BIRZEIT UNIVERSITY

Discrete Mathematic and Application Comp233

CHAPTER 7

Functions

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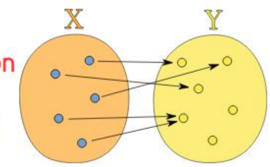
SECTION 7.1

Functions Defined on General Sets

Formal Definition of a Function

A function relates each element of a set with exactly one element of another set (possibly the same set).



Domain, Codomain and Range

There are special names for **what can go into**, and **what can come out** of a function:

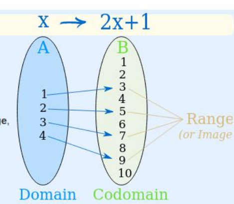
- What can go **into** a function is called the **Domain**
- What **may possibly come out** of a function is called the **Codomain**
- What **actually comes out** of a function is called the **Range**

Example

- The set "A" is the **Domain**,
- The set "B" is the **Codomain**,
- And the set of elements that get pointed to in B (the actual values produced by the function) are the **Range**, also called the Image.

And we have:

- Domain: {1, 2, 3, 4}
- Codomain: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
- Range: {3, 5, 7, 9}



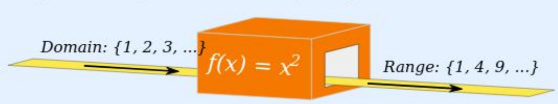
Part of the Function

Now, what comes **out** *(the Range)* depends on what we put **in** *(the Domain)* ...

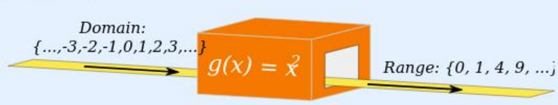
... but **WE** can define the Domain!

In fact the Domain is an essential part of the function. Change the Domain and we have a different function.

Example: a simple function like f(x) = x^2 can have the **domain** (what goes in) of just the counting numbers {1,2,3,...}, and the **range** will then be the set {1,4,9,...}



And another function g(x) = x^2 can have the domain of integers {...,-3,-2,-1,0,1,2,3,...}, in which case the range is the set {0,1,4,9,...}



Functions Defined on General Sets

We have already defined a function as a certain type of relation. The following is a restatement of the definition of function that includes additional terminology associated with the concept.

• Definition

A **function f from a set X to a set Y**, denoted $f: X \to Y$, is a relation from X, the **domain**, to Y, the **co-domain**, that satisfies two properties: (1) every element in X is related to some element in Y, and (2) no element in X is related to more than one element in Y. Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f: x \to y$. The unique element to which f sends x is denoted

$f(x)$ and is called **f of x**, or
the output of f for the input x, or
the value of f at x, or
the image of x under f.

The set of all values of f taken together is called the *range of f* or the *image of X under f*. Symbolically,

$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}.$$

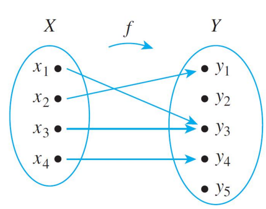
Given an element y in Y, there may exist elements in X with y as their image. If $f(x) = y$, then x is called **a preimage of y** or **an inverse image of y**. The set of all inverse images of y is called *the inverse image of y*. Symbolically,

$$\textbf{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$

Arrow Diagrams

We have known that if X and Y are finite sets, you can define a function f from X to Y by drawing an arrow diagram.

You make a list of elements in X and a list of elements in Y, and draw an arrow from each element in X to the corresponding element in Y, as shown in Figure 7.1.1.



This arrow diagram does define a function because

1. Every element of X has an arrow coming out of it.
2. No element of X has two arrows coming out of it that point to two different elements of Y.

Figure 7.1.1

Example 2 – *A Function Defined by an Arrow Diagram*

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram in Figure 7.1.3.

a. Write the domain and co-domain of f.

b. Find $f(a)$, $f(b)$, *and* $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2?
Is b an inverse image of 3?

e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.

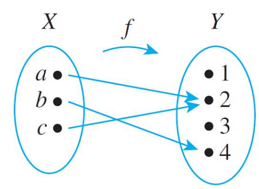


Figure 7.1.3

Example 2 – *Solution*

a. domain of $f = \{a, b, c\}$, co-domain of $f = \{1, 2, 3, 4\}$

b. $f(a) = 2,\ f(b) = 4,\ f(c) = 2$

c. range of $f = \{2, 4\}$

d. Yes, No

e. inverse image of $2 = \{a, c\}$
inverse image of $4 = \{b\}$
inverse image of $1 = \emptyset$ (*since no arrows point to* 1)

f. $\{(a, 2), (b, 4), (c, 2)\}$

Arrow Diagrams

In Example 2 there are no arrows pointing to the 1 or the 3.

