Set theory chapter 6

defintion: $A \subseteq B \iff \forall x$, if $x \in A$ then $x \in B$

Two sets A, B

$A \not\subseteq B \iff \forall x$, if $x \in A$ then $x \notin B$.

A proper subset A is proper subset of B

$\iff$ 1) $A \subseteq B$ and

2) there is at least one element in B that is not in A.

Ex:

$A = \{1\}$, $B = \{1, \{1\}\}$

a) Is $A \subseteq B$?

b) Is A a proper subset?

a) Yes: $A \subseteq B$ ✓

a)

b) Yes, b) there is atleast one element in subset

$B \Rightarrow \{1\}$

Proof and disproof:

Let X, Y ~~are~~ sets. To prove $X \subseteq Y$

(1) $\cancel{x}$ is phac element in set X.

(2) to show that x is in set Y.

$x \in Y$.

Ex: Define sets A, B as the following

$A = \{ m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z} \}$

$B = \{ n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z} \}$

Outline Suppose x is pbac element in set A

⋮

Show that x is an element of set B.

Proof: Suppose that x is pbac element of set A

[we have to show x is an element of set B, by definition of B this mean $x = 3 \cdot$ [some Integer]]

$\Rightarrow$ Given by that $x = 6r + 12$, can we express x as 3. (some Integer)

Yes: $x = 3(2r+4)$, let $t = 2r+4$

t is integer why?

because $(*, +)$ is closed

Check $x \stackrel{?}{=} 3 \cdot \underbrace{\text{some}}_{t}$

Set B

$$\underline{3 \cdot (s.I)} = 3(t) = 3(2r+4) = 6r+12 = x$$

$$\underline{\underline{A \subseteq B}}$$

Is $B \subseteq A$? Counter example

$$3 \cdot (s.I) \Rightarrow \underbrace{\underline{3}}_{x} = 3 \cdot * 1$$

$$x = 6r+12 \Rightarrow 3 = 6r+12$$

$$1 = 2r+4$$

$\therefore \underline{r} = -3/2$ it's not integer $\notin \mathbb{Z}$

Def: $\underline{\underline{A = B}}$? A, B are sets ?

$$A = B \iff A \subseteq B \text{ and } B \subseteq A$$

Ex: $A = \{ m \in \mathbb{Z} \mid m = 2a \text{ for some integer } a \}$

$B = \{ n \in \mathbb{Z} \mid n = 2b - 2, \text{ for some integer } b \}$

Is $A = B$? Yes/NO

$A \subseteq B$; $B \subseteq A$

Proof:

Suppose that x is phac element in set A.

we have to show x is an element of set B.

from defitinon of B, $x = 2b - 2$ for same integer b

$\Rightarrow$ Can we express ? X expressed as $2(S.I) - 2$?

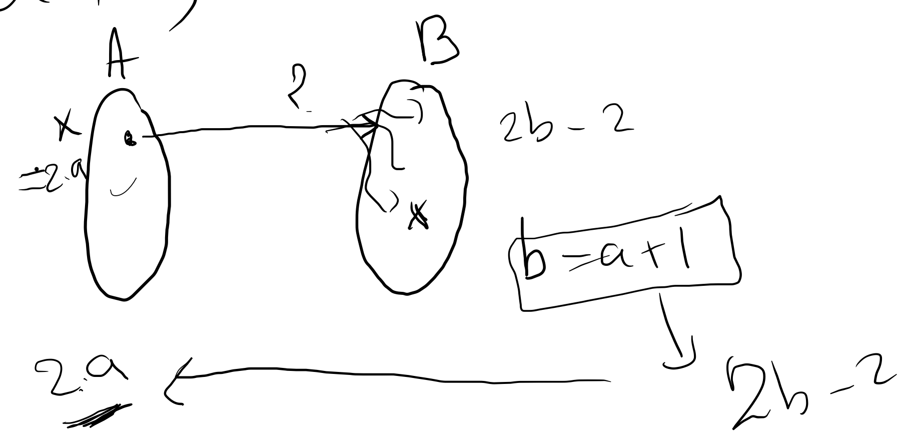
X(set A) ? J(X in B)

