BIRZEIT UNIVERSITY

Discrete Mathematic and Application Comp233

CHAPTER 5

SEQUENCES,MATHEMATICAL INDUCTION, AND RECURSION

Instructor
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Sequences

Imagine that a person decides to count his ancestors. He has two parents, four grandparents, eight great- grandparents, and so forth, These numbers can be written in a row as

2, 4, 8, 16, 32, 64, 128,…

The symbol "…" is called an *ellipsis*. It is shorthand for "and so forth."

To express the pattern of the numbers, suppose that each is labeled by an integer giving its position in the row.

Position in the row	1	2	3	4	5	6	7 . . .
Number of ancestors	2	4	8	16	32	64	128 . . .

Sequences

The number corresponding to position 1 is 2, which equals 2^1. The number corresponding to position 2 is 4, which equals 2^2.

For positions 3, 4, 5, 6, and 7, the corresponding numbers are 8, 16, 32, 64, and 128, which equal 2^3, 2^4, 2^5, 2^6, and 2^7, respectively.

For a general value of k, let A_k be the number of ancestors in the kth generation back. The pattern of computed values strongly suggests the following for each k:

$$A_k = 2^k.$$

Sequences

- **Definition**

A **sequence** is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

We typically represent a sequence as a set of elements written in a row. In the sequence denoted

$$a_m, a_{m+1}, a_{m+2}, \ldots, a_n,$$

each individual element a_k (read "a sub k") is called a **term**.

Sequences

The k in a_k is called a **subscript** or **index**, m (which may be any integer) is the subscript of the **initial term**, and n (which must be greater than or equal to m) is the subscript of the **final term**. The notation

$$a_m, a_{m+1}, a_{m+2}, \ldots$$

denotes an **infinite sequence**. An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k.

The following example shows that it is possible for two different formulas to give sequences with the same terms.

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Example 1 – *Finding Terms of Sequences Given by Explicit Formulas*

Define sequences a_1, a_2, a_3,… and b_2, b_3, b_4,… by the following explicit formulas:

$$a_k = \frac{k}{k+1} \quad \text{for all integers } k \geq 1,$$

$$b_i = \frac{i-1}{i} \quad \text{for all integers } i \geq 2.$$

Compute the first five terms of both sequences.

Solution:

$$a_1 = \frac{1}{1+1} = \frac{1}{2} \qquad b_2 = \frac{2-1}{2} = \frac{1}{2}$$

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Example 1 – *Solution*

cont'd

$$a_2 = \frac{2}{2+1} = \frac{2}{3} \qquad b_3 = \frac{3-1}{3} = \frac{2}{3}$$

$$a_3 = \frac{3}{3+1} = \frac{3}{4} \qquad b_4 = \frac{4-1}{4} = \frac{3}{4}$$

$$a_4 = \frac{4}{4+1} = \frac{4}{5} \qquad b_5 = \frac{5-1}{5} = \frac{4}{5}$$

$$a_5 = \frac{5}{5+1} = \frac{5}{6} \qquad b_6 = \frac{6-1}{6} = \frac{5}{6}$$

As you can see, the first terms of both sequences are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$ can be shown that all terms of both sequences are identical.

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Summation Notation

Consider again the example in which $A_k = 2^k$ represents the number of ancestors a person has in the kth generation back. **What is the total number of ancestors for the past six generations**?

The answer is

$$A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 126.$$

It is convenient to use a shorthand notation to write such sums.

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Summation Notation

In 1772 the French mathematician Joseph Louis Lagrange introduced the capital Greek letter sigma, Σ , to denote the word *sum* (or *summation*), and defined the summation notation as follows:

• Definition

If m and n are integers and $m \leq n$, the symbol $\sum_{k=m}^{n} a_k$, read the **summation from k equals m to n of a-sub-k**, is the sum of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We say that $a_m + a_{m+1} + a_{m+2} + \ldots + a_n$ is the **expanded form** of the sum, and we write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

We call k the **index** of the summation, m the **lower limit** of the summation, and n the **upper limit** of the summation.

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Example 4 – *Computing Summations*

Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, and $a_5 = 2$. Compute the following:

a. $\sum_{k=1}^{5} a_k$ **b.** $\sum_{k=2}^{2} a_k$ **c.** $\sum_{k=1}^{2} a_{2k}$

Solution:

a.

$$\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5$$

$$= (-2) + (-1) + 0 + 1 + 2$$

$$= 0$$

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Example 4 – *Solution*

cont'd

b. $\sum_{k=2}^{2} a_k = a_2$

$= -1$

c. $\sum_{k=1}^{2} a_{2k} = a_{2\cdot 1} + a_{2\cdot 2}$

$= a_2 + a_4$

$= -1 + 1$

$= 0$

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Example 6 – *Changing from Summation Notation to Expanded Form*

Write the following summation in expanded form:

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1}.$$

Solution:

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1}$$

$$= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

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Example 7 – *Changing from Expanded Form to Summation Notation*

Express the following using summation notation:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}.$$

Solution:

The general term of this summation can be expressed as $\frac{k+1}{n+k}$ for integers *k* from 0 to *n*.

Hence

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{k=0}^{n} \frac{k+1}{n+k}.$$

Summation Notation

A more mathematically precise definition of summation, called a *recursive definition*, is the following:

If *m* is any integer, then

$$\sum_{k=m}^{m} a_k = a_m \quad \text{and} \quad \sum_{k=m}^{n} a_k = \sum_{k=m}^{n-1} a_k + a_n \quad \text{for all integers } n > m.$$

When solving problems, it is often useful to rewrite a summation using the recursive form of the definition, either by separating off the final term of a summation or by adding a final term to a summation.

Example 9 – *Separating Off a Final Term and Adding On a Final Term*

a. Rewrite $\sum_{i=1}^{n+1} \frac{1}{i^2}$ by separating off the final term.

b. Write $\sum_{k=0}^{n} 2^k + 2^{n+1}$ as a single summation.

Solution:

a. $$\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$$

b. $$\sum_{k=0}^{n} 2^k + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$$

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Example 10 – *A Telescoping Sum*

Some sums can be transformed into telescoping sums, which then can be rewritten as a simple expression.

