جامعة بيرزيت
BIRZEIT UNIVERSITY

Discrete Mathematic and Application Comp233

CHAPTER 6

SET THEORY

Instructor
Murad Njoum

1

Subsets: Proof and Disproof

We begin by rewriting what it means for a set *A* to be a subset of a set *B* as a formal universal conditional statement:

$$A \subseteq B \quad \Leftrightarrow \quad \forall x, \text{ if } x \in A \text{ then } x \in B.$$

The negation is, therefore, existential:

$$A \not\subseteq B \quad \Leftrightarrow \quad \exists x \text{ such that } x \in A \text{ and } x \notin B.$$

A ***proper subset*** **جزئي** of a set is a subset that is not equal to its containing set. Thus

A is a **proper subset** of *B* $\Leftrightarrow$
(1) $A \subseteq B$, and
(2) there is at least one element in *B* that is not in *A*.

Instructor : Murad Njoum

2

Example 1 – *Testing Whether One Set Is a Subset of Another*

Let $A = \{1\}$ and $B = \{1, \{1\}\}$.

a. Is $A \subseteq B$?

b. If so, is A a proper subset of B?

Solution:

a. Because $A = \{1\}$, A has only one element, namely the symbol 1. This element is also one of the elements in set B. Hence every element in A is in B, and so $A \subseteq B$.

b. B has two distinct elements, the symbol 1 and the set $\{1\}$ whose only element is 1.

Since $1 \neq \{1\}$, the set $\{1\}$ is not an element of A, and so there is an element of B that is not an element of A. **Hence *A* is a proper subset of *B*.**

Subsets: Proof and Disproof

Because we define what it means for one set to be a subset of another by means of a universal conditional statement, we can use the method of direct proof to establish a subset relationship.

Such a proof is called an ***element argument*** and is the fundamental proof technique of set theory.

Element Argument: The Basic Method for Proving That One Set Is a Subset of Another

Let sets X and Y be given. To prove that $X \subseteq Y$,

1. **suppose** that x is a particular but arbitrarily chosen element of X,
2. **show** that x is an element of Y.

Example 2 – *Proving and Disproving Subset Relations*

Define sets *A* and *B* as follows:

$$A = \{m \in \mathbf{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbf{Z}\}$$
$$B = \{n \in \mathbf{Z} \mid n = 3s \text{ for some } s \in \mathbf{Z}\}.$$

a. Outline a proof that $A \subseteq B$.

b. Prove that $A \subseteq B$.

c. Disprove that $B \subseteq A$.

Example 2 – *Solution*

a. Proof Outline:

Suppose *x* is a particular but arbitrarily chosen element of *A*.

.
.
.

Therefore, *x* is an element of *B*.

b. Proof:

Suppose *x* is a particular but arbitrarily chosen element of *A*.

[We must show that $x \in B$. By definition of B, this means we must show that $x = 3 \cdot$(some integer).]

Example 2 – *Solution* cont'd

By definition of *A*, there is an integer *r* such that
$x = 6r + 12$.

[Given that $x = 6r + 12$, *can we express* x *as* $3 \cdot$(*some integer*)?
I.e., does $6r + 12 = 3 \cdot$(*some integer*)? *Yes,* $6r + 12 = 3 \cdot (2r + 4)$.*]*

Let $s = 2r + 4$.

[We must check that s is an integer.]

Then *s* is an integer because products and sums of integers are integers.

[Now we must check that $x = 3s$.*]*

Example 2 – *Solution* cont'd

Also $3s = 3(2r + 4) = 6r + 12 = x$,
Thus, by definition of *B*, *x* is an element of *B*,

[which is what was to be shown].

c. To disprove a statement means to show that it is false, and to show it is false that $B \subseteq A$, you must find an element of *B* that is not an element of *A*.

By the definitions of *A* and *B*, this means that you must find an integer *x* of the form $3 \cdot$ (some integer) that cannot be written in the form $6 \cdot$ (some integer) + 12.

Example 2 – *Solution*

cont'd

A little experimentation reveals that various numbers do the job. For instance, you could let ***x* = 3.**

Then $x \in B$ because **3 = 3 · 1, but *x* ∉ *A*** because there is no integer *r* such that **3 = 6*r* + 12.** For if there were such an integer, then

$$
\begin{aligned}
& 6r + 12 = 3 && \text{by assumption} \\
\Rightarrow\quad & 2r + 4 = 1 && \text{by dividing both sides by 3} \\
\Rightarrow\quad & 2r = 3 && \text{by subtracting 4 from both sides} \\
\Rightarrow\quad & r = 3/2 && \text{by dividing both sides by 2,}
\end{aligned}
$$

but 3/2 is not an integer. Thus $3 \in B$ but $3 \notin A$, and so $B \not\subseteq A$.

Set Equality

We have known that by the axiom of extension, sets *A* and *B* are equal if, and only if, they have exactly the same elements.

We restate this as a definition that uses the language of subsets.

• Definition

Given sets A and B, A **equals** B, written $\boldsymbol{A = B}$, if, and only if, every element of A is in B and every element of B is in A.

Symbolically:

$$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

Example 3 – *Set Equality*

This version of the definition of equality implies the following:

To know that a set A equals a set B, you must know that $A \subseteq B$ and you must also know that $B \subseteq A$.

Define sets A and B as follows:

$$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$$
$$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$$

Is $A = B$?

Solution:

Yes. To prove this, both subset relations $A \subseteq B$ and $B \subseteq A$ must be proved.

Example 3 – *Solution* cont'd

Part 1, Proof That $A \subseteq B$:

Suppose x is a particular but arbitrarily chosen element of A.

[We must show that $x \in B$. By definition of B, this means we must show that $x = 2 \cdot$ (some integer) $- 2$.]

By definition of A, there is an integer a such that $x = 2a$.

[Given that $x = 2a$, can x also be expressed as $2 \cdot$ (some integer) $- 2$? i.e., is there an integer, say b, such that $2a = 2b - 2$? Solve for b to obtain $b = (2a + 2)/2 = a + 1$. Check to see if this works.]

Example 3 – *Solution*

cont'd

Let $b = a + 1$.

[First check that b is an integer.]

Then b is an integer because it is a sum of integers.

[Then check that $x = 2b - 2$.]

Also $2b - 2 = 2(a + 1) - 2 = 2a + 2 - 2 = 2a = x$,

Thus, by definition of B, x is an element of B

[which is what was to be shown].

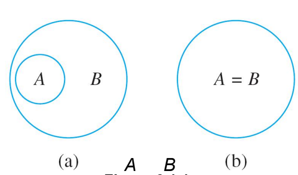
Part 2, Proof That *B* *A*:

Similarly we can prove that B A. Hence $A = B$.

Venn Diagrams

If sets A and B are represented as regions in the plane, relationships between A and B can be represented by pictures, called **Venn diagrams,** that were introduced by the British mathematician John Venn in 1881.