$\Leftrightarrow \quad 2a = 2b - 2 \Rightarrow \quad a = b - 1 \Rightarrow \boxed{b = a + 1}$

(in set A)

Check $\quad x \overset{?}{=} 2(b) - 2$

$= 2(a+1) - 2 = 2a + \cancel{2} - \cancel{2} = 2a = x$



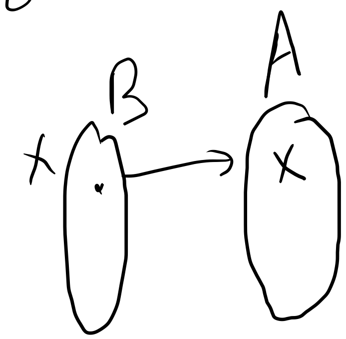
$A \subseteq B$

Condition

$\mathbb{Z}$ integer

Suppose that x is phac element in set B. We want to show x is an element of set A. From definition of A $x = 2a$ for some integer a.

Can we expresss $2b - 2 = 2a$?

$\Rightarrow$ 

$2b-2 = 2a \Rightarrow \boxed{b-1=a}$

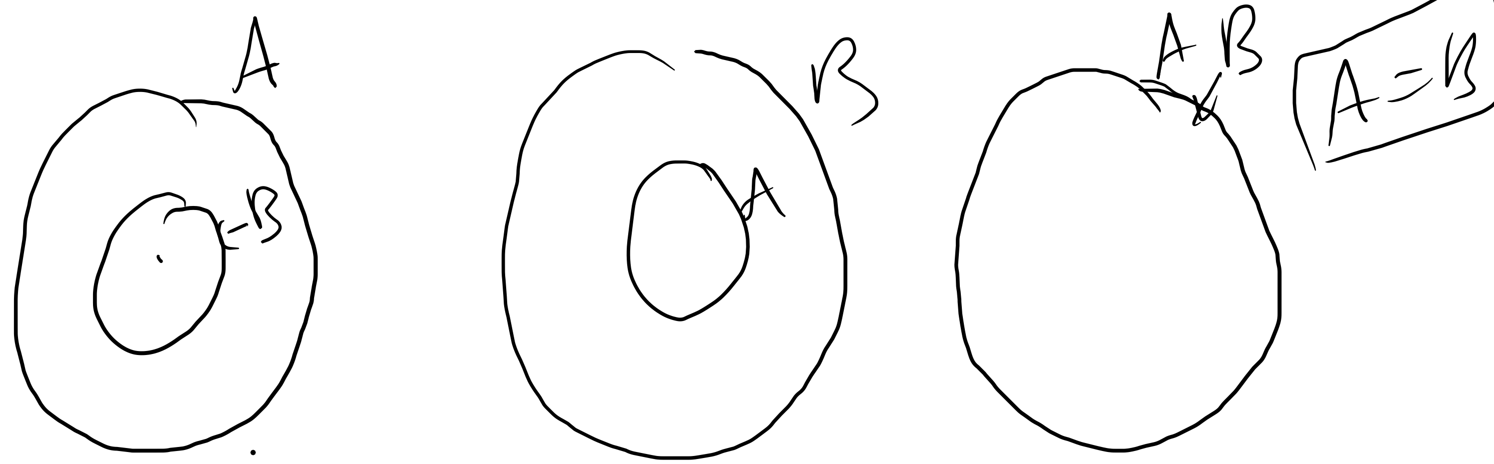
$\Rightarrow x = 2a = 2(b-1) = \underbrace{2b-2} = x$ (set B)

