Chapter 9
Counting & Probability

9.1 Basics of Probability and Counting

9.2 Possibility Trees and the Multiplication Rule

9.3 Counting Elements of Disjoint Sets: Addition Rule

9.5 Counting Subsets of a Set: Combinations

9.6 r-Combinations with Repetition Allowed

In this Lecture

We will learn:

- Part 1: **Probability and Sample Space**
- Part 2: **Counting in Sub lists**

Tossing Coins

Tossing two coins and observing whether 0, 1, or 2 heads are obtained.

What are the chances of having 0,1,2 heads?

Tossing Coins

- Tossing two coins 50times and observing whether 0, 1, or 2 heads are obtained.
- What are the chances of having 0,1,2 heads?

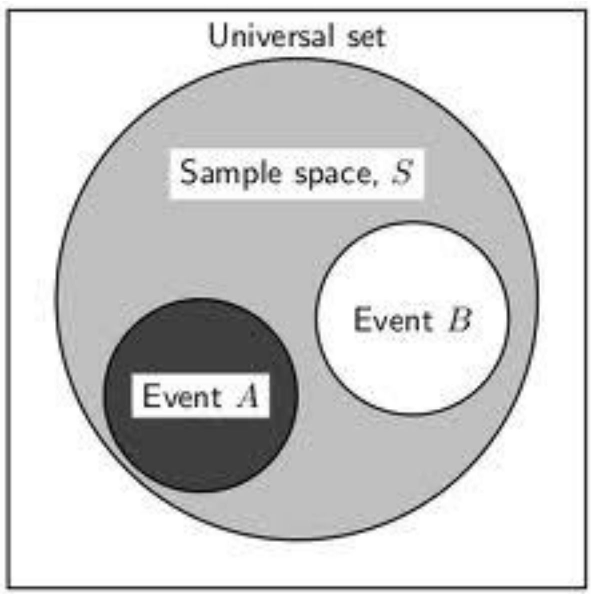
Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	𝍸 𝍸 |	11	22%
1 head obtained	𝍸 𝍸 𝍸 𝍸 𝍸 ||	27	54%
0 heads obtained	𝍸 𝍸 ||	12	24%

Sample Space

الفراغ العيني

• Definition

A **sample space** is the set of all possible outcomes of a random process or experiment. An **event** is a subset of a sample space.



Sample Space

In case an experiment has finitely many outcomes and all outcomes are **equally** likely to occur, the ***probability*** of an event (set of outcomes) is just the ratio of the number of outcomes in the event to the total number of outcomes

Equally Likely Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability of *E***, denoted ***P*(*E*)**, is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S}.$$

- **Notation**

For any finite set A, $N(A)$ denotes the number of elements in A.

$$P(E) = \frac{N(E)}{N(S)}.$$

Probabilities for a Deck of Cards

52 car

diamonds (♦)
hearts (♥)
clubs (♣)
spades (♠)

a. What is the sample space of outcomes?

➔ the 52 cards in the deck.

b. What is the event that the chosen card is a black face card?

➔ *E* = {J♣, Q♣, K♣, J♠, Q♠, K♠}

c. What is the probability that the chosen card is a black face card?

$$P(E) = \frac{N(E)}{N(S)} = \frac{6}{52} \cong 11.5\%.$$

Rolling a Pair of Dice

a. Write the sample space *S* of possible outcomes (using compact notion).

S = {1 1, 1 2, 1 3, 1 4, 1 5, 1 6, 2 1, 2 2, 2 3, 2 4, 2 5, 2 6, 3 1, 3 2, 3 3, 3 4, 3 5, 3 6, 4 1, 4 2, 4 3, 4 4, 4 5, 4 6, 5 1, 5 2, 5 3, 5 4, 5 5, 5 6, 6 1, 6 2, 6 3, 6 4, 6 5, 6 6}.

b. write the event *E* that the numbers showing face up have a sum of 6 and find the probability of this event.

E = {1 5, 2 4, 3 3, 4 2, 5 1}.

$$P(E) = \frac{N(E)}{N(S)} = \frac{5}{36}.$$

Part 2

Counting the Elements of a List

Some counting problems are as simple as counting the elements of a list. E.g., how many integers are there from 5 through 12?

list:	5	6	7	8	9	10	11	12
	↕	↕	↕	↕	↕	↕	↕	↕
count:	1	2	3	4	5	6	7	8

12-5+1=8=#of elements

Theorem 9.1.1 The Number of Elements in a List

If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

Part 2

Counting the Elements of a Sublist

- How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

100 101 102 103 104 105 106 107 108 109 110 ⋯ 994 995 996 997 998 999

$5 \cdot 20$ (100), $5 \cdot 21$ (105), $5 \cdot 22$ (110), $5 \cdot 199$ (995)

From the sketch it is clear that there are as many three-digit integers that are multiples of 5 as there are integers from 20 to 199 inclusive. By Theorem 9.1.1, **there are 199 − 20 + 1,** such integers.

Hence there are 180 three-digit integers that are divisible by 5.

- What is the probability that a randomly chosen three-digit integer is divisible by 5?

Sample space: 999 – 100 + 1 = 900.

P(E)=180/900 = 1/5.

By Theorem 9.1.1 the total number of integers from 100 through 999 is 999 − 100 + 1 = 900. By part (a), 180 of these are divisible by 5. Hence the probability that a randomly chosen three-digit integer is divisible by 5 is 180/900 = 1/5.

Counting & Probability

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Counting

9.2 Possibility Trees and the Multiplication Rule

In this lecture:

- ❑ Part 1: **Possibility Trees**
- ❑ Part 2: **Multiplication Rule**
- ❑ Part 3: **Permutations**

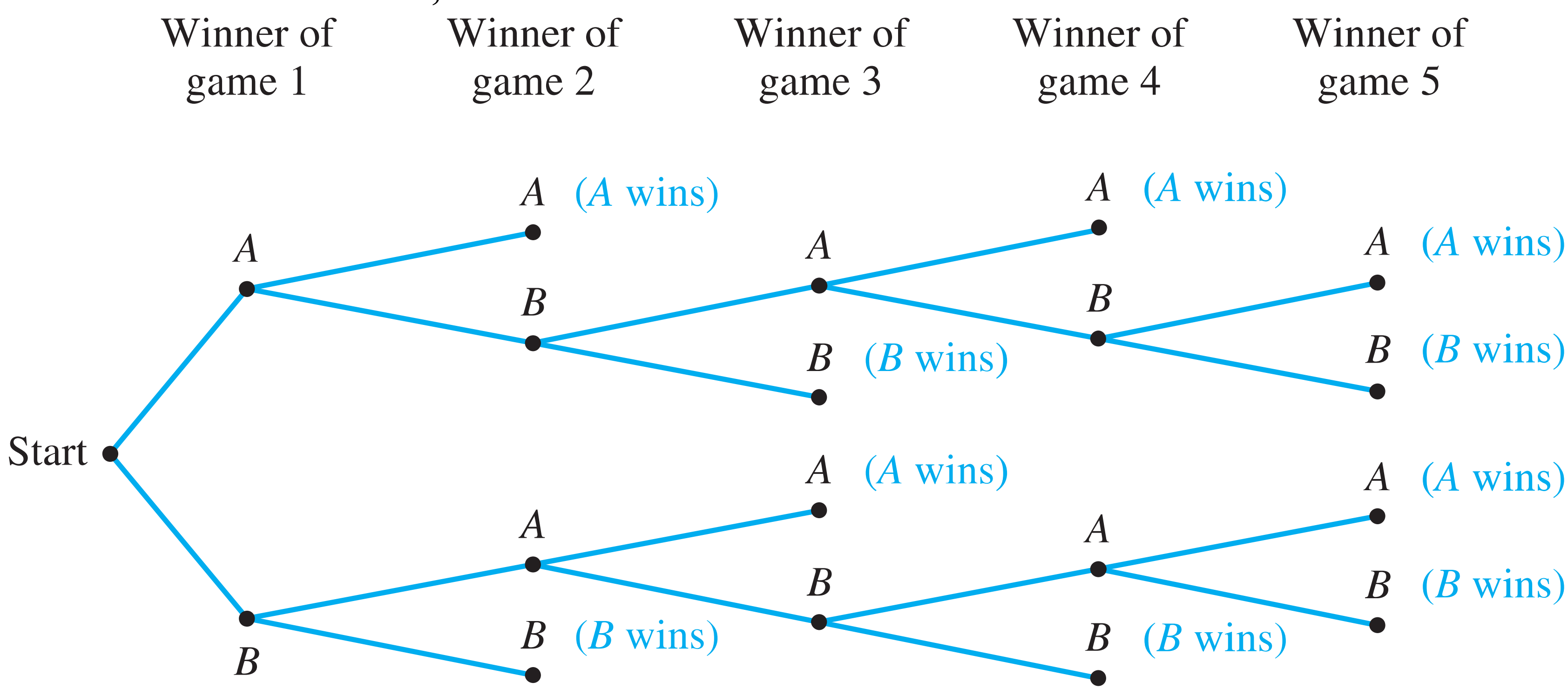
Possibility Trees

Barcelona **(A)** and Real Madrid **(B)** are to play with each other repeatedly until one of them wins two games in a row or a total of three games

a- How many ways can the tournament be played?

A–A, *A–B–A–A*, *A–B–A–B–A*, *A–B–A–B–B*, *A–B–B*,
B–A–A, *B–A–B–A–A*, *B–A–B–A–B*, *B–A–B–B*, and *B–B*.

*** In five cases *A* wins, and in the other five *B* wins.**



Possibility Trees

Barcelona **(A)** and Real Madrid **(B)** are to play with each other repeatedly until one of them wins two games in a row or a total of three games

b- Assuming that all the ways of playing the tournament are equally likely, what is the probability that five games are needed to determine the tournament winner?

Since all the possible ways of playing the tournament listed in part (a) are assumed to be equally likely, and the listing shows that five games are needed in four different cases (A–B–A–B–A, A–B–A–B–B, B–A–B–A–B, and B–A–B–A–A),

the probability that five games are needed is 4/10 = 2/5 = 40%.

Possibility Trees

We have 4 computers (A,B,C,D) and 3 printers (X,Y,Z). Each of these printers is connected with each of the computers. *Suppose you want to print something through one of the computers, How many possibilities for you have?*

$3+3+3+3 = 4 \cdot 3 = 12.$

Possibility Trees

A person buying a personal computer system is offered a choice of three models of the basic unit, two models of keyboard, and two models of printer.

How many distinct systems can be purchased?

Possibility Trees

Notices that representing the possibilities in a tree structure is a useful tool for tracking all possibilities in situations in which events happen in order.