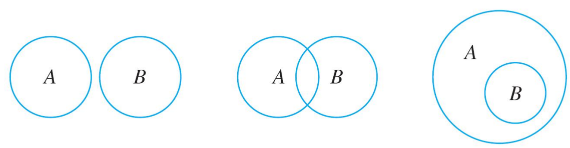
For instance, the relationship $A \subseteq B$ can be pictured in one of two ways, as shown in Figure 6.1.1.



A *B*
Figure 6.1.1

Venn Diagrams

The relationship $A \not\subseteq B$ can be represented in three different ways with Venn diagrams, as shown in Figure 6.1.2.



(a) (b) (c)

$A \not\subseteq B$

Figure 6.1.2

Example 4 – *Relations among Sets of Numbers*

Since **Z, Q, and R** denote the sets of **integers**, **rational numbers**, and **real numbers**, respectively, **Z** is a subset of **Q** because every integer is rational (any integer *n* can be written in the form $\frac{n}{1}$).

Q is a **subset** of **R** because every **rational** number is **real** (any rational number can be represented as a length on the number line).

Z is a **proper** subset of **Q** because there are **rational** numbers that are **not integers** (for example, $\frac{1}{2}$).

Example 4 – *Relations among Sets of Numbers* cont'd

Q is a proper subset of **R** because there are real numbers that are not rational (for example, $\sqrt{2}$).

This is shown diagrammatically in Figure 6.1.3.

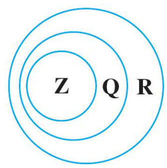


Figure 6.1.3

Operations on Sets

• Definition

Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$**,** is the set of all elements that are in at least one of A or B.
2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$**,** is the set of all elements that are common to both A and B.
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $\boldsymbol{B - A}$**,** is the set of all elements that are in B and not A.
4. The **complement** of A, denoted $\boldsymbol{A^c}$**,** is the set of all elements in U that are not in A.

Symbolically:

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\},$$
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\},$$
$$A^c = \{x \in U \mid x \notin A\}.$$

Operations on Sets

Venn diagram representations for union, intersection, difference, and complement are shown in Figure 6.1.4.

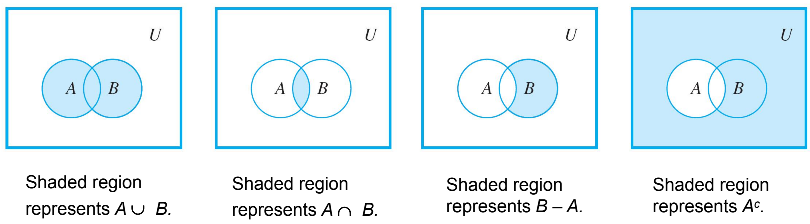


Figure 6.1.4

Example 5 – *Unions, Intersections, Differences, and Complements*

Let the universal set be the set $U = \{a, b, c, d, e, f, g\}$ and let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$. Find $A \cup B$, $A \cap B$, $B - A$, and A^c.

Solution:

$$A \cup B = \{a, c, d, e, f, g\}$$

$$A \cap B = \{e, g\}$$

$$B - A = \{d, f\}$$

$$A^c = \{b, d, f\}$$

Operations on Sets

There is a convenient notation for subsets of real numbers that are intervals.

• Notation

Given real numbers a and b with $a \le b$:

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\} \qquad [a, b] = \{x \in \mathbf{R} \mid a \le x \le b\}$$
$$(a, b] = \{x \in \mathbf{R} \mid a < x \le b\} \qquad [a, b) = \{x \in \mathbf{R} \mid a \le x < b\}.$$

The symbols ∞ and $-\infty$ are used to indicate intervals that are unbounded either on the right or on the left:

$$(a, \infty) = \{x \in \mathbf{R} \mid x > a\} \qquad [a, \infty) = \{x \in \mathbf{R} \mid x \ge a\}$$
$$(-\infty, b) = \{x \in \mathbf{R} \mid x < b\} \qquad [-\infty, b) = \{x \in \mathbf{R} \mid x \le b\}.$$

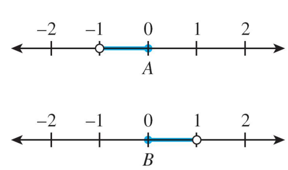
Observe that the notation for the interval (a, b) is identical to the notation for the ordered pair (a, b). However, context makes it unlikely that the two will be confused.

Example 6 – *An Example with Intervals*

Let the universal set be the set **R** of all real numbers and let

$$A = (-1, 0] = \{x \in \mathbf{R} \mid -1 < x \le 0\} \text{ and } B = [0, 1) = \{x \in \mathbf{R} \mid 0 \le x < 1\}.$$

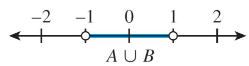
These sets are shown on the number lines below.



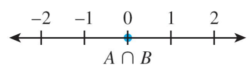
Find $A \cup B$, $A \cap B$, $B - A$, and A^c.

Example 6 – *Solution*

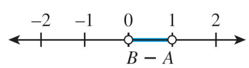
$A \cup B = \{x \in \mathbf{R} \mid x \in (-1,\ 0] \text{ or } x \in [0,\ 1)\} = \{x \in \mathbf{R} \mid x \in (-1,\ 1)\} = (-1,\ 1).$



$A \cap B = \{x \in \mathbf{R} \mid x \in (-1,\ 0] \text{ and } x \in [0,\ 1)\} = \{0\}.$



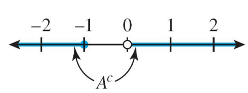
$B - A = \{x \in \mathbf{R} \mid x \in [0,\ 1) \text{ and } x \notin (-1,\ 0]\} = \{x \in \mathbf{R} \mid 0 < x < 1\} = (0, 1)$



Example 6 – *Solution*

cont'd

$A^c = \{x \in \mathbf{R} \mid \text{it is not the case that } x \in (-1,\ 0]\}$



$= \{x \in \mathbf{R} \mid \text{it is not the case that } (-1 < x \text{ and } x \leq 0)\}$ — by definition of the double inequality

$= \{x \in \mathbf{R} \mid x \leq -1 \text{ or } x > 0\} = (-\infty,\ -1] \cup (0,\ \infty)$ — by De Morgan's law

Operations on Sets

The definitions of unions and intersections for more than two sets are very similar to the definitions for two sets.

• Definition

Unions and Intersections of an Indexed Collection of Sets
Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

An alternative notation for $\bigcup_{i=0}^{n} A_i$ is $A_0 \cup A_1 \cup \ldots \cup A_n$,

alternative notation for $\bigcap_{i=0}^{n} A_i$ is $A_0 \cap A_1 \cap \ldots \cap A_n$.

Example 7 – *Finding Unions and Intersections of More than Two Sets*

For each positive integer *i*, let

$$A_i = \left\{x \in \mathbf{R} \mid -\tfrac{1}{i} < x < \tfrac{1}{i}\right\} = A_i = \left(-\tfrac{1}{i}, \tfrac{1}{i}\right).$$

a. Find $A_1 \cup A_2 \cup A_3$ and $A_1 \cap A_2 \cap A_3$.

b. Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.