for $\underline{\text{Integer}}$

$\therefore A \subseteq B$, and $B \subseteq A$ $\#\#$ $A = B$

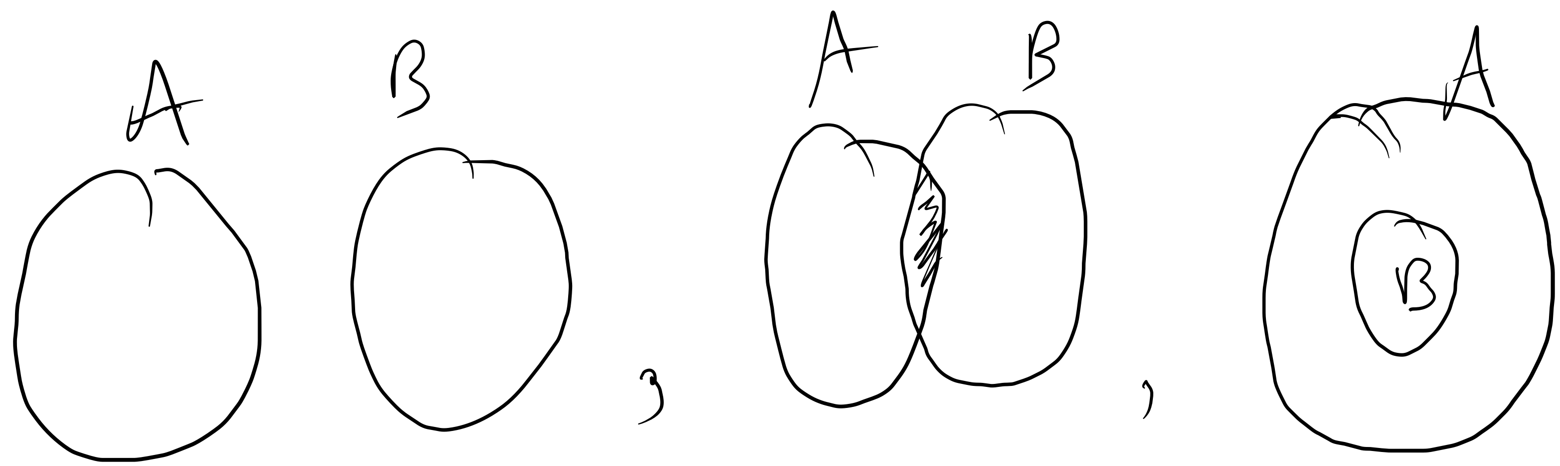
Venn Diagrams! (A, B are sets)

$A \subseteq B$



$B \subseteq A$ $\quad$ $A \subseteq B$ $\quad$ identical

$A \not\subseteq B$:



$A \not\subseteq B$, $A \not\subseteq B$, $A \not\subseteq B$

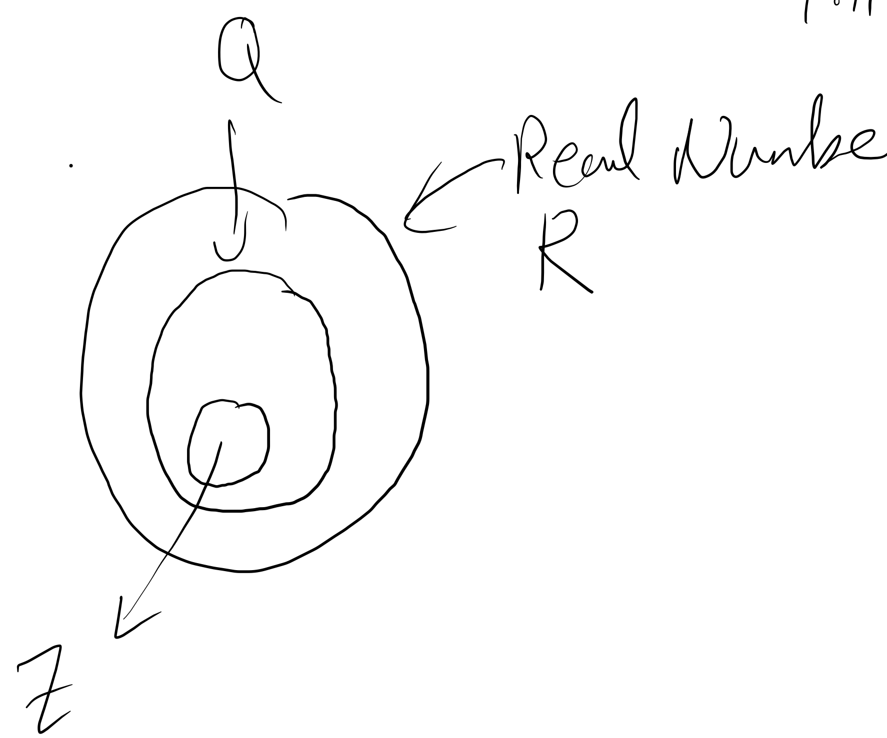
Notes :