Counting

9.2 Possibility Trees and the Multiplication Rule

In this lecture:

- Part 1: **Possibility Trees**
- Part 2: **Multiplication Rule**
- Part 3: **Permutations**

The Multiplication Rule

Theorem 9.2.1 The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,

the second step can be performed in n_2 ways *[regardless of how the first step was performed]*,

$\vdots$

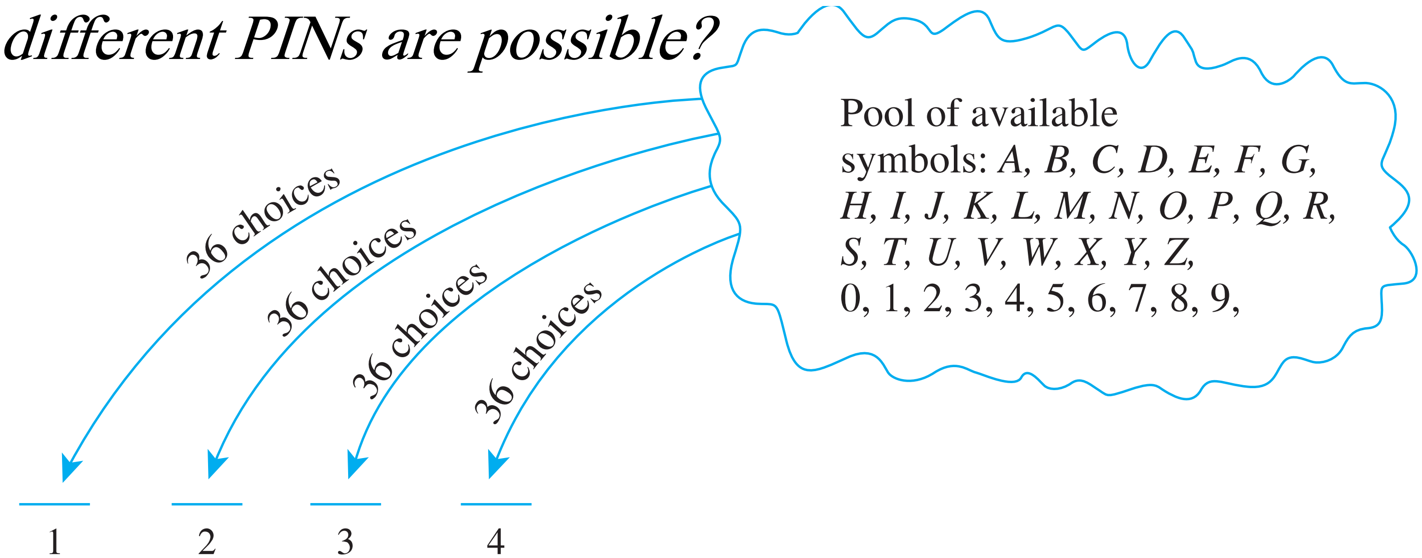
the kth step can be performed in n_k ways *[regardless of how the preceding steps were performed]*,

then the entire operation can be performed in $n_1 n_2 \cdots n_k$ ways.

Counting Example 1

A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the 10 digits, with repetition allowed.

How many different PINs are possible?



Step 1: Choose the first symbol.

Step 2: Choose the second symbol.

Step 3: Choose the third symbol.

Step 4: Choose the fourth symbol.

$36 \cdot 36 \cdot 36 \cdot 36 = 36^4 = 1{,}679{,}616$ PINs in all.

Counting Example 1

A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the 10 digits, with repetition **not** allowed.

How many different PINs are possible?

$$36 \cdot 35 \cdot 34 \cdot 33 = 1{,}413{,}720$$

what is the probability that a PIN chosen at random contains no repeated symbol?

$$\frac{1{,}413{,}720}{1{,}679{,}616} \cong .8417$$

Counting Example 2

for i := 1 **to** 4

    for j := 1 **to** 3

        [*Statements in body of inner loop. None contain branching statements that lead out of the inner loop.*]

    next j

next i

How many times this statement will be executed?

Counting Example 3

Suppose A_1, A_2, A_3, and A_4 are sets with n_1, n_2, n_3, and n_4 elements, respectively.

How many elements in $A_1 \times A_2 \times A_3 \times A_4$

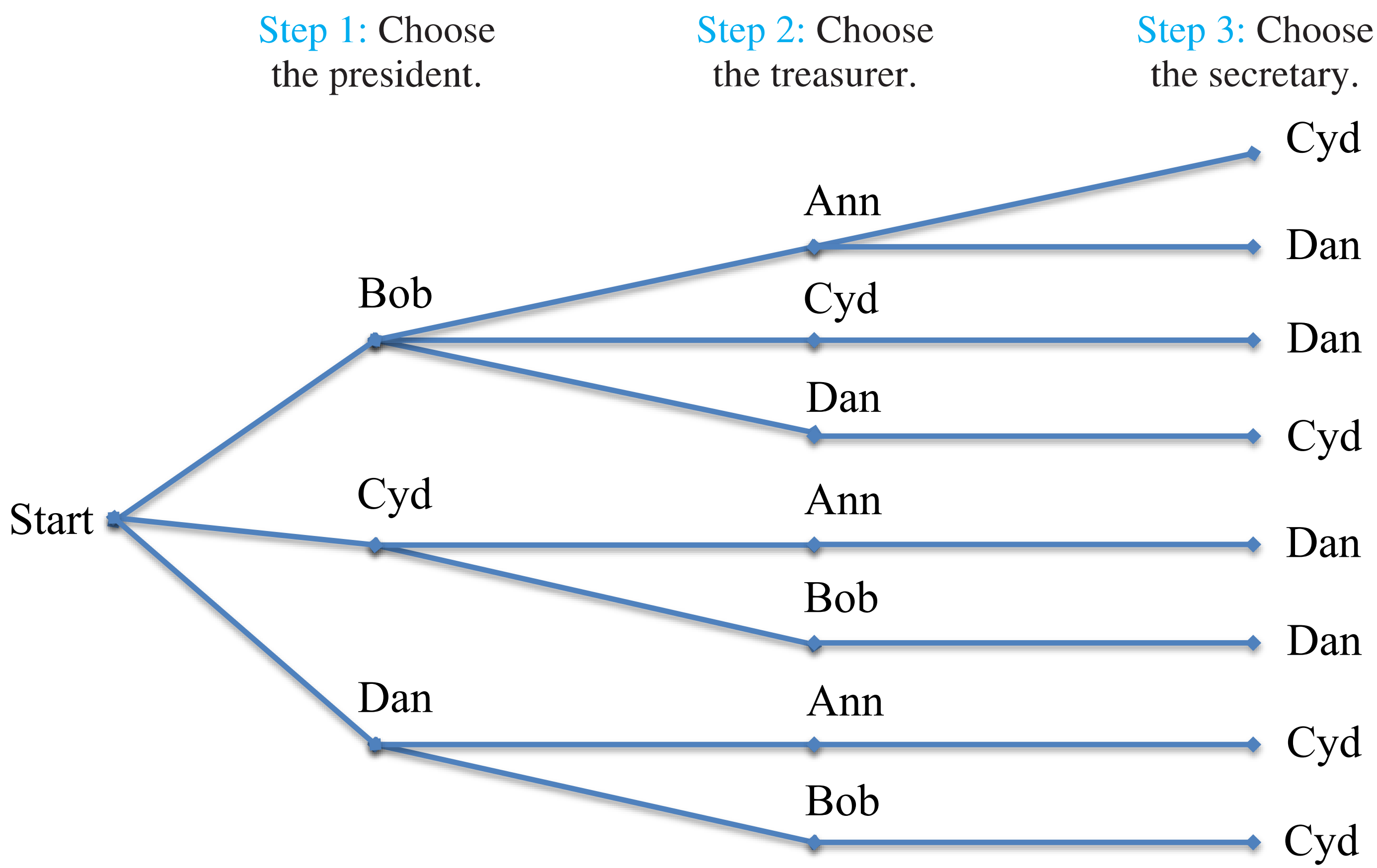
Solution: Each element in $A_1 \times A_2 \times A_3 \times A_4$ is an ordered 4-tuple of the form (a_1,a_2,a_3,a_4)

By the multiplication rule, there are $n_1 n_2 n_3 n_4$ ways to perform the entire operation. Therefore, there are $n_1 n_2 n_3 n_4$ distinct 4-tuples in $A_1 \times A_2 \times A_3 \times A_4$

Counting Example 4

Three officers—a president, a treasurer, and a secretary—are to be chosen from among four people: Ann, Bob, Cyd, and Dan. Suppose that, Ann cannot be president and either Cyd or Dan must be secretary. How many ways can the officers be chosen?

$3 \cdot 3 \cdot 2 = 18$



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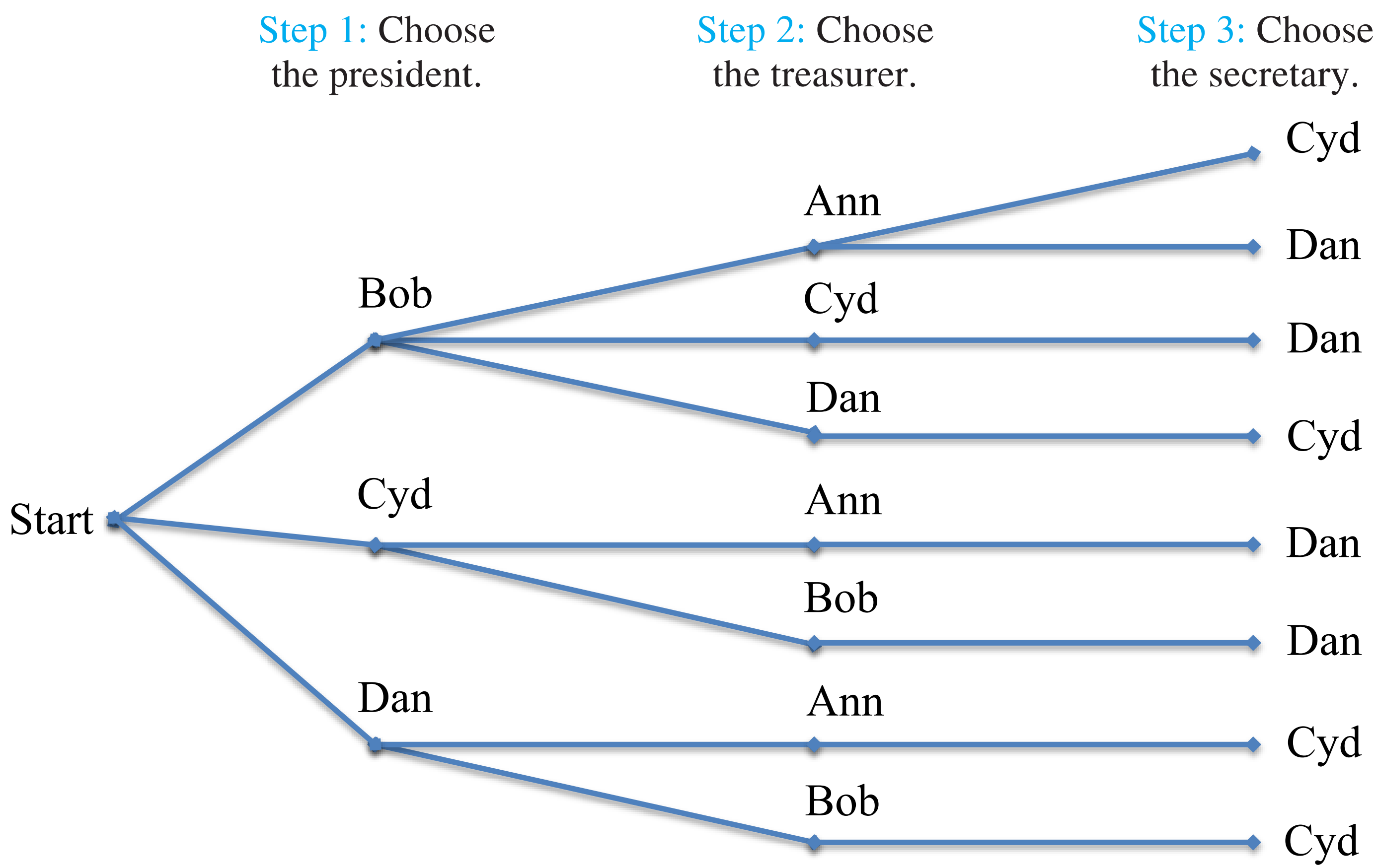