Solution:

a. $A_1 \cup A_2 \cup A_3 = \{x \in \mathbf{R} \mid x$ is in at least one of the intervals $(-1, 1)$, or $\left(-\tfrac{1}{2}, \tfrac{1}{2}\right)$, or $\left(-\tfrac{1}{3}, \tfrac{1}{3}\right)\}$

Example 7 – *Solution*

cont'd

$= \{x \in \mathbf{R} \mid -1 < x < 1\}$ because all the elements in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{3}, \frac{1}{3}\right)$ are in $(-1, 1)$

$= (-1, 1)$

$A_1 \cap A_2 \cap A_3 = \{x \in \mathbf{R} \mid x \text{ is in all of the intervals } (-1, 1), \text{ and } \left(-\frac{1}{2}, \frac{1}{2}\right), \text{ and } \left(-\frac{1}{3}, \frac{1}{3}\right)\}$

$= \left\{x \in \mathbf{R} \mid -\frac{1}{3} < x < \frac{1}{3}\right\}$ because $\left(-\frac{1}{3}, \frac{1}{3}\right) \subseteq \left(-\frac{1}{2}, \frac{1}{2}\right) \subseteq (-1, 1)$

$= \left(-\frac{1}{3}, \frac{1}{3}\right)$

Example 7 – *Solution*

cont'd

b. $\bigcup_{i=1}^{\infty} A_i = \{x \in \mathbf{R} \mid x \text{ is in at least one of the intervals } \left(-\frac{1}{i}, \frac{1}{i}\right), \text{ where } i \text{ is a positive integer}\}$

$= \{x \in \mathbf{R} \mid -1 < x < 1\}$ because all the elements in every interval $\left(-\frac{1}{i}, \frac{1}{i}\right)$ are in $(-1, 1)$

$= (-1, 1)$

$\bigcap_{i=1}^{\infty} A_i = \{x \in \mathbf{R} \mid x \text{ is in all of the intervals } \left(-\frac{1}{i}, \frac{1}{i}\right), \text{ where } i \text{ is a positive integer}\}$

$= \{0\}$ because the only element in every interval is 0

The Empty Set

We have stated that a set is defined by the elements that compose it. This being so, can there be a set that has no elements? It turns out that it is convenient to allow such a set.

Because it is unique, we can give it a special name. We call it the **empty set** (or **null set**) and denote it by the symbol Ø.

Thus $\{1, 3\} \cap \{2, 4\} = Ø$ and $\{x \in \mathbf{R} \mid x^2 = -1\} = Ø$.

Instructor : Murad Njoum

29

Example 8 – *A Set with No Elements*

Describe the set $D = \{x \in \mathbf{R} \mid 3 < x < 2\}$.

Solution:

We have known that $a < x < b$ means that $a < x$ and $x < b$. So D consists of all real numbers that are both greater than 3 and less than 2.

Since there are no such numbers, D has no elements and so $D = Ø$.

Instructor : Murad Njoum

30

Partitions of Sets

In many applications of set theory, sets are divided up into nonoverlapping (or *disjoint*) pieces. Such a division is called a *partition*.

• Definition

Two sets are called **disjoint** if, and only if, they have no elements in common. Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Are A and B disjoint?

Solution:

Yes. By inspection A and B have no elements in common, or, in other words, $\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$.

Partitions of Sets

• Definition

Sets $A_1, A_2, A_3 \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all $i, j = 1, 2, 3, \ldots$

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

Example 10 – *Mutually Disjoint Sets*

a. Let $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$. Are A_1, A_2, and A_3 mutually disjoint?

b. Let $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$. Are B_1, B_2, and B_3 mutually disjoint?

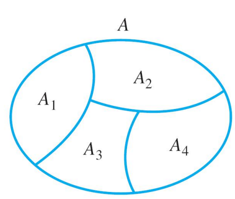
Solution:

a. Yes. A_1 and A_2 have no elements in common, A_1 and A_3 have no elements in common, and A_2 and A_3 have no elements in common.

b. No. B_1 and B_3 both contain 4.

Partitions of Sets

Suppose A, A_1, A_2, A_3, and A_4 are the sets of points represented by the regions shown in Figure 6.1.5.



A Partition of a Set

Figure 6.1.5

Then A_1, A_2, A_3, and A_4 are subsets of A, and

$A = A_1 \cup A_2 \cup A_3 \cup A_4$.

Partitions of Sets

Suppose further that boundaries are assigned to the regions representing A_2, A_3, and A_4 in such a way that these sets are mutually disjoint.

Then A is called a *union of mutually disjoint subsets*, and the collection of sets $\{A_1, A_2, A_3, A_4\}$ is said to be a *partition* of A.

• Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i
2. The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

Example 11 – *Partitions of Sets*

a. Let $A = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A?

b. Let **Z** be the set of all integers and let

$$T_0 = \{n \in \mathbf{Z} \mid n = 3k, \text{ for some integer } k\},$$
$$T_1 = \{n \in \mathbf{Z} \mid n = 3k + 1, \text{ for some integer } k\}, \text{ and}$$
$$T_2 = \{n \in \mathbf{Z} \mid n = 3k + 2, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_2\}$ a partition of **Z**?

Example 11 – *Solution*

a. Yes. By inspection, $A = A_1 \cup A_2 \cup A_3$ and the sets A_1, A_2, and A_3 are mutually disjoint.

b. Yes. By the quotient-remainder theorem, every integer n can be represented in exactly one of the three forms

$$n = 3k \quad \text{or} \quad n = 3k + 1 \quad \text{or} \quad n = 3k + 2,$$

for some integer k.

This implies that no integer can be in any two of the sets T_0, T_1, or T_2. So T_0, T_1, and T_2 are mutually disjoint.

It also implies that every integer is in one of the sets T_0, T_1, or T_2. So $\mathbf{Z} = T_0 \cup T_1 \cup T_2$.

Power Sets

There are various situations in which it is useful to consider the set of all subsets of a particular set.

The **power set axiom** guarantees that this is a set.

• Definition

Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Example 12 – *Power Set of a Set*

Find the power set of the set {*x*, *y*}. That is, find $\mathscr{P}$ ({*x*, *y*}).

Solution:

({*x*, *y*}) $\mathscr{P}$ the set of all subsets of {*x*, *y*}. We know that Ø is a subset of every set, and so Ø ∈ $\mathscr{P}$ ({*x*, *y*}).

Also any set is a subset of itself, so {*x*, *y*} ∈ $\mathscr{P}$ ({*x*, *y*}). The only other subsets of {*x*, *y*} are {*x*} and {*y*}, so

$$\mathscr{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}.$$

Cartesian Products

• Definition

Let n be a positive integer and let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple, $(x_1, x_2, \ldots, x_n)$,** consists of $x_1, x_2, \ldots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n. An ordered 2-tuple is called an **ordered pair,** and an ordered 3-tuple is called an **ordered triple.**

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.

Symbolically:

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \quad \Leftrightarrow \quad x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

In particular,

$$(a, b) = (c, d) \quad \Leftrightarrow \quad a = c \text{ and } b = d.$$

Example 13 – *Ordered n-tuples*

a. Is $(1, 2, 3, 4) = (1, 2, 4, 3)$?

b. Is $\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right)$?

