جامعة بيرزيت

BIRZEIT UNIVERSITY

Discrete Mathematic and Application Comp233

CHAPTER 3

THE LOGIC OF QUANTIFIED STATEMENTS

Instructor
Murad Njoum

Predicates and Quantified Statements I

In logic, predicates المسندات can be obtained by removing some or all of the nouns from a statement. For instance, let **P** stand for "**is a student at IT College**" and let **Q** stand for "**is a student at**." Then both *P* and *Q* are **predicate symbols**.

The sentences "**x is a student at IT College**" and "*x* is a student at *y*" are symbolized as *P*(*x*) and as *Q*(*x*, *y*) respectively, where *x* and *y* are *predicate variables* that take values in appropriate sets.

When concrete values are substituted in place of predicate variables, a statement results.

Predicates and Quantified Statements I

For simplicity, we define a *predicate* to be a predicate symbol together with suitable predicate variables. In some other treatments of logic, such objects are referred to as **propositional functions** or **open sentences**.

• Definition

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Predicates and Quantified Statements I

When an element in the domain of the variable of a one-variable predicate is substituted for the variable, the resulting statement is either true or false. The set of all such elements that make the predicate true is called the ***truth set* of the predicate.**

• Definition

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted

$$\{x \in D \mid P(x)\}.$$

Example 2 – *Finding the Truth Set of a Predicate*

Let ***Q*(*n*)** be the predicate "***n* is a factor of 8**." Find the truth set of ***Q*(*n*)** if

a. the domain of *n* is the set $\mathbf{Z}^+$ of all positive integers

b. the domain of *n* is the set **Z** of all integers.

Solution:

a. The truth set is {1, 2, 4, 8} because these are exactly the positive integers that divide 8 evenly.

b. The truth set is {1, 2, 4, 8, −1, −2, −4, −8} because the negative integers −1, −2, −4, and −8 also divide into 8 without leaving a remainder.

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The Universal Quantifier:

One sure way to change predicates into statements is to assign specific values to all their variables.

For example, if *x* represents the number 35, the sentence "*x* is (evenly) divisible by 5" is a true statement since 35 = 5 · 7. Another way to obtain statements from predicates is to add **quantifiers.**محددو الكمية

Quantifiers are words that refer to quantities such as "**some**" or "**all**" and tell for how many elements a given predicate is true.

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The Universal Quantifier:

The symbol ∀ denotes "**for all**" and is called the **universal quantifier**.

The domain of the predicate variable is generally indicated between the ∀ symbol and the variable name or immediately following the variable name. Some other expressions that can be used instead of *for all are for every, for arbitrary, for any, for each,* and *given any*.

The Universal Quantifier:

Sentences that are quantified universally are defined as statements by giving them the truth values specified in the following definition:

• Definition

Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for every x in D. It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D. A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Example 3 – *Truth and Falsity of Universal Statements*

a. Let D = {1, 2, 3, 4, 5}, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Show that this statement is true.

b. Consider the statement

$$\forall x \in \mathbf{R}, x^2 \geq x.$$

Find a counterexample to show that this statement is false.

Example 3 – *Solution*

a. Check that "$x^2 \geq x$" is true for each individual x in D.

$$1^2 \geq 1, \qquad 2^2 \geq 2, \qquad 3^2 \geq 3, \qquad 4^2 \geq 4, \qquad 5^2 \geq 5.$$

Hence " x D, $x^2 \geq x$" is true.

b. *Counterexample:* Take x = $\frac{1}{2}$. Then x is in **R** (since $\frac{1}{2}$ is a real number) and

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \ngeq \frac{1}{2}.$$

Hence " x **R**, $x^2 \geq x$" is false.

The Universal Quantifier:

The technique used to show the truth of the **universal** statement in Example 3(a) is called the **method of exhaustion**.

It consists of **showing the truth** of the predicate separately **for each individual element of the domain**.

This method can, in theory, be used whenever the domain of the predicate **variable is finite**.

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The Existential Quantifier:

The symbol denotes "there exists" and is called the **existential quantifier.** For example, the sentence "There is a student in Math 140" can be written as

a person *p* such that *p* is a student in Math 140,

or, more formally,

p P such that p is a student in Math 140,

where ***P*** is the set of all people. The domain of the predicate variable is generally indicated either between the symbol and the variable name or immediately following the variable name.

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The Existential Quantifier:

The words *such that* are inserted just before the predicate. Some other expressions that can be used in place of *there exists* are *there is a*, *we can find a*, *there is at least one*, *for some*, and *for at least one*.