This tree is not homogenous, thus we cannot use the multiplication rule!!

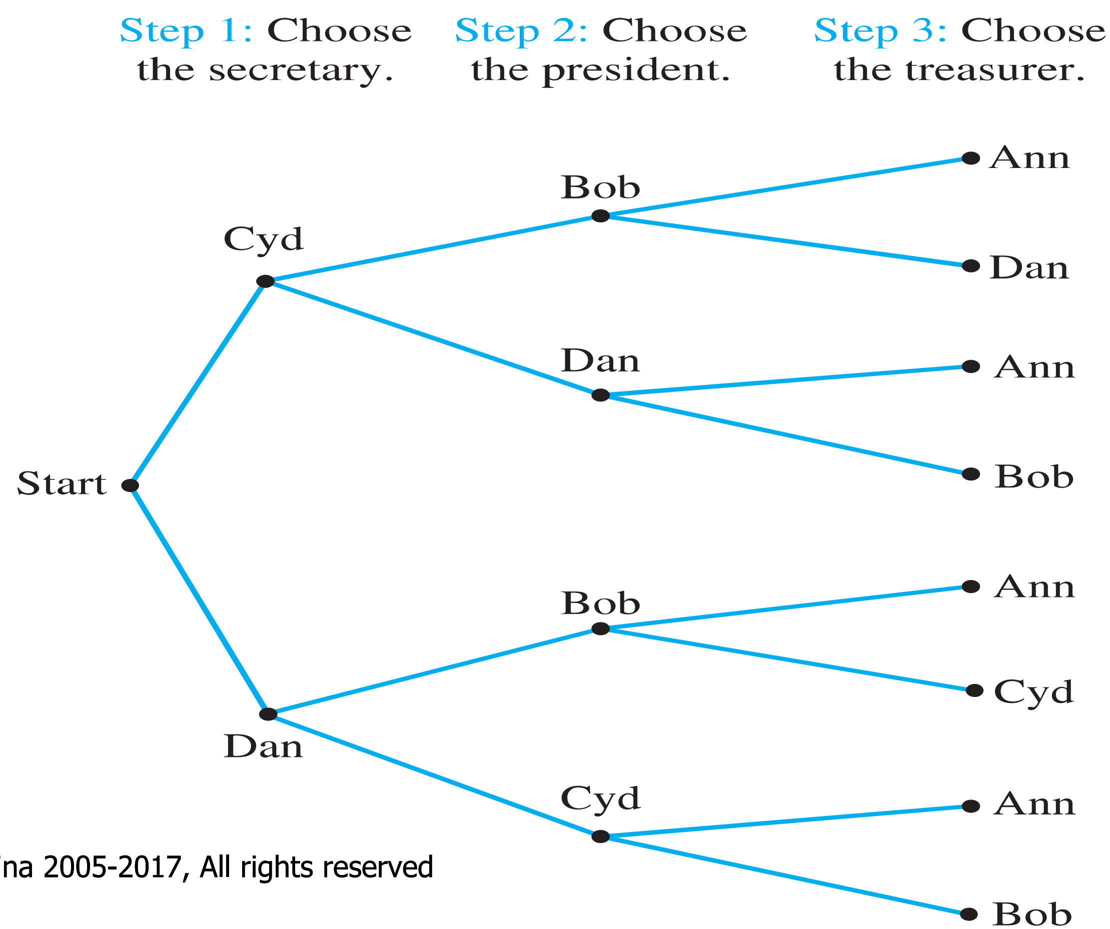
Better Idea?

Counting Example 4—reorder the steps to get the correct number of ways by the multiplication rule

تم انتخاب أربعة طلاب لنادي الكلية: (Ann, Bob, Cyd, Dan). نريد اختيار رئيس، أمين صندوق، وسكرتير. لا يمكن ل Ann ان تكون رئيساً، والسكرتير اما ان يكون Dan او Cyd كم تشكيلة ممكنة؟

2.2.2 = 8

We should be smart to represent our problem in a way to be able to use the multiplication rule



Counting

9.2 Possibility Trees and the Multiplication Rule

In this lecture:

- ❑ Part 1: **Possibility Trees**
- ❑ Part 2: **Multiplication Rule**
- ❑ Part 3: **Permutations**

Permutations

التباديل : عدد التشكيلات الممكنة لمجموعة جزئية من العناصر منتقاة من مجموعة كلية من العناصر مع مراعاة لأهمية تسلسل العناصر في تشكيلات المجموعة الجزئية

كم كلمة من خمس حروف ممكن ان نكون اذا كان لدينا عشرة حروف؟

كانت القاعدة التي تمكن من حساب عدد التبديلات لمجموعة ما، معروفة لدى الهنديين على الأقل في حوالي عام 1150م.

Permutations

A **permutation** of a set of objects is an ordering of the objects in a row.

For example, the set of elements $\{a, b, c\}$ has six permutations.

$$abc \quad acb \quad cba \quad bac \quad bca \quad cab$$

Generally, given a set of n objects, how many permutations does the set have? Imagine forming a permutation as an n-step operation:

Step 1: Choose an element to write **first**.

Step 2: Choose an element to write **second**

. . .

Step *n*: Choose an element to write ***n*th**.

Permutations

by the multiplication rule, there are

$$n(n-1)(n-2)\cdots 2\cdot 1 = n!$$

ways to perform the entire operation.

Theorem 9.2.2

For any integer n with $n \geq 1$, the number of permutations of a set with n elements is $n!$.

Example 1

How many ways can the letters in the word *COMPUTER* be arranged in a row?

$8! = 40{,}320$

How many ways can the letters in the word *COMPUTER* be arranged if the letters *CO* must remain next to each other (in order) as a unit?

$7! = 5{,}040$

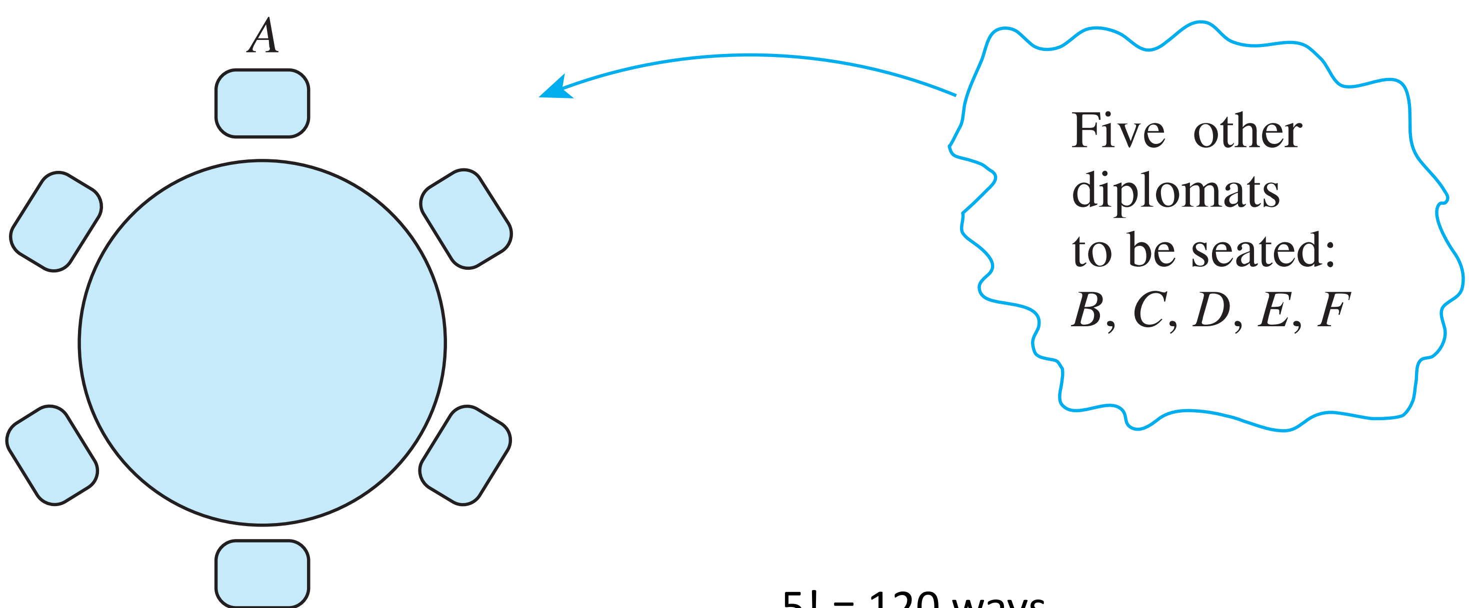
If letters of the word *COMPUTER* are randomly arranged in a row, what is the probability that the letters *CO* remain next to each other (in order) as a unit?

When the letters are arranged randomly in a row, the total number of arrangements is 40,320 by part (a), and the number of arrangements with the letters *CO* next to each other (in order) as a unit is 5,040.

$$\frac{5{,}040}{40{,}320} = \frac{1}{8} = 12.5\%.$$

Example 2

كيف يمكن توزيع ستة دبلوماسيين حول طاولة مستديرة



5! = 120 ways

لانها مستديرة، ثبت واحدة، ويبقى خمسة يمكن تبديلها

Permutations of Selected Elements

Given the set $\{a, b, c\}$, there are six ways to select two letters from the set and write them in order.

$$ab \quad ac \quad ba \quad bc \quad ca \quad cb$$

Each such ordering of two elements of $\{a, b, c\}$ is called a 2-*permutation* of $\{a, b, c\}$.

