COMP 233 Discrete Mathematics

Chapter 8

Relations

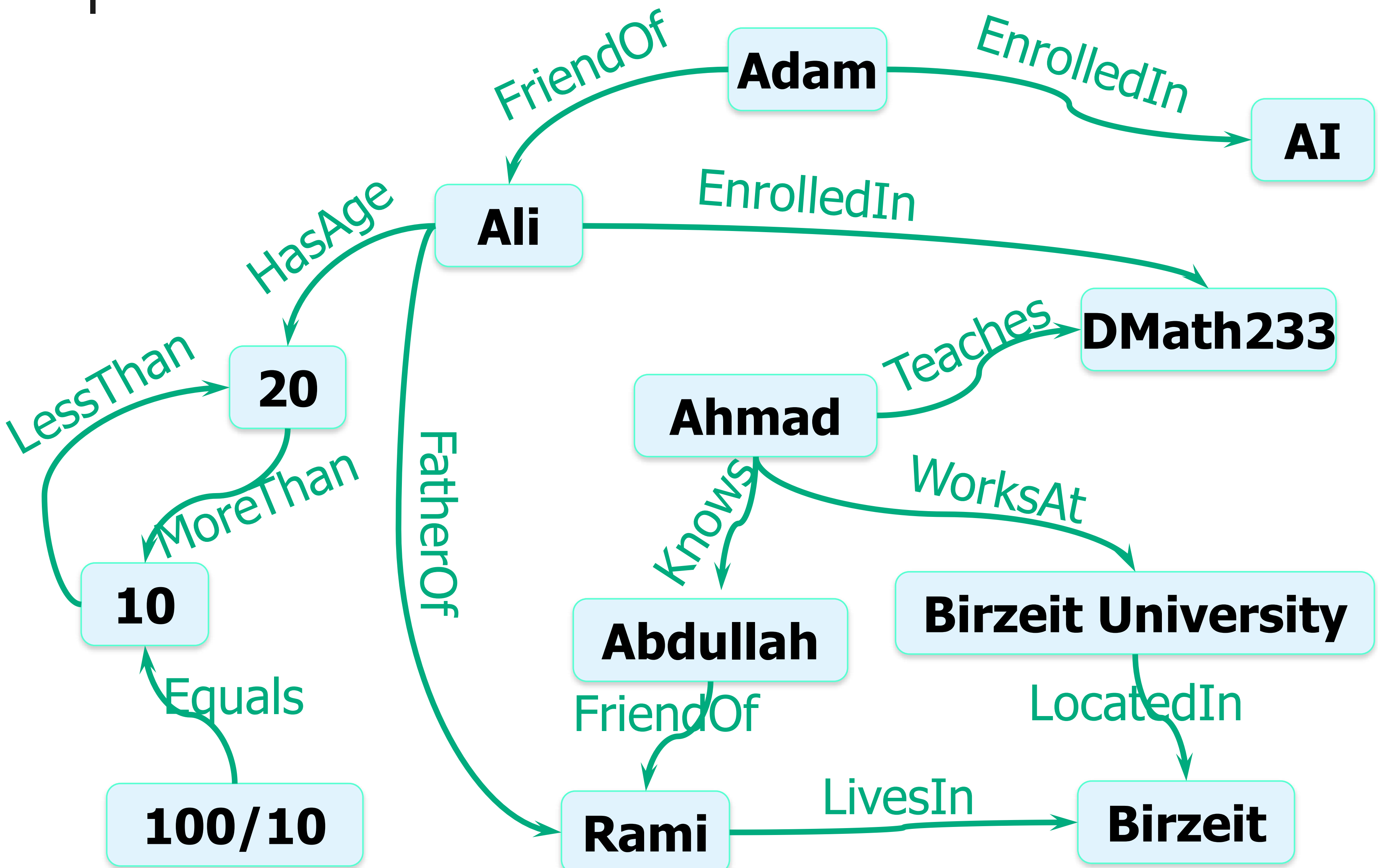
Relations

8.1 Introduction to Relations

In this lecture:

- ❑ Part 1: What is a Relation
- ❑ Part 2: Inverse of a Relation
- ❑ Part 3: Directed Graphs
- ❑ Part 4: *n*-ary Relations
- ❑ Part 5: Relational Databases

What is a Relation?



Outline

In this chapter we discuss the mathematics of relations defined on sets, focusing on ways to represent relations and exploring various properties they may have.

- **Binary (n-ary) Relations**
  - Representations of binary relations
    - set of ordered pairs, arrow diagram, directed graphs
- **Properties of binary relations**
  - reflexivity, symmetry, transitivity
- **Equivalence relations**
  - equivalence classes
- Inverse relations
- Proving Properties of Relations on Sets
  - Equality relation, less then, divides, …

Cartesian Product of Sets

Definition: Given any sets A and B, we define the **Cartesian product of *A* and B**, denoted ***A*** $\times$ **B**, to be the set of all ordered pairs (a,b) where a is in A and b is in B.

In symbols: $A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$

"tuple"

Example: Let A = {1, 3, 5} and let B = {u, v}.
Find A × B.

Solution:

$A \times B = \{(1,u),(1,v),(3,u),(3,v),(5,u),(5,v)\}$

Definition of Binary Relation

Definition: A binary relation R from a set A to a set B is a subset of $A \times B$. Given an ordered pair (a, b) in $A \times B$, we say that a is related to b, written $a\ R\ b$, if, and only if, $(a, b) \in R$. In symbols:

$$a\ R\ b \iff (a, b) \in R$$

Example of a Binary Relation

Ex: Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$.
Define a binary relation R from A to B as follows:

$$a\,R\,b \Leftrightarrow a > b.$$

a. Is $2\,R\,4$? No — Is $5\,R\,4$? Yes — Is $(7,2) \in R$? Yes

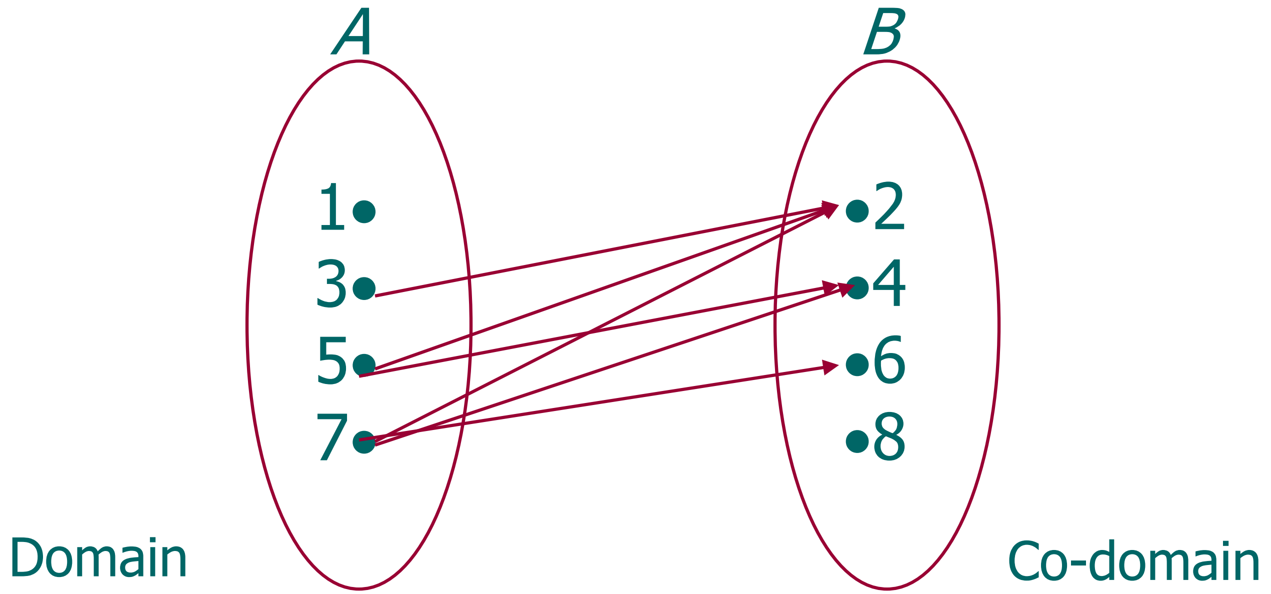
b. Write R as a set of ordered pairs.

$$R = \{(3,2), (5,2), (5,4), (7,2), (7,4), (7,6)\}$$

Example of a Binary Relation

c. Draw an "arrow diagram" to represent R, where

$R = \{(3,2), (5,2), (5,4), (7,2), (7,4), (7,6)\}$.