For instance, observe that

By **Partial fraction**: $\frac{1}{k(k+1)} = \frac{A}{k} + \frac{B}{(k+1)}$, after solving A=1,B=-1

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}.$$

Use this identity to find a simple expression for $\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

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Example 10 – *Solution*

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}.$$

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Product Notation

The notation for the product of a sequence of numbers is analogous to the notation for their sum. The Greek capital letter pi, Π, denotes a product. For example,

$$\prod_{k=1}^{5} a_k = a_1 a_2 a_3 a_4 a_5.$$

• Definition

If m and n are integers and $m \leq n$, the symbol $\prod_{k=m}^{n} a_k$, read the **product from k equals m to n of a-sub-k**, is the product of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

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Example 11 – *Computing Products*

A recursive definition for the product notation is the following: If *m* is any integer, then

$$\prod_{k=m}^{m} a_k = a_m \quad \text{and} \quad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n \quad \text{for all integers } n > m.$$

Compute the following products:

a. $\prod_{k=1}^{5} k$

b. $\prod_{k=1}^{1} \frac{k}{k+1}$

Solution:

a. $\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

b. $\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$

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Properties of Summations and Products

The following theorem states general properties of summations and products.

Theorem 5.1.1

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$
2. $c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ generalized distributive law
3. $\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k).$

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Example 12 – *Using Properties of Summation and Product*

Let $a_k = k + 1$ and $b_k = k - 1$ for all integers k. Write each of the following expressions as a single summation or product:

a. $\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k$ **b.** $\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right)$

Solution:

a.

$$\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(k+1) + 2 \cdot \sum_{k=m}^{n}(k-1)$$ by substitution

$$= \sum_{k=m}^{n}(k+1) + \sum_{k=m}^{n} 2 \cdot (k-1)$$ by Theorem 5.1.1 (2)

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Example 12 – *Using Properties of Summation and Product*

$$= \sum_{k=m}^{n}((k+1) + 2 \cdot (k-1))$$ by Theorem 5.1.1 (1)

$$= \sum_{k=m}^{n}(3k-1)$$ by algebraic simplification

b.

$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \left(\prod_{k=m}^{n}(k+1)\right) \cdot \left(\prod_{k=m}^{n}(k-1)\right)$$ by substitution

$$= \prod_{k=m}^{n}(k+1) \cdot (k-1)$$ by Theorem 5.1.1 (3)

$$= \prod_{k=m}^{n}(k^2-1)$$ by algebraic simplification

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Change of Variable

Observe that $$\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$$

and also that $$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2.$$

Hence $$\sum_{k=1}^{3} k^2 = \sum_{i=1}^{3} i^2.$$

This equation illustrates the fact that the symbol used to represent the index of a summation can be replaced by any other symbol as long as the replacement is made in each location where the symbol occurs.

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Change of Variable

As a consequence, the index of a summation is called a dummy variable.

A **dummy variable** is a symbol that derives its entire meaning from its local context. Outside of that context (both before and after), the symbol may have another meaning
entirely.

A general procedure to transform the first summation into the second is illustrated in Example 13.

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Example 13 – *Transforming a Sum by a Change of Variable*

Transform the following summation by making the specified change of variable.

summation: $\sum_{k=0}^{6} \frac{1}{k+1}$ *change of variable:* $j = k + 1$

Solution:

First calculate the lower and upper limits of the new summation:

$$\text{When } k = 0, \quad j = k + 1 = 0 + 1 = 1.$$

Thus the new sum goes from $j = 1$ to $j = 7$.

$$\text{When } k = 6, \quad j = k + 1 = 6 + 1 = 7.$$

Example 13 – *Solution* cont'd

Next calculate the general term of the new summation. You will need to replace each occurrence of *k* by an expression in *j* :

$$\text{Since } j = k + 1, \text{ then } k = j - 1.$$

$$\text{Hence } \frac{1}{k+1} = \frac{1}{(j-1)+1} = \frac{1}{j}.$$

Finally, put the steps together to obtain

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j}.$$

Example 14 – *When the Upper Limit Appears in the Expression to Be Summed*

Sometimes it is necessary to shift the limits of one summation in order to add it to another.

A general procedure for making such a shift when the upper limit is part of the summand is illustrated in the next example.

a. Transform the following summation by making the $j = k - 1$ specified change of variable.

summation: $\sum_{k=1}^{n+1} \left(\frac{k}{n+k} \right)$ *change of variable:*

b. Transform the summation obtained in part (a) by changing all *j*'s to *k*'s.

Example 14 – *Solution*

a. When $k = 1$, then $j = k - 1 = 1 - 1 = 0$. (So the new lower limit is 0.)
When $k = n + 1$, then $j = k - 1 = (n + 1) - 1 = n$. (So the new upper limit is n.)

Since $j = k - 1$, then $k = j + 1$. Also note that n is a constant as far as the terms of the sum are concerned.

It follows that

and so the general term of the new summation is

$$\frac{k}{n+k} = \frac{j+1}{n+(j+1)}$$

Example 14 – *Solution*

cont'd

Therefore,

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{j=0}^{n} \frac{j+1}{n+(j+1)}. \quad 5.1.3$$

b. Changing all the j's to k's in the right-hand side of equation (5.1.3) gives

$$\sum_{j=0}^{n} \frac{j+1}{n+(j+1)} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)} \quad 5.1.4$$

Combining equations (5.1.3) and (5.1.4) results in

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}.$$

Factorial and "*n* Choose *r*" Notation

The product of all consecutive integers up to a given integer occurs so often in mathematics that it is given a special notation—*factorial* notation.

• Definition

For each positive integer n, the quantity $\boldsymbol{n}$ **factorial** denoted $\boldsymbol{n!}$**,** is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1.$$

Zero factorial, denoted $0!$, is defined to be 1:

$$0! = 1.$$

Factorial and "*n* Choose *r*" Notation

A recursive definition for factorial is the following: Given any nonnegative integer *n*,

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1. \end{cases}$$

The next example illustrates the usefulness of the recursive definition for making computations.

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Example 16 – *Computing with Factorials*

Simplify the following expressions:

a. $\frac{8!}{7!}$ **b.** $\frac{5!}{2! \cdot 3!}$ **c.** $\frac{1}{2! \cdot 4!} + \frac{1}{3! \cdot 3!}$

d. $\frac{(n+1)!}{n!}$ **e.** $\frac{n!}{(n-3)!}$

Solution:

a. $\frac{8!}{7!} = \frac{8 \cdot \cancel{7!}}{\cancel{7!}} = 8$

b. $\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{2! \cdot \cancel{3!}} = \frac{5 \cdot 4}{2 \cdot 1} = 10$

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Example 16 – *Solution*

cont'd

c. $\dfrac{1}{2!\cdot 4!} + \dfrac{1}{3!\cdot 3!} = \dfrac{1}{2!\cdot 4!}\cdot\dfrac{3}{3} + \dfrac{1}{3!\cdot 3!}\cdot\dfrac{4}{4}$ by multiplying each numerator and denominator by just what is necessary to obtain a common denominator

$= \dfrac{3}{3\cdot 2!\cdot 4!} + \dfrac{4}{3!\cdot 4\cdot 3!}$ by rearranging factors