Solution:

a. No. By definition of equality of ordered 4-tuples,

$$(1,\ 2,\ 3,\ 4) = (1,\ 2,\ 4,\ 3) \Leftrightarrow 1 = 1,\ 2 = 2,\ 3 = 4,\ \text{and } 4 = 3$$

But $3 \neq 4$, and so the ordered 4-tuples are not equal.

Example 13 – *Solution*

cont'd

b. Yes. By definition of equality of ordered triples,

$$\left(3, (-2)^2, \frac{1}{2}\right) = \left(\sqrt{9}, 4, \frac{3}{6}\right) \quad \Leftrightarrow \quad 3 = \sqrt{9} \text{ and } (-2)^2 = 4 \text{ and } \frac{1}{2} = \frac{3}{6}.$$

Because these equations are all true, the two ordered triples are equal.

Cartesian Products

• Definition

Given sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$ denoted $\mathbf{A_1 \times A_2 \times \ldots \times A_n}$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$.

Symbolically:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

In particular,

$$A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1 \text{ and } a_2 \in A_2\}$$

is the Cartesian product of A_1 and A_2.

Example 14 – *Cartesian Products*

Let $A_1 = \{x, y\}$, $A_2 = \{1, 2, 3\}$, and $A_3 = \{a, b\}$.

a. Find $A_1 \times A_2$. **b.** Find $(A_1 \times A_2) \times A_3$.

c. Find $A_1 \times A_2 \times A_3$.

Solution:

a. $A_1 \times A_2 = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$

b. The Cartesian product of A_1 and A_2 is a set, so it may be used as one of the sets making up another Cartesian product. This is the case for $(A_1 \times A_2) \times A_3$.

Example 14 – *Solution*

cont'd

$$(A_1 \times A_2) \times A_3 = \{(u, v) \mid u \in A_1 \times A_2 \text{ and } v \in A_3\}$$ by definition of Cartesian product

$$= \{((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a),$$

$$((y, 2), a), ((y, 3), a), ((x, 1), b), ((x, 2), b), ((x, 3), b),$$

$$((y, 1), b), ((y, 2), b), ((y, 3), b)\}$$

c. The Cartesian product $A_1 \times A_2 \times A_3$ is superficially similar to, but is not quite the same mathematical object as, $(A_1 \times A_2) \times A_3$. $(A_1 \times A_2) \times A_3$ is a set of ordered pairs of which one element is itself an ordered pair, whereas $A_1 \times A_2 \times A_3$ is a set of ordered triples.

Example 14 – *Solution*

cont'd

By definition of Cartesian product,

$$A_1 \times A_2 \times A_3 = \{(u, v, w) \mid u \in A_1, v \in A_2, \text{and } w \in A_3\}$$

$$= \{(x, 1, a), (x, 2, a), (x, 3, a), (y, 1, a), (y, 2, a),$$

$$(y, 3, a), (x, 1, b), (x, 2, b), (x, 3, b), (y, 1, b),$$

$$(y, 2, b), (y, 3, b)\}.$$

Properties of Sets

We begin by listing some set properties that involve subset relations.

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets A and B,

$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets A, B, and C,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

Properties of Sets

Procedural versions of the definitions of the other set operations are derived similarly and are summarized below.

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. $x \in X \cup Y \quad \Leftrightarrow \quad x \in X \text{ or } x \in Y$
2. $x \in X \cap Y \quad \Leftrightarrow \quad x \in X \text{ and } x \in Y$
3. $x \in X - Y \quad \Leftrightarrow \quad x \in X \text{ and } x \notin Y$
4. $x \in X^c \quad \Leftrightarrow \quad x \notin X$
5. $(x, y) \in X \times Y \quad \Leftrightarrow \quad x \in X \text{ and } y \in Y$

Example 1 – *Proof of a Subset Relation*

Prove Theorem 6.2.1(1)(a): For all sets *A* and *B*,
$A \cap B \subseteq A$.

Solution:
We start by giving a proof of the statement and then explain how you can obtain such a proof yourself.

Example 1 – *Solution* cont'd

Proof:
Suppose *A* and *B* are any sets and suppose *x* is any element of $A \cap B$.

Then $x \in A$ and $x \in B$ by definition of intersection.

In particular, $x \in A$.

Thus $A \cap B \subseteq A$.

Set Identities

An **identity** is an equation that is universally true for all elements in some set. For example, the equation
a + *b* = *b* + *a* is an identity for real numbers because it is true for all real numbers *a* and *b*.

The collection of set properties in the next theorem consists entirely of set identities. That is, they are equations that are true for all sets in some universal set.

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

Set Identities

cont'd

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

Set Identities

cont'd

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

$\cup$	union
$\cap$	intersection
$\mathbb{U}$	universal Set
$\in$	belongs to
$\notin$	does not belong to
$\subset$	proper subset of
$\subseteq$	subset of or is contained in
$\not\subset$	not a proper subset of
$\not\subseteq$	not a subset of or is not contained in
A' (or) A^c	complement of A
$\varnothing$ (or) { }	empty set or null set or void set
$n(A)$	number of elements in the set A
$P(A)$	power set of A
Δ	symmetric difference

Proving Set Identities

As we have known,

Two sets are equal each is a subset of the other.

The method derived from this fact is the most basic way to prove equality of sets.

Basic Method for Proving That Sets Are Equal

Let sets X and Y be given. To prove that $X = Y$:

1. Prove that $X \subseteq Y$.
2. Prove that $Y \subseteq X$.

Example 2 – *Proof of a Distributive Law*

Prove that for all sets *A*, *B*, and *C*,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Solution:

The proof of this fact is somewhat more complicated than the proof in Example 1, so we first derive its logical structure, then find the core arguments, and end with a formal proof as a summary.

Example 2 – *Solution*

cont'd

As in Example 1, the statement to be proved is universal, and so, by the method of generalizing from the generic particular, the proof has the following outline:

Starting Point: Suppose A, B, and C are arbitrarily chosen sets.

To Show: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Example 2 – *Solution*

cont'd

Now two sets are equal if, and only if, each is a subset of the other.

Hence, the following two statements must be proved:

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

and $$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C).$$

Showing the first containment requires showing that

$$\forall x, \text{ if } x \in A \cup (B \cap C) \text{ then } x \in (A \cup B) \cap (A \cup C).$$

Example 2 – *Solution*

cont'd

Showing the second containment requires showing that

$\forall x$, if $x \in (A \cup B) \cap (A \cup C)$ then $x \in A \cup (B \cap C)$.

Note that both of these statements are universal. So to prove the first containment, you

suppose you have any element x in $A \cup (B \cap C)$,

and then you

show that $x \in (A \cup B) \cap (A \cup C)$.

Example 2 – *Solution*

cont'd

And to prove the second containment, you

suppose you have any element x in $(A \cup B) \cap (A \cup C)$,

and then you

show that $x \in A \cup (B \cap C)$.