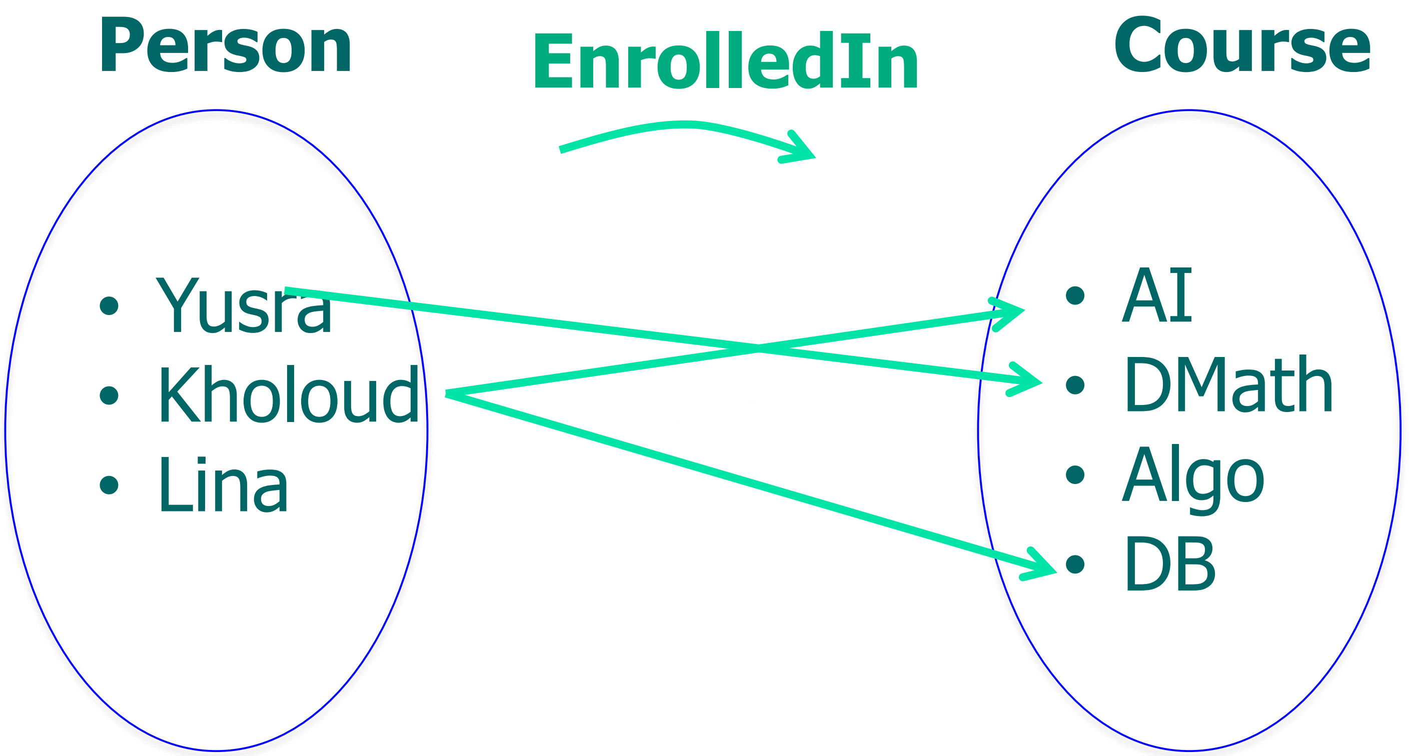


Note: An arrow diagram can be used to define a relation.

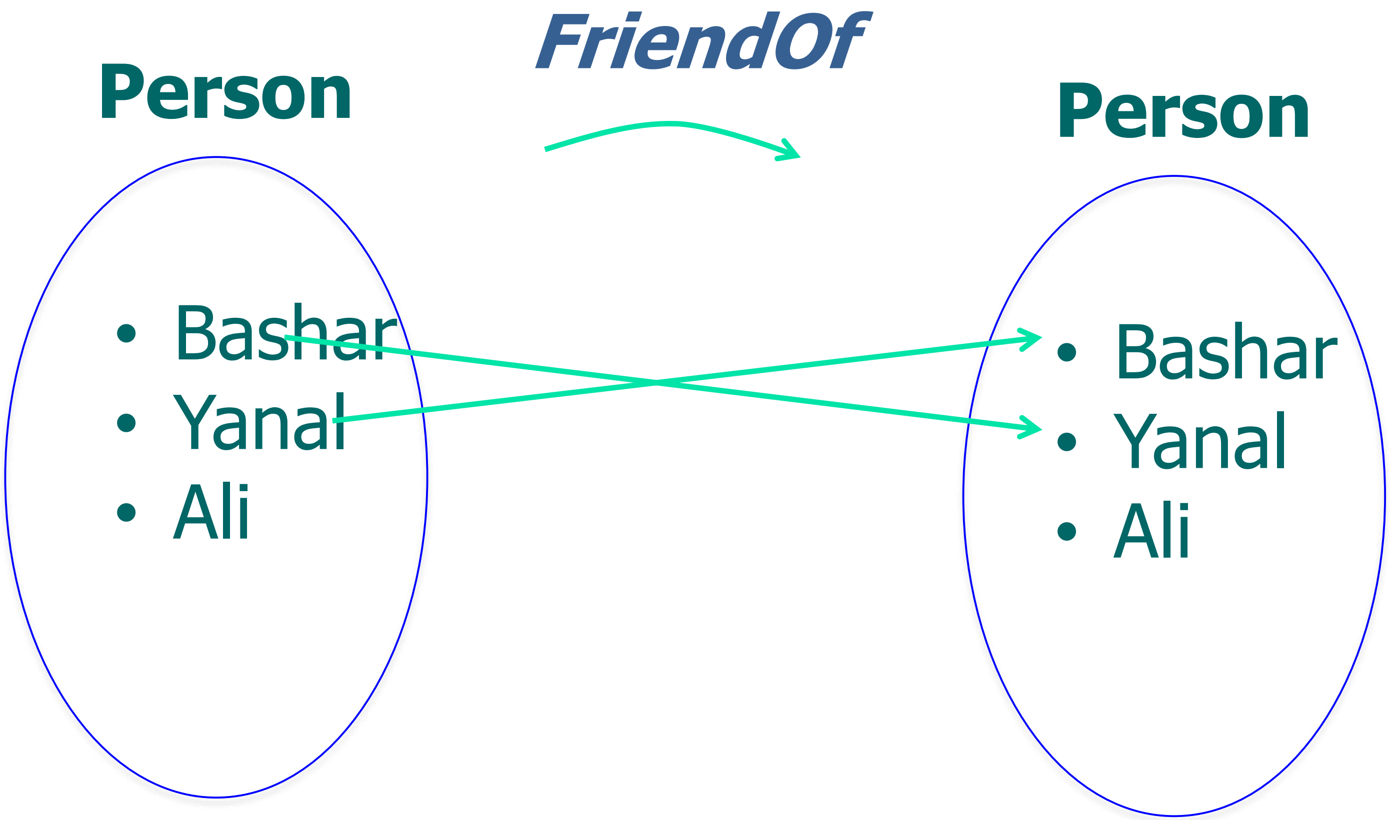
Representing Relations1: ordered pairs

- *R* = {(3,2), (5,2), (5,4), (7,2), (7,4), (7,6)}.
- *EnrolledIn* ={(Ali, COMP233), (Sana,ENG231)}
- *FriendOf* = {(Bashar, Ynal), (Ynal, Bashar)}