اي مجموع التبديلات التي يمكن ان ننتقي بها افراد المجموعة مع مراعاة الترتيب

- **Definition**

An ***r*-permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $\boldsymbol{P(n, r)}$.

How many permutations in $P(n, r)$?

Permutations of Selected Elements

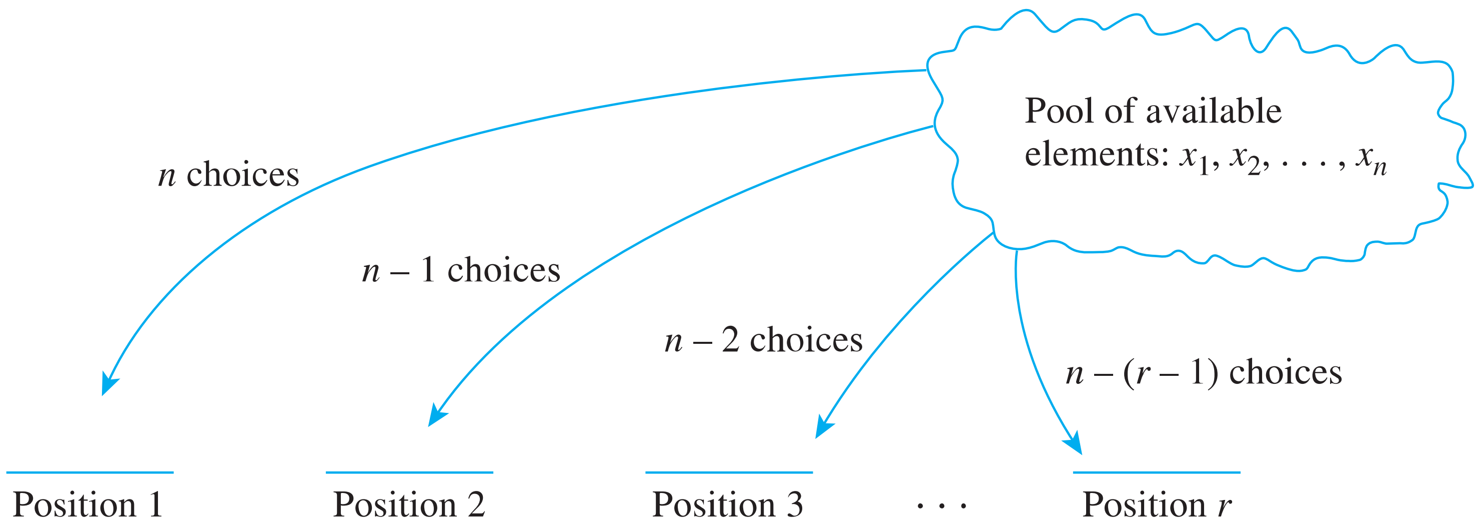
Theorem 9.2.3

If n and r are integers and $1 \le r \le n$, then the number of r-permutations of a set of n elements is given by the formula

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) \qquad \text{first version}$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!} \qquad \text{second version.}$$



Example 3

a. Evaluate $P(5, 2)$.

$$P(5,2) = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 20$$

b. How many 4-permutations are there of a set of 7 objects?

$$P(7,4) = \frac{7!}{(7-4)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 \cdot 5 \cdot 4 = 840.$$

c. How many 5-permutations are there of a set of 5 objects?

$$P(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5! = 120.$$

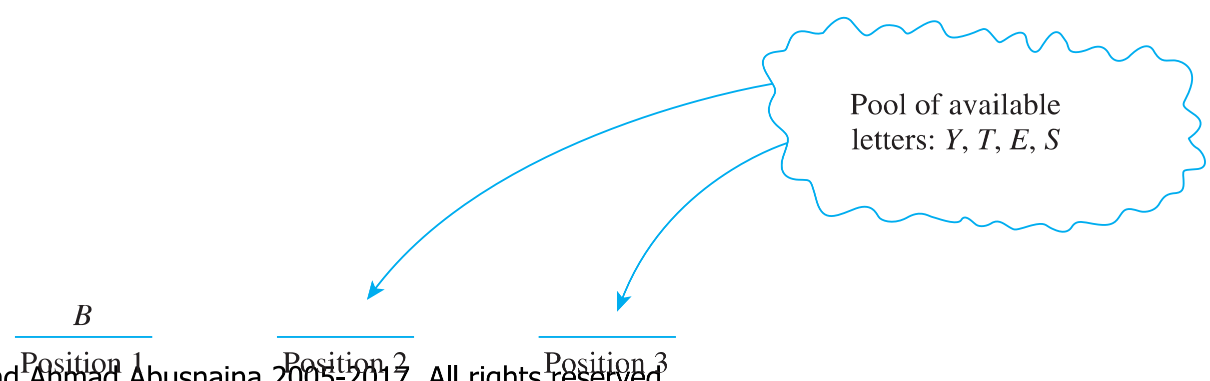
Example 4

How many different ways can 3 of the letters of the word *BYTES* be chosen and written in a row?

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5\cdot 4\cdot 3\cdot \not{2}\cdot \not{1}}{\not{2}\cdot \not{1}} = 5\cdot 4\cdot 3 = 60.$$

How many different ways can this be done if the first letter must be *B*?

$$P(4,2) = \frac{4!}{(4-2)!} = \frac{4\cdot 3\cdot \not{2}\cdot \not{1}}{\not{2}\cdot \not{1}} = 4\cdot 3 = 12.$$



Example 5

Prove that for all integers $n \geq 2$,

$$P(n, 2) + P(n, 1) = n^2.$$

$$P(n, 2) = \frac{n!}{(n-2)!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = n(n-1)$$

and

$$P(n, 1) = \frac{n!}{(n-1)!} = \frac{n \cdot \cancel{(n-1)!}}{\cancel{(n-1)!}} = n.$$

Hence

$$P(n, 2) + P(n, 1) = n \cdot (n-1) + n = n^2 - n + n = n^2,$$

Counting

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9.3 Counting Elements of Disjoint Sets: Addition Rule

In this lecture:

- ❑ Part 1: **Addition Rule**
- ❑ Part 2: **Difference Rule**
- ❑ Part 2: **Inclusion Rule**

Apply these rules to count elements of union and disjoint sets

Additional Rule

e.g., Number of students in this class = Number of Girls + Number of boys, in this class

Theorem 9.3.1 The Addition Rule

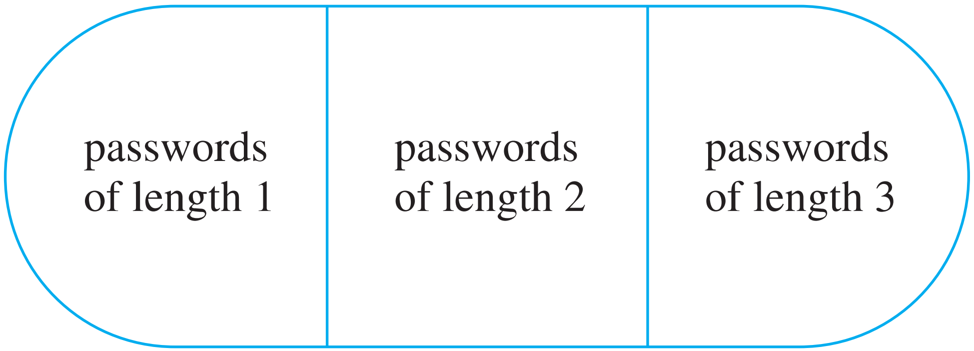
Suppose a finite set A equals the union of k distinct mutually disjoint subsets A_1, $A_2, \ldots, A_k$. Then

$$N(A) = N(A_1) + N(A_2) + \cdots + N(A_k).$$

The number of elements in a union of mutually disjoint finite sets equals the sum of the number of elements in each of the component sets.

Exercise

A password consists of from 1, 2, or 3 letters chosen from {a..z} with repetitions allowed. How many different passwords are possible?



Exercise

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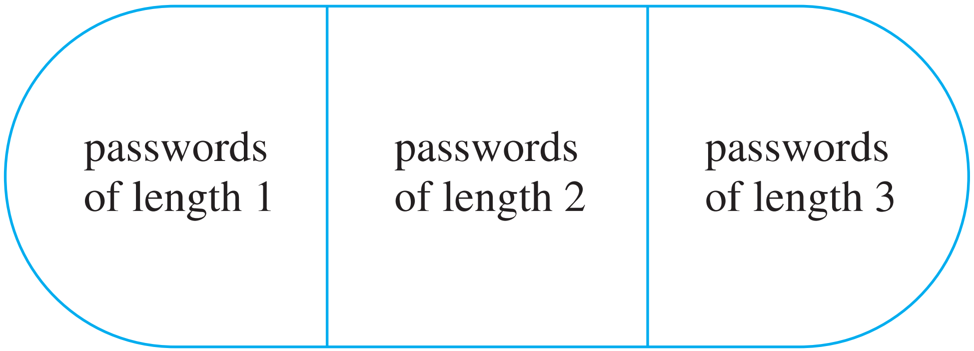


Number of passwords of length 1 = 26 (because there are 26 letters in the alphabet)

Number of passwords of length 2 = 26^2 (two-step process in which there are 26 ways to perform each step)

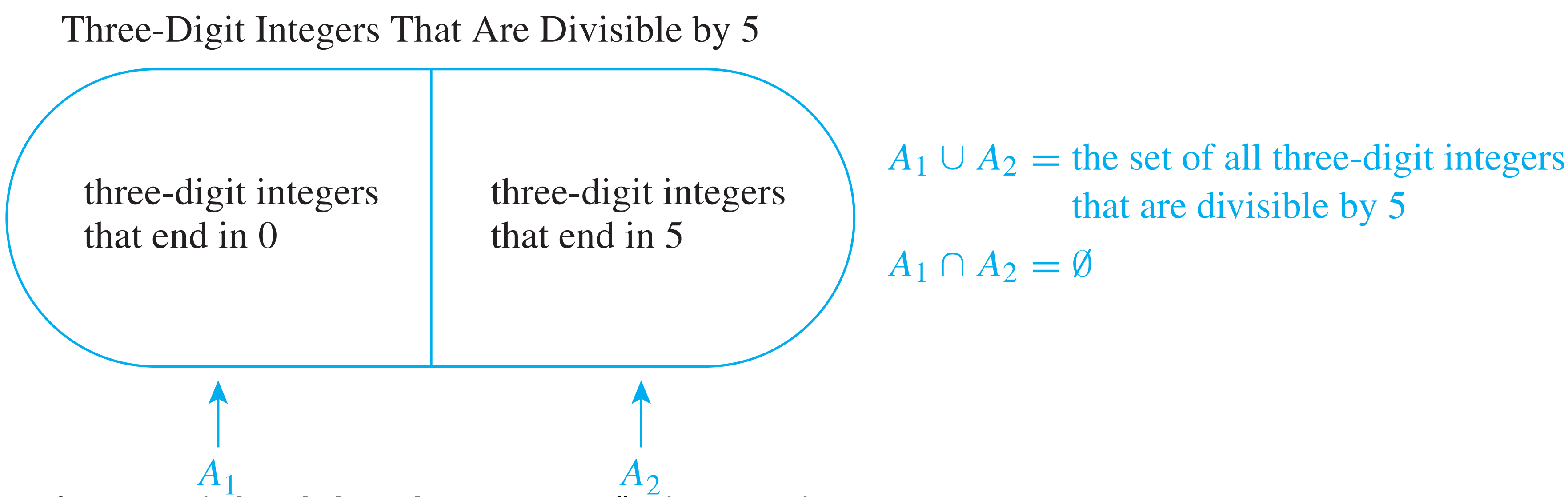
Number of passwords of length 3 = 26^3

Total = $26 + 26^2 + 26^3$ = 18,278.