$\mathbb{Z}$, $\Rightarrow$ $\{Z, Z^+, Z^-\}$
integers

R : Real numbers $\{\sqrt{2}, \frac{1}{\sqrt{3}}\}$

Q : Rational Numbers : $\frac{\text{integer}}{\text{integer}} = \frac{2}{1}, \frac{7}{1}, \frac{14}{2}, \frac{21}{4}$

Venn Diagram :



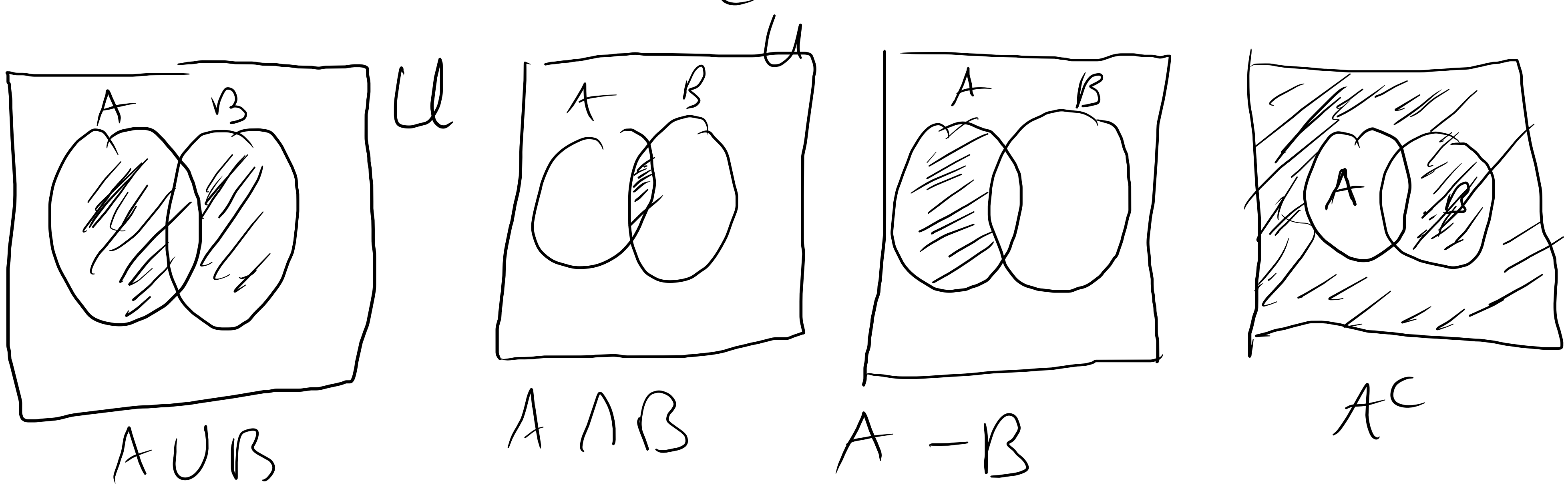
Operation

$$A \cup B = \{ x \in U \mid x \in A \text{ or } x \in B \}$$

$$A \cap B = \{ x \in U \mid x \in A \text{ and } x \in B \}$$

$$A - B = \{ x \in U \mid x \in A \text{ and } x \notin B \}$$

$$A^C = \{ x \in U \mid x \notin A \}$$



Operator Sets:

$$(a,b) = \{x \in R \mid a < x < b\}$$

$$[a,b) = \{x \in R \mid a \leq x < b\}$$

$$(a,b] = \{x \in R \mid a < x \leq b\}$$

$$[a,b] = \{x \in R \mid a \leq x \leq b\}$$

$$(a,\infty) = \{x \in R \mid x > a\}$$

$$[a,\infty) = \{x \in R \mid x \geq a\}$$

$$(-\infty,a) = \{x \in R \mid x < a\}$$

$$(-\infty,a] = \{x \in R \mid x \leq a\}$$

Ex:

$A = (-1, 0] = \{x \in R \mid -1 < x \leq 0\}$

$B = [0, 1) = \{x \in R \mid 0 \leq x < 1\}$

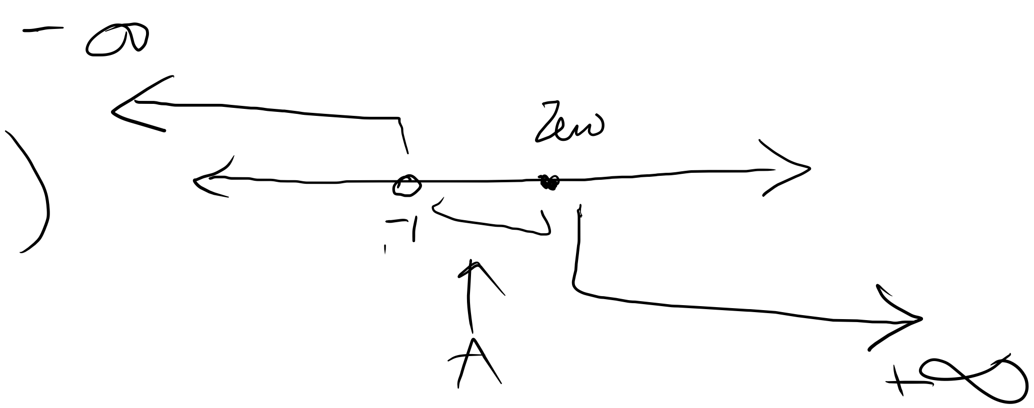
Find $A \cup B$, $A \cap B$, $B - A$, A^c ?

$A \cup B = (-1, 0] \cup [0, 1) = (-1, 1) = \{x \in R \mid -1 < x < 1\}$

$A \cap B = (-1, 0] \cap [0, 1) = \{0\} = \{x \in R \mid x = 0\}$

$B - A = (\underline{0}, 1)$

$A^c = (-\infty, -1] \cup (0, \infty)$



$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i, \text{ at least } i = 0, 1, 2, \dots, n\}$$

$$\hookrightarrow \bigcup_{i=0}^{5} A_i = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ at least nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \dots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integer } i\}$$

Ex: $A_i = \{ x \in \mathbb{R} \mid -\frac{1}{i} < x < \frac{1}{i} \}$

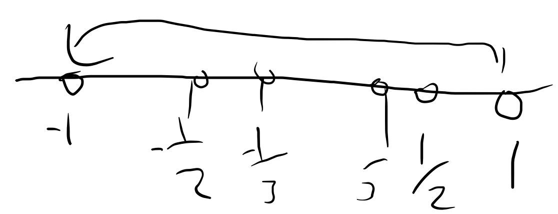
$$A_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$$

a) Find $A_1 \cup A_2 \cup A_3$ and $A_1 \cap A_2 \cap A_3$

b) Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$

Solution: $A_1 \cup A_2 \cup A_3 = \{ x \in \mathbb{R} \mid$ x is at least one of the intervals $(-1, 1)$ or $(-\frac{1}{2}, \frac{1}{2})$ or $(-\frac{1}{3}, \frac{1}{3})$ $\}$

$$= (-1, 1)$$



$$A_1 \cap A_2 \cap A_3 = \left(-\frac{1}{3}, \frac{1}{3}\right)$$

$$\bigcup_{i=1}^{\infty} A_i = \{x \in R \mid x \text{ is at least one of the intervals } (-\tfrac{1}{i}, \tfrac{1}{i})\}$$