In a sentence such as " integers *m* and *n* such that *m* + *n* = *m* · *n*," the symbol is understood to refer to both *m* and *n*.

• Definition

Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D$ such that $Q(x)$." It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D. It is false if, and only if, $Q(x)$ is false for all x in D.

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Example 4 – *Truth and Falsity of Existential Statements*

a. Consider the statement

m **Z**$^+$ such that $m^2 = m$.

Show that this statement is true.

b. Let *E* = {5, 6, 7, 8} and consider the statement

m *E* such that $m^2 = m$.

Show that this statement is false.

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Example 4 – *Solution*

a. Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m. Hence " m **Z** such that $m^2 = m$" is true.

b. Note that $m^2 = m$ is not true for any integers m from 5 through 8:

$$5^2 = 25 \neq 5, \quad 6^2 = 36 \neq 6, \quad 7^2 = 49 \neq 7, \quad 8^2 = 64 \neq 8.$$

Thus " m *E* such that $m^2 = m$" is false.

Formal Versus Informal Language

It is important to be able to translate **from formal to informal** language when trying to make sense of mathematical concepts that are new to you.
It is equally important to be able to translate from informal to formal language when thinking out a complicated problem.

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol or .

a. $\forall x \in \mathbf{R}, x^2 \geq 0.$

b. $\forall x \in \mathbf{R}, x^2 \neq -1.$

c. $\exists m \in \mathbf{Z}^+$ such that $m^2 = m.$

Example 5 – *Solution*

a. All real numbers have nonnegative squares.

Or: Every real number has a nonnegative square.

Or: Any real number has a nonnegative square.

Or: The square of each real number is nonnegative.

b. All real numbers have squares that are not equal to −1.

Or: No real numbers have squares equal to −1.
(The words *none are* or *no . . . are* are equivalent to the words *all are not*.)

Example 5 – *Solution* cont'd

c. There is a positive integer whose square is equal to itself.

Or: We can find at least one positive integer equal to its own square.

Or: Some positive integer equals its own square.

Or: Some positive integers equal their own squares.

Universal Conditional Statements

A reasonable argument can be made that the most important form of statement in mathematics is the **universal conditional statement:**

$$x, \text{ if } P(x) \text{ then } Q(x).$$

Familiarity with statements of this form is essential if you are to learn to speak mathematics.

Example 8 – *Writing Universal Conditional Statements Informally*

Rewrite the following statement informally, without quantifiers or variables.

$$x \quad \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4.$$

Solution:

If a real number is greater than 2 then its square is greater than 4.

Or: Whenever a real number is greater than 2, its square is greater than 4.

Or: The square of any real number greater than 2 is greater than 4.

Or: The squares of all real numbers greater than 2 are greater than 4.

Equivalent Forms of Universal and Existential Statements

Equivalent Forms of Universal and Existential Statements

Observe that the two statements “ real numbers *x*, if *x* is an integer then *x* is rational” and “ integers x, x is rational” mean the same thing.

Both have informal translations “All integers are rational.” In fact, a statement of the form

$$\forall x \in U, \text{if } P(x) \text{ then } Q(x)$$

can always be rewritten in the form

$$\forall x \in D, Q(x)$$

by **narrowing *U*** to be the domain *D* consisting of all values of the variable *x* that make *P*(*x*) true.

Equivalent Forms of Universal and Existential Statements

Conversely, a statement of the form

$$\forall x \in D, Q(x)$$

can be rewritten as

$$\forall x, \text{if } x \text{ is in } D \text{ then } Q(x).$$

Rewrite the following statement in the two forms “ *x*, if _______ then _______” and “ _______*x*, ________”:

All squares are rectangles.

Solution: *x*, if *x* is a square then *x* is a rectangle.

squares *x*, *x* is a rectangle.

Equivalent Forms of Universal and Existential Statements

Similarly, a statement of the form

“ x such that p(x) and Q(x)”

can be rewritten as

“ x $\in$D such that Q(x),”

where *D* is the set of all *x* for which *P*(*x*) is true.

Example 11 – *Equivalent Forms for Existential Statements*

A **prime number** is an integer **greater than 1** whose only positive integer factors are itself and 1. Consider the statement "**There is an integer that is both prime and even**."

Let Prime(*n*) be "*n* is prime" and Even(*n*) be "*n* is even." Use the notation Prime(*n*) and Even(*n*) to rewrite this statement in the following two forms:

a. *n* such that ______ ______ .

b. ______ *n* such that ______.

a. *n* such that Prime(*n*) Even(*n*).

b. Two answers: a prime number *n* such that Even(*n*).

an even number *n* such that Prime(*n*).