Representing Relations2: Arrow diagram

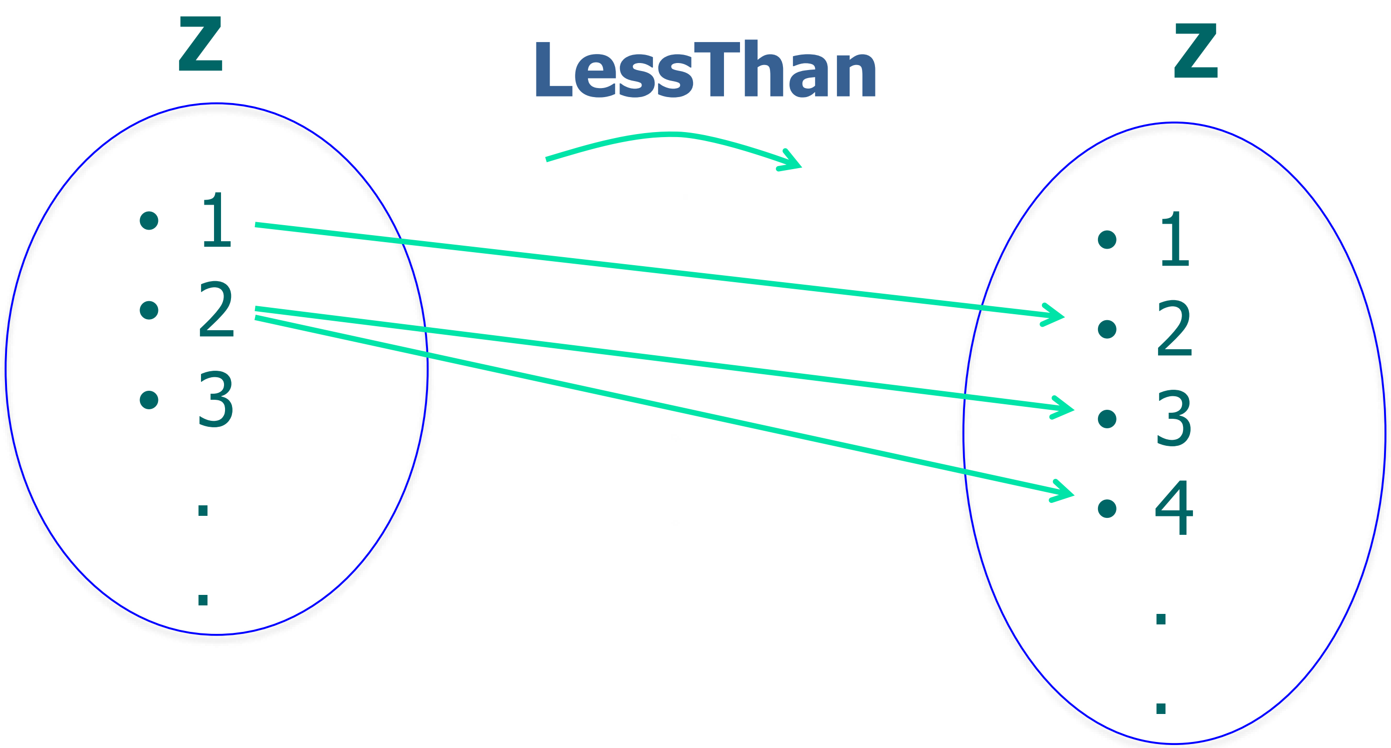


Representing Relations2: Arrow diagram

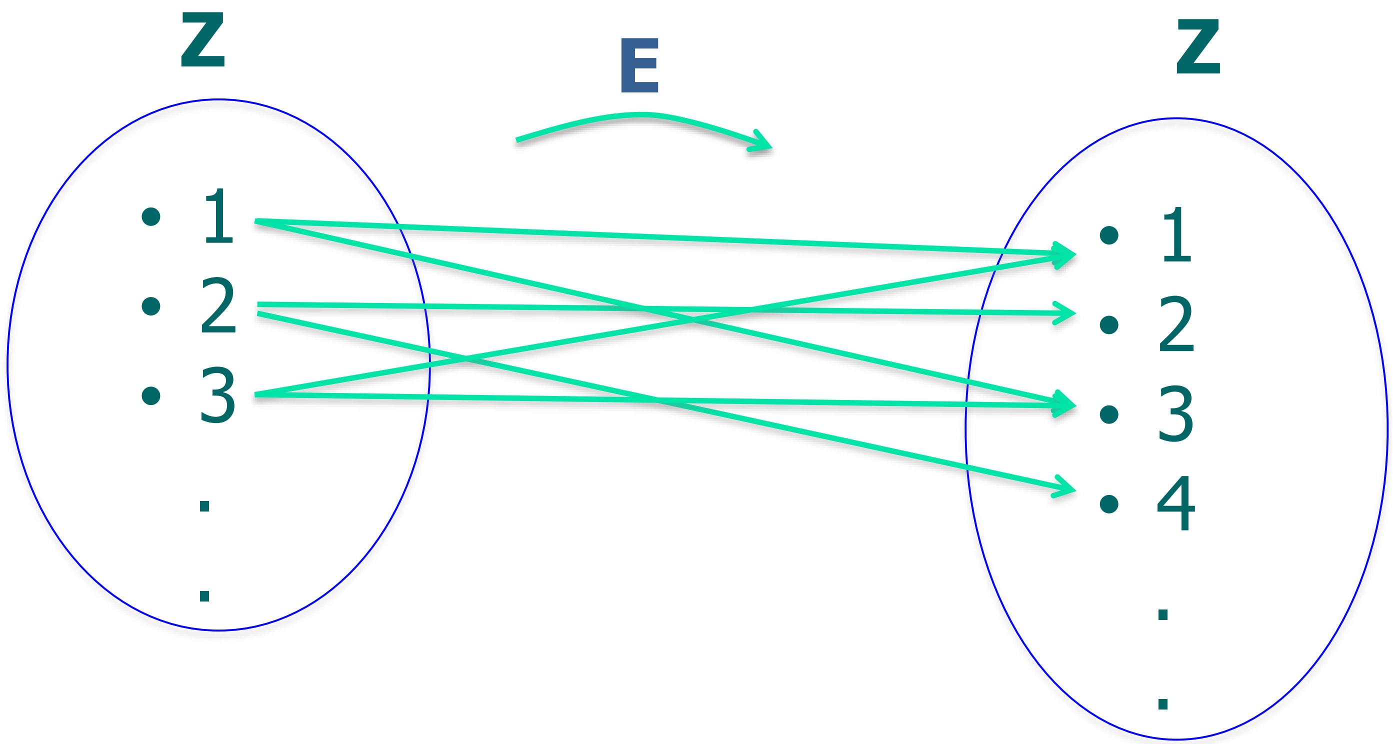


FriendOf = {(Bashar, Ynal), (Ynal, Bashar)}

Representing Relations2: Arrow diagram



Representing Relations2: Arrow diagram



Let E be a relation from **Z** to **Z** as follows:

For all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$, $m\ E\ n \Leftrightarrow m - n$ is even.

Example

Define a relation E from **Z** to **Z** as follows: For all $(m,n) \in$ **Z** × **Z**, $mEn \Leftrightarrow m-n$ is even.

a. Is $4\ E\ 0$? Is $2\ E\ 6$? Is $3\ E\ (-3)$? Is $5\ E\ 2$?

b. List five integers that are related by E to 1.

c. Prove that if n is any odd integer, then $n\ E\ 1$.

a. Yes,$4\ E\ 0$ because $4-0=4$ and 4 is even.
 Yes, $2\ E\ 6$ because $2-6=-4$ and -4 is even.
 Yes, $3\ E\ (-3)$ because $3-(-3)=6$ and 6 is even.
 No, $5\ \ E\ 2$ because $5-2=3$ and 3 is not even.

b. 1 because$1-1=0$ is even, 3 because $3-1=2$ is even, 5 because $5-1=4$ is even, -1 because $-1-1=-2$ is even, -3 because $-3-1=-4$ is even.

Example

Define a relation E from **Z** to **Z** as follows: For all $(m,n) \in \mathbf{Z} \times \mathbf{Z}$, $mEn \Leftrightarrow m-n$ is even.

a. Is $4\ E\ 0$? Is $2\ E\ 6$? Is $3\ E\ (-3)$? Is $5\ E\ 2$?

b. List five integers that are related by E to 1.

c. Prove that if n is any odd integer, then $n\ E\ 1$.

c. Suppose n is any odd integer.
Then $n = 2k + 1$ for some integer k.
By definition of E, $n\ E\ 1$ if, and only if, $n - 1$ is even.
By substitution,
$n - 1 = (2k + 1) - 1 = 2k$,
and since k is an integer, $2k$ is even.
Hence $n\ E\ 1$ *[as was to be shown]*.

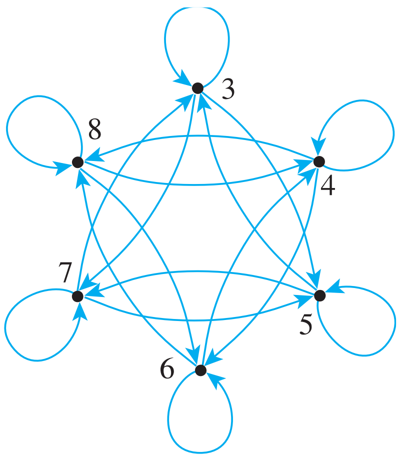
Directed Graphs of a relation from a set to itself

When a relation *R* is defined *on* a set *A*, the arrow diagram of the relation can be modified so that it becomes a **directed graph.**

For all points *x* and *y* in *A*, there is an arrow from

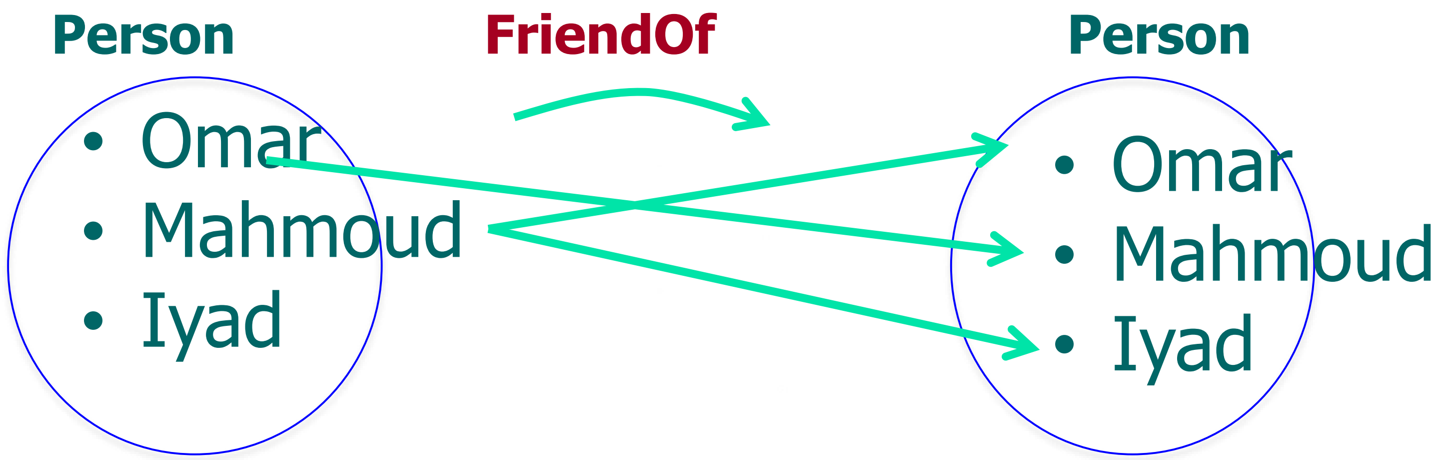
x to $y \Leftrightarrow \quad x\,R\,y \quad \Leftrightarrow \quad (x,y)\ R$.

Definition. A relation on a set A is a relation from A to A.



It is important to distinguish clearly between a relation and the set on which it is defined.

So far two representation styles



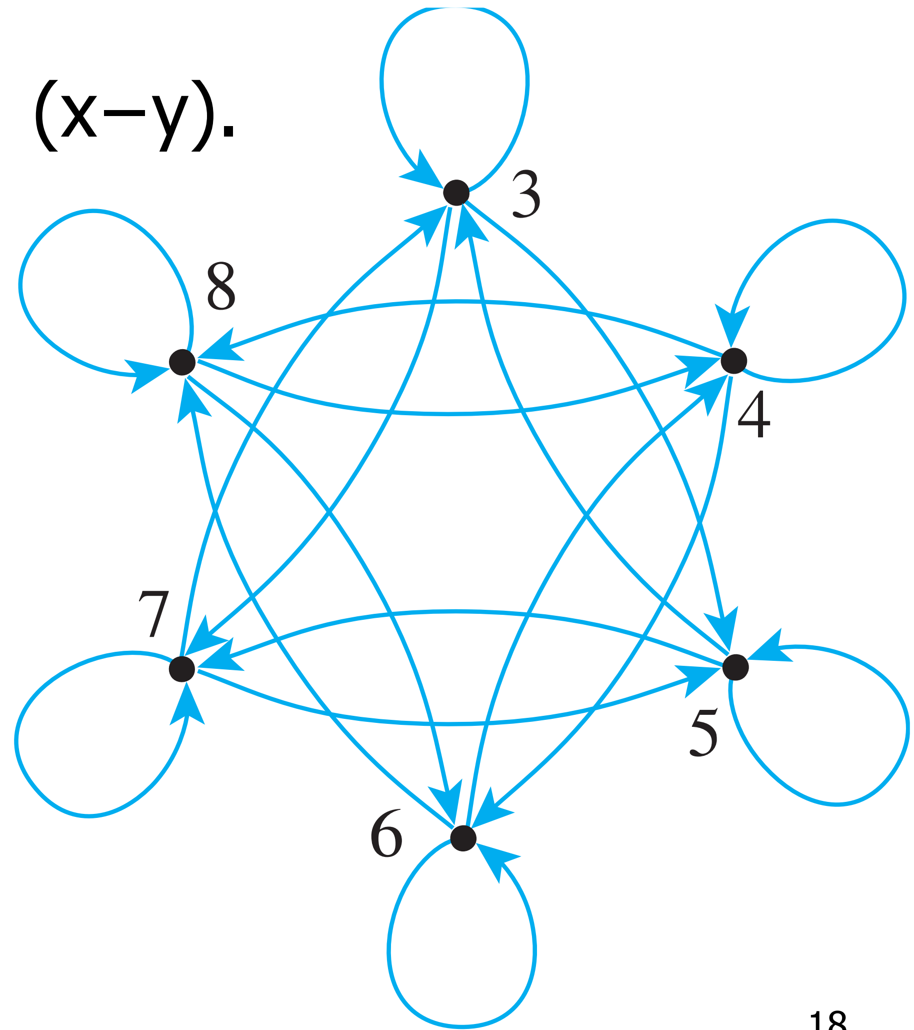
***FriendOf* = {(Omar, Mahmoud), (Mahmoud,Omar), (Mahmoud, Iyad)}**

How would you Represent relations on As a directed graph?