Example 2 – *Solution*

cont'd

In Figure 6.2.1, the structure of the proof is illustrated by the kind of diagram that is often used in connection with structured programs.

Suppose A, B, and C are sets. *[Show* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. *That is, show* $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ *and* $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.*]*

> Show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. *[That is, show* $\forall x$, *if* $x \in A \cup (B \cap C)$ *then* $x \in (A \cup B) \cap (A \cup C)$.*]*
>
> > Suppose $x \in A \cup (B \cap C)$. *[Show* $x \in (A \cup B) \cap (A \cup C)$.*]*
> >
> > ⋮
> >
> > Thus $x \in (A \cup B) \cap (A \cup C)$.
>
> Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Figure 6.2.1

Example 2 – *Solution*

cont'd

> Show $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. *[That is, show* $\forall x$, *if* $x \in (A \cup B) \cap (A \cup C)$ *then* $x \in A \cup (B \cap C)$.*]*
>
> > Suppose $x \in (A \cup B) \cap (A \cup C)$. *[Show* $x \in A \cup (B \cap C)$.*]*
> >
> > ⋮
> >
> > Thus $x \in A \cup (B \cap C)$.
>
> Hence $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Thus $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$.

Figure 6.2.1 (continued)

The analysis in the diagram reduces the proof to two concrete tasks: filling in the steps indicated by dots in the two center boxes of Figure 6.2.1.

Proving Set Identities

Theorem 6.2.2(3)(a) A Distributive Law for Sets

For all sets A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Proof: Suppose A, B, and C are any sets.

(1) *Proof that* $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$:

Let $x \in A \cup (B \cap C)$. *[We must show that* $x \in$ __(a)__.*]*

By definition of $\cup$, $x \in$ __(b)__ or $x \in B \cap C$.

Case 1 **($x \in A$):** Since $x \in A$, then both statements $x \in A \cup B$ and $x \in A \cup C$ are true by definition of $\cup$. Hence $x \in (A \cup B) \cap (A \cup C)$ by definition of $\cap$.

Case 2 **($x \in B \cap C$):** Since $x \in B \cap C$, then $x \in B$ and $x \in C$ by definition of $\cap$. Since $x \in B$, then $x \in A \cup B$ by definition of $\cup$. Similarly, since $x \in C$, then $x \in A \cup C$ by definition of $\cup$. Hence $x \in (A \cup B) \cap (A \cup C)$ by definition of $\cap$.

Therefore, in both cases 1 and 2, $x \in (A \cup B)$ __(c)__ $(A \cup C)$.

Because x could be any element in $A \cup (B \cap C)$, this argument shows that every element of $A \cup (B \cap C)$ is in $(A \cup B) \cap (A \cup C)$. Hence,

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

by definition of __(d)__.

(2) *Proof that* $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$:

Let $x \in (A \cup B) \cap (A \cup C)$. *[We must show that* $x \in A \cup (B \cap C)$.*]*

We consider the two cases: $x \in A$ and $x \notin A$.*

Case 1 **($x \in A$):** In this case, because x is in A, we can conclude immediately that $x \in A \cup (B \cap C)$ by definition of $\cup$.

Case 2 **($x \notin A$):** In this case, we know that $x \in (A \cup B) \cap (A \cup C)$. Thus, by definition of __(a)__, $x \in A \cup B$ and $x \in A \cup C$.

Because x is in $A \cup B$, then x is in at least one of A or B, and since x is not in A, then x is in B. Similarly, because x is in $A \cup C$, then x is in at least one of A or C, and since x is not in A, then x is in C.

It follows that $x \in B$ __(b)__ $x \in C$, and, thus, $x \in B \cap C$ by definition of $\cap$.

Since $x \in B \cap C$, then by definition of __(c)__, $x \in A \cup (B \cap C)$.

Therefore, in both cases 1 and 2, $x \in A \cup (B \cap C)$.

Because x could be any element in $(A \cup B) \cap (A \cup C)$, this argument shows that every element of $(A \cup B) \cap (A \cup C)$ is in $A \cup (B \cap C)$. Hence, $(A \cup B) \cap (A \cup C)$ __(d)__ $A \cup (B \cap C)$. Thus, $(A \cup B) \cap (A \cup C)$ __(d)__ $A \cup (B \cap C)$ by definition of subset.

(3) *Conclusion:* Since both subset relations have been proved, it follows, by definition of set equality, that __(a)__.

Solution

(1) a. $(A \cup B) \cap (A \cup C)$ b. A c. $\cap$ d. subset

(2) a. $\cap$ b. and c. $\cup$ d. $\subseteq$

(3) a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proving Set Identities

Suppose *A* and *B* are arbitrarily chosen sets.

Theorem 6.2.2(9)(a) A De Morgan's Law for Sets

For all sets A and B, $(A \cup B)^c = A^c \cap B^c$.

Proof: Suppose A and B are sets.

Proof that $(A \cup B)^c \subseteq A^c \cap B^c$:

[We must show that $\forall x$, *if* $x \in (A \cup B)^c$ *then* $x \in A^c \cap B^c$.*]*

Suppose $x \in (A \cup B)^c$. *[We must show that* $x \in A^c \cap B^c$.*]* By definition of complement,

$$x \notin A \cup B.$$

Now to say that $x \notin A \cup B$ means that

it is false that (x is in A or x is in B).

By De Morgan's laws of logic, this implies that

x is not in A and x is not in B,

which can be written

$$x \notin A \quad \text{and} \quad x \notin B.$$

Hence $x \in A^c$ and $x \in B^c$ by definition of complement. It follows, by definition of intersection, that $x \in A^c \cap B^c$ *[as was to be shown]*. So $(A \cup B)^c \subseteq A^c \cap B^c$ by definition of subset.

Proof that $A^c \cap B^c \subseteq (A \cup B)^c$:

[We must show that $\forall x$, *if* $x \in A^c \cap B^c$ *then* $x \in (A \cup B)^c$.*]*

Suppose $x \in A^c \cap B^c$. *[We must show that* $x \in (A \cup B)^c$.*]* By definition of intersection, $x \in A^c$ and $x \in B^c$, and by definition of complement,

$$x \notin A \quad \text{and} \quad x \notin B.$$

In other words,

x is not in A and x is not in B.

By De Morgan's laws of logic this implies that

it is false that (x is in A or x is in B),

which can be written

$$x \notin A \cup B$$

by definition of union. Hence, by definition of complement, $x \in (A \cup B)^c$ *[as was to be shown]*. It follows that $A^c \cap B^c \subseteq (A \cup B)^c$ by definition of subset.

Conclusion: Since both set containments have been proved, $(A \cup B)^c = A^c \cap B^c$ by definition of set equality.