Implicit Quantification

Mathematical writing contains many examples of **implicitly quantified statements**. Some occur, through the presence of the word a or an. Others occur in cases where the general context of a sentence supplies part of its meaning.

For example, in an algebra course in which the letter *x* is always used to indicate a real number, the predicate

If $x > 2$ then $x^2 > 4$

is interpreted to mean the same as the statement

real numbers *x*, if $x > 2$ then $x^2 > 4$.

Implicit Quantification

Mathematicians often use a **double arrow** to indicate **implicit quantification** symbolically.

For instance, they might express the above statement as

$$x > 2 \quad x^2 > 4.$$

• Notation

Let $P(x)$ and $Q(x)$ be predicates and suppose the common domain of x is D.

- The notation $\boldsymbol{P(x) \Rightarrow Q(x)}$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$, or, equivalently, $\forall x$, $P(x) \rightarrow Q(x)$.
- The notation $\boldsymbol{P(x) \Leftrightarrow Q(x)}$ means that $P(x)$ and $Q(x)$ have identical truth sets, or, equivalently, $\forall x$, $P(x) \leftrightarrow Q(x)$.

Example 12 – *Using and*

Let

Q(n) be "n is a factor of 8,"
R(n) be "n is a factor of 4,"
S(n) be "n < 5 and n ≠ 3,"

and suppose the domain of *n* is $\mathbf{Z}^+$, the set of positive integers.

Use the and symbols to indicate true relationships among ***Q***(***n***), ***R***(***n***), **and** ***S***(***n***).

Example 12 – *Solution*

1. As noted in Example 2, the truth set of $Q(n)$ is {1, 2, 4, 8} when the domain of n is $\mathbf{Z}^+$. By similar reasoning the truth set of $R(n)$ is {1, 2, 4}.

Thus it is true that every element in the truth set of $R(n)$ is in the truth set of $Q(n)$, or, equivalently,

n in Z^+, $R(n) \rightarrow Q(n)$.

So $R(n)$ $Q(n)$, or, equivalently

n is a factor of 4 n is a factor of 8.

Example 12 – *Solution*

cont'd

2. The truth set of **$S(n)$ is {1, 2, 4},** which is identical to the truth set of $R(n)$, or, equivalently,

n in Z+, R(n) ↔ S(n).

So $R(n)$ $S(n)$, or, equivalently,

n is a factor of 4 $n < 5$ and $n \neq 3$.

Moreover, since every element in the truth set of $S(n)$ is in the truth set of $Q(n)$, or, equivalently,

n in Z+, S(n) → Q(n), then S(n) Q(n), or, equivalently,

$n < 5$ and $n \neq 3$ n is a factor of 8.

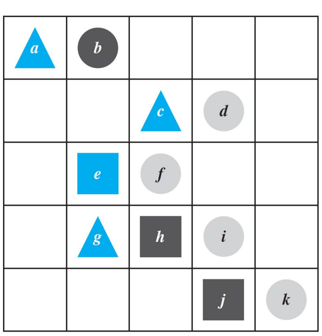
Tarski's World

Tarski's World is a computer program developed by information *scientists Jon Barwise and John Etchemendy* to help teach the principles of logic.

It is described in their book *The Language of First-Order Logic*, which is accompanied by a CD-Rom containing the program Tarski's World, named after the great logician Alfred Tarski.

Example 13 – *Investigating Tarski's World*

The program for Tarski's World provides pictures of blocks of various sizes, shapes, and colors, which are located on a grid. Shown in Figure 3.1.1 is a picture of an arrangement of objects in a two-dimensional Tarski world.



The configuration can be described using logical operators and—for the two-dimensional version—notation such as Triangle(x), meaning "x is a triangle," *Blue*(y), meaning "y is blue," and RightOf(x, y), meaning "x is to the right of y (but possibly in a different row)."

Individual objects can be given names such as *a*, *b*, or *c*.

Figure 3.1.1

Example 13 – *Investigating Tarski's World*

cont'd

Determine the truth or falsity of each of the following statements. The domain for all variables is the set of objects in the Tarski world shown above.

a. t, Triangle(t) → Blue(t).

b. x, Blue(x) → Triangle(x).

c. y such that Square(y) RightOf(d, y).

d. z such that Square(z) Gray(z).

Example 13 – *Solution*

a. This statement is **true**: All the triangles are blue.

b. This statement is **false.** As a counterexample, note that *e* is blue and it is not a triangle.

c. This statement is **true** because *e* and *h* are both square and *d* is to their right.

d. This statement is **false:** All the squares are either blue or black.