Exercise

How many three-digit integers (i.e., integers from 100 to 999 inclusive) are divisible by 5? (using the addition rule)

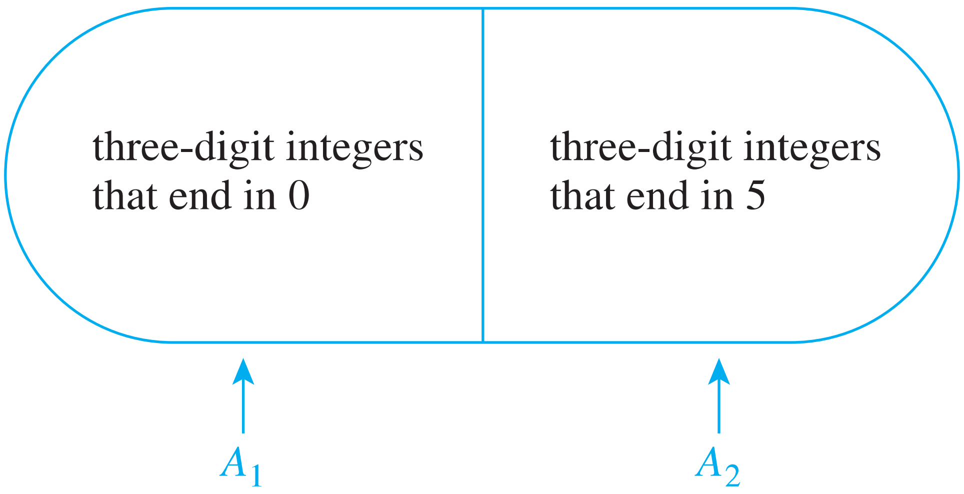
Integers that are divisible by 5 end either in 5 or in 0. Thus the set of all three-digit integers that are divisible by 5 can be split into two mutually disjoint subsets A_1 and A_2



Exercise

How many three-digit integers (i.e., integers from 100 to 999 inclusive) are divisible by 5? (using the addition rule)

Three-Digit Integers That Are Divisible by 5



$$\begin{bmatrix}\text{the number of} \\ \text{three-digit integers} \\ \text{that are divisible by 5}\end{bmatrix} = N(A_1) + N(A_2) = 90 + 90 = 180.$$

Counting

9.3 Counting Elements of Disjoint Sets: Addition Rule

In this lecture:

- ❑ Part 1: **Addition Rule**
- ❑ Part 2: **Difference Rule**
- ❑ Part 2: **Inclusion Rule**

Apply these rules to count elements of union and disjoint sets

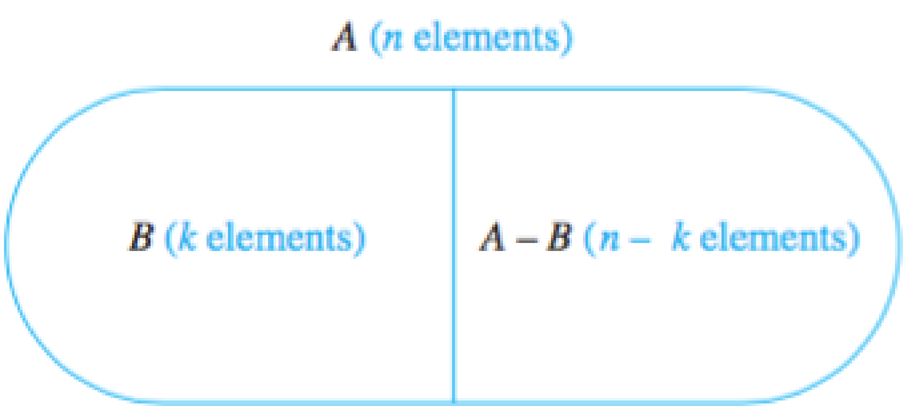
The Difference Rule

Number of students without girls =
number of all students – number of girls

Theorem 9.3.2 The Difference Rule

If A is a finite set and B is a subset of A, then

$$N(A - B) = N(A) - N(B).$$



Exercise

The PIN codes are made from exactly four symbols chosen from the 26 letters and the 10 digits, with repetitions allowed.

a) How many PINs contain repeated symbols?

$$1{,}679{,}616 - 1{,}413{,}720 = 265{,}896$$

Exercise

The PIN codes are made from exactly four symbols chosen from the 26 letters and the 10 digits, with repetitions allowed.

- **If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?**

One way

$$\frac{265{,}896}{1{,}679{,}616} \cong 0.158 = 15.8\%.$$

Another way

$$P(S-A) = \frac{N(S-A)}{N(S)}$$ by definition of probability in the equally likely case

$$= \frac{N(S)-N(A)}{N(S)}$$ by the difference rule

$$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)}$$ by the laws of fractions

$$= 1 - P(A)$$ by definition of probability in the equally likely case

$$\cong 1 - 0.842$$ by Example 9.2.4

$$\cong 0.158 = 15.8\%$$

Exercise

The PIN codes are made from exactly four symbols chosen from the 26 letters and the 10 digits, with repetitions allowed.

- **If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?**

Formula for the Probability of the Complement of an Event

If S is a finite sample space and A is an event in S, then

$$P(A^c) = 1 - P(A).$$

Another way

$$= \frac{N(S) - N(A)}{N(S)}$$ by the difference rule

$$= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)}$$ by the laws of fractions

$$= 1 - P(A)$$ by definition of probability in the equally likely case

$$\cong 1 - 0.842$$ by Example 9.2.4

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Counting

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In this lecture:

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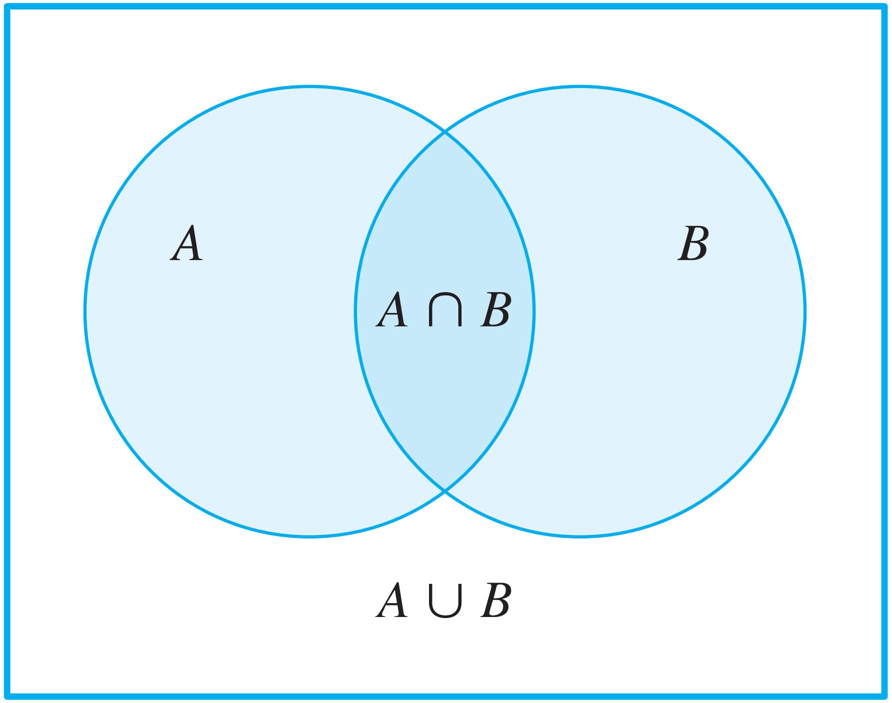
Apply these rules to count elements of union and disjoint sets

The Inclusion/Exclusion Rule

- Until now, we learned to count union of sets that they are disjoint.

- Now, we learn how to count elements in a union of sets when some of the sets **overlap** (i.e., they are not disjoint)



Exercise

➢ How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

Exercise

➢ How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

3s

1 2 3 4 5 6 ... 996 997 998 999

$3 \cdot 1$ $3 \cdot 2$ $3 \cdot 332$ $3 \cdot 333$

5s

1 2 3 4 5 6 7 8 9 10 ... 995 996 997 998 999 1,000

$5 \cdot 1$ $5 \cdot 2$ $5 \cdot 199$ $5 \cdot 200$

Overlap

1 2 ... 15 ... 30 ... 975 ... 990 ... 999 1,000

$15 \cdot 1$ $15 \cdot 2$ $15 \cdot 65$ $15 \cdot 66$

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$
$$= 333 + 200 - 66 = 467$$

Exercise

- How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

Exercise

- How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

1,000 – 467 = 533

The Inclusion/Exclusion Rule

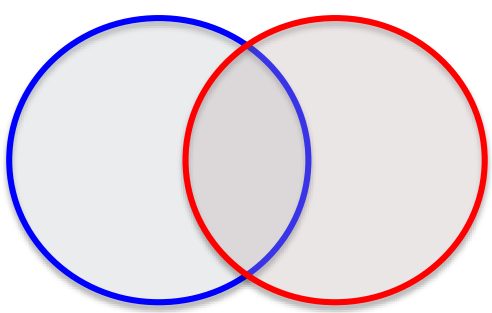
Theorem 9.3.3 The Inclusion/Exclusion Rule for Two or Three Sets

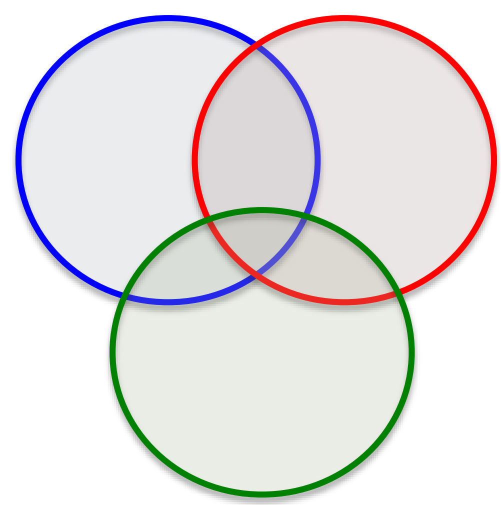
If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$





Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

- How many students did not take any of the three courses?

Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

- How many students did not take any of the three courses?

$$50-47 = 3.$$

Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

- How many students took all three courses?

P = the set of students who took precalculus

C = the set of students who took calculus

J = the set of students who took Java.

Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

- How many students took all three courses?

P = the set of students who took precalculus

C = the set of students who took calculus

J = the set of students who took Java.

$$N(P \cup C \cup J) =$$

$$N(P) + N(C) + N(J) - N(P \cap C) - N(P \cap J) - N(C \cap J) + N(P \cap C \cap J)$$

$$47 = 30 + 26 + 18 - 9 - 16 - 8 + N(P \cap C \cap J).$$

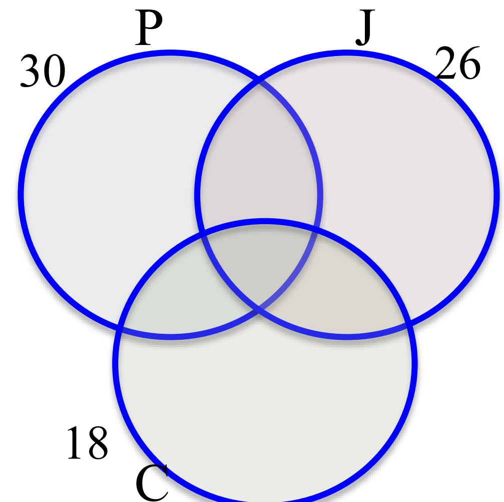
$$N(P \cap C \cap J) = 6.$$

Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

- How many students took precalculus and calculus but not Java?



Exercise

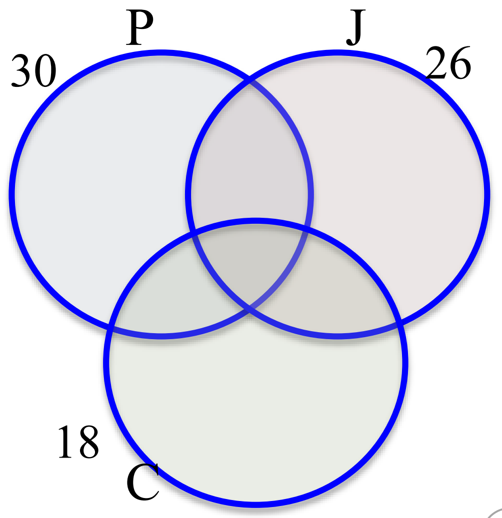
Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➢ How many students took precalculus and calculus but not Java?

$= (N(P \cap C)) - (N(P \cap C \cap J)) = ?$

$9 - 6 = 3$

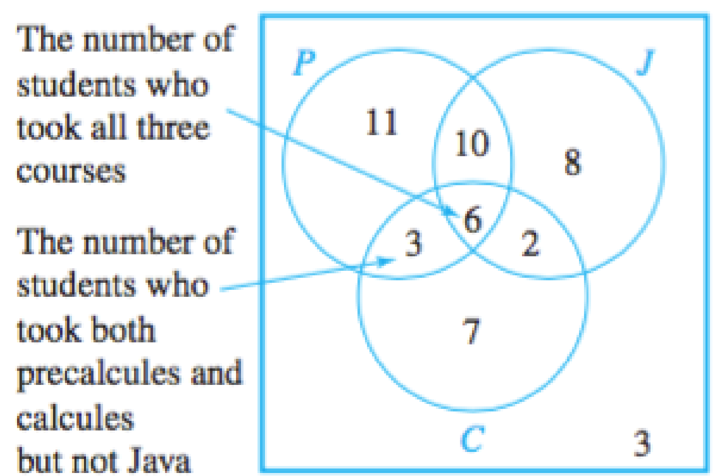


Exercise

Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

- How many students took precalculus but neither calculus nor Java?



Exercise

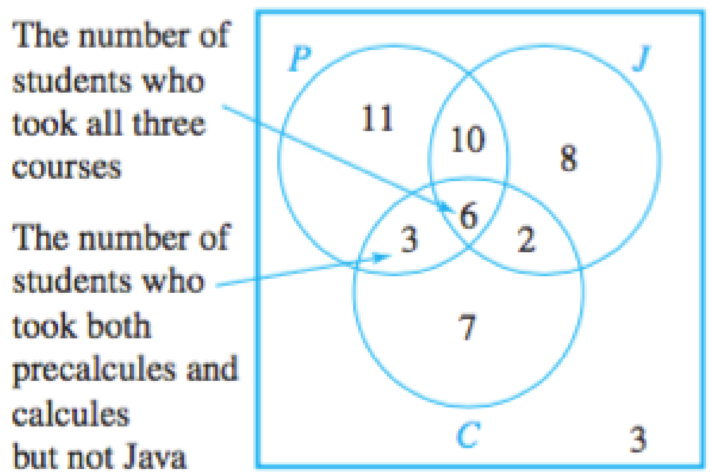
Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➢ How many students took precalculus but neither calculus nor Java?

N(P) – (N(P∩C)) – N(P∩J)) + N(P∩C∩J) =?

30 – 9 – 16 + 6 = 11



Counting

9.1 Basics of Probability and Counting

9.2 Possibility Trees and the Multiplication Rule

9.3 Counting Elements of Disjoint Sets: Addition Rule

9.5 Counting Subsets of a Set: Combinations

9.6 r-Combinations with Repetition Allowed

Counting

9.5 Counting Subsets of a Set: Combinations

In this lecture:

- ❑ Part 1: **Permutation versus Combinations**
- ❑ Part 2: **How to Calculate Combinations**
- ❑ Part 3: **Permutations of a Set with Repeated Elements**

Apply these rules to count elements of union and disjoint sets

Counting Subsets of a Set: Combinations (التوافيق)

Suppose 5 members of a group of 12 are to be chosen to work as a team. *How many distinct five-person teams can be selected?*

كم فريق من 5 اشخاص يمكننا ان نكون من بين 12 شخص؟

Ordering is not important, as the result is a **set**.

Permutation (التباديل) Vs. Combinations (التوافيق)

An **ordered** selection of r elements from a set of n elements is ***an r-permutation*** *P(n, r)* of the set.

➔ How many 2-permutations we can produce from {a,b,c,d}

= P(4,2)

An **unordered** selection of r elements from a set of n elements is the same as a subset of size r or an ***r-combination*** of the set.

➔ How many 2-combinations (subsets) we can produce from {a,b,c,d

= $\binom{4}{2}$

Counting Subsets of a Set: Combinations (التوافيق)

- **Definition**

Let n and r be nonnegative integers with $r \leq n$. An ***r*-combination** of a set of n elements is a subset of r of the n elements. As indicated in Section 5.1, the symbol

$$\binom{n}{r},$$

which is read "n choose r," denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

Example 1

Let $S = \{\text{Ann, Bob, Cyd, Dan}\}$. Each committee consisting of three of the four people in S is a 3-combination of S.

List all such 3-combinations of S.

{Bob, Cyd, Dan} leave out Ann

{Ann, Cyd, Dan} leave out Bob

{Ann, Bob, Dan} leave out Cyd

{Ann, Bob, Cyd} leave out Dan.

What is $\binom{4}{3}$?

= 4.

Example 2

How many unordered selections of 2 elements can be made from the set {0, 1, 2, 3}?

{0, 1}, {0, 2}, {0, 3}	subsets containing 0
{1, 2}, {1, 3}	subsets containing 1 but not already listed
{2, 3}	subsets containing 2 but not already listed.

Thus $\binom{4}{2} = 6,$

How to calculate $\binom{n}{r}$

How to calculate $\binom{n}{r}$

$$P(n, r) = \binom{n}{r} \cdot r!.$$

$$\binom{n}{r} = \frac{P(n, r)}{r!}.$$

$$\binom{n}{r} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}.$$

Theorem 9.5.1

The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n, r)}{r!} \qquad \text{first version}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{second version}$$

where n and r are nonnegative integers with $r \leq n$.

How to calculate $\binom{n}{0}$

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1$$

Exercise 1

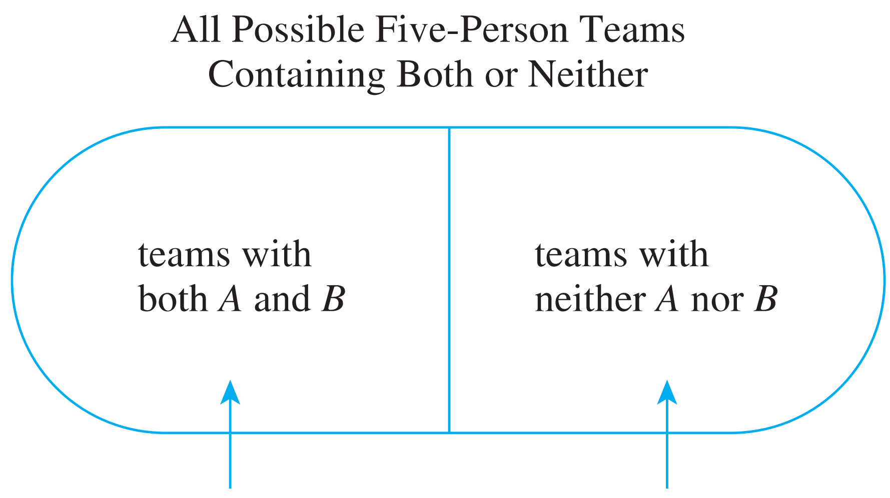
Suppose 5 members of a group of 12 are to be chosen to work as a team. ***How many distinct five-person teams can be selected?***

$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot 7!} = 11 \cdot 9 \cdot 8 = 792.$$

Exercise 2

Suppose 5 members of a group of 12 are to be chosen to work as a team.

Suppose two members of the group of 12 insist on working as a pair - any team must contain either both or neither. How many five-person teams can be formed?