$= \dfrac{3}{3!\cdot 4!} + \dfrac{4}{3!\cdot 4!}$ because $3\cdot 2! = 3!$ and $4\cdot 3! = 4!$

$= \dfrac{7}{3!\cdot 4!}$ by the rule for adding fractions with a common denominator

$= \dfrac{7}{144}$

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Example 16 – *Solution*

cont'd

d. $\dfrac{(n+1)!}{n!} = \dfrac{(n+1)\cdot \cancel{n!}}{\cancel{n!}}$

$= n + 1$

e. $\dfrac{n!}{(n-3)!} = \dfrac{n\cdot(n-1)\cdot(n-2)\cdot\cancel{(n-3)!}}{\cancel{(n-3)!}}$

$= n\cdot(n-1)\cdot(n-2)$

$= n^3 - 3n^2 + 2n$

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Factorial and "*n* Choose *r*" Notation

An important use for the factorial notation is in calculating values of quantities, called *n choose r*, that occur in many branches of mathematics, especially those connected with the study of counting techniques and probability.

• Definition

Let n and r be integers with $0 \le r \le n$. The symbol

$$\binom{n}{r}$$

is read "***n* choose *r***" and represents the number of subsets of size r that can be chosen from a set with n elements.

Observe that the definition implies that $\binom{n}{r}$ will always be an integer because it is a number of subsets.

Factorial and "*n* Choose *r*" Notation

The computational formula:

• Formula for Computing $\binom{n}{r}$

For all integers n and r with $0 \le r \le n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Many electronic calculators have keys for computing values of $\binom{n}{r}$. These are denoted in various ways such as nCr, $C(n, r)$, ${}^nC_r$, and $C_{n,r}$.

The letter C is used because the quantities $\binom{n}{r}$ are also called *combinations* التوافيق. Sometimes they are referred to as *binomial coefficients* because of the connection with the binomial theorem.

Example 17 – *Computing* $\binom{n}{r}$ *by Hand*

Use the formula for computing $\binom{n}{r}$ to evaluate the following expressions:

a. $\binom{8}{5}$ **b.** $\binom{4}{0}$ **c.** $\binom{n+1}{n}$

Solution:

a. $$\binom{8}{5} = \frac{8!}{5!(8-5)!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(5\cdot 4\cdot 3\cdot 2\cdot 1)\cdot(\cdot 3\cdot 2\cdot 1)}$$

$$= 56.$$

always cancel common factors before multiplying

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Example 17 – *Solution*

cont'd

b. $$\binom{4}{4} = \frac{4!}{4!(4-4)!}$$

$$= \frac{4!}{4!0!}$$

$$= \frac{4\cdot 3\cdot 2\cdot 1}{(4\cdot 3\cdot 2\cdot 1)(1)}$$

$$= 1$$

The fact that 0! = 1 makes this formula computable. It gives the correct value because a set of size 4 has exactly one subset of size 4, namely itself.

c. $$\binom{n+1}{n} = \frac{(n+1)!}{n!((n+1)-n)!} = \frac{(n+1)!}{n!1!} = \frac{(n+1)\cdot n!}{n!} = n+1$$

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Sequences in Computer Programming

An important data type in computer programming consists of finite sequences. In computer programming contexts, these are usually referred to as *one-dimensional arrays*. For example, consider a program that analyzes the wages paid to a sample of 50 workers. Such a program might compute the **average wage** and the difference between each individual wage and the average.

This would require that each wage be stored in memory for retrieval later in the calculation.

To avoid the use of entirely separate variable names for all of the 50 wages, each is written as a term of a one-dimensional array:

$$W[1], W[2], W[3], \ldots, W[50].$$

Example 18 – *Dummy Variable in a Loop*

The index variable for a **for-next** loop is a dummy variable. For example, the following three algorithm segments all produce the same output:

```
1. for i := 1 to n
       print a[i]
   next i
```

```
2. for j := 0 to n − 1
       print a[j + 1]
   next j
```

```
3. for k := 2 to n + 1
       print a[k − 1]
   next k
```

Sequences in Computer Programming

The recursive definitions for summation, product, and factorial lead naturally to computational algorithms.

For instance, here are two sets of pseudocode to find the sum of *a*[1], *a*[2], …, *a*[*n*].

The one on the left exactly mimics تقليد the recursive definition by initializing the sum to equal *a*[1]; the one on the right initializes the sum to equal 0.

Sequences in Computer Programming

In both cases the output is $\sum_{k=1}^{n} a[k]$.

```
s := a[1]
for k := 2 to n
    s := s + a[k]
next k
```

```
s := 0
for k := 1 to n
    s := s + a[k]
next k
```

Mathematical Induction I

Mathematical induction is one of the more recently developed techniques of proof in the history of mathematics.

It is used to check conjectures التخمينات about the outcomes of processes that occur repeatedly and according to definite patterns.

In general, mathematical induction is a method for proving that a property defined for integers *n* is true for all values of *n* that are greater than or equal to some initial integer.

Mathematical Induction I

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

$$\text{for all integers } n \geq a,\ P(n)$$

is true.

The validity of proof by mathematical induction is generally taken as an axiom. That is why it is referred to as the *principle* of mathematical induction rather than as a theorem.

Mathematical Induction I

Proving a statement by mathematical induction is a two-step process. The first step is called the *basis step*, and the second step is called the *inductive step*.

Method of Proof by Mathematical Induction

Consider a statement of the form, "For all integers $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This supposition is called the **inductive hypothesis.***]*

Then

show that $P(k+1)$ is true.

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Example 1 – *Sum of the First n Integers*

Use mathematical induction to prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \quad \text{for all integers } n \geq 1.$$

Solution:

To construct a proof by induction, you must first identify the property $P(n)$. In this case, $P(n)$ is the equation

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}. \quad \leftarrow \text{the property } (P(n))$$

[To see that P(n) is a sentence, note that its subject is "the sum of the integers from 1 to n" and its verb is "equals."]

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Example 1 – *Solution*

cont'd

In the basis step of the proof, you must show that the property is true for $n = 1$, or, in other words that $P(1)$ is true.

Now $P(1)$ is obtained by substituting 1 in place of n in $P(n)$. The left-hand side of $P(1)$ is the sum of all the successive integers starting at 1 and ending at 1. This is just 1. Thus $P(1)$ is

$$1 = \frac{1(1+1)}{2}.$$

← basis ($P(1)$)

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Example 1 – *Solution*

cont'd

Of course, this equation is true because the right-hand side is

$$\frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = 1,$$

which equals the left-hand side.