Negations of Quantified Statements

The general form of the negation of a universal statement follows immediately from the definitions of negation and of the truth values for universal and existential statements.

Theorem 3.2.1 Negation of a Universal Statement

The negation of a statement of the form

$$\forall x \text{ in } D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

Symbolically, $\sim(\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } \sim Q(x).$

Negations of Quantified Statements

Thus

The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

Note that when we speak of **logical equivalence for quantified statements,** we mean that the statements always have identical truth values no matter what predicates are substituted for the predicate symbols and no matter what sets are used for the domains of the predicate variables.

Negations of Quantified Statements

The general form for the negation of an existential statement follows immediately from the definitions of negation and of the truth values for existential and universal statements.

Theorem 3.2.2 Negation of an Existential Statement

The negation of a statement of the form

$$\exists x \text{ in } D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall x \text{ in } D, \sim Q(x).$$

Thus

Symbolically, $\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x)$.

The negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").

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Example 1 – *Negating Quantified Statements*

Write formal negations for the following statements:

a. primes *p*, *p* is odd.

b. a triangle *T* such that the sum of the angles of *T* equals 200° .

Solution:

a. By applying the rule for the negation of a statement, you can see that the answer is

a prime p such that p is not odd.

b. By applying the rule for the negation of a statement, you can see that the answer is

triangles *T*, the sum of the angles of *T* does not equal 200° .

Negations of Universal Conditional Statements

Negations of universal conditional statements are of special importance in mathematics.

The form of such negations can be derived from facts that have already been established.

By definition of the negation of a *for all* statement,

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \rightarrow Q(x)). \quad 3.2.1$$

But the negation of an if-then statement is logically equivalent to an *and* statement. More precisely,

$$\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x). \quad 3.2.2$$

Negations of Universal Conditional Statements

Substituting (3.2.2) into (3.2.1) gives

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

Written less symbolically, this becomes

Negation of a Universal Conditional Statement

$$\sim(\forall x, \text{ if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x).$$

Example 4 – *Negating Universal Conditional Statements*

Write a formal negation for statement (a) and an informal negation for statement (b).

a. people *p*, if *p* is blond then *p* has blue eyes.

b. If a computer program has more than 100,000 lines, then it contains a bug.

Solution:

a. a person *p* such that *p* is blond and *p* does not have blue eyes.

b. There is at least one computer program that has more than 100,000 lines and does not contain a bug.

The Relation among , , , and

The **negation** of a *for all* statement is a *there exists* statement, and the negation of a *there exists* statement is a *for all* statement.

These facts are analogous to **De Morgan's** laws, which state that the negation of an *and* statement is an *or* statement and that the negation of an *or* statement is an *and* statement.

This similarity is not accidental. In a sense, **universal** statements are generalizations of **and statements**, and **existential** statements are generalizations of **or statements.**

The Relation among , , , and

If $Q(x)$ is a predicate and the domain D of x is the set $\{x_1, x_2, \ldots, x_n\}$, then the statements

$$\forall x \in D, Q(x)$$

and

$$Q(x_1) \wedge Q(x_2) \wedge \cdots \wedge Q(x_n)$$

are logically equivalent.

Similarly, if $Q(x)$ is a predicate and $D = \{x_1, x_2, \ldots, x_n\}$, then the statements

$$\exists x \in D \text{ such that } Q(x)$$

and

$$Q(x_1) \vee Q(x_2) \vee \cdots \vee Q(x_n)$$

are logically equivalent.

Variants of Universal Conditional Statements

We have known that a conditional statement has a contrapositive, a converse, and an inverse.

The definitions of these terms can be extended to universal conditional statements.

• Definition

Consider a statement of the form: $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

Example 5 – *Contrapositive, Converse, and Inverse of a Universal Conditional Statement*

Write a **formal and an informal** contrapositive, converse, and inverse for the following statement:

Statement is:

If a real number is greater than 2, then its square is greater than 4.

Solution:

The formal version of this statement is

x R, if $x > 2$ then $x^2 > 4$.

Example 5 – *Solution* cont'd

Contrapositive: x **R**, if $x^2 \leq 4$ then $x \leq 2$.
Or: If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.

Converse: x **R**, if $x^2 > 4$ then $x > 2$.
Or: If the square of a real number is greater than 4, then the number is greater than 2.

Inverse: x **R**, if $x \leq 2$ then $x^2 \leq 4$.
Or: If a real number is less than or equal to 2, then the square of the number is less than or equal to 4.