Example

Let A = {3, 4, 5, 6, 7, 8} and define a relation R on A as follows:

For all x, y ∈ A, x R y ⇔ 2 | (x−y).



Draw an arrow from one point to another ⇔ the first point is related to the second.

Exercise

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows:

For all $x, y \in A$, $x \, R \, y \Leftrightarrow 3 \mid (x-y)$.

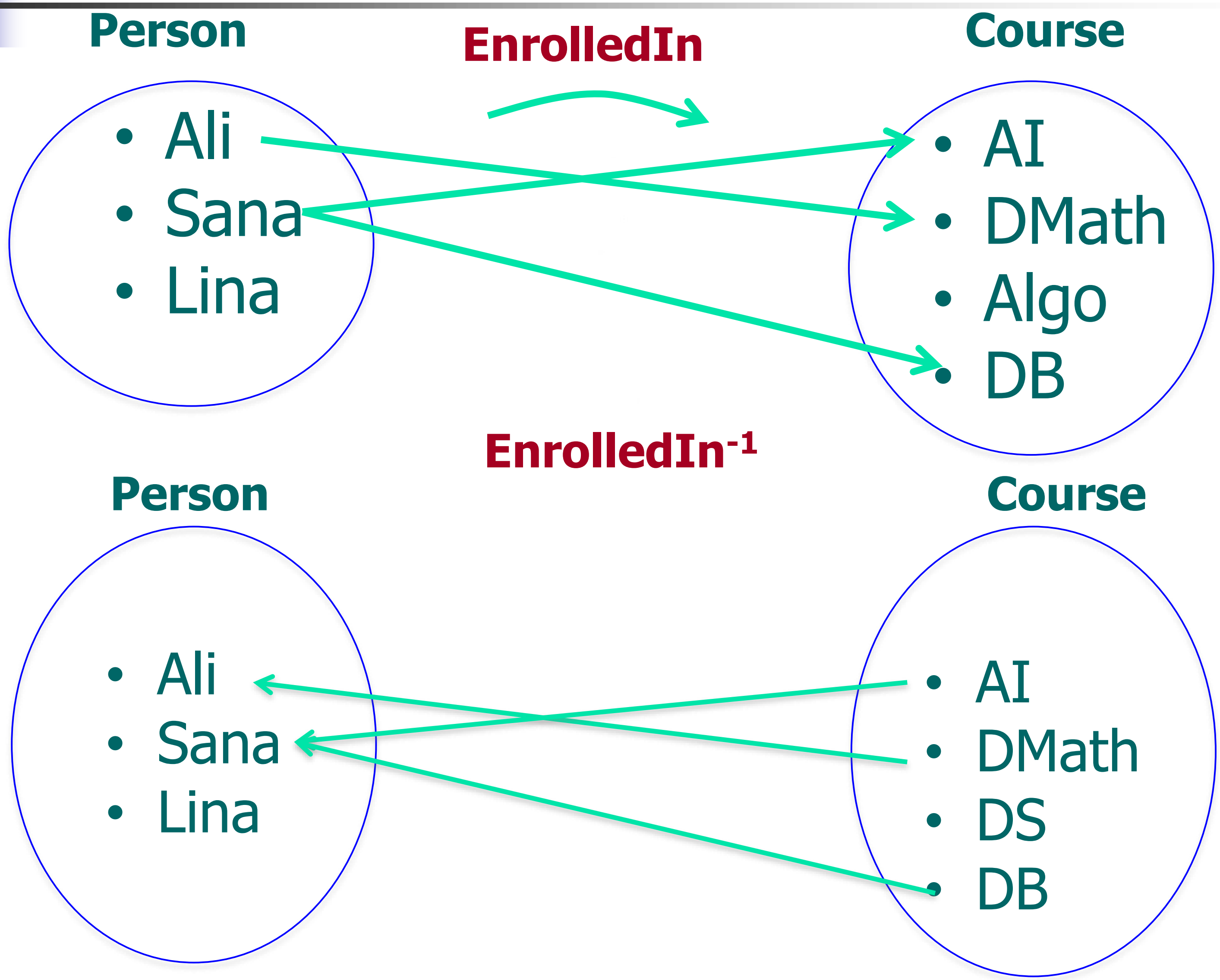
Inverse Relation R^{-1}

Let R be a relation from A to B.
Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}.$$

For all $x \in A$ and $y \in B$, $(y,x) \in R^{-1} \Leftrightarrow (x,y) \in R$.

Recall


Person
EnrolledIn
Course
Ali
Sana
Lina
AI
DMath
Algo
DB
EnrolledIn-1
Person
Course
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DB

Exercises

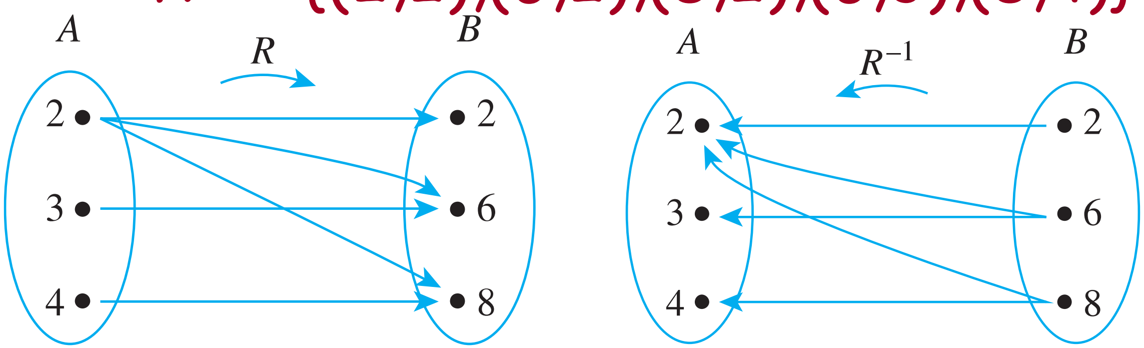
Let $A = \{2,3,4\}$ and $B = \{2,6,8\}$ and let R be the "divides" relation from A to B: For all $(x, y) \in A \times B$,

$$x\, R\, y \Leftrightarrow x \mid y \qquad x \text{ divides } y.$$

State explicitly which ordered pairs are in R and R^{-1}, and draw arrow diagrams for R and R^{-1}

$R = \{(2,2),(2,6),(2,8),(3,6),(4,8)\}$

$R^{-1} = \{(2,2),(6,2),(8,2),(6,3),(8,4)\}$



Describe R^{-1} in words:

For all $(y, x) \in B \times A$, $y\, R^{-1}\, x \Leftrightarrow y$ is a multiple of x.

Inverse Relation R^{-1}

What would be the inverse of the following relations in English

$SonOf^{-1}$ = ?

$WifeOf^{-1}$ = ?

$WorksAt^{-1}$ = ?

$EnrolledIn^{-1}$ = ?

$PresidentOf^{-1}$ = ?

$BrotherOf^{-1}$ = ?

$SisterOf^{-1}$ = ?

N-ary Relations

- **Definition**

Given sets $A_1, A_2, \ldots, A_n$, an ***n*-ary relation** R on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The special cases of 2-ary, 3-ary, and 4-ary relations are called **binary, ternary,** and **quaternary relations,** respectively.

N-ary Relations

EnrolledIn(Ali, Dmath) EnrolledIn(Sami, DB)	Binary (2-ary)
Enrollment(Sami, DB, 99)	Ternary (3-ary)
Enrollment(Sami, DB, 99, 2014)	Quaternary (4-ary)
Enrollment(Sami, DB, 99, 2014,F)	5-ary
$R(a_1, a_2, a_3, \ldots.., a_n)$	*n*-ary

Relations

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- ❑ Part 5: **Relational Databases**

Relational Databases

Let $A1$ be a set of positive integers,
$A2$ a set of alphabetic character strings,
$A3$ a set of numeric character strings,
$A4$ a set of alphabetic character strings.
Define a quaternary relation R on $A1 \times A2 \times A3 \times A4$ as follows:
$(a1, a2, a3, a4) \in R \Leftrightarrow$ a patient with patient ID number $a1$, name $a2$, was admitted on date $a3$, with primary diagnosis $a4$.

Patient(ID, Name, Date, **Diagnosis)**

(011985, John Schmidt, 020710, asthma)
(574329, Tak Kurosawa, 114910, pneumonia)
(466581, Mary Lazars, 103910, appendicitis)
(008352, Joan Kaplan, 112409, gastritis)
(011985, John Schmidt, 021710, pneumonia)
(244388, Sarah Wu, 010310, broken leg)
(778400, Jamal Baskers, 122709, appendicitis)

Relational Databases

Simplified version of a database that might be used in a hospital

Define R on $A1 \times A2 \times A3 \times A4$ as follows:

$(a1, a2, a3, a4) \in R \Leftrightarrow$ a patient with patient ID number $a1$, named $a2$, was admitted on date $a3$, with primary diagnosis $a4$.

Each table in the database represents a **Relation**

Patient

ID	Name	Date	Diagnosis
(011985,	John Schmidt,	020710,	asthma)
(574329,	Tak Kurosawa,	114910,	pneumonia)
(466581,	Mary Lazars,	103910,	appendicitis)
(008352,	Joan Kaplan,	112409,	gastritis)
(011985,	John Schmidt,	021710,	pneumonia)
(244388,	Sarah Wu,	010310,	broken leg)
(778400,	Jamal Baskers,	122709,	appendicitis)

Each row in the table is called **tuple**

Relational Databases

Simplified version of a database that might be used in a hospital

Define R on $A1 \times A2 \times A3 \times A4$ as follows:

$(a1, a2, a3, a4) \in R \Leftrightarrow$ a patient with patient ID number $a1$, named $a2$, was admitted on date $a3$, with primary diagnosis $a4$.