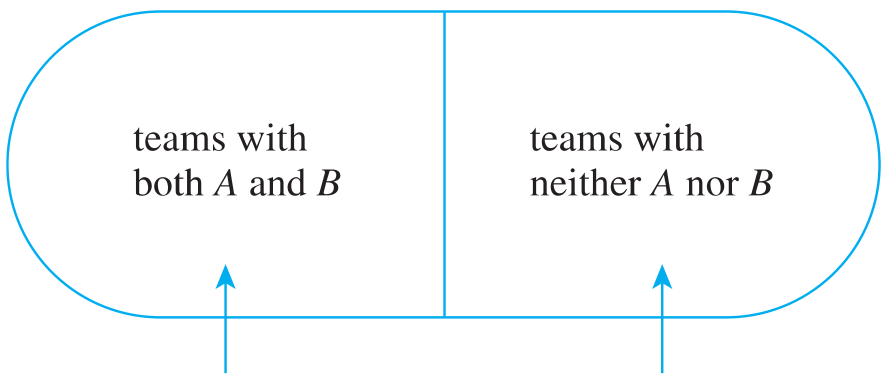


Exercise 2

Suppose 5 members of a group of 12 are to be chosen to work as a team.

Suppose two members of the group of 12 insist on working as a pair - any team must contain either both or neither. How many five-person teams can be formed?

All Possible Five-Person Teams
Containing Both or Neither



There are $\binom{10}{3} = 120$ of these. There are $\binom{10}{5} = 252$ of these.

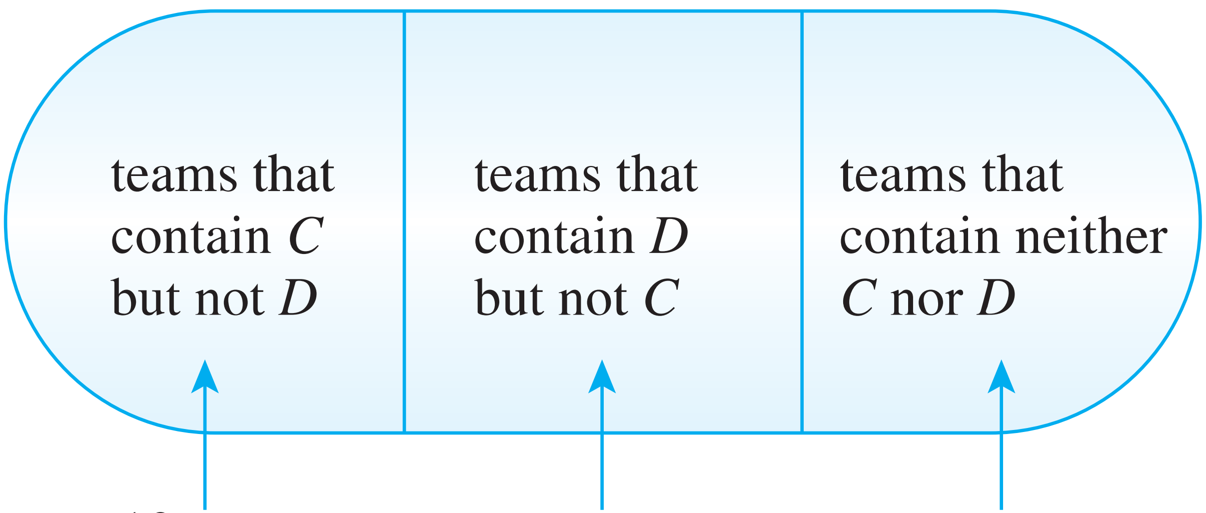
So the total number of teams that contain either both A and B or neither A nor B is $120 + 252 = 372$.

Exercise 3

Suppose 5 members of a group of 12 are to be chosen to work as a team.

Suppose 2 members of the group don't get along and refuse to work together on a team. How many five-person teams can be formed?

All Possible Five-Person Teams
That Do Not Contain Both C and D



$\binom{10}{4} = 210$ $\binom{10}{4} = 210$ $\binom{10}{5} = 252$

So the total number of teams that do not contain both C and D is $210 + 210 + 252 = 672$.

Exercise 4

Suppose the group of 12 consists of 5 men and 7 women.
How many 5-person teams can be chosen that consist of 3 men and 2 women?

{A, B, C, D, E, **m, n, o, p, q, s, t, r**}

⇓

$\{x_1, x_2, x_3, y_1, y_2\}$

$$\left[\begin{array}{l}\text{number of teams of five that}\\ \text{contain three men and two women}\end{array}\right] = \binom{5}{3}\binom{7}{2} = \frac{5!}{3!2!}\cdot\frac{7!}{2!5!}$$

$$= \frac{7\cdot 6\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{\cancel{3}\cdot \cancel{2}\cdot 1\cdot \cancel{2}\cdot 1\cdot \cancel{2}\cdot 1}$$

$$= 210.$$

Exercise 5

Suppose the group of 12 consists of 5 men and 7 women.
How many 5-person teams contain at **least** one man?

$$\begin{bmatrix}\text{number of teams}\\ \text{with at least}\\ \text{one man}\end{bmatrix} = \begin{bmatrix}\text{total number}\\ \text{of teams}\\ \text{of five}\end{bmatrix} - \begin{bmatrix}\text{number of teams}\\ \text{of five that do not}\\ \text{contain any men}\end{bmatrix}$$

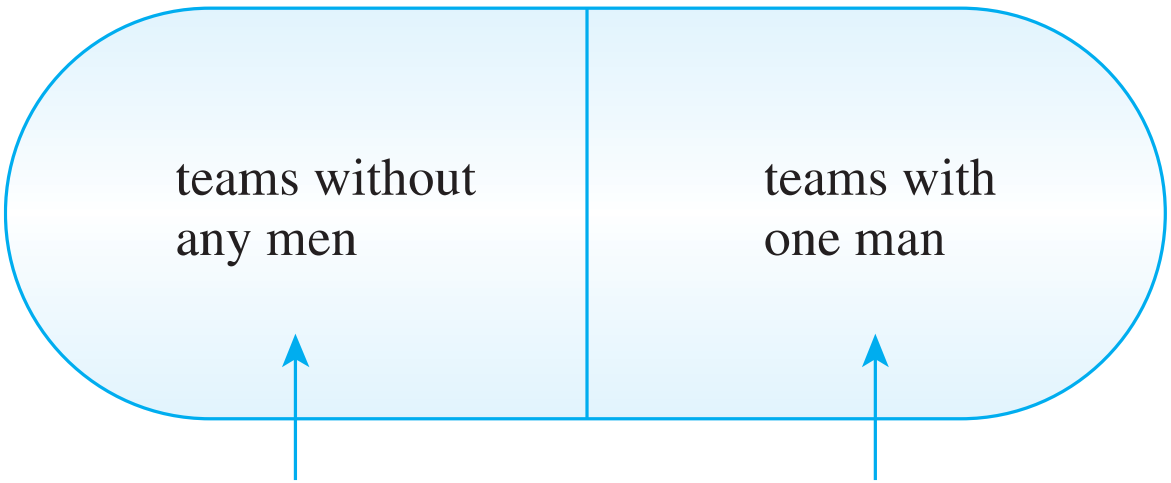
$$= \binom{12}{5} - \binom{7}{5} = 792 - \frac{7!}{5!\cdot 2!}$$

$$= 792 - \frac{7\cdot \overset{3}{\cancel{6}}\cdot \cancel{5}!}{\cancel{5}!\cdot \cancel{2}\cdot 1} = 792 - 21 = 771.$$

Exercise 6

Suppose the group of 12 consists of 5 men and 7 women.
How many 5-person teams contain at **most** one man?

Teams with At Most One Man



There are $\binom{5}{0}\binom{7}{5} = 21$ of these.

There are $\binom{5}{1}\binom{7}{4} = 175$ of these.

So the total number of teams with at most one man is 21 + 175 = 196.

Counting

9.4 Counting Subsets of a Set: Combinations

In this lecture:

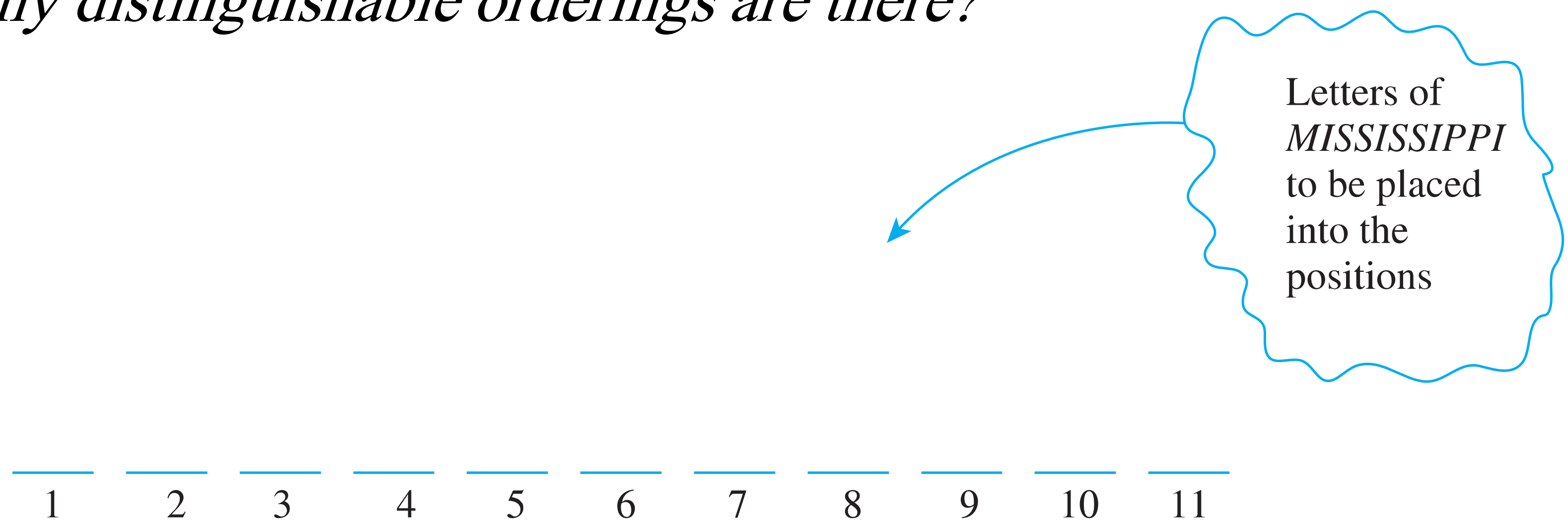
- ❑ Part 1: **Permutation versus Combinations**
- ❑ Part 2: **How to Calculate Combinations**
- ❑ Part 3: **Permutations of a Set with Repeated Elements**

Apply these rules to count elements of union and disjoint sets

Permutations of a Set with Repeated Elements

Consider various ways of ordering the letters in the word *MISSISSIPPI*: *IIMSSPISSIP*, *ISSSPMIIPIS*, and so on.

How many distinguishable orderings are there?



Step 1: Choose a subset of four positions for the *S*'s.

Step 2: Choose a subset of four positions for the *I*'s.

Step 3: Choose a subset of two positions for the *P*'s.

Step 4: Choose a subset of one position for the *M*.