This illustrates the fact that although each element of the domain of a function must have an arrow pointing out from it, there can be elements of the co-domain to which no arrows point.

Note also that there are two arrows pointing to the 2—one coming from *a* and the other from *c*.

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Arrow Diagrams

Earlier we have given a test for determining whether two functions with the same domain and co-domain are equal, saying that the test results from the definition of a function as a binary relation.

We formalize this justification in Theorem 7.1.1.

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

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Example 3 – *Equality of Functions*

a. Let J_3 = {0, 1, 2}, and define functions *f* and *g* from J_3 to J_3 as follows: For all *x* in J_3,

$$f(x) = (x^2 + x + 1)\ mod\ 3 \quad \text{and} \quad g(x) = (x + 2)^2\ mod\ 3.$$

Does *f* = *g*?

b. Let *F*: **R** → **R** and *G*: **R** → **R** be functions. Define new functions *F* + *G*: **R** → **R** and *G* + *F*: **R** → **R** as follows: For all *x* ∈ **R,**

$$(F + G)(x) = F(x) + G(x) \quad \text{and} \quad (G + F)(x) = G(x) + F(x).$$

Does *F* + *G* = *G* + *F*?

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Example 3 – *Solution*

a. Yes, the table of values shows that *f*(*x*) = *g*(*x*) for all *x* in J_3.

x	x^2+x+1	$f(x) = (x^2+x+1)\ mod\ 3$	$(x+2)^2$	$g(x) = (x+2)^2\ mod\ 3$
0	1	$1\ mod\ 3 = 1$	4	$4\ mod\ 3 = 1$
1	3	$3\ mod\ 3 = 0$	9	$9\ mod\ 3 = 0$
2	7	$7\ mod\ 3 = 1$	16	$16\ mod\ 3 = 1$

b. Again the answer is yes. For all real numbers *x*,

$$\begin{aligned} (F + G)(x) &= F(x) + G(x) && \text{by definition of } F + G \\ &= G(x) + F(x) && \text{by the commutative law for addition of real numbers} \\ &= (G + F)(x) && \text{by definition of } G + F \end{aligned}$$

Hence *F* + *G* = *G* + *F*.

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Example 4 – *The Identity Function on a Set*

Given a set X, define a function I_X from X to X by

for all x in X.

$$I_X(x) = x$$

The function I_X is called the **identity function on *X*** because it sends each element of X to the element that is identical to it. Thus the identity function can be pictured as a machine that sends each piece of input directly to the output chute without changing it in any way.

Let X be any set and suppose that a_{ij}^k and $\phi(z)$ are elements of X. Find $I_X\left(a_{ij}^k\right)$ and $I_X(\phi(z))$

Whatever is input to the identity function comes out unchanged, so

$I_X\left(a_{ij}^k\right) = a_{ij}^k$ and $I_X(\phi(z)) = \phi(z)$.

Examples of Functions

• Definition Logarithms and Logarithmic Functions

Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base *b* of *x*,** written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically,

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

The **logarithmic function with base *b*** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$.

Find the following:

a. $\log_3 9$ **b.** $\log_2\left(\frac{1}{2}\right)$ **c.** $\log_{10}(1)$

d. $\log_2(2^m)$ (m is any real number) **e.** $2^{\log_2 m}$ $(m > 0)$

Example 8 – *The Logarithmic Function with Base b*

Solution:

a. $\log_3 9 = 2$ because $3^2 = 9$.

b. $\log_2 \left(\frac{1}{2}\right) = -1$ because $2^{-1} = \frac{1}{2}$.

c. $\log_{10}(1) = 0$ because $10^0 = 1$.

d. $\log_2(2^m) = m$ because the exponent to which 2 must be raised to obtain 2^m is m.

e. $2^{\log_2 m} = m$ because $\log_2 m$ is the exponent to which 2 must be raised to obtain m.

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Examples of Functions

We have known that if *S* is a nonempty, finite set of characters, then a **string over *S*** is a finite sequence of elements of *S*.

The number of characters in a string is called the **length** of the string. The **null string over *S*** is the "string" with no characters.

It is usually denoted ϵ and is said to have length 0.

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Example 9 – *Encoding and Decoding Functions*

Digital messages consist of finite sequences of 0's and 1's. When they are communicated across a transmission channel, they are frequently coded in special ways to reduce the possibility that they will be garbled by interfering noise in the transmission lines.

For example, suppose a message consists of a sequence of 0's and 1's. A simple way to encode the message is to write each bit three times. Thus the message 00101111

would be encoded as 000000111000111111111111.

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Example 9 – *Encoding and Decoding Functions* cont'd

The receiver of the message decodes it by replacing each section of three identical bits by the one bit to which all three are equal.

Let *A* be the set of all strings of 0's and 1's, and let *T* be the set of all strings of 0's and 1's that consist of consecutive triples of identical bits.