Variants of Universal Conditional Statements

Let *P*(*x*) and *Q*(*x*) be any predicates, let *D* be the domain of *x*, and consider the statement

$$\forall x \in D, \text{if } P(x) \text{ then } Q(x)$$

and its contrapositive

$$\forall x \in D, \text{if } \sim Q(x) \text{ then } \sim P(x).$$

Any particular *x* in *D* that makes "if *P*(*x*) then *Q*(*x*)" true also makes "if ~*Q*(*x*) then ~*P*(*x*)" true (by the logical equivalence between $p \to q$ and $\sim q \to \sim p$).

Variants of Universal Conditional Statements

It follows that the sentence "If *P*(*x*) then *Q*(*x*)" is true for all *x* in *D* if, and only if, the sentence "If ~*Q*(*x*) then ~*P*(*x*)" is true for all *x* in *D*.

Thus we write the following and say that a universal conditional statement is logically equivalent to its contrapositive:

$$\forall x \in D, \text{if } P(x) \text{ then } Q(x) \equiv \forall x \in D, \text{if } \sim Q(x) \text{ then } \sim P(x)$$

Variants of Universal Conditional Statements

In Example 3.2.5 we noted that the statement

x **R**, if $x > 2$ then $x^2 > 4$

has the **converse**

x **R**, if $x^2 > 4$ then $x > 2$.

Observe that the statement is true whereas its **converse is false** (since, for instance, $(-3)^2 = 9 > 4$ but $-3 \not> 2$).

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Variants of Universal Conditional Statements

This shows that a universal conditional statement may have a different truth value from its converse.

Hence a universal conditional statement is not logically equivalent to its converse.

This is written in symbols as follows:

$$\forall x \in D, \text{if } P(x) \text{ then } Q(x) \not\equiv \forall x \in D, \text{if } Q(x) \text{ then } P(x).$$

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Necessary and Sufficient Conditions, Only If

The definitions of *necessary, sufficient,* and *only if* can also be extended to apply to universal conditional statements.

• Definition

- "$\forall x, r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$."
- "$\forall x, r(x)$ is a **necessary condition** for $s(x)$" means "$\forall x$, if $\sim r(x)$ then $\sim s(x)$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$." **(contrapositive)**
- "$\forall x, r(x)$ **only if** $s(x)$" means "$\forall x$, if $\sim s(x)$ then $\sim r(x)$" or, equivalently, "$\forall x$, if $r(x)$ then $s(x)$."

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Example 6 – *Necessary and Sufficient Conditions*

Rewrite the following statements as **quantified conditional** statements. Do not use the word *necessary* or *sufficient*.

a. Squareness is a sufficient condition for rectangularity.

b. Being at least 35 years old is a necessary condition for being President of the Palestine.

Solution:

a. A formal version of the statement is

***x*, if *x* is a square, then *x* is a rectangle.**

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Example 6 – *Solution*

cont'd

Or, in informal language:

If a figure is a square, then it is a rectangle.

b. Using formal language, you could write the answer as

people *x*, if *x* is younger than 35, then *x* cannot be President of Palestine.

Or, by the equivalence between a statement and its **contrapositive**:

people *x*, **if x is President of Palestine, then x is at least 35 years old.**

Statements with Multiple Quantifiers

When a statement contains more than one **quantifier المحدد**, we imagine the actions suggested by the quantifiers as being performed in the order in which the quantifiers occur.

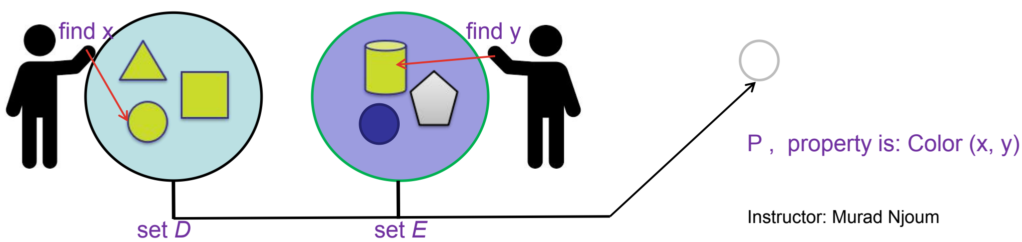
For instance, consider a statement of the form

***x* in set *D*, *y* in set *E* such that *x* and *y* satisfy property *P*(*x*, *y*).**

Statements with Multiple Quantifiers

To show that such a statement is true, you must be able to meet the following challenge:

- Imagine that someone is allowed to choose any element whatsoever from the set *D*, and imagine that the person gives you that element. Call it *x*.
- The challenge for you is to find an element y in E so that the person's x and your y, taken together, satisfy property P(x, y).



