \overline{a}) && \underline{\quad(c)\quad} \\
&= (a + a) \cdot 1 && \underline{\quad(d)\quad} \\
&= a + a && \underline{\quad(e)\quad}.
\end{aligned}
$$

Solution

a. because 0 is an identity for $+$

b. by the complement law for $\cdot$

c. by the distributive law for $+$ over $\cdot$

d. by the complement law for $+$

e. because 1 is an identity for $\cdot$

Exercises

1. For all a in B, $a \cdot a = a$.

   Proof: Let a be any element of B. Then

$$\begin{aligned} a &= a \cdot 1 && \underline{\text{(a)}} \\ &= a \cdot (a + \overline{a}) && \underline{\text{(b)}} \\ &= (a \cdot a) + (a \cdot \overline{a}) && \underline{\text{(c)}} \\ &= (a \cdot a) + 0 && \underline{\text{(d)}} \\ &= a \cdot a && \underline{\text{(e)}}. \end{aligned}$$

2. For all a in B, $a + 1 = 1$.

   Proof: Let a be any element of B. Then

$$\begin{aligned} a + 1 &= a + (a + \overline{a}) && \underline{\text{(a)}} \\ &= (a + a) + \overline{a} && \underline{\text{(b)}} \\ &= a + \overline{a} && \text{by Example 6.4.2} \\ &= 1 && \underline{\text{(c)}}. \end{aligned}$$

3. For all a and b in B, $(a + b) \cdot a = a$.

   Proof: Let a and b be any elements of B. Then

$$\begin{aligned} (a + b) \cdot a &= a \cdot (a + b) && \underline{\text{(a)}} \\ &= a \cdot a + a \cdot b && \underline{\text{(b)}} \\ &= a + a \cdot b && \underline{\text{(c)}} \\ &= a \cdot 1 + a \cdot b && \underline{\text{(d)}} \\ &= a \cdot (1 + b) && \underline{\text{(e)}} \\ &= a \cdot (b + 1) && \underline{\text{(f)}} \\ &= a \cdot 1 && \text{by exercise 2} \\ &= a && \underline{\text{(g)}}. \end{aligned}$$

Class Exercise - 3

Given sets A and B, what would you suppose and what would you show to prove that $(A \cap B) \cap B^c = \varnothing$?

In general: How do you show that a set equals the empty set?

Answer: Show that the set has no elements. Go by contradiction. Suppose the set has an element. Show that this supposition leads to a contradiction.

Class Exercises

1. Given sets A, B, and C, what would you suppose and what would you show to prove that $(A \cap B) \cup C \subseteq A \cap (B \cup C)$?

2. True or false? Justify your answer.

For all sets A, B, and C, $(A \cap B) \cup C = A \cap (B \cup C)$.

3. Given sets A and B, what would you suppose and what would you show to prove that $(A \cap B) \cap B^c = \emptyset$?

Example

Prove: For all sets A, B, and C, $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

Proof: Let A, B, and C be any sets. Then

$$
\begin{aligned}
(A \cap B) \cup C &= C \cup (A \cap B) && \text{by \_\_?\_\_\_\_\_\_} \\
&= (C \cup A) \cap (C \cup B) && \text{by \_\_?\_\_\_\_\_\_} \\
&= (A \cup C) \cap (B \cup C) && \text{by \_\_?\_\_\_\_\_\_.}
\end{aligned}
$$

Cite a property from Theorem 6.2.2 for every step of the proof.