Proving Set Identities

Theorem 6.2.3 Intersection and Union with a Subset

For any sets A and B, if $A \subseteq B$, then

$$\text{(a) } A \cap B = A \quad \text{and} \quad \text{(b) } A \cup B = B.$$

Proof:

Part (a): Suppose A and B are sets with $A \subseteq B$. To show part (a) we must show both that $A \cap B \subseteq A$ and that $A \subseteq A \cap B$. We already know that $A \cap B \subseteq A$ by the inclusion of intersection property. To show that $A \subseteq A \cap B$, let x be any element in A. *[We must show that x is in $A \cap B$.]* But, because of the hypothesis that $A \subseteq B$, we can conclude that x is also in B by definition of subset. Hence

$$x \in A \quad \text{and} \quad x \in B,$$

and thus

$$x \in A \cap B$$

by definition of intersection *[as was to be shown]*.

Proof:

Part (b): The proof of part (b) is left as an exercise.

The Empty Set

The crucial fact is that the negation of a universal statement is existential: If a set B is not a subset of a set A, then there exists an element in B that is not in A. But if B has no elements, then no such element can exist.

Theorem 6.2.4 A Set with No Elements Is a Subset of Every Set

If E is a set with no elements and A is any set, then $E \subseteq A$.

If E is a set with no elements and A is any set, then to say that $E \subseteq A$ is the same as saying that

$$\forall\, x, \text{ if } x \in E, \text{ then } x \in A.$$

The Empty Set

But since *E* has no elements, this conditional statement is vacuously true.

How many sets with no elements are there? Only one.

Corollary 6.2.5 Uniqueness of the Empty Set

There is only one set with no elements.

Suppose you need to show that a certain set equals the empty set. By Corollary 6.2.5 it suffices to show that the set has no elements.

The Empty Set

For since there is only one set with no elements (namely ∅), if the given set has no elements, then it must equal ∅.

Element Method for Proving a Set Equals the Empty Set

To prove that a set X is equal to the empty set ∅, prove that X has no elements. To do this, suppose X has an element and derive a contradiction.

Example 5 – *A Proof for a Conditional Statement*

Prove that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.

Solution:

Since the statement to be proved is both universal and conditional, you start with the method of direct proof:

Suppose A, B, and C are arbitrarily chosen sets that satisfy the condition: $A \subseteq B$ and $B \subseteq C^c$.

Show that $A \cap C = \emptyset$.

Example 5 – *Solution*

cont'd

Since the conclusion to be shown is that a certain set is empty, you can use the principle for proving that a set equals the empty set.

A complete proof is shown below.

Proposition 6.2.6

For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$.

Example 5 – *Solution*

cont'd

Proof:

Suppose A, B, and C are any sets such that $A \subseteq B$ and $B \subseteq C^c$. We must show that $A \cap C = \varnothing$. Suppose not. That is, suppose there is an element x in $A \cap C$.

By definition of intersection, $x \in A$ and $x \in C$. Then, since $A \subseteq B$, $x \in B$ by definition of subset. Also, since $B \subseteq C^c$, then $x \in C^c$ by definition of subset again. It follows by definition of complement that $x \notin C$. Thus $x \in C$ and $x \notin C$, which is a contradiction.

So the supposition that there is an element x in $A \cap C$ is false, and thus $A \cap C = \varnothing$ *[as was to be shown].*

Disproving an Alleged Set Property

We have known that to show a universal statement is false, it suffices to find one example (called a counterexample) for which it is false.

Example 1 – *Finding a Counterexample for a Set Identity*

Is the following set property true?

For all sets *A*, *B*, and *C*, $(A - B) \cup (B - C) = A - C.$

Solution:

Observe that the property is true if, and only if,

the given equality holds for *all* sets *A*, *B*, and *C*.

So it is false if, and only if,

there are sets *A*, *B*, and *C* for which the equality does *not* hold.

Example 1 – *Solution*

cont'd

One way to solve this problem is to picture sets *A*, *B*, and *C* by drawing a Venn diagram such as that shown in
Figure 6.3.1.

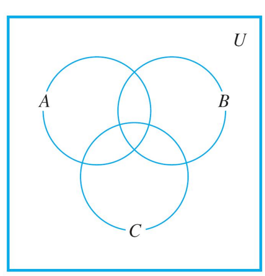


Figure 6.3.1

Example 1 – *Solution*

cont'd

If you assume that any of the eight regions of the diagram may be empty of points, then the diagram is quite general.

Find and shade the region corresponding to
$(A - B) \cup (B - C)$. Then shade the region corresponding to $A - C$. These are shown in Figure 6.3.2.

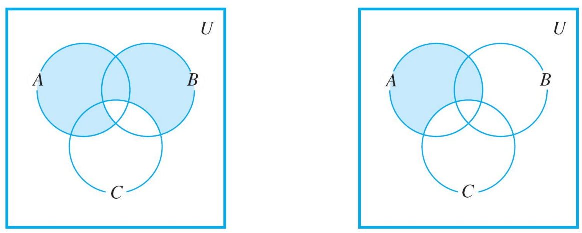


Figure 6.3.2

Example 1 – *Solution*

cont'd

Comparing the shaded regions seems to indicate that the property is false.

For instance, if there is an element in B that is not in either A or C then this element would be in $(A - B) \cup (B - C)$ (because of being in B and not C) but it would not be in
$A - C$ since $A - C$ contains nothing outside A.

Similarly, an element that is in both A and C but not B would be in $(A - B) \cup (B - C)$ (because of being in A and not B), but it would not be in $A - C$ (because of being in both A and C).

Example 1 – *Solution*

cont'd

Construct a concrete counterexample in order to confirm your answer and make sure that you did not make a mistake either in drawing or analyzing your diagrams.

One way is to put one of the integers from 1–7 into each of the seven subregions enclosed by the circles representing *A*, *B*, and *C*.

If the proposed set property had involved set complements, it would also be helpful to label the region outside the circles, and so we place the number 8 there.

Example 1 – *Solution*

cont'd

(See Figure 6.3.3.) Then define discrete sets *A*, *B*, and *C* to consist of all the numbers in their respective subregions.

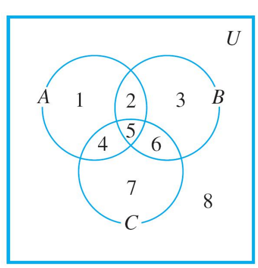


Figure 6.3.3

Example 1 – *Solution*

cont'd

Counterexample 1:

Let $A = \{1, 2, 4, 5\}$, $B = \{2, 3, 5, 6\}$, and $C = \{4, 5, 6, 7\}$.

Then

$$A - B = \{1, 4\}, \quad B - C = \{2, 3\}, \quad \text{and} \quad A - C = \{1, 2\}.$$

Hence

$$(A - B) \cup (B - C) = \{1, 4\} \cup \{2, 3\} = \{1, 2, 3, 4\},$$

$$\text{whereas} \quad A - C = \{1, 2\}.$$

Since $\{1, 2, 3, 4\} \neq \{1, 2\}$, we have that $(A - B) \cup (B - C) \neq A - C$.

Instructor : Murad Njoum

81

Problem-Solving Strategy

How can you discover whether a given universal statement about sets is true or false? There are two basic approaches: the optimistic and the pessimistic.

In the optimistic approach, you simply plunge in and start trying to prove the statement, asking yourself, "What do I need to show?" and "How do I show it?"