$$= (-\tfrac{1}{1}, \tfrac{1}{1}) \cup (-\tfrac{1}{2}, \tfrac{1}{2}) \cup (-\tfrac{1}{3}, \tfrac{1}{3}) \text{------} (-\tfrac{1}{\infty}, \tfrac{1}{\infty})$$

$$= (-1, 1)$$

$$\bigcap_{i=1}^{\infty} A_i = \{x \in R \mid x \text{ is } \underline{\underline{\text{all}}} \text{ of the intervals } (-\tfrac{1}{i}, \tfrac{1}{i}) \; i \text{ is positive}\}$$

$$= (-1, 1) \cap (-\tfrac{1}{2}, \tfrac{1}{2}) \cap (-\tfrac{1}{3}, \tfrac{1}{3}) \cap \text{----} (-\tfrac{1}{\infty}, \tfrac{1}{\infty})$$

$$= \{0\}$$

Empty set:

1) $\overset{A}{\{1,3\}} \cap \overset{B}{\{2,4\}} = \emptyset$

2) $\{x \in R \mid x^2 = -1\} = \emptyset$

3) $\{x \in R \mid 3 < x < 2\} = \emptyset$

A, B is disjoint $\iff A \cap B = \emptyset$

$A = \{1,3,5\}$, $B = \{2,4,6\}$

$A \cap B = \emptyset \rightarrow$ disjoint

mutually disjoint $A_i \cap A_i = \emptyset \quad i \neq j$

Ex: a) $A_1 = \{3,5\}$, $A_2 = \{1,4,6\}$, $A_3 = \{2\}$

A_1, A_2, A_3 mutually disjoint?

b) $B_1 = \{2,4,6\}$, $B_2 = \{3,7\}$, $B_3 = \{4,5\}$

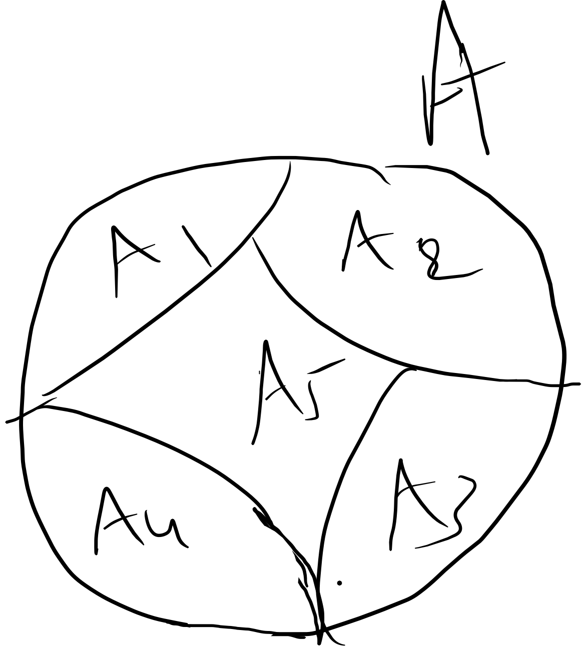
are B's mutally disjoint?

$$A_i \cap A_j = \phi \quad i \neq j$$

a) $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, $A_2 \cap A_3 = \phi$

$\therefore$ A_1, A_2, A_3 are mutally disjoint.

b) $B_1 \cap B_2 = \phi$, $B_1 \cap B_3 = \{4\} \neq \phi$ not mutally disjoint



mutually disjoint

Partion: 1) A is the union of All A_i

2) Sets A_1, A_2, A_3 --- are multally disjoint

Ex: $A = \{1,2,3,4,5,6\} = A_1 = \{1,2\}$ $A_2 = \{3,4\}$

$A_3 = \{5,6\}$

$A_1 \cap A_2 = \phi$, $A_2 \cap A_3 = \phi$, $A_1 \cap A_3 = \phi$

and $A_1 \cup A_2 \cup A_3 = A$

Chapter 6

Lecture 2 — Set Theory

Mutually disjoint:

Ex:

$T_0 = \{n \in Z \mid n = 3k, \text{ for some integer } k\}$

$T_1 = \{n \in Z \mid n = 3k+1, \text{ for some integer } k\}$

$T_2 = \{n \in Z \mid n = 3k+2, \text{ " " " }\}$

$T_0 \cap T_1 \cap T_2 \overset{??}{=}$ disjoint

QRT

$n = dq + r \qquad 0 \leq r < d,$

$\{3k, 3k+1, 3k+2\}, \ldots \quad \{\underbrace{0, 1, 2}_{k=0}, \underbrace{3, 4, 5}_{k=1}, \ldots\}$

$T_0 \cap T_1 \cap T_2 = \phi$ M. disjoint

Power set: Denoted by $\mathcal{P}(A)$: power of set A.

→ it's set of all subset of A.

Ex: Set $A = \{x, y\}$, find $P(A)$?

$$P(A) = P(\{x, y\}) = \{\{x\}\{y\} \{x, y\}, \phi\}$$

Note: # of elements in Power set 2^N → # of element in set A

Ex: $A = \{1, 2, 3\}$, $P(A)$?

$$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

$2^3 = 8$

Cartesian Products:

$(x_1, x_2, x_3, \ldots, x_n) = (y_1, y_2, y_3, \ldots, y_n)$

iff $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, ...

In particular $(a, b) = (c, d) \iff a = c$, and $b = d$

Ex: Is $(1, 2, 3, 4) = (1, 2, 4, 3)$, NO

$1 = 1$, $2 = 2$, $3 \neq 4$, $4 \neq 3$

Ex: $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$ ✓ True

$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$

Cartesian of A_1 and A_2

Ex: Let $A_1 = \{x, y\}$ $A_2 = \{1, 2, 3\}$ $A_3 = \{a, b\}$

Find a) $A_1 \times A_2$ b) $(A_1 \times A_2) \times A_3$ c) $A_1 \times A_2 \times A_3$

a) $A_1 \times A_2 = \{(x,1), (x,2), (x,3), (y,1), (y,2), (y,3)\}$

a) $(A_1 \times A_2) \times A_3 = \{((x,1),a), ((x,1),b), ((x,2),a), ((x,2),b),$
$((x,3),a), ((x,3),b), ((y,1),a), ((y,1),b),$
$((y,2),a), ((y,2),b), ((y,3),a), ((y,3),b)\}$

c) $A_1 \times A_2 \times A_3 = \{(x,1,a), (x,1,b), (x,2,a), (x,2,b),$
$(x,3,a), (x,3,b), (y,1,a), \ldots \}$

Properties of Sets:

1) $A \cap B \subseteq A$ and $A \cap B \subseteq B$ Inclusion of Intersection

2) $A \subseteq A \cup B$ and $B \subseteq A \cup B$ Inclusion of Union.

3) If $A \subseteq B$, $B \subseteq C$, then $A \subseteq C$, Transitive

Prove that $A \cap B \subseteq A$, A, and B are any sets

Suppose that x is ~~p~~ ~~bac~~ element of $A \cap B$, then $x \in A$ and $x \in B$ by defintion of intersection In particular $x \in A$, thus $A \cap B \subseteq A$

Set identities:

1) $A \cup B = B \cup A$
$A \cap B = B \cap A$ } Commutative laws

2) $(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$ } Associative laws

3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ } Distributive laws

4) $A \cup \phi = A$ $\quad$ $A \cap U = A$ $\quad$ Complements law
$A \cup A^c = U$ $\quad$ $A \cap A^c = \phi$

5) $(A^{c})^{c} = A$ $\quad$ Double complement
$A \cup A = A$ $\quad$ $A \cap A = A$ $\quad$ Idempotent

6) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$ Demergans law

7) $A \cup (A \cap B) = A$
$A \cap (A \cup B) = A$ } absorption law

8) complement $\cup = \phi$, $\phi^c = \cup$

9) $A - B = A \cap B^c$ $\Rightarrow$ difference law