Permutations of a Set with Repeated Elements

Consider various ways of ordering the letters in the word *MISSISSIPPI*: *IIMSSPISSIP, ISSSPMIIPIS*, and so on.

How many distinguishable orderings are there?

$$\begin{bmatrix}\text{number of ways to} \\ \text{position all the letters}\end{bmatrix} = \binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1}$$

$$= \frac{11!}{4!\cancel{7!}} \cdot \frac{\cancel{7!}}{4!\cancel{3!}} \cdot \frac{\cancel{3!}}{2!\cancel{1!}} \cdot \frac{\cancel{1!}}{1!\cancel{0!}}$$

$$= \frac{11!}{4! \cdot 4! \cdot 2! \cdot 1!} = 34{,}650.$$

Permutations of a Set with Repeated Elements

Theorem 9.5.2 Permutations with sets of Indistinguishable Objects

Suppose a collection consists of n objects of which

n_1 are of type 1 and are indistinguishable from each other
n_2 are of type 2 and are indistinguishable from each other
$\vdots$
n_k are of type k and are indistinguishable from each other,

and suppose that $n_1 + n_2 + \cdots + n_k = n$. Then the number of distinguishable permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$
$$= \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}.$$

Double Counting and common mistakes

Read Some tips about counting from the book

Counting

9.1 Basics of Probability and Counting

9.2 Possibility Trees and the Multiplication Rule

9.3 Counting Elements of Disjoint Sets: Addition Rule

9.5 Counting Subsets of a Set: Combinations

9.6 r-Combinations with Repetition Allowed

What is this section about?

In this chapter we discussed how to count the numbers of **ways of choosing *k* elements from *n***

	Strings Order Matters	Sets Order Doesn't Matter
Repetition Allowed	n^k **Select 4-digites PIN, from 0-9** =10.10.10.10 = 10^4 = 10000	**?** **Select 10 cans, from 4 types of drinks** =?
Repetition not Allowed	$P(n,k)$ **Select 4-digites PIN, from 0-9** 10!/6!=10.9.8.7 = 5040	$\binom{n}{k}$ **Select 4-position team, from 10 people** =10!/4!.6! = 5040/24 = 210

r-Combinations with Repetition Allowed

Examples:

- buy 20 drinks of Cola, 7up, or Fanta. How many ways?
- select a committee of 3 people, from 10 persons, but one person may play one or more roles.

Given a set on n elements $\{X_1, X_2, \ldots, X_n\}$

Choose r element $[X_{i1}, X_{i2}, \ldots X_{ik}]$

With **repetition** allowed, and **unordered**.

• Definition

An ***r*-combination with repetition allowed**, or **multiset of size *r***, chosen from a set X of n elements is an unordered selection of elements taken from X with repetition allowed. If $X = \{x_1, x_2, \ldots, x_n\}$, we write an r-combination with repetition allowed, or multiset of size r, as $[x_{i_1}, x_{i_2}, \ldots, x_{i_r}]$ where each x_{i_j} is in X and some of the x_{i_j} may equal each other.

Example

Find the number of 3-combinations with repetition allowed, or multi sets of size 3, that can be selected from {1, 2, 3, 4}

[1, 1, 1] ; [1, 1, 2]; [1, 1, 3]; [1, 1, 4]
[1, 2, 2] ; [1, 2, 3]; [1, 2, 4];
[1, 3, 3] ; [1, 3, 4]; [1, 4, 4];
[2, 2, 2] ; [2, 2, 3]; [2, 2, 4];
[2, 3, 3] ; [2, 3, 4]; [2, 4, 4];
[3, 3, 3] ; [3, 3, 4]; [3, 4, 4];
[4, 4, 4]

} 20 ways

→ How to calculate this automatically?

→ Can we "see" this multiset problem as a string problem?

Calculating r-Combinations with Repetition Allowed

Consider the numbers 1, 2, 3, and 4 as categories and imagine choosing a total of three numbers from the categories with multiple selections from any category allowed.

Calculating r-Combinations with Repetition Allowed

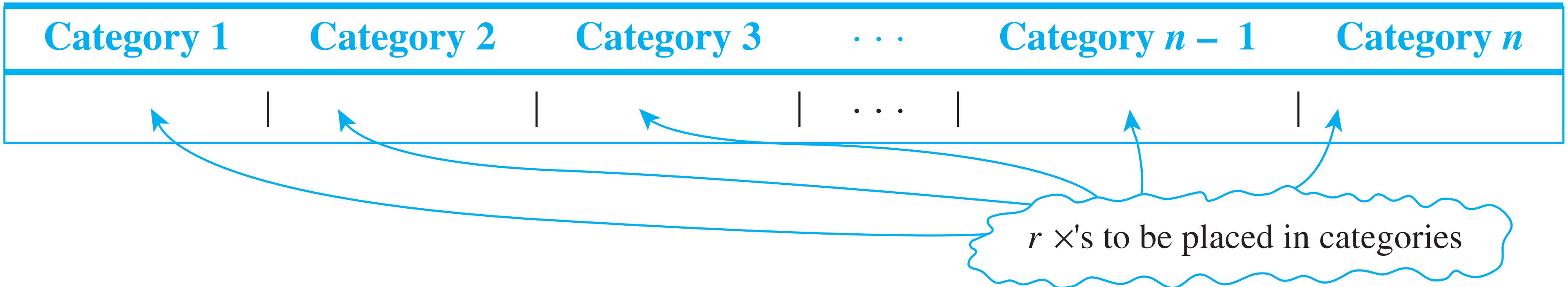
Consider the numbers 1, 2, 3, and 4 as categories and imagine choosing a total of three numbers from the categories with multiple selections from any category allowed.

Category 1	Category 2	Category 3	Category 4	Result of the Selection
|	× |	|	× ×	1 from category 2 2 from category 4
× |	|	× |	×	1 each from categories 1, 3, and 4
× × × |	|	|		3 from category 1

× × | | × | means [1,1,3]

The problem now became like selecting 3 positions out of 6, because once 3 positions have been chosen for the ×'s, the |'s are placed in the remaining 3 positions, which is $\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3 \cdot 2 \cdot 1 \cdot 3!} = 20$

Calculating r-Combinations with Repetition Allowed



$n-1$ vertical bars (to separate the n categories)
r crosses (to represent the r elements to be chosen).

Theorem 9.6.1

The number of r-combinations with repetition allowed (multisets of size r) that can be selected from a set of n elements is

$$\binom{r+n-1}{r}.$$

This equals the number of ways r objects can be selected from n categories of objects with repetition allowed.

Exercise 1.a

A person giving a party wants to buy 15 cans of drinks. He shops at a store that sells 5 different types of soft drinks.

How many different selections of cans of 15 soft drinks can he make?

Can be represented by a string of $5 - 1 = 4$ vertical bars (to separate the categories of soft drinks) and 15 crosses (to represent the cans selected). For instance,

$$\times \times \times \mid \times \times \times \times \times \times \times \mid \quad \mid \times \times \times \mid \times \times$$

$$\binom{15+5-1}{15} = \binom{19}{15} = \frac{19 \cdot \overset{6}{\cancel{18}} \cdot 17 \cdot \overset{2}{\cancel{16}} \cdot \cancel{15!}}{\cancel{15!} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 3{,}876.$$

Exercise 1.b

A person giving a party wants to buy 15 cans of drinks. He shops at a store that sells 5 different types of soft drinks.

If sprite *is one of the types of soft drink, how many different selections include at least 6 cans of **sprite** ?*

Thus we need to select 9 cans from the 5 types.
The nine additional cans can be represented as 9 ×'s and 4 |'s.

$$\binom{9+4}{9} = \binom{13}{9} = \frac{13 \cdot \cancel{12} \cdot 11 \cdot \overset{5}{\cancel{10}} \cdot \cancel{9!}}{\cancel{9!} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 715.$$

Exercise 2

Counting Triples (i, j, k) with $1 \le i \le j \le k \le n$

If n is a positive integer, how many triples of integers from 1 through n can be formed in which the elements of the triple are written in increasing order but are not necessarily distinct? In other words, how many triples of integers (i, j, k) are there with $1 \le i \le j \le k \le n$?

* Any triple of integers (i, j, k) with $1 \le i \le j \le k \le n$ can be represented as a string of $n - 1$ vertical bars and three crosses, with the positions of the crosses indicating which three integers from 1 to n are included in the triple. The table below illustrates this for $n = 5$.

Category 1	2	3	4	5	Result of the Selection
|	|	× × |	|	×	(3, 3, 5)
× |	× |	|	× |		(1, 2, 4)

$$\binom{3+(n-1)}{3} = \binom{n+2}{3} = \frac{(n+2)!}{3!(n+2-3)!}$$

$$= \frac{(n+2)(n+1)n\cancel{(n-1)!}}{3!\cancel{(n-1)!}} = \frac{n(n+1)(n+2)}{6}.$$

Exercise

Counting Iterations of a Loop

How many times will the innermost loop be iterated when the algorithm segment below is implemented and run?

for $k := 1$ **to** n

  for $j := 1$ **to** k

    for $i := 1$ **to** j

      [*Statements in the body of the inner loop, none containing branching statements that lead outside the loop*]

    next i

  next j

next k

$$\binom{3+(n-1)}{3} = \frac{n(n+1)(n+2)}{6}$$

Exercise

The Number of Integral Solutions of an Equation

How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 10$ if x_1, x_2, x_3, and x_4 are nonnegative integers?

Categories x_1 x_2 x_3 x_4	Solution to the equation $x_1 + x_2 + x_3 + x_4 = 10$
× × | × × × × × | | × × ×	$x_1 = 2,\quad x_2 = 5,\quad x_3 = 0,\quad$ and $\quad x_4 = 3$
× × × × | × × × × × × | |	$x_1 = 4,\quad x_2 = 6,\quad x_3 = 0,\quad$ and $\quad x_4 = 0$

$$\binom{10+3}{10} = \binom{13}{10} = \frac{13!}{10!(13-10)!} = \frac{13 \cdot 12 \cdot 11 \cdot \cancel{10!}}{\cancel{10!} \cdot 3 \cdot 2 \cdot 1} = 286.$$

Exercise

Additional Constraints on the Number of Solutions

How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 10$ if each $x_i \geq 1$?

$$\binom{6+3}{6} = \binom{9}{6} = \frac{9!}{6!(9-6)!} = \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6! \cdot 3 \cdot 2 \cdot 1} = 84.$$

Start by putting one cross in each of the four categories, then distribute the remaining six crosses among the categories

Summary

In this chapter we discussed how to count the numbers of **ways of choosing *k* elements from *n***

	Order Matters	Order Doesn't Matter
Repetition Allowed	n^k	$\binom{k+n-1}{k}$
Repetition not Allowed	$P(n,k)$	$\binom{n}{k}$