In the inductive step, you assume that $P(k)$ is true, for a particular but arbitrarily chosen integer k with $k \geq 1$. *[This assumption is the inductive hypothesis.]*

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Example 1 – *Solution*

cont'd

You must then show that *P*(*k* + 1) is true. What are *P*(*k*) and *P*(*k* + 1)? *P*(*k*) is obtained by substituting *k* for every *n* in *P*(*n*).

Thus *P*(*k*) is

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}. \quad \leftarrow \text{inductive hypothesis } (P(k))$$

Similarly, *P*(*k* + 1) is obtained by substituting the quantity (*k* + 1) for every *n* that appears in *P*(*n*).

Example 1 – *Solution*

cont'd

Thus *P*(*k* + 1) is

$$1 + 2 + \cdots + (k+1) = \frac{(k+1)((k+1)+1)}{2},$$

or, equivalently,

$$1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}. \quad \leftarrow \text{to show } (P(k+1))$$

Example 1 – *Solution*

cont'd

Now the inductive hypothesis is the supposition that $P(k)$ is true. How can this supposition be used to show that $P(k + 1)$ is true? $P(k + 1)$ is an equation, and the truth of an equation can be shown in a variety of ways.

One of the most straightforward is to use the inductive hypothesis along with algebra and other known facts to transform separately the left-hand and right-hand sides until you see that they are the same.

Example 1 – *Solution*

cont'd

In this case, the left-hand side of $P(k + 1)$ is

$$1 + 2 + \cdots + (k + 1),$$

which equals

$$(1 + 2 + \cdots + k) + (k + 1)$$

The next-to-last term is k because the terms are successive integers and the last term is $k + 1$.

But by substitution from the inductive hypothesis,

$$(1 + 2 + \cdots + k) + (k + 1) = \frac{k(k+1)}{2} + (k + 1)$$

since the inductive hypothesis says that $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

Example 1 – *Solution*

cont'd

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

by multiplying the numerator and denominator of the second term by 2 to obtain a common denominator

$$= \frac{k^2+k}{2} + \frac{2k+2}{2}$$

by multiplying out the two numerators

$$= \frac{k^2+3k+2}{2}$$

by adding fractions with the same denominator and combining like terms.

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Example 1 – *Solution*

cont'd

So the left-hand side of *P*(*k* + 1) is $\frac{k^2+3k+2}{2}$

Now the right-hand side of *P*(*k* + 1) is $\frac{(k+1)(k+2)}{2} = \frac{k^2+3k+2}{2}$ by multiplying out the numerator.

Thus the two sides of *P*(*k* + 1) are equal to each other, and so the equation *P*(*k* + 1) is true.

This discussion is summarized as follows:

Theorem 5.2.2 Sum of the First *n* Integers

For all integers $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

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Example 1 – *Solution*

cont'd

Proof (by mathematical induction):

Let the property *P*(*n*) be the equation

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \quad \leftarrow P(n)$$

Show that *P*(1) is true:

To establish *P*(1), we must show that

$$1 = \frac{1(1+1)}{2} \quad \leftarrow \quad P(1)$$

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Example 1 – *Solution*

cont'd

But the left-hand side of this equation is 1 and the right-hand side is

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

also. Hence *P*(1) is true.

Show that for all integers *k* ≥ 1, if *P*(*k*) is true then *P*(*k* + 1) is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer k ≥ 1. That is:] Suppose that *k* is any integer with *k* ≥ 1 such that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \quad \leftarrow P(k) \text{ inductive hypothesis}$$

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Example 1 – *Solution*

cont'd

[We must show that P(k + 1) is true. That is:] We must show that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)[(k+1)+1]}{2},$$

or, equivalently, that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}. \quad \leftarrow P(k+1)$$

[We will show that the left-hand side and the right-hand side of P(k + 1) are equal to the same quantity and thus are equal to each other.]

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Example 1 – *Solution*

cont'd

The left-hand side of $P(k + 1)$ is

$$1 + 2 + 3 + \cdots + (k+1)$$

$$= 1 + 2 + 3 + \cdots + k + (k+1)$$ by making the next-to-last term explicit

$$= \frac{k(k+1)}{2} + (k+1)$$ by substitution from the inductive hypothesis

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k^2+k}{2} + \frac{2k+2}{2}$$

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Example 1 – *Solution*

cont'd

$$= \frac{k^2 + 3k + 2}{2} \qquad \text{by algebra.}$$

And the right-hand side of *P*(*k* + 1) is

$$\frac{(k+1)(k+2)}{2} = \frac{k^2 + 3k + 2}{2}.$$

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Example 1 – *Solution*

cont'd

Thus the two sides of *P*(*k* + 1) are equal to the same quantity and so they are equal to each other. Therefore the equation *P*(*k* + 1) is true *[as was to be shown]*.

[Since we have proved both the basis step and the inductive step, we conclude that the theorem is true.]

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Mathematical Induction I

• Definition Closed Form

If a sum with a variable number of terms is shown to be equal to a formula that does not contain either an ellipsis or a summation symbol, we say that it is written **in closed form**.

For example, writing $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ expresses the sum $1 + 2 + 3 + \cdots + n$ in closed form.

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Example 2 – *Applying the Formula for the Sum of the First n Integers*

a. Evaluate $2 + 4 + 6 + \cdots + 500$.

b. Evaluate $5 + 6 + 7 + 8 + \cdots + 50$.

c. For an integer $h \geq 2$, write $1 + 2 + 3 + \cdots + (h - 1)$ in closed form.

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Example 2 – *Solution*

a. $2 + 4 + 6 + \cdots + 500 = 2 \cdot (1 + 2 + 3 + \cdots + 250)$

$$= 2 \cdot \left(\frac{250 \cdot 251}{2}\right)$$ by applying the formula for the sum of the first n integers with $n = 250$

$$= 62{,}750.$$

b. $5 + 6 + 7 + 8 + \cdots + 50 = (1 + 2 + 3 + \cdots + 50) - (1 + 2 + 3 + 4)$

$$= \frac{50 \cdot 51}{2} - 10$$ by applying the formula for the sum of the first n integers with $n = 50$

$$= 1{,}265$$

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Example 2 – *Solution* cont'd

c. $1 + 2 + 3 + \cdots + (h - 1) = \frac{(h-1) \cdot [(h-1) + 1]}{2}$ by applying the formula for the sum of the first n integers with $n = h - 1$

$$= \frac{(h-1) \cdot h}{2}$$ since $(h - 1) + 1 = h$.

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Mathematical Induction I

In a **geometric sequence,** each term is obtained from the preceding one by multiplying by a constant factor.

If the first term is 1 and the constant factor is r, then the sequence is 1, r, r^2, r^3, . . . , r^n, ...

The sum of the first n terms of this sequence is given by the formula

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

for all integers $n \geq 0$ and real numbers r not equal to 1.