Example 1 – *Truth of a Statement in a Tarski World*

Consider the Tarski world shown in Figure 3.3.1.

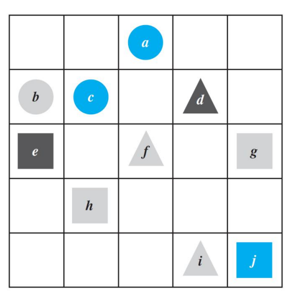


Figure 3.3.1

Show that the following statement is **true** in this world:

For all triangles *x*, there is a square *y* such that *x* and *y* have the same color.

Example 1 – *Solution*

The statement says that no matter which triangle someone gives you, you will be able to find a square of the same color. There are only three triangles, *d*, *f*, and *i*.

The following table shows that for each of **these triangles a square of the same color can be found.**

Given $x =$	choose $y =$	and check that y is the same color as x.
d	e	yes •
f or i	h or g	yes •

Statements with Multiple Quantifiers

Now consider a statement containing both and , where the comes before the :

an *x* in *D* such that *y* in *E*, *x* and *y* satisfy property *P*(*x*, *y*).

To show that a statement of this form is true:

You must find one single element (call it *x*) in *D* with the following property:

- After you have found your *x*, someone is allowed to choose any element whatsoever from *E*. The person challenges you by giving you that element. Call it *y*.
- Your job is to show that your *x* together with the person's *y* satisfy property *P*(*x*, *y*).

Statements with Multiple Quantifiers

Here is a summary of the convention for interpreting statements with two different quantifiers:

Interpreting Statements with Two Different Quantifiers

If you want to establish the truth of a statement of the form

x in *D*,   *y* in *E* such that *P*(*x*, *y*)

your challenge is to allow someone else to pick whatever element *x* in *D* they wish and then you must find an element *y* in *E* that "works" for that particular *x*.

If you want to establish the truth of a statement of the form

***x* in *D* such that   *y* in *E*, *P*(*x*, *y*)**

your job is to find one particular *x* in *D* that will "work" no matter what *y* in *E* anyone might choose to challenge you with.

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Example 3 – *Interpreting Multiply-Quantified Statements*

A college cafeteria line has **four stations**: salads, main courses, desserts تحلاية, and beverages مشروبات.

The salad station offers a choice of green **salad or fruit salad**; the main course station offers **spaghetti or fish**; the dessert station offers **pie or cake** كعكة أو فطيرة ; and the beverage station offers **milk, soda, or coffee**. Three students, Ahmad, Tamer, and Yusra, go through the line and make the following choices:

Ahmad: green salad, spaghetti, pie, milk

Tamer : fruit salad, fish, pie, cake, milk, coffee

Yusra: spaghetti, fish, pie, soda

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Example 3 – *Interpreting Multiply-Quantified Statements*

cont'd

These choices are illustrated in Figure 3.3.2.

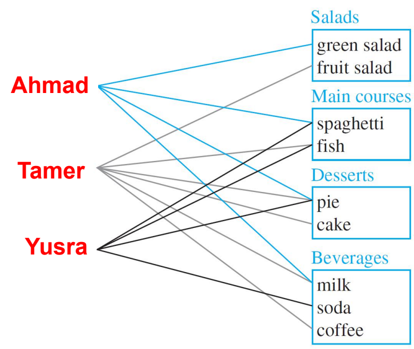

Figure 3.3.2

Example 3 – *Interpreting Multiply-Quantified Statements*

cont'd

Write each of following statements **informally** and find its **truth value**.

a. an item *I* such that students *S*, *S* chose *I*.

b. a student *S* such that items *I*, *S* chose *I*.

c. a student *S* such that stations *Z*, an item *I* in *Z* such that *S* chose *I*.

d. students *S* and stations *Z*, an item *I* in *Z* such that *S* chose *I*.

Example 3 – *Solution*

a. There is an item that was chosen by every student.

This is true; every student chose pie.

b. There is a student who chose every available item.

This is false; no student chose all nine items.

c. There is a student who chose at least one item from every station.

This is true; both Ahmad and Tamer chose at least one item from every station.

d. Every student chose at least one item from every station.

This is false; Yusra did not choose a salad.

Example 4 – *Translating Multiply-Quantified Statements from Informal to Formal Language*

The **reciprocal** تبادلي of a real number *a* is a real number *b* such that *ab* = 1. The following two statements are true.
Rewrite them formally using quantifiers and variables:

a. Every nonzero real number has a reciprocal.

b. There is a real number with no reciprocal.