Each table in the database represents a Relation

Patient

ID	Name	Date	Diagnosis
(0			
(5			
(4			
(0			
(0			
(2			
(7			

Each row in the table is called tuple

- Notice that **Tables** in this way are called **Relations**.
- Information stored in this way is called a "**Relational Database**"

Relations

8.2 Properties of Relations

In this lecture:

- Part 1: **Properties: Reflexivity, Symmetry, Transitivity**
- Part 2: Proving Properties of Relations
- Part 3: Transitive Closure

Properties of Relations

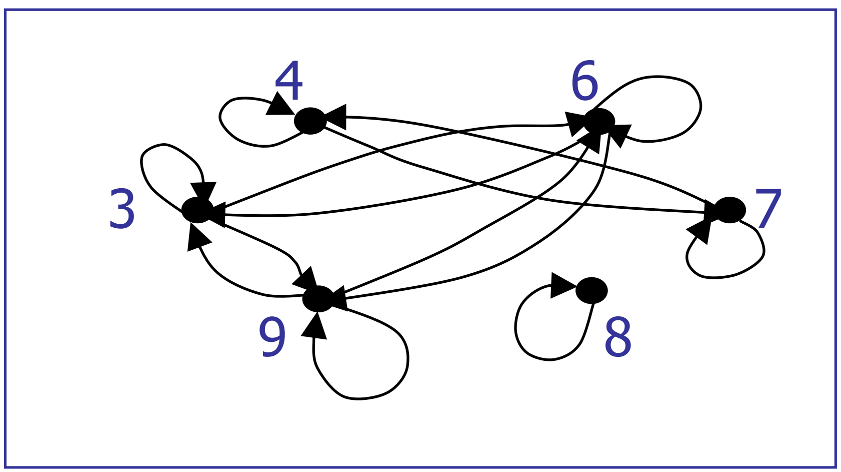
Definition: Let A be a set and let R be a binary relation "on" A.
(i.e., R is a binary relation from A to A).

R is **reflexive** $\Leftrightarrow \forall\ x$ in A, $x\ R\ x$.

R is **symmetric** $\Leftrightarrow \forall\ x$ and y in A, if $x\ R\ y$ then $y\ R\ x$.

R is **transitive** $\Leftrightarrow \forall\ x$, y, and z in A, if $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$.

R is an **equivalence relation** $\Leftrightarrow$ R is reflexive, symmetric, and transitive.



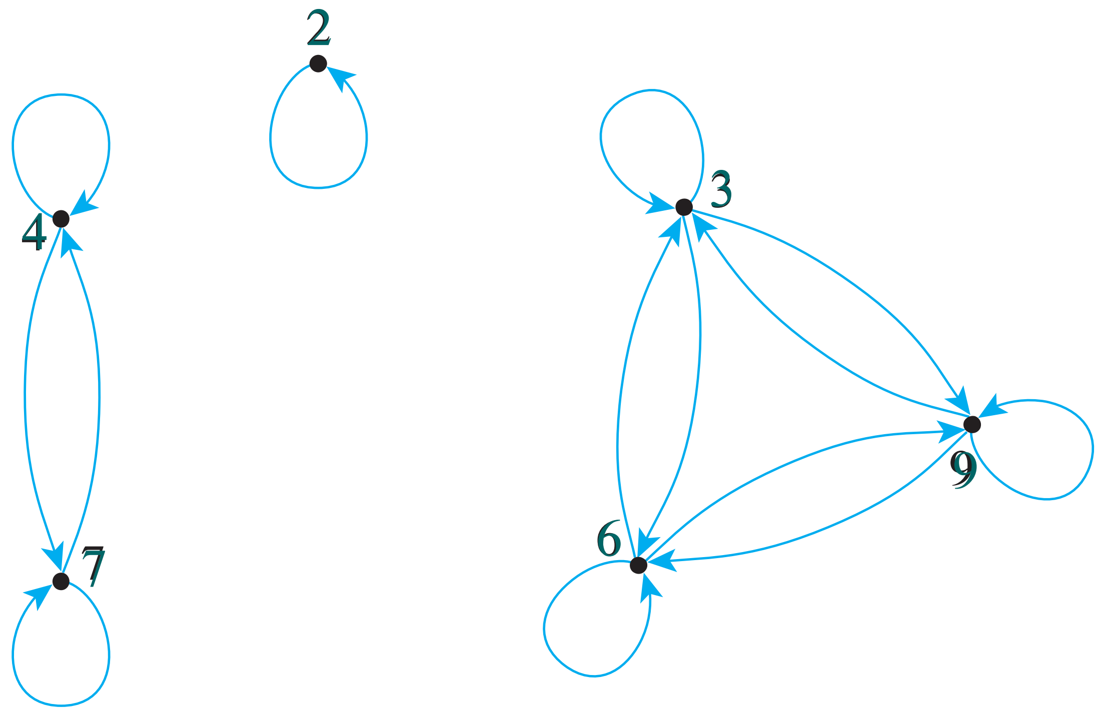
Example: Consider the binary relation S defined on the set {3, 4, 6, 7, 8, 9} with directed graph shown at the left.

a. Is S reflexive?
b. Is S symmetric?
c. Is S transitive?
d. Is S an equivalence relation?

Exercise

Let $A = \{2,3,4,6,7,9\}$ and define a relation R on A as:
For all $x, y \in A$, $x \, R \, y \Leftrightarrow 3 | (x - y)$.

Is R Reflexive? ✓ Symmetric? ✓ Transitive? ✓

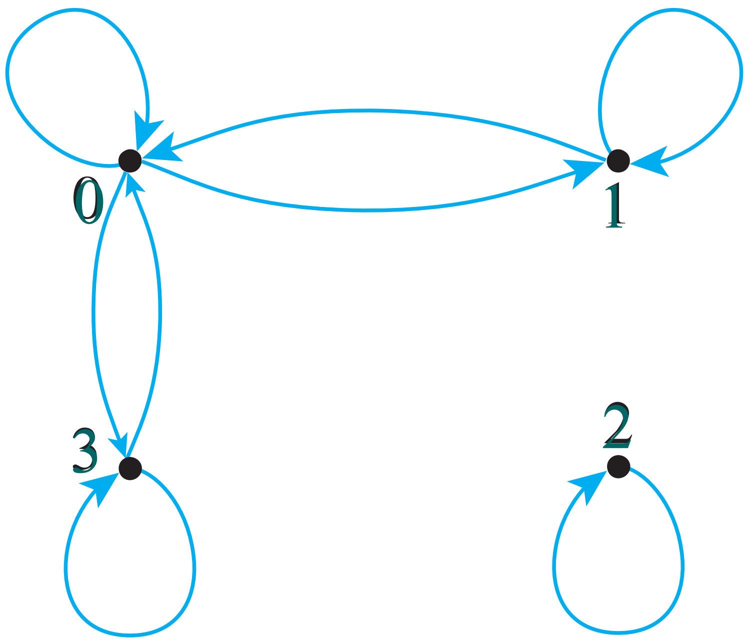


Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:

$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$

Is R Reflexive? ☑ Symmetric? ☑ Transitive? ☒

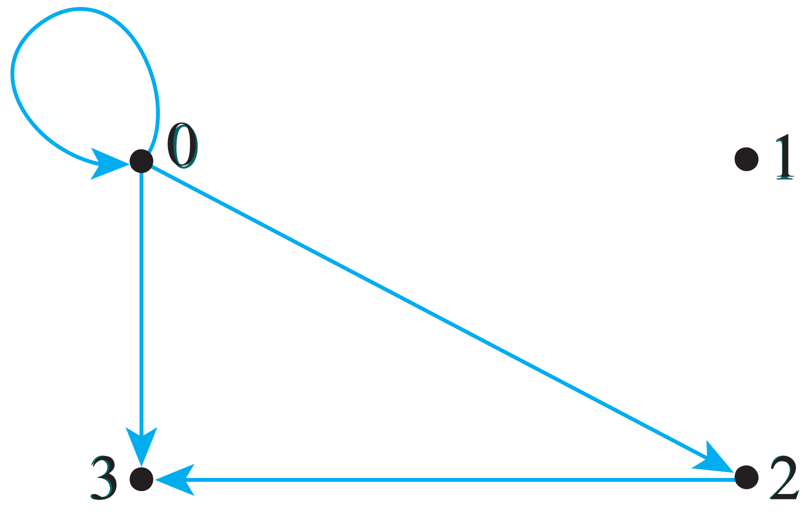


Exercise

Let $A = \{0, 1, 2, 3\}$ and define relation R on A as:

$R = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$

Is R Reflexive? ☒ *Symmetric?* ☒ *Transitive?* ☑

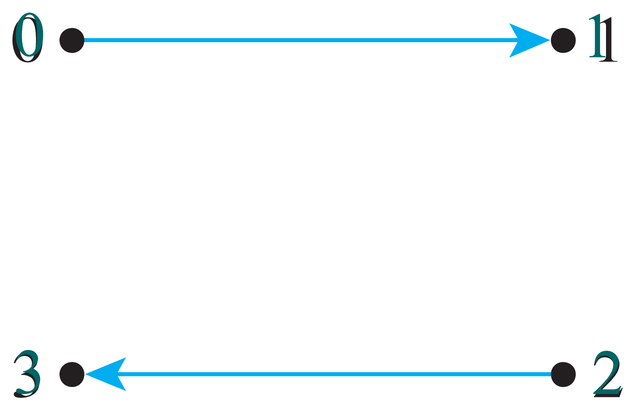


Exercise

Let A = {0, 1, 2, 3} and define relation R on A as:

R ={(0,1),(2,3)}

Is R Reflexive? ✗ *Symmetric?* ✗ *Transitive?* ✓



R is transitive by default because it is *not* not transitive!

Recall: Truth Table for →

In Logic (& Math, CS, etc.): The only time a statement of the form **if *p* then *q*** is false is when the hypothesis (p) is true and the conclusion (q) is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Note: When the hypothesis of an if-then statement is false, we say that the if-then statement is "vacuously true" or "true by default." In other words, it is true because it is not false.