Mathematical Induction I

The expanded form of the formula is

$$r^0 + r^1 + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1},$$

and because $r^0 = 1$ and $r^1 = r$, the formula for $n \geq 1$ can be rewritten as

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}.$$

Example 3 – *Sum of a Geometric Sequence*

Prove that $\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$ for all integers $n \geq 0$ and all real numbers r except 1.

Solution:

In this example the property $P(n)$ is again an equation, although in this case it contains a real variable r:

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}. \quad \leftarrow \text{the property } (P(n))$$

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Example 3 – *Solution*

cont'd

Because r can be any real number other than 1, the proof begins by supposing that r is a particular but arbitrarily chosen real number not equal to 1.

Then the proof continues by mathematical induction on n, starting with $n = 0$.

In the basis step, you must show that $P(0)$ is true; that is, you show the property is true for $n = 0$.

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Example 3 – *Solution* cont'd

So you substitute 0 for each n in $P(n)$:

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1} - 1}{r - 1}. \quad \leftarrow \text{basis } (P(0))$$

In the inductive step, you suppose k is any integer with $k \geq 0$ for which $P(k)$ is true; that is, you suppose the property is true for $n = k$.

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Example 3 – *Solution* cont'd

So you substitute k for each n in $P(n)$:

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1}. \quad \leftarrow \text{inductive hypothesis } (P(k))$$

Then you show that $P(k + 1)$ is true; that is, you show the property is true for $n = k + 1$.

So you substitute $k + 1$ for each n in $P(n)$:

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1} - 1}{r - 1},$$

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Example 3 – *Solution*

cont'd

Or, equivalently,

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2} - 1}{r - 1}. \qquad \leftarrow \text{to show } (P(k+1))$$

In the inductive step for this proof we use another common technique for showing that an equation is true:

We start with the left-hand side and transform it
step-by-step into the right-hand side using the inductive hypothesis together with algebra and other known facts.

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Example 3 – *Solution*

cont'd

Theorem 5.2.3 Sum of a Geometric Sequence

For any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Proof (by mathematical induction):
Suppose *r* is a particular but arbitrarily chosen real number that is not equal to 1, and let the property *P*(*n*) be the equation

$$\sum_{i=0}^{n} r^i = \frac{r^{i+1} - 1}{r - 1} \qquad \leftarrow P(n)$$

We must show that *P*(*n*) is true for all integers $n \geq 0$. We do this by mathematical induction on *n*.

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Example 3 – *Solution* cont'd

Show that *P*(0) is true:

To establish *P*(0), we must show that

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1}-1}{r-1} \qquad \leftarrow P(0)$$

The left-hand side of this equation is $r^0 = 1$ and the right-hand side is

$$\frac{r^{0+1}-1}{r-1} = \frac{r-1}{r-1} = 1$$

also because $r^1 = r$ and $r \neq 1$. Hence *P*(0) is true.

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Example 3 – *Solution* cont'd

Show that for all integers *k* ≥ 0, if *P*(*k*) is true then *P*(*k* + 1) is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer k ≥ 0. That is:]

Let *k* be any integer with *k* ≥ 0, and suppose that

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1}-1}{r-1} \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

[We must show that P(k + 1) is true. That is:] We must show that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1}-1}{r-1}.$$

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Example 3 – *Solution*

cont'd

Or, equivalently, that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2}-1}{r-1}. \qquad \leftarrow P(k+1)$$

[We will show that the left-hand side of P(k + 1) equals the right-hand side.] The left-hand side of $P(k + 1)$ is

$$\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1}$$ by writing the $(k+1)$st term separately from the first k terms

$$= \frac{r^{k+1}-1}{r-1} + r^{k+1}$$ by substitution from the inductive hypothesis

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Example 3 – *Solution*

cont'd

$$= \frac{r^{k+1}-1}{r-1} + \frac{r^{k+1}(r-1)}{r-1}$$ by multiplying the numerator and denominator of the second term by $(r-1)$ to obtain a common denominator

$$= \frac{(r^{k+1}-1) + r^{k+1}(r-1)}{r-1}$$ by adding fractions

$$= \frac{r^{k+1}-1+r^{k+2}-r^{k+1}}{r-1}$$ by multiplying out and using the fact that $r^{k+1}\cdot r = r^{k+1}\cdot r^1 = r^{k+2}$

$$= \frac{r^{k+2}-1}{r-1}$$ by canceling the r^{k+1}'s.

which is the right-hand side of $P(k + 1)$ *[as was to be shown.]*

[Since we have proved the basis step and the inductive step, we conclude that the theorem is true.]

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Proving an Equality

The proofs of the basis and inductive steps in Examples 1 and 3 illustrate two different ways to show that an equation is true:

(1) transforming the left-hand side and the right-hand side independently until they are seen to be equal, and

(2) transforming one side of the equation until it is seen to be the same as the other side of the equation.

Sometimes people use a method that they believe proves equality but that is actually invalid.

Proving an Equality

For example, to prove the basis step for Theorem 5.2.3, they perform the following steps:

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1} - 1}{r - 1}$$

$$r^0 = \frac{r^1 - 1}{r - 1}$$

$$1 = \frac{r - 1}{r - 1}$$

$$1 = 1$$

Proving an Equality

The problem with this method is that starting from a statement and deducing a true conclusion does not prove that the statement is true.

A true conclusion can also be deduced from a false statement. For instance, the steps below show how to deduce the true conclusion that 1 = 1 from the false statement that 1 = 0:

$$1 = 0 \qquad \leftarrow \text{false}$$

$$0 = 1$$

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Proving an Equality

$$1 + 0 = 0 + 1$$

$$1 = 1 \qquad \leftarrow \text{true}$$

When using mathematical induction to prove formulas, be sure to use a method that avoids invalid reasoning, both for the basis step and for the inductive step.

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Example 4 – *Applying the Formula for the Sum of a Geometric Sequence*

In each of (**a**) and (**b**) below, assume that *m* is an integer that is greater than or equal to 3. Write each of the sums in closed form.

a. $1 + 3 + 3^2 + \cdots + 3^{m-2}$

b. $3^2 + 3^3 + 3^4 + \cdots + 3^m$

Solution:

a.

$$1 + 3 + 3^2 + \cdots + 3^{m-2} = \frac{3^{(m-2)+1} - 1}{3 - 1}$$ by applying the formula for the sum of a geometric sequence with $r = 3$ and $n = m - 2$

$$= \frac{3^{m-1} - 1}{2}.$$

b.

$$3^2 + 3^3 + 3^4 + \cdots + 3^m = 3^2 \cdot (1 + 3 + 3^2 + \cdots + 3^{m-2})$$ by factoring out 3^2

$$= 9 \cdot \left(\frac{3^{m-1} - 1}{2}\right)$$ by part (a).