The number 0 has no reciprocal.

Solution:

a. nonzero real numbers *u*, a real number *v* such that *uv* = 1.

b. a real number *c* such that real numbers *d*, $cd \neq$ 1.

Negations of Multiply-Quantified Statements

We apply these laws to find

~(*x* in *D*, *y* in *E* such that *P*(*x*, *y*))

by moving in stages from left to right along the sentence.

First version of negation: *x* in *D* such that ~(*y* in *E* such that *P*(*x*, *y*)).

Final version of negation: ***x* in *D* such that *y* in *E*, ~*P*(*x*, *y*).**

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Negations of Multiply-Quantified Statements

Similarly, to find

~(*x* in *D* such that *y* in *E*, *P*(*x*, *y*)),

we have

First version of negation: *x* in *D*,~(*y* in *E*, *P*(*x*, *y*)).

Final version of negation: ***x* in *D*, *y* in *E* such that ~*P*(*x*, *y*)**.

These facts can be summarized as follows:

Negations of Multiply-Quantified Statements

$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y).$

$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) \equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y).$

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Example 8 – *Negating Statements in a Tarski World*

Refer to the Tarski world of Figure 3.3.1.

Write a **negation** for each of the following statements, and determine which is true, the given statement or its negation.

a. For all squares *x*, there is a circle *y* such that *x* and *y* have the same color.

b. There is a triangle *x* such that for all squares *y*, *x* is to the right of *y*.

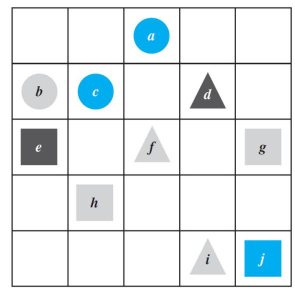

Figure 3.3.1

Example 8(a) – *Solution*

First version of negation: a square *x* such that
~(a circle *y* such that *x* and *y* have the same color).

Final version of negation: a square *x* such that
circles *y*, *x* and *y* do not have the same color.

The negation is true. Square *e* is black and no circle is black, so there is a square that does not have the same color as any circle.

Example 8(b) – *Solution*

cont'd

First version of negation: triangles *x*,~ (squares *y*, *x* is to the right of *y*).

Final version of negation:

triangles *x*, a square *y* such that *x* is not to the right of *y*.

The negation is true because no matter what triangle is chosen, it is not to the right of square *g* (or square *j*).

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Order of Quantifiers

Consider the following two statements:

people *x*, a person *y* such that *x* loves *y*.

a person *y* such that people *x*, *x* loves *y*.

Note that except for the order of the quantifiers, these statements **are identical.**

However, the first means that given any person, it is possible to find someone whom that person loves, whereas the second means that there is one amazing individual who is loved by all people.

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Order of Quantifiers

The two sentences illustrate an extremely important property about multiply-quantified statements:

In a statement containing both ∀ and ∃, changing the order of the quantifiers usually changes the meaning of the statement.

Interestingly, however, if one quantifier immediately follows another quantifier *of the same type*, then the order of the quantifiers does not affect the meaning.

Example 9 – *Quantifier Order in a Tarski World*

Look again at the Tarski world of Figure 3.3.1. Do the following two statements have the same truth value?

a. For every square *x* there is a triangle *y* such that *x* and *y* have different colors.

b. There exists a triangle *y* such that for every square *x*, *x* and *y* have different colors.

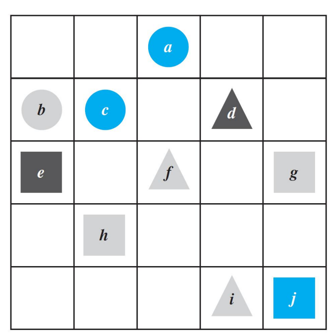


Figure 3.3.1

Example 9 – *Solution*

Statement (a) says that if someone gives you one of the squares from the Tarski world, you can find a triangle that has a different color. **This is true**.

If someone gives you square *g* or *h* (which are gray), you can use triangle *d* (which is black); if someone gives you square *e* (which is black), you can use either triangle *f* or triangle *i* (which are both gray); and if someone gives you square *j* (which is blue), you can use triangle *d* (which is black) or triangle *f* or *i* (which are both gray).

Example 9 – *Solution* cont'd

Statement (b) says that there is one particular triangle in the Tarski world that has a different color from every one of the squares in the world. **This is false**.

Two of the triangles are gray, but they cannot be used to show the truth of the statement because the Tarski world contains gray squares.