Remark that the transitivity condition is **vacuously true** for T . To see this, observe that the transitivity condition says that $\forall x,y,z \in A$, if $[(x,y)\in T \wedge (y,z)\in T]$ then $[(x,z)\in T]$

Exercises

1. A = {BZU students}. Define R on A by:

$$x \, R \, y \Leftrightarrow x \text{ lives within 1 mile of } y.$$

Is R reflexive? Is R symmetric? Is R transitive?

Is R an equivalence relation?

2. $A = \{0, 1, 2, 3\}$. Define R on A by:

$$R = \{(1,3), (2,3)\}$$

Is R reflexive? Is R symmetric? Is R transitive?
Is R an equivalence relation?

Equivalence Relation

علاقة تكافؤ

Definition

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

➔ The relation induced by a partition is an equivalence relation

Example

Let X be the set of all nonempty subsets of {1, 2, 3}. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation R on X as follows: For all A and B in X,

$$A\ R\ B \Leftrightarrow \text{the least element of } A \text{ equals the least element of } B.$$

Prove that R is an equivalence relation on X.

R *is reflexive*: Suppose A is a nonempty subset of {1, 2, 3}.
[We must show that A **R** *A.]*
It is true to say that the least element of A equals the least element of A.
Thus, by definition of R, A **R** A.

Example

Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation R on X as follows: For all A and B in X,

$A\ R\ B \Leftrightarrow$ the least element of A equals the least element of B.

Prove that R is an equivalence relation on X.

R *is reflexive.*

R *is symmetric*:

Suppose A and B are nonempty subsets of $\{1, 2, 3\}$ and A **R** B.

*[We must show that B **R** A.]*

Since A **R** B, the least element of A equals the least element of B.

But this implies that the least element of B equals the least element of A, and so, by definition of **R**, B **R** A.

Example

Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Define a relation R on X as follows: For all A and B in X,

$$A\ R\ B \Leftrightarrow \text{the least element of } A \text{ equals the least element of } B.$$

Prove that R is an equivalence relation on X.

R *is reflexive.*

R *is symmetric.*

R *is transitive*:

Suppose A, B, and C are nonempty subsets of $\{1, 2, 3\}$,
A **R** B, and $B\ R\ C$.
*[We must show that A **R** C.]*
Since A **R** B, the least element of A equals the least element of B
since B **R** C, the least element of B equals the least element of C.
Thus the least element of A equals the least element of C, and so, by definition of **R**, A **R** C.

Relations

8.2 Properties of Relations

In this lecture:

- Part 1: Properties: Reflexivity, Symmetry, Transitivity
- Part 2: **Proving Properties of Relations**
- Part 3: Transitive Closure

Proving Properties on Relations on Infinite Sets

Until now we discussed relations on Finite Sets

What about relations on infinite Sets? We need to proof!

Outline of proof.
To prove a relation is reflexive, symmetric, or transitive, first write down what is to be proved, in **First Order Logic**.

For instance, for symmetry
$\forall x, y \in A$, if $x\ R\ y$ then $y\ R\ x$.

Then use **direct methods** of proving

Recall: Definition of Relation Properties and their consequences

Let A be a set and let R be a binary relation on A. Complete the following sentences.

R is not reflexive $\Leftrightarrow$ there is an element x in A such that $x \not R x$ *[that is, such that* $(x, x) \notin R$*]*.

R is not symmetric $\Leftrightarrow$ there are elements x and y in A such that $x R y$ but $y \not R x$ *[that is, such that* $(x, y) \in R$ *but* $(y, x) \notin R$*]*.

R is not transitive $\Leftrightarrow$ there are elements x, y and z in A such that $x R y$ and $y R z$ but $x \not R z$ *[that is, such that* $(x, y) \in R$ *and* $(y, z) \in R$ *but* $(x, z) \notin R$*]*.

Properties of Less than

Define a relation R on R (the set of all real numbers) as follows:
For all x, y ∈ R, x R y ⇔ x < y.

Is R Reflexive? Symmetric? Transitive?

Solution

R is not reflexive:
R is reflexive if, and only if, ∀x ∈ R, x R x.
By definition of R, this means that ∀x ∈R, x<x.
But this is false: ∃x ∈R such that x is not less than x.
As a counterexample, let x = 0.
$0 \nless 0$

Properties of Less than

Define a relation R on R (the set of all real numbers) as follows:
For all x, y ∈ R, $x\ R\ y \Leftrightarrow x < y$.

Is R Reflexive? Symmetric? Transitive?

Solution

R is not symmetric:
R is symmetric if, and only if, $\forall x, y \in R$, if x R y then y R x. .
By definition of R, this means that $\forall x, y \in R$, if $x < y$ then $y < x$.
But this is false: $\exists x, y \in R$ such that $x < y$ and y not $< x$.
As a counterexample, let $x = 0$ and $y = 1$.
$0 < 1$ but $1 \nless 0$.

Properties of Less than

Define a relation R on R (the set of all real numbers) as follows:
For all x, y ∈ R, $x \, R \, y \Leftrightarrow x < y$.

Is R Reflexive? Symmetric? Transitive?

Solution

R is transitive:

R is transitive if, and only if, for all x, y, z ∈ R, if x R y and y R z then x R z.
By definition of R,
this means that ∀x, y, z ∈ R, if $x < y$ and $y < z$, then $x < z$. But this statement is true by the transitive law of order for real numbers (Appendix A, T18).

Properties of Less than

Define a relation R on R (the set of all real numbers) as follows:
For all x, y ∈ R, x R y ⇔ x < y.

Is R Reflexive? Symmetric? Transitive?

Solution

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By definition of R, this means that ∀x ∈R, x<x.
But this is false: ∃x ∈R such that x is not less than x.
As a counterexample, let x = 0.

R is not symmetric: R is symmetric if, and only if, ∀x, y ∈ R, if x R y then y R x. By definition of R, this means that ∀x, y ∈ R, if x < y then y < x. But this is false: ∃x, y ∈ R such that x < y and y not < x. As a counterexample, let x = 0 and y = 1. 0 < 1 but 1 not < 0.

R is transitive: R is transitive if, and only if, for all x, y, z ∈ R, if x R y and y R z then x R z. By definition of R, this means that ∀x, y, z ∈ R, if x < y and y < z, then x < z. But this statement is true by the transitive law of order for real numbers (Appendix A, T18).

Properties of Congruence Modulo 3

$$m\ T\ n \Leftrightarrow 3|(m-n).$$

Is R Reflexive? *Symmetric?* *Transitive?*

For all $m \in \mathbf{Z}$, $3|(m-m)$.

Suppose m is a particular but arbitrarily chosen integer. *[We must show that m T m.]*

Now, $m-m = 0$.
But $3 \mid 0$ since $0 = 3 \cdot 0$.
Hence $3|(m-m)$.
Thus, by definition of T, $m\ T\ m$
[as was to be shown].

Properties of Congruence Modulo 3

$$m\ T\ n \Leftrightarrow 3|(m-n).$$

Is R Reflexive? *Symmetric?* *Transitive?*

For all $m, n \in \mathbf{Z}$, **if** $3|(m-n)$ then $3|(n-m)$.

Suppose *m* and *n* are particular but arbitrarily chosen integers that satisfy the condition *m T n.*

[We must show that n T m.]

- By definition of *T*, since *m T n* then 3 | (m − *n).*
- By definition of "divides" this means that *m* − *n* = 3k, for some integer *k.*
- Multiplying both sides by −1 gives *n* − *m* = 3(−k).
- Since −k is an integer, this equation shows that 3 | (n − *m).*
- Hence, by definition of *T*, *n T m [as was to be shown].*

Properties of Congruence Modulo 3

$$m \, T \, n \Leftrightarrow 3|(m-n).$$

Is R Reflexive? *Symmetric?* *Transitive?*

For all $m, n \in \mathbf{Z}$, **if** $3|(m-n)$ and $3|(n-p)$ then $3|(m-p)$.

Suppose *m, n,* and *p* are particular but arbitrarily chosen integers that satisfy the condition $m \, T \, n$ and $n \, T \, p$.
[We must show that m T p.]

- By definition of T, since m T n *and* $n \, T \, p$, *then* $3|(m-n)$ *and* $3|(n-p)$.
- *By* definition of "divides" this means that $m - n = 3r$ and $n - p = 3s$, for some integers r and s.
- Adding the two equations gives $(m-n)+(n-p)=3r+3s$, and simplifying gives that $m - p = 3(r + s)$. Since $r + s$ is an integer, this equation shows that $3|(m - p)$.
- Hence, by definition of *T*, $m \, T \, p$ *[as was to be shown].*

D is the "divides" relation on $\mathbf{Z}^+$: For all positive integers m and n, $m \; D \; n \Leftrightarrow m \mid n$.

Define a relation Q on $\mathbf{R}$ as follows: For all real numbers x and y, $x \; Q \; y \Leftrightarrow x - y$ is rational.

Properties of Equality

Define a relation R on **R** (the set of all real numbers) as follows:
For all real numbers x and y. $x \, R \, y \Leftrightarrow x = y$.

Is R Reflexive? *Symmetric?* *Transitive?*

***R is reflexive*:**
R is reflexive if, and only if, the following statement is true:
For all $x \in \mathbf{R}$, $x \, R \, x$.
And since $x \, R \, x$ just means that $x = x$,
this is the same as saying For all $x \in \mathbf{R}$, $x = x$.
Which is true; every real number is equal to it

Properties of Equality

Define a relation R on **R** (the set of all real numbers) as follows:
For all real numbers x and y. $x\,R\,y \Leftrightarrow x = y$.

Is R Reflexive? Symmetric? Transitive?

***R is symmetric*:** R is symmetric if, and only if, the following statement is true:
For all $x,y \in$ **R**, **if** $x\,R\,y$ then $y\,R\,x$.

By definition of R, $x\,R\,y$ means that $x = y$
and $y\,R\,x$ means that $y = x$.
Hence R is symmetric if, and only if,
For all $x,y \in$ **R**, **if** $x = y$ then $y = x$.

This statement is true; if one number is equal to a second, then the second is equal to the first.

Properties of Equality

Define a relation R on **R** (the set of all real numbers) as follows: For all real numbers x and y. $x \, R \, y \Leftrightarrow x = y$.