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Deducing Additional Formulas

As with the formula for the sum of the first *n* integers, there is a way to think of the formula for the sum of the terms of a geometric sequence that makes it seem simple and intuitive. Let

$$S_n = 1 + r + r^2 + \cdots + r^n.$$

Then

$$rS_n = r + r^2 + r^3 + \cdots + r^{n+1},$$

and so

$$rS_n - S_n = (r + r^2 + r^3 + \cdots + r^{n+1}) - (1 + r + r^2 + \cdots + r^n)$$

$$= r^{n+1} - 1.$$ 5.2.1

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Deducing Additional Formulas

But

$$rS_n - S_n = (r-1)S_n. \qquad 5.2.2$$

Equating the right-hand sides of equations (5.2.1) and (5.2.2) and dividing by $r - 1$ gives

$$S_n = \frac{r^{n+1} - 1}{r - 1}.$$

This derivation of the formula is attractive and is quite convincing. However, it is not as logically airtight as the proof by mathematical induction.

Deducing Additional Formulas

To go from one step to another in the previous calculations, the argument is made that each term among those indicated by the ellipsis (. . .) has such-and-such an appearance and when these are canceled such-and-such occurs.

But it is impossible actually to see each such term and each such calculation, and so the accuracy of these claims cannot be fully checked.

With mathematical induction it is possible to focus exactly on what happens in the middle of the ellipsis and verify without doubt that the calculations are correct.

Mathematical Induction II

In natural science courses, deduction and induction are presented as alternative modes of thought—deduction being to infer a conclusion from general principles using the laws of logical reasoning, and induction being to enunciate a general principle after observing it to hold in a large number of specific instances.

In this sense, then, *mathematical* induction is not inductive but deductive.

Once proved by mathematical induction, a theorem is known just as certainly as if it were proved by any other mathematical method.

Mathematical Induction II

Inductive reasoning, in the natural sciences sense, *is* used in mathematics, but only to make conjectures, not to prove them.

For example, observe that

$$1 - \frac{1}{2} = \frac{1}{2}$$

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = \frac{1}{3}$$

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = \frac{1}{4}$$

Mathematical Induction II

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)=\frac{1}{k},$$

then by substitution

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{k}\right)\left(1-\frac{1}{k+1}\right)=\frac{1}{k}\left(1-\frac{1}{k+1}\right)$$
$$=\frac{1}{k}\left(\frac{k+1-1}{k+1}\right)$$
$$=\frac{1}{k}\left(\frac{k}{k+1}\right)=\frac{1}{k+1}.$$

Thus mathematical induction makes knowledge of the general pattern a matter of mathematical certainty rather than vague conjecture.

Example 1 – *Proving a Divisibility Property*

Use mathematical induction to prove that for all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

Solution:
As in the previous proofs by mathematical induction, you need to identify the property *P*(*n*).

In this example, *P*(*n*) is the sentence

$2^{2n} - 1$ is divisible by 3. ← the property ($P(n)$)

Example 1 – *Solution*

cont'd

By substitution, the statement for the basis step, $P(0)$, is

$2^{2 \cdot 0} - 1$ is divisible by 3. ← basis ($P(0)$)

The supposition for the inductive step, $P(k)$, is

$2^{2k} - 1$ is divisible by 3, ← inductive hypothesis ($P(k)$)

and the conclusion to be shown, $P(k + 1)$, is

$2^{2(k+1)} - 1$ is divisible by 3. ← to show ($P(k + 1)$)

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Example 1 – *Solution*

cont'd

We know that an integer m is divisible by 3 if, and only if, $m = 3r$ for some integer r.

Now the statement $P(0)$ is true because
$2^{2 \cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0$, which is divisible by 3 because $0 = 3 \cdot 0$.

To prove the inductive step, you suppose that k is any integer greater than or equal to 0 such that $P(k)$ is true.

This means that $2^{2k} - 1$ is divisible by 3. You must then prove the truth of $P(k + 1)$. Or, in other words, you must show that $2^{2(k+1)} - 1$ is divisible by 3.

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Example 1 – *Solution*

cont'd

But

$$2^{2(k+1)} - 1 = 2^{2k+2} - 1$$

$$= 2^{2k} \cdot 2^2 - 1 \quad \text{by the laws of exponents}$$

$$= 2^{2k} \cdot 4 - 1.$$

The aim is to show that this quantity, $2^{2k} \cdot 4 - 1$, is divisible by 3. Why should that be so? By the inductive hypothesis,
$2^{2k} - 1$ is divisible by 3, and $2^{2k} \cdot 4 - 1$ resembles $2^{2k} - 1$.

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Example 1 – *Solution*

cont'd

Observe what happens, if you subtract $2^{2k} - 1$ from $2^{2k} \cdot 4 - 1$:

$$\underbrace{2^{2k} \cdot 4 - 1}_{\text{divisible by 3?}} - \underbrace{(2^{2k} - 1)}_{\text{divisible by 3}} = \underbrace{2^{2k} \cdot 3.}_{\text{divisible by 3}}$$

Adding $2^{2k} - 1$ to both sides gives

$$\underbrace{2^{2k} \cdot 4 - 1}_{\text{divisible by 3?}} = \underbrace{2^{2k} \cdot 3}_{\text{divisible by 3}} + \underbrace{2^{2k} - 1.}_{\text{divisible by 3}}$$

Both terms of the sum on the right-hand side of this equation are divisible by 3; hence the sum is divisible by 3.

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Example 1 – *Solution*

cont'd

Therefore, the left-hand side of the equation is also divisible by 3, which is what was to be shown.

This discussion is summarized as follows:

Proposition 5.3.1

For all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

Proof (by mathematical induction):

Let the property *P*(*n*) be the sentence "$2^{2n} – 1$ is divisible by 3."

$$2^{2n} - 1 \text{ is divisible by } 3. \quad \leftarrow P(n)$$

Example 1 – *Solution*

cont'd

Show that *P*(0) is true:

To establish *P*(0), we must show that

$$2^{2 \cdot 0} - 1 \text{ is divisible by } 3. \quad \leftarrow P(0)$$

But

$$2^{2 \cdot 0} - 1 = 2^0 - 1 = 1 - 1 = 0$$

and 0 is divisible by 3 because $0 = 3 \cdot 0$.

Hence *P*(0) is true.