The only other triangle is black, but it cannot be used either because there is a black square in the Tarski world.

Thus one of the statements is true and the other is false, and so they have opposite truth values.

Example 10 – *Formalizing Statements in a Tarski World*

Consider once more the Tarski world of Figure 3.3.1:

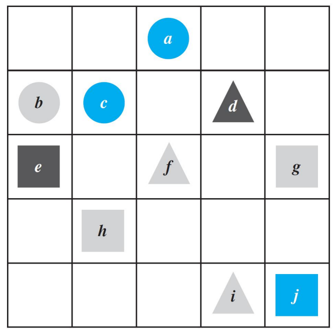


Figure 3.3.1

Example 10 – *Formalizing Statements in a Tarski World* cont'd

Let **Triangle(x), Circle(x), and Square(x)** mean "*x* is a triangle," "*x* is a circle," and "*x* is a square"; let Blue(*x*), Gray(*x*), and Black(*x*) mean "*x* is blue," "*x* is gray," and "*x* is black";

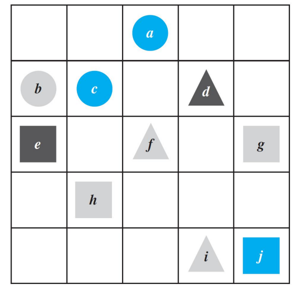
let **RightOf(*x*, *y*)**, **Above(x, y)**, and **SameColorAs(x, y)** mean "*x* is to the right of *y*," "*x* is above *y*," and "*x* has the same color as *y*"; and use the notation **x = y** to denote the predicate "*x* is equal to *y*".

Let the common domain *D* of all variables be the set of all the objects in the Tarski world.

Example 10 – *Formalizing Statements in a Tarski World*

cont'd

Use formal, logical notation to write each of the following statements, and write a formal negation for each statement.



a. For all circles *x*, *x* is above *f*.

b. There is a square *x* such that *x* is black.

c. For all circles *x*, there is a square *y* such that *x* and *y* have the same color.

d. There is a square *x* such that for all triangles *y*, *x* is to right of *y*.

Example 10(a) – *Solution*

Statement:

x(Circle(*x*) →Above(*x*, *f*)).

Negation:

~(*x*(Circle(*x*) → Above(*x*, *f*))

≡ *x* ~ (Circle(*x*) → Above(*x*, *f*))

by the law for negating a statement

≡ *x*(Circle(*x*) ~Above(*x*, *f*))

by the law of negating an if-then statement

Example 10(b) – *Solution*

cont'd

Statement:

x(Square(*x*)   Black(*x*)).

Negation:

~(*x*(Square(*x*)   Black(*x*))

≡ *x* ~ (Square(*x*)   Black(*x*))

by the law for negating a   statement

≡ *x*(~Square(*x*)   ~Black(*x*))

by De Morgan's law

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Example 10(c) – *Solution*

cont'd

Statement:

x(Circle(*x*) → *y*(Square(*y*)   SameColor(*x*, *y*))).

Negation:

~(*x*(Circle(*x*) → y(Square(*y*)   SameColor(*x*, *y*))))

≡ *x* ~ (Circle(*x*) → *y*(Square(*y*)   SameColor(*x*, *y*)))

by the law for negating a   statement

≡ *x*(Circle(*x*)   ~(*y*(Square(*y*)   SameColor(*x*, *y*))))

by the law for negating an if-then statement

≡ *x*(Circle(*x*)   *y*(~(Square(*y*)   SameColor(*x*, *y*))))

by the law for negating a   statement

≡ *x*(Circle(*x*)   *y*(~Square(*y*)   ~SameColor(*x*, *y*)))

by De Morgan's law

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Example 10(d) – *Solution*

cont'd

Statement:

x(Square(*x*)   *y*(Triangle(*y*) → RightOf(*x*, *y*))).

Negation:

~(*x*(Square(*x*)   *y*(Triangle(*y*) → RightOf(*x*, *y*))))

≡ x ~ (Square(x)   y(Triangle(x) → RightOf(x, y)))

by the law for negating a   statement

≡ *x*(~Square(*x*)   ~(*y*(Triangle(*y*) → RightOf(*x*, *y*))))

by De Morgan's law

≡ *x*(~Square(*x*)   *y*(~(Triangle(*y*) → RightOf(*x*, *y*))))

by the law for negating a   statement

≡ *x*(~Square(*x*)   *y*(Triangle(*y*)   ~RightOf(*x*, *y*)))

by the law for negating an if-then statement