Is R Reflexive? *Symmetric?* *Transitive?*

R is transitive: R is transitive if, and only if, the following statement is true: For all $x,y,z \in$ **R**, **if** $x \, R \, y$ and $y \, R \, z$ then $x \, R \, z$.

By definition of R, $x \, R \, y$ means that $x = y$, $y \, R \, z$ means that $y = z$, and $x \, R \, z$ means that $x = z$. Hence R is transitive iff the following statement is true: For all $x,y,z \in$ **R**, **if** $x=y$ and $y=z$ then $x=z$.

This statement is true: If one real number equals a second and the second equals a third, then the first equals the third.

Relations

8.2 Properties of Relations

In this lecture:

- ❑ Part 1: Properties: Reflexivity, Symmetry, Transitivity
- ❑ Part 2: Proving Properties of Relations
- ❑ Part 3: **Transitive Closure**

Transitive Closure of a Relation

A relation fails to be transitive because it fails to contain certain ordered pairs. For example, if (1, 3) and (3, 4) are in a relation *R*, then the pair (1, 4) ***must*** be in *R* if *R* is to be transitive.

To obtain a transitive relation from one that is not transitive, it is **necessary** to add ordered pairs.

The relation obtained by adding the least number of ordered pairs to ensure transitivity is called the ***transitive closure*** of the relation.

Formal definition, the transitive closure of a relation is the smallest transitive relation that contains the relation.

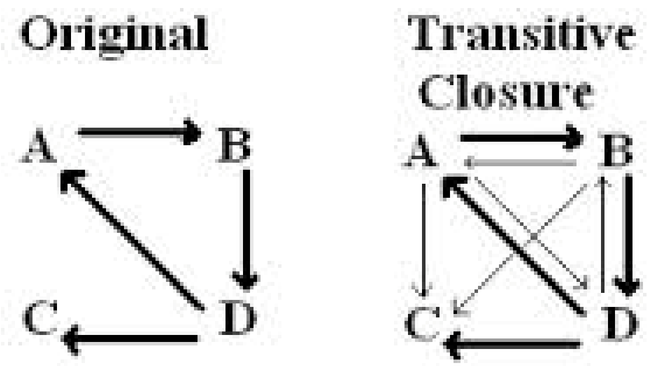
Transitive Closure of a Relation

The **smallest** transitive relation that contains the relation.

• Definition

Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the following three properties:

1. R^t is transitive.
2. $R \subseteq R^t$.
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$.



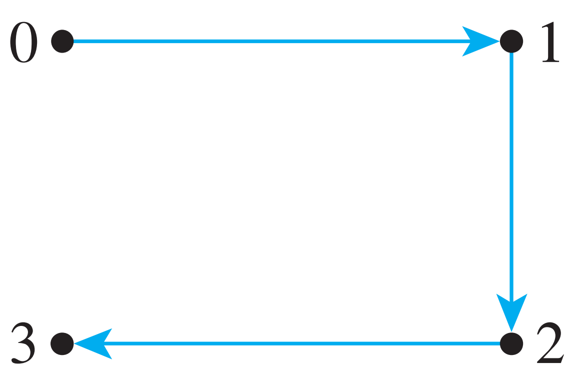
Exercise

Let $A = \{0, 1, 2, 3\}$ and consider the relation R defined on A as:

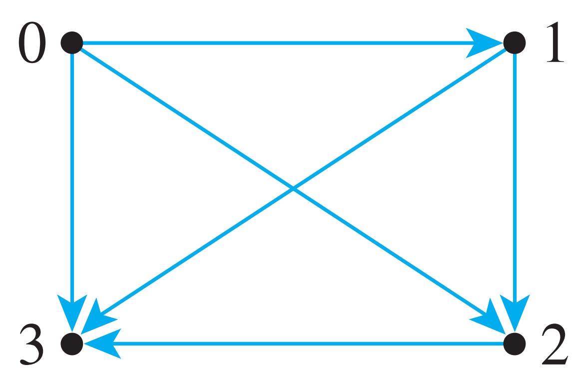
$R = \{(0, 1), (1, 2), (2, 3)\}$.

Find the transitive closure of R.

$R^t = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$.



R



R^t

Relations

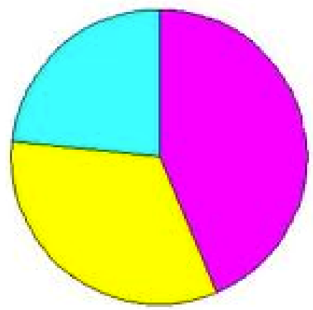
In this lecture:

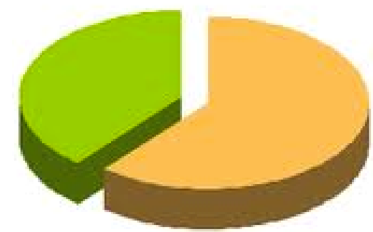
- Part 1: **Partitioned Sets**
- Part 2: **Equivalence Classes**
- Part 3: **Equivalence Relation**

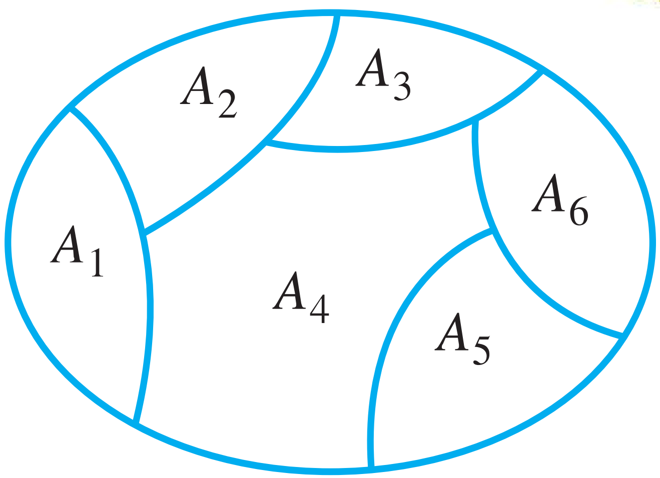
Partitioned Sets

Sets can be partitioned into disjoint sets

A **partition** of a set A is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is A.







تقسيم جامع مانع

Total (جامع)

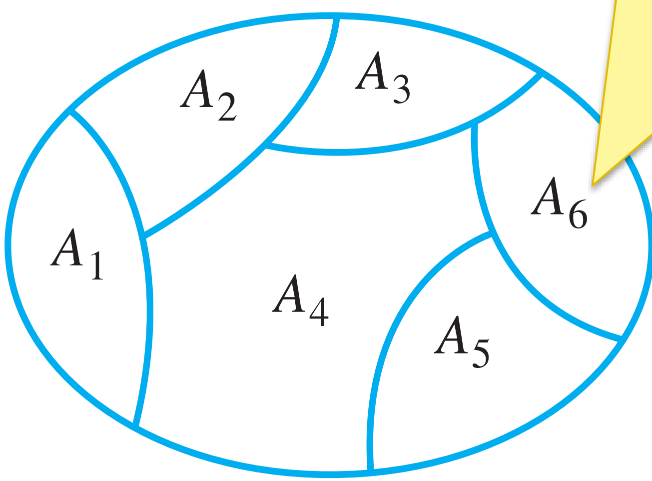
$A_i \cup A_2 \cup \ldots \cup A_6 = A$

Disjoint (مانع)

$A_i \cap A_j = \phi$, whenever $i \neq j$

Relations Induced by a Partition

A **relation induced by a partition**, is a relation between two elements in the same partition.



تقسيم جامع مانع

Total (جامع)

$A_i \cup A_2 \cup \ldots \cup A_6 = A$

Disjoint (مانع)

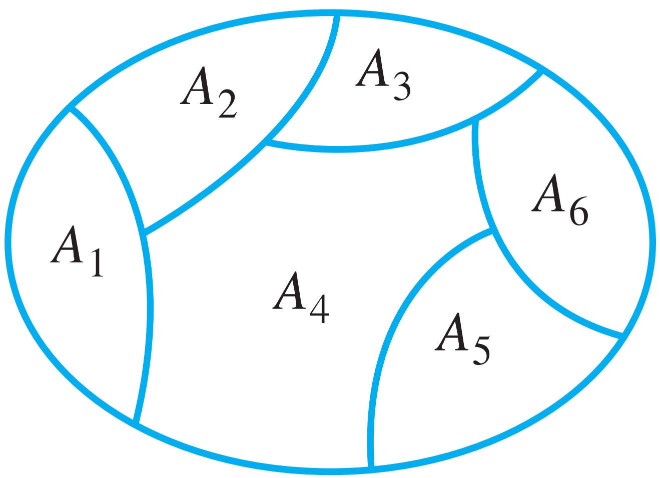
$A_i \cap A_j = \phi$, whenever $i \neq j$

Relations Induced by a Partition

- **Definition**

Given a partition of a set A, the **relation induced by the partition,** R, is defined on A as follows: For all $x, y \in A$,

$$x \; R \; y \quad \Leftrightarrow \quad \text{there is a subset } A_i \text{ of the partition such that both } x \text{ and } y \text{ are in } A_i.$$



تقسيم جامع مانع

Total (جامع)

$A_i \cup A_2 \cup \ldots \cup A_6 = A$

Disjoint (مانع)

$A_i \cap A_j = \phi$, whenever $i \neq j$

Example

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:

$\{0, 3, 4\}, \{1\}, \{2\}$.

Find the relation R induced by this partition.

Since {0, 3, 4} is a subset of the partition,

	$0 \; R \; 3$	because both 0 and 3 are in {0, 3, 4},	
	$3 \; R \; 0$	because both 3 and 0 are in {0, 3, 4},	
	$0 \; R \; 4$	because both 0 and 4 are in {0, 3, 4},	
	$4 \; R \; 0$	because both 4 and 0 are in {0, 3, 4},	
	$3 \; R \; 4$	because both 3 and 4 are in {0, 3, 4},	and
	$4 \; R \; 3$	because both 4 and 3 are in {0, 3, 4}.	
Also,	$0 \; R \; 0$	because both 0 and 0 are in {0, 3, 4}	
	$3 \; R \; 3$	because both 3 and 3 are in {0, 3, 4},	and
	$4 \; R \; 4$	because both 4 and 4 are in {0, 3, 4}.	