Example 1 – *Solution* cont'd

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k + 1)$ is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer $k \geq 0$. That is:]

Let k be any integer with $k \geq 0$, and suppose that

$$2^{2k} - 1 \text{ is divisible by } 3. \quad \leftarrow P(k) \text{ inductive hypothesis}$$

By definition of divisibility, this means that

$$2^{2k} - 1 = 3r \quad \text{for some integer } r.$$

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Example 1 – *Solution* cont'd

[We must show that P(k + 1) is true. That is:] We must show that

$$2^{2(k+1)} - 1 \text{ is divisible by } 3. \quad \leftarrow P(k+1)$$

But

$$\begin{aligned} 2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\ &= 2^{2k} \cdot 2^2 - 1 \quad \text{by the laws of exponents} \\ &= 2^{2k} \cdot 4 - 1 \\ &= 2^{2k}(3 + 1) - 1 \end{aligned}$$

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Example 1 – *Solution*

cont'd

$$= 2^{2k} \cdot 3 + (2^{2k} - 1)$$ by the laws of algebra

$$= 2^{2k} \cdot 3 + 3r$$ by inductive hypothesis

$$= 3(2^{2k} + r)$$ by factoring out the 3.

But $(2^{2k} + r)$ is an integer because it is a sum of products of integers, and so, by definition of divisibility, $2^{2(k+1)} - 1$ is divisible by 3 *[as was to be shown]*.

[Since we have proved the basis step and the inductive step, we conclude that the proposition is true.]

Example 2 – *Proving an Inequality*

Use mathematical induction to prove that for all integers $n \geq 3$,

$$2n + 1 < 2^n.$$

Solution:
In this example the property *P*(*n*) is the inequality

$$2n + 1 < 2^n.$$ ← the property $(P(n))$

By substitution, the statement for the basis step, *P*(3), is

$$2 \cdot 3 + 1 < 2^3.$$ ← basis $(P(3))$

Example 2 – *Solution*

cont'd

The supposition for the inductive step, $P(k)$, is

$$2k + 1 < 2^k, \quad \leftarrow \text{inductive hypothesis } (P(k))$$

and the conclusion to be shown is

$$2(k + 1) + 1 < 2^{k+1}. \quad \leftarrow \text{to show } (P(k + 1))$$

To prove the basis step, observe that the statement $P(3)$ is true because $2 \cdot 3 + 1 = 7$, $2^3 = 8$, and $7 < 8$.

Example 2 – *Solution*

cont'd

To prove the inductive step, suppose the inductive hypothesis, that $P(k)$ is true for an integer $k \geq 3$.

This means that $2k + 1 < 2^k$ is assumed to be true for a particular but arbitrarily chosen integer $k \geq 3$.

Then derive the truth of $P(k + 1)$. Or, in other words, show that the inequality is true. But by multiplying ou$2(k + 1) + 1 < 2^{k+1}$

$$2(k + 1) + 1 = 2k + 3 = (2k + 1) + 2, \quad 5.3.1$$

and by substitution from the inductive hypothesis,

$$(2k + 1) + 2 < 2^k + 2. \quad 5.3.2$$

Example 2 – *Solution*

cont'd

Hence $$(2k+1)+2 < 2^k + 2.$$ The left-most part of equation (5.3.1) is less than the right-most part of inequality (5.3.2).

If it can be shown that $2^k + 2$ is less than 2^{k+1}, then the desired inequality will have been proved.

But since the quantity 2^k can be added to or subtracted from an inequality without changing its direction,

$$2^k + 2 < 2^{k+1} \quad \Leftrightarrow \quad 2 < 2^{k+1} - 2^k = 2^k(2-1) = 2^k.$$

And since multiplying or dividing an inequality by 2 does not change its direction,

$$2 < 2^k \quad \Leftrightarrow \quad 1 = \frac{2}{2} < \frac{2^k}{2} = 2^{k-1}$$ by the laws of exponents.

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Example 2 – *Solution*

cont'd

This last inequality is clearly true for all $k \geq 2$. Hence it is true that $$2(k+1)+1 < 2^{k+1}$$

This discussion is made more flowing (but less intuitive) in the following formal proof:

Proposition 5.3.2

For all integers $n \geq 3$, $2n + 1 < 2^n$.

Proof (by mathematical induction):

Let the property $P(n)$ be the inequality

$$2n + 1 < 2^n. \qquad \leftarrow P(n)$$

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Example 2 – *Solution* cont'd

Show that *P*(3) is true:

To establish *P*(3), we must show that

$$2 \cdot 3 + 1 < 2^3. \qquad \leftarrow P(3)$$

But

$$2 \cdot 3 + 1 = 7 \quad \text{and} \quad 2^3 = 8 \quad \text{and} \quad 7 < 8.$$

Hence *P*(3) is true.

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Example 2 – *Solution* cont'd

Show that for all integers $k \geq 3$, if *P*(*k*) is true then *P*(*k* + 1) is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer $k \geq 3$. That is:]

Suppose that *k* is any integer with $k \geq 3$ such that

$$2k + 1 < 2^k. \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

[We must show that P(k + 1) is true. That is:] We must show that

$$2(k + 1) + 1 < 2^{(k+1)}.$$

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Example 2 – *Solution*

cont'd

Or, equivalently,

$$2k + 3 < 2^{(k+1)}. \qquad \leftarrow P(k+1)$$

But

$$2k + 3 = (2k + 1) + 2 \qquad \text{by algebra}$$

$$< 2^k + 2^k \qquad \text{because } 2k + 1 < 2^k \text{ by the inductive hypothesis and because } 2 < 2^k \text{ for all integers } k \geq 2$$

- $2k + 3 < 2 \cdot 2^k = 2^{k+1}$ by the laws of exponents.

[This is what we needed to show.]

[Since we have proved the basis step and the inductive step, we conclude that the proposition is true.]

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Mathematics in Programming

Example : Finding the sum of a geometric series

Prove that these codes will return the same output.

```
int n, r, sum=0;
int i;
scanf("%d",&n);
scanf("%d",&r);

if(r != 1)  {
   for(i=0 ; i<=n ; i++)  {
          sum = sum + pow(r,i);
   }
   printf("%d\n", sum);
}
```

```
int n, r, sum=0;
scanf("%d",&n);
scanf("%d",&r);

if(r != 1) {
       sum=((pow(r,n+1))-1)/(r-1);
       printf("%d\n", sum);
 }
```

Mathematics in Programming
Proving Divisibility Property

What will the output of this program be for any input n?

```
int n;
scanf("%d",&n);

if(n >= 0) {
      if( (pow(2,(2*n))  - 1) %3 == 0)\\ does 2^2n -1 | 3??
\\
        printf("this property is true");
      else
        printf("this property isn't true");
}
```


Applying the Formula for the Sum of a Geometric Sequence

In each of (a) and (b) below, assume that m is an integer that is greater than or equal to 3. Write each of the sums in closed form.