In the pessimistic approach, you start by searching your mind for a set of conditions that must be fulfilled to construct a counterexample.

With either approach you may have clear sailing and be immediately successful or you may run into difficulty.

82

"Algebraic" Proofs of Set Identities

Let U be a universal set and consider the power set of U, $\mathscr{P}(U)$. The set identities given in Theorem 6.2.2 hold for all elements of $\mathscr{P}(U)$.

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

   (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.

2. *Associative Laws:* For all sets A, B, and C,

   (a) $(A \cup B) \cup C = A \cup (B \cup C)$ and
   (b) $(A \cap B) \cap C = A \cap (B \cap C)$.

3. *Distributive Laws:* For all sets, A, B, and C,

   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

"Algebraic" Proofs of Set Identities cont'd

4. *Identity Laws:* For all sets A,

   (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$.

5. *Complement Laws:*

   (a) $A \cup A^c = U$ and (b) $A \cap A^c = \emptyset$.

6. *Double Complement Law:* For all sets A,

   $(A^c)^c = A$.

7. *Idempotent Laws:* For all sets A,

   (a) $A \cup A = A$ and (b) $A \cap A = A$.

8. *Universal Bound Laws:* For all sets A,

   (a) $A \cup U = U$ and (b) $A \cap \emptyset = \emptyset$.

9. *De Morgan's Laws:* For all sets A and B,

   (a) $(A \cup B)^c = A^c \cap B^c$ and (b) $(A \cap B)^c = A^c \cup B^c$.

10. *Absorption Laws:* For all sets A and B,

    (a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$.

11. *Complements of U and $\emptyset$:*

    (a) $U^c = \emptyset$ and (b) $\emptyset^c = U$.

12. *Set Difference Law:* For all sets A and B,

    $A - B = A \cap B^c$.

Example 3 – *Solution*

cont'd

Construct an algebraic proof that for all sets A and B, $\boldsymbol{A - (A \cap B) = A - B}$.

$= (A \cap A^c) \cup (A \cap B^c)$	by the distributive law
$= \emptyset \cup (A \cap B^c)$	by the complement law
$= (A \cap B^c) \cup \emptyset$	by the commutative law for $\cup$
$= A \cap B^c$	by the identity law for $\cup$
$= A - B$	by the set difference law.

"Algebraic" Proofs of Set Identities

Once a certain number of identities and other properties have been established, new properties can be derived from them algebraically without having to use element method arguments.

It turns out that only identities (1–5) of Theorem 6.2.2 are needed to prove any other identity involving only unions, intersections, and complements.

"Algebraic" Proofs of Set Identities

With the addition of identity (12), the set difference law, any set identity involving unions, intersections, complements, and set differences can be established.

To use known properties to derive new ones, you need to use the fact that such properties are universal statements. Like the laws of algebra for real numbers, they apply to a wide variety of different situations.

Assume that all sets are subsets of $\mathscr{P}(U)$, *then*, for instance, one of the distributive laws states that
for all sets *A*, *B*, and *C*, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Example 3 – *Deriving a Set Identity Using Properties of ∅*

Construct an algebraic proof that for all sets *A* and *B*,

$$A - (A \cap B) = A - B.$$

Cite a property from Theorem 6.2.2 for every step of the proof.

Solution:
Suppose *A* and *B* are any sets. Then

$$A - (A \cap B) = A \cap (A \cap B)^c \qquad \text{by the set difference law}$$

$$= A \cap (A^c \cup B^c) \qquad \text{by De Morgan's laws}$$

Formalizing Statements in Set Theory

All smart students

Student ∩ Smart

Between sets

∩ and

and

Between predicate and propositions

Students who are not Smart

Student ∩ $Smart^c$ / Student - Smart

There are no smart students from Palestine

Palestinian ∩ Student ∩ Smart =

There are no smart students from Palestine among the winners

Winner ∩ Palestinian ∩ Student ∩ Smart =

All Palestinian Americans except Women

(American ∩ Palestinian) – Women / American ∩ Palestinian ∩ $women^c$

SECTION 6.4

Boolean Algebras

Boolean Algebras, Russell's Paradox, and the Halting Problem

Table 6.4.1 summarizes the main features of the logical equivalences from Theorem 2.1.1 and the set properties from Theorem 6.2.2. Notice how similar the entries in the two columns are.

Logical Equivalences	Set Properties
For all statement variables p, q, and r:	For all sets A, B, and C:
a. $p \vee q \equiv q \vee p$ b. $p \wedge q \equiv q \wedge p$	a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$ b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$	a. $A \cup (B \cup C) \equiv A \cup (B \cup C)$ b. $A \cap (B \cap C) \equiv A \cap (B \cap C)$
a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	a. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$
a. $p \vee \mathbf{c} \equiv p$ b. $p \wedge \mathbf{t} \equiv p$	a. $A \cup \emptyset = A$ b. $A \cap U = A$

Instructor : Murad Njoum

Table 6.4.1

Boolean Algebras, Russell's Paradox, and the Halting Problem

Logical Equivalences	Set Properties
a. $p \vee \sim p \equiv \mathbf{t}$ b. $p \wedge \sim p \equiv \mathbf{c}$	a. $A \cup A^c = U$ b. $A \cap A^c = \emptyset$
$\sim(\sim p) \equiv p$	$(A^c)^c = A$
a. $p \vee p \equiv p$ b. $p \wedge p \equiv p$	a. $A \cup A = A$ b. $A \cap A = A$
a. $p \vee \mathbf{t} \equiv \mathbf{t}$ b. $p \wedge \mathbf{c} \equiv \mathbf{c}$	a. $A \cup U = U$ b. $A \cap \emptyset = \emptyset$
a. $\sim(p \vee q) \equiv \sim p \wedge \sim q$ b. $\sim(p \wedge q) \equiv \sim p \vee \sim q$	a. $(A \cup B)^c = A^c \cap B^c$ b. $(A \cap B)^c = A^c \cup B^c$
a. $p \vee (p \wedge q) \equiv p$ b. $p \wedge (p \vee q) \equiv p$	a. $A \cup (A \cap B) \equiv A$ b. $A \cap (A \cup B) \equiv A$
a. $\sim \mathbf{t} \equiv \mathbf{c}$ b. $\sim \mathbf{c} \equiv \mathbf{t}$	a. $U^c = \emptyset$ b. $\emptyset^c = U$

Table 6.4.1 (continued)

Instructor : Murad Njoum

Boolean Algebras, Russell's Paradox, and the Halting Problem

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6. *Double negative law:*	$\sim(\sim p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* **t** *and* **c**:	$\sim\mathbf{t} \equiv \mathbf{c}$	$\sim\mathbf{c} \equiv \mathbf{t}$

Boolean Algebras, Russell's Paradox, and the Halting Problem

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

   (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.

2. *Associative Laws:* For all sets A, B, and C,

   (a) $(A \cup B) \cup C = A \cup (B \cup C)$ and
   (b) $(A \cap B) \cap C = A \cap (B \cap C)$.

3. *Distributive Laws:* For all sets, A, B, and C,

   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. *Identity Laws:* For all sets A,

   (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$.