Example

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:

$$\{0, 3, 4\}, \{1\}, \{2\}.$$

Find the relation R induced by this partition.

Since {1} is a subset of the partition,

$1\ R\ 1$ because both 1 and 1 are in {1},

and since {2} is a subset of the partition,

$2\ R\ 2$ because both 2 and 2 are in {2}.

Hence

$R = \{(0,0),(0,3),(0,4),(1,1),(2,2),(3,0), (3,3),(3,4),(4,0),(4,3),(4,4)\}.$

Relations Induced by a Partition

Theorem 8.3.1

Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

Relations

8.3 Equivalence Relations

In this lecture:

- ❑ Part 1: **Partitioned Sets**
- ❑ Part 2: **Equivalence Relation**
- ❑ Part 3: **Equivalence Classes**

Equivalence Class

- **Definition**

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of *a*,** denoted **[*a*]** and called the **class of *a*** for short, is the set of all elements x in A such that x is related to a by R.

In symbols:

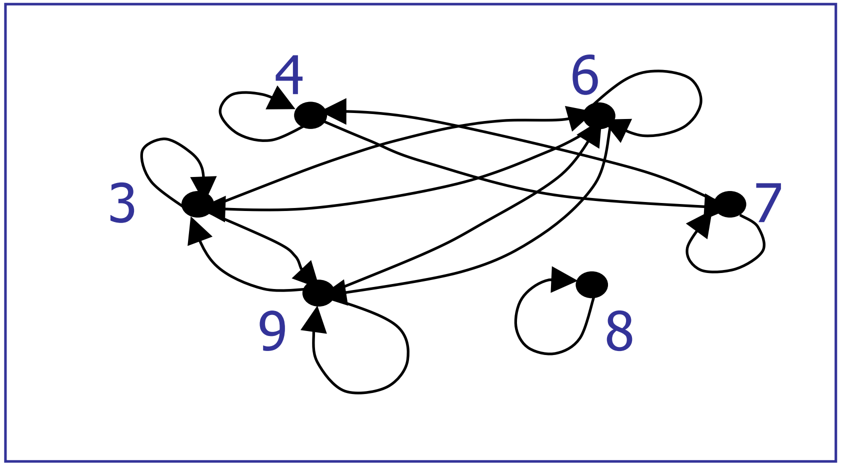
$$[a] = \{x \in A \mid x \; R \; a\}$$

$$\text{for all } x \in A, \quad x \in [a] \quad \Leftrightarrow \quad x \; R \; a.$$

Equivalence Class of an Element

Definition: Let *R* be an equivalence relation from *A* to *A*, and suppose *a* is any element of *A*.
The **equivalence class of *a***, denoted by **[*a*]**, is defined as follows:
$[a] = \{x \in A \mid x \text{ is related to } a \text{ by } R\}$.

Example: Consider the binary relation *S* defined on the set {3, 4, 6, 7, 8, 9} with directed graph shown below. Find [3], [4], [6], [7], [8], and [9].



[3] = {3, 6, 9}
[4] = {4, 7}
[6] = {3, 6, 9}
[7] = {4, 7}
[8] = {8}
[9] = {3, 6, 9}

What are the *distinct* equivalence classes of this relation?

{3, 6, 9}, {4, 7}, {8}

Example

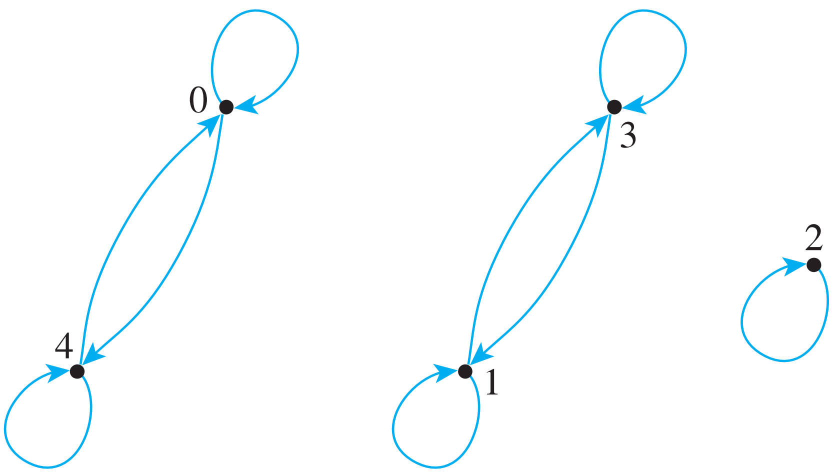
Let $A = \{0,1,2,3,4\}$ and define a relation R on A as :
$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$.

Find the distinct equivalence classes of R.

Example

Let $A = \{0,1,2,3,4\}$ and define a relation R on A as :
$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$.

Find the distinct equivalence classes of R.



$[0] = \{x \in A \mid x \; R \; 0\} = \{0, 4\}$

$[1] = \{x \in A \mid x \; R \; 1\} = \{1, 3\}$

$[2] = \{x \in A \mid x \; R \; 2\} = \{2\}$

$[3] = \{x \in A \mid x \; R \; 3\} = \{1, 3\}$

$[4] = \{x \in A \mid x \; R \; 4\} = \{0, 4\}$

$[0] = [4]$ and $[1] = [3]$.

Thus the *distinct* equivalence classes of the relation are
$\{0, 4\}$, $\{1, 3\}$, and $\{2\}$.

Equivalence Class

Lemma 8.3.2

Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If $a \; R \; b$, then $[a] = [b]$.

Lemma 8.3.3

If A is a set, R is an equivalence relation on A, and a and b are elements of A, then

$$\text{either} \quad [a] \cap [b] = \emptyset \quad \text{or} \quad [a] = [b].$$

- **Definition**

Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is any element a such that $[a] = S$.

Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set **Z** of all integers. That is, for all integers m and n,

$$m\ R\ n \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n \pmod 3.$$

Describe the distinct equivalence classes of R.

For each integer a,

$$\begin{aligned}[a] &= \{x \in \mathbf{Z} \mid x\ R\ a\} \\ &= \{x \in \mathbf{Z} \mid 3 \mid (x-a)\} \\ &= \{x \in \mathbf{Z} \mid x - a = 3k, \text{ for some integer } k\}.\end{aligned}$$

Therefore

$$[a] = \{x \in \mathbf{Z} \mid x = 3k + a, \text{ for some integer } k\}.$$

Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set $\mathbf{Z}$ of all integers. That is, for all integers m and n,

$$mRn \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n (\text{mod} 3).$$

In particular:

$$
\begin{aligned}
[0] &= \{x \in \mathbf{Z} \mid x = 3k + 0, \text{ for some integer } k\} \\
&= \{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\} \\
&= \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\}, \\
[1] &= \{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\} \\
&= \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\}, \\
[2] &= \{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\} \\
&= \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}.
\end{aligned}
$$

Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set **Z** of all integers. That is, for all integers m and n,

$$mRn \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n(\text{mod}3).$$

Now since 3 R 0, then by Lemma 8.3.2,

$$[3] = [0].$$

More generally, by the same reasoning,

$$[0] = [3] = [-3] = [6] = [-6] = \ldots, \text{ and so on.}$$

Similarly,

$$[1] = [4] = [-2] = [7] = [-5] = \ldots, \text{ and so on.}$$

And

$$[2] = [5] = [-1] = [8] = [-4] = \ldots, \text{ and so on.}$$

Congruence Modulo 3

Let R be the relation of congruence modulo 3 on the set **Z** of all integers. That is, for all integers m and n,

$$mRn \Leftrightarrow 3|(m-n) \Leftrightarrow m \equiv n \pmod 3.$$

Notice that every integer is in class [0], [1], or [2]. Hence the distinct equivalence classes are

$$\{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\},$$

$$\{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\},$$

$$\{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}.$$

Congruence Modulo

- **Definition**

Let m and n be integers and let d be a positive integer. We say that $\boldsymbol{m}$ **is congruent to** $\boldsymbol{n}$ **modulo** $\boldsymbol{d}$ and write

$$\boldsymbol{m \equiv n \ (\text{mod } d)}$$

if, and only if,

$$d \mid (m - n).$$

Symbolically:

$$m \equiv n \ (\text{mod } d) \quad \Leftrightarrow \quad d \mid (m - n)$$

Congruence Modulo

Determine which of the following congruences are true and which are false.

a. $12 \equiv 7 (\text{mod} 5)$

b. $6 \equiv -8 (\text{mod} 4)$

c. $3 \equiv 3 (\text{mod} 7)$

a. True. $12 - 7 = 5 = 5 \cdot 1$. Hence $5 \mid (12 - 7)$, and so $12 \equiv 7 \pmod 5$.

b. False. $6 - (-8) = 14$, and $4 \nmid 14$ because $14 \neq 4 \cdot k$ for any integer k. Consequently, $6 \not\equiv -8 \pmod 4$.

c. True. $3 - 3 = 0 = 7 \cdot 0$. Hence $7 \mid (3 - 3)$, and so $3 \equiv 3 \pmod 7$.

Exercise

Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero.
Symbolically,

$$A = \mathbf{Z} \times (\mathbf{Z} - \{0\}).$$

Define a relation R on A as follows: For all $(a, b), (c, d) \in A$,

$$(a,b) R (c,d) \Leftrightarrow ad = bc.$$

Describe the distinct equivalence classes of R

For example, the class (1,2):

$$[(1, 2)] = \{(1, 2), (-1, -2), (2, 4), (-2, -4), (3, 6), (-3, -6), \ldots\}$$

since $\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6}$ and so forth.

Worksheet #5

Q#1

Let $A = \{-1, 1, 2, 4\}$ and $B = \{1, 2\}$ and define relations R and S from A to B as follows:

For all $(x, y) \in A \times B$,

$x\ R\ y \Leftrightarrow |x| = |y|$ and

$x\ S\ y \Leftrightarrow x - y$ is even.

State explicitly which ordered pairs are in $A \times B$, R, S.

Worksheet #5

Q#2

determine whether the given relation is **reflexive**, **symmetric**, **transitive**, or **none of these**. Justify your answers.

D is the "divides" relation on $\mathbf{Z}^+$:
For all positive integers
m and n, $m\ D\ n \Leftrightarrow m \mid n$.