a. $1 + 3 + 3^2 + \cdots + 3^{m-2}$

b. $3^2 + 3^3 + 3^4 + \cdots + 3^m$

Solution

a. $$1 + 3 + 3^2 + \cdots + 3^{m-2} = \frac{3^{(m-2)+1} - 1}{3 - 1}$$ by applying the formula for the sum of a geometric sequence with $r = 3$ and $n = m - 2$

$$= \frac{3^{m-1} - 1}{2}$$

b. $$3^2 + 3^3 + 3^4 + \cdots + 3^m = 3^2 \cdot (1 + 3 + 3^2 + \cdots + 3^{m-2})$$ by factoring out 3^2

$$= 9 \cdot \left(\frac{3^{m-1} - 1}{2}\right)$$ by part (a). ■

Mathematics in Programming

Example : Finding the sum of a integers

Same Question: Prove that these programs prints the same results in case n ≥ *1*

```
For (i=1, i≤n; i++)
    S=S+i;
Print ("%d", S);
```

```
S=(n(n+1))/2
Print ("%d",S);
```


Outline a proof by math induction for the statement:

For all integers $n \geq 0$, $5^n - 1$ is divisible by 4.

Proof by mathematical induction:

Let the property P(*n*) be the sentence

$5^n - 1$ is divisible by 4. ← *the property P(n)*

Show that the property is true for $n = 0$:

We must show that $5^0 - 1$ is divisible by 4.

But $5^0 - 1 = 1 - 1 = 0$, and 0 is divisible by 4 because $0 = 4 \cdot 0$.

Show that for all integers $k \geq 0$, if the property is true for $n = k$, then it is true for $n = k + 1$:

Let k be an integer with $k \geq 0$, and **suppose** that *[the property is true for n = k.*

$5^k - 1$ is divisible by 4. ← *inductive hypothesis*

We must **show** that *P(k + 1) is true.*

$5^{k+1} - 1$ is divisible by 4.

Scratch Work for proving that

For all integers $n \geq 0$, $5^n - 1$ is divisible by 4.

$$\begin{aligned} 5^{k+1} - 1 &= 5^k \cdot 5 - 1 \\ &= 5^k \cdot (4 + 1) - 1 \\ &= 5^k \cdot 4 + 5^k \cdot 1 - 1 \\ &= 5^k \cdot 4 + (5^k - 1) \end{aligned}$$

Note: Each of these terms is divisible by 4.

So: $5^{k+1} - 1 = 5^k \cdot 4 + 4 \cdot r$ *(where r is an integer)*

$= 4 \cdot (5^k + r)$

$(5^k + r)$ is an integer because it is a sum of products of integers, and so, by definition of divisibility $5^{k+1} - 1$ is divisible by 4.

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Proving a Divisibility Property

For all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3.

$3 \mid 2^{2n} - 1$ ← $P(n)$

Basis Step: *Show that P(0) is true.*

P(0): $2^{2.0} - 1 = 2^0 - 1 = 1 - 1 = 0$ as $3 \mid 0$, thus $P(0)$ is true.

Inductive Step: *Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k + 1)$ is also true:*

Suppose: $2^{2k} - 1$ is divisible by 3. ← $P(k)$ inductive hypothesis

$2^{2k} - 1 = 3r$ for some integer r.

We want to prove $2^{2(k+1)} - 1$ is divisible by 3. ← $P(k+1)$

$2^{2(k+1)} - 1 = 2^{2k+2} - 1$ by the laws of exponents

$= 2^{2k} \cdot 2^2 - 1 = 2^{2k} \cdot 4 - 1$

$= 2^{2k}(3 + 1) - 1 = 2^{2k} \cdot 3 + (2^{2k} - 1) = 2^{2k} \cdot 3 + 3r$

$= 3(2^{2k} + r)$ Which is integer

so, by definition of divisibility, $2^{2(k+1)} - 1$ is divisible by 3

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Proving Inequality

For all integers $n \geq 3$, $2n + 1 < 2^n$

Let P(n) be $2n+1<2^n$

Basis Step: *Show that P(3) is true.* P(3): $2.3+1 < 2^3$ which is true.

Inductive Step: *Show that for all integers $k \geq 3$, if P(k) is true then P(k + 1) is also true*:

Suppose: $2k+1<2^k$ is true ← $P(k)$ inductive hypothesis

$2(k+1) +1 < 2^{k+1}$ ← $P(k+1)$

$2k+3 = (2k+1) +2$ by algebra

$< 2^k + 2^k$ as $2k + 1 < 2^k$ by the hypothesis and because $2 < 2^k$ $(k \geq 2)$

$2k + 3 < 2 \cdot 2^k = 2^{k+1}$

[This is what we needed to show.]

Proving a Property of a Sequence

Define a sequence $a_1, a_2, a_3 \ldots$ as follows:

$a_1 = 2$

$a_k = 5a_{k-1}$ for all integers $k \geq 2$.

$\mathbf{a_1} = \mathbf{2}$

$\mathbf{a_2} = 5a_{2-1} = 5a_1 = 5 \cdot 2 = \mathbf{10}$

$\mathbf{a_3} = 5a_{3-1} = 5a_2 = 5 \cdot 10 = 50$

$a_4 = 5a_{4-1} = 5a_3 = 5 \cdot 50 = \mathbf{250}$

Property ➔ The terms of the sequence satisfy the equation $a_n = 2 \cdot 5^{n-1}$

Proving a Property of a Sequence

Prove this property:
$a_n = 2 \cdot 5^{n-1}$ for all integers $n \geq 1$

Basis Step: *Show that P(1) is true.* $a_1 = 2 \cdot 5^{1-1} - 1 = 2 \cdot 5^0 - 1 = 2$

Inductive Step: *Show that for all integers $k \geq 1$, if P(k) is true then P(k + 1) is also true:*

Suppose: $a_k = 2 \cdot 5^{k-1}$ ← *P*(*k*) inductive hypothesis

$a_{k+1} = 2 \cdot 5^k$ ← *P*(*k*+1)

$= 5a_{(k+1)-1}$ by definition of $a_1, a_2, a_3 \ldots$

$= 5a_k$

$= 5 \cdot (2 \cdot 5^{k-1})$ by the hypothesis

$= 2 \cdot (5 \cdot 5^{k-1})$

$= 2 \cdot 5^k$

[This is what we needed to show.]

Exercise

9. $7^n - 1$ is divisible by 6, for each integer $n \geq 0$.

23. **a.** $n^3 > 2n + 1$, for each integer $n \geq 2$.
b. $n! > n^2$, for each integer $n \geq 4$.

H **13.** For any integer $n \geq 0$, $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.

25. A sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for each integer $k \geq 1$. Show that $b_n > 4n$ for every integer $n \geq 0$.