5. *Complement Laws:*

   (a) $A \cup A^c = U$ and (b) $A \cap A^c = \emptyset$.

Boolean Algebras, Russell's Paradox, and the Halting Problem

cont'd

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

Boolean Algebras, Russell's Paradox, and the Halting Problem

If you let (*or*) correspond to $\cup$ (union), (*and*) correspond to $\cap$ (intersection), **t** (a tautology) correspond to *U* (a universal set), **c** (a contradiction) correspond to Ø (the empty set), and ~ (negation) correspond to ^{c} (complementation), then you can see that the structure of the set of statement forms with operations and is essentially identical to the structure of the set of subsets of a universal set with operations $\cup$ and $\cap$.

In fact, both are special cases of the same general structure, known as a *Boolean algebra*.

Boolean Algebras, Russell's Paradox, and the Halting Problem

In this section we show how to derive the various properties associated with a Boolean algebra from a set of just five axioms.

• Definition: Boolean Algebra

A **Boolean algebra** is a set B together with two operations, generally denoted $+$ and $\cdot$, such that for all a and b in B both $a+b$ and $a \cdot b$ are in B and the following properties hold:

1. *Commutative Laws:* For all a and b in B,

$$\text{(a) } a+b=b+a \quad \text{and} \quad \text{(b) } a \cdot b = b \cdot a.$$

2. *Associative Laws:* For all a, b, and c in B,

$$\text{(a) } (a+b)+c=a+(b+c) \quad \text{and} \quad \text{(b) } (a \cdot b) \cdot c = a \cdot (b \cdot c).$$

3. *Distributive Laws:* For all a, b, and c in B,

$$\text{(a) } a+(b \cdot c)=(a+b) \cdot (a+c) \quad \text{and} \quad \text{(b) } a \cdot (b+c) = (a \cdot b)+(a \cdot c).$$

4. *Identity Laws:* There exist distinct elements 0 and 1 in B such that for all a in B,

$$\text{(a) } a+0=a \quad \text{and} \quad \text{(b) } a \cdot 1 = a.$$

5. *Complement Laws:* For each a in B, there exists an element in B, denoted $\overline{a}$ and called the **complement** or **negation** of a, such that

$$\text{(a) } a+\overline{a}=1 \quad \text{and} \quad \text{(b) } a \cdot \overline{a} = 0.$$

Boolean Algebras, Russell's Paradox, and the Halting Problem

In any Boolean algebra, the complement of each element is unique, the quantities 0 and 1 are unique, and identities analogous to those in Theorem 2.1.1 and Theorem 6.2.2 can be deduced.

Theorem 6.4.1 Properties of a Boolean Algebra

Let B be any Boolean algebra.

1. *Uniqueness of the Complement Law:* For all a and x in B, if $a+x=1$ and $a \cdot x = 0$ then $x=\overline{a}$.
2. *Uniqueness of* 0 *and* 1: If there exists x in B such that $a+x=a$ for all a in B, then $x=0$, and if there exists y in B such that $a \cdot y = a$ for all a in B, then $y=1$.
3. *Double Complement Law:* For all $a \in B$, $\overline{(\overline{a})}=a$.

Boolean Algebras, Russell's Paradox, and the Halting Problem

cont'd

4. *Idempotent Law:* For all $a \in B$,

$$\text{(a) } a + a = a \quad \text{and} \quad \text{(b) } a \cdot a = a.$$

5. *Universal Bound Law:* For all $a \in B$,

$$\text{(a) } a + 1 = 1 \quad \text{and} \quad \text{(b) } a \cdot 0 = 0.$$

6. *De Morgan's Laws:* For all a and $b \in B$,

$$\text{(a) } \overline{a + b} = \overline{a} \cdot \overline{b} \quad \text{and} \quad \text{(b) } \overline{a \cdot b} = \overline{a} + \overline{b}.$$

7. *Absorption Laws:* For all a and $b \in B$,

$$\text{(a) } (a + b) \cdot a = a \quad \text{and} \quad \text{(b) } (a \cdot b) + a = a.$$

8. *Complements of 0 and 1:*

$$\text{(a) } \overline{0} = 1 \quad \text{and} \quad \text{(b) } \overline{1} = 0.$$

Part 1: Uniqueness of the Complement Law

Suppose a and x are particular, but arbitrarily chosen, elements of B that sati following hypothesis: $a + x = 1$ and $a \cdot x = 0$. Then

$x = x \cdot 1$	because 1 is an identity for $\cdot$
$= x \cdot (a + \overline{a})$	by the complement law for $+$
$= x \cdot a + x \cdot \overline{a}$	by the distributive law for $\cdot$ over $+$
$= a \cdot x + x \cdot \overline{a}$	by the commutative law for $\cdot$
$= 0 + x \cdot \overline{a}$	by hypothesis
$= a \cdot \overline{a} + x \cdot \overline{a}$	by the complement law for $\cdot$
$= (\overline{a} \cdot a) + (\overline{a} \cdot x)$	by the commutative law for $\cdot$
$= \overline{a} \cdot (a + x)$	by the distributive law for $\cdot$ over $+$
$= \overline{a} \cdot 1$	by hypothesis
$= \overline{a}$	because 1 is an identity for $\cdot$.

Boolean Algebras, Russell's Paradox, and the Halting Problem

You may notice that all parts of the definition of a Boolean algebra and most parts of Theorem 6.4.1 contain paired statements. For instance, the distributive laws state that for all a, b, and c in B,

(a) $a + (b \cdot c) = (a + b) \cdot (a + c)$ and
(b) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$,

and the identity laws state that for all a in B,

(a) $a + 0 = a$ and (b) $a \cdot 1 = a$.

Boolean Algebras, Russell's Paradox, and the Halting Problem

Note that each of the paired statements can be obtained from the other by interchanging all the + and · signs and interchanging 1 and 0. Such interchanges transform any Boolean identity into its **dual** identity.

It can be proved that the dual of any Boolean identity is also an identity. This fact is often called the **duality principle** for a Boolean algebra.

Example 1 – *Proof of the Double Complement Law*

Prove that for all elements a in a Boolean algebra

$B, \overline{(\overline{a})} = a.$

Solution:

Start by supposing that *B* is a Boolean algebra and *a* is any element of *B*. The basis for the proof is the uniqueness of the complement law: that each element in *B* has a unique complement that satisfies certain equations with respect to it.

So if *a* can be shown to satisfy those equations with respect to $\overline{a}$, then *a* must be the complement of $\overline{a}$.

Example 1 – *Solution* cont'd

Theorem 6.4.1(3) Double Complement Law

For all elements a in a Boolean algebra B, $\overline{(\overline{a})} = a$.

Proof:

Suppose *B* is a Boolean algebra and *a* is any element of *B*. Then

$$\overline{a} + a = a + \overline{a} \quad \text{by the commutative law}$$

$$= 1 \quad \text{by the complement law for 1}$$

and

$$\overline{a} \cdot a = a \cdot \overline{a} \quad \text{by the commutative law}$$