The encoding and decoding processes described above are actually functions from *A* to *T* and from *T* to *A*.

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Example 9 – *Encoding and Decoding Functions*

cont'd

The encoding function *E* is the function from *A* to *T* defined as follows: For each string $s \in A$,

E(*s*) = the string obtained from *s* by replacing each bit of *s* by the same bit written three times.

The decoding function *D* is defined as follows: For each string $t \in T$,

D(*t*) = the string obtained from *t* by replacing each consecutive triple of three identical bits of *t* by a single copy of that bit.

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Example 9 – *Encoding and Decoding Functions*

cont'd

The advantage of this particular coding scheme is that it makes it possible to do a certain amount of error correction when interference in the transmission channels has introduced errors into the stream of bits.

If the receiver of the coded message observes that one of the sections of three consecutive bits that should be identical does not consist of identical bits, then one bit differs from the other two.

In this case, if errors are rare, it is likely that the single bit that is different is the one in error, and this bit is changed to agree with the other two before decoding.

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Boolean Functions

We have discussed earlier that how to find input/output tables for certain digital logic circuits.

Any such input/output table defines a function in the following way: The elements in the input column can be regarded as ordered tuples of 0's and 1's; the set of all such ordered tuples is the domain of the function.

The elements in the output column are all either 0 or 1; thus {0, 1} is taken to be the co-domain of the function. The relationship is that which sends each input element to the output element in the same row.

Example 11 – *A Boolean Function*

• Definition

An (***n*-place**) **Boolean function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}$, which is denoted $\{0, 1\}^n$. Thus $f: \{0, 1\}^n \to \{0, 1\}$.

Consider the three-place Boolean function defined from the set of all 3-tuples of 0's and 1's to {0, 1} as follows: For each triple (x_1, x_2, x_3) of 0's and 1's,

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3)\ mod\ 2.$$

Describe *f* using an input/output table.

Solution:

$$f(1, 1, 1) = (1 + 1 + 1)\ mod\ 2 = 3\ mod\ 2 = 1$$

$$f(1, 1, 0) = (1 + 1 + 0)\ mod\ 2 = 2\ mod\ 2 = 0$$

Example 11 – *Solution*

cont'd

The rest of the values of *f* can be calculated similarly to obtain the following table.

Input			Output
x_1	x_2	x_3	$(x_1 + x_2 + x_3)$ *mod* 2
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

Checking Whether a Function Is Well Defined

It can sometimes happen that what appears to be a function defined by a rule is not really a function at all. To give an example, suppose we wrote, "Define a function
$f : \mathbf{R} \to \mathbf{R}$ by specifying that for all real numbers *x*,

$$f(x) \text{ is the real number } y \text{ such that } x^2 + y^2 = 1.$$

There are two distinct reasons why this description does not define a function. For almost all values of *x*, either (1) there is no *y* that satisfies the given equation or (2) there are two different values of *y* that satisfy the equation.

Checking Whether a Function Is Well Defined

For instance, when $x = 2$, there is no real number y such that $2^2 + y^2 = 1$, and when $x = 0$, both $y = -1$ and $y = 1$ satisfy the equation $0^2 + y^2 = 1$.

In general, we say that a "function" is **not well defined** if it fails to satisfy at least one of the requirements for being a function.

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Example 12 – *A Function That Is Not Well Defined*

We know that **Q** represents the set of all rational numbers. Suppose you read that a function $f: \mathbf{Q} \to \mathbf{Z}$ is to be defined by the formula

$$f\left(\frac{m}{n}\right) = m \quad \text{for all integers } m \text{ and } n \text{ with } n \neq 0.$$

That is, the integer associated by f to the number $\frac{m}{n}$ is m. Is f well defined? Why?

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Example 12 – *Solution*

The function *f* is not well defined.

The reason is that fractions have more than one representation as quotients of integers.

For instance, $\frac{1}{2} = \frac{3}{6}$. Now if *f* were a function, then the definition of a function would imply that $f\left(\frac{1}{2}\right) = \left(\frac{3}{6}\right)$ since $\frac{1}{2} = \frac{3}{6}$.

Example 12 – *Solution* cont'd

But applying the formula for *f*, you find that

$$f\left(\frac{1}{2}\right) = 1 \quad \text{and} \quad f\left(\frac{3}{6}\right) = 3,$$

and so

$$f\left(\frac{1}{2}\right) \neq f\left(\frac{3}{6}\right).$$

This contradiction shows that *f* is not well defined and, therefore, is not a function.

Note that the phrase *well-defined function* is actually redundant; for a function to be well defined really means that it is worthy of being called a function.

Functions Acting on Sets

Given a function from a set X to a set Y, you can consider the set of images in Y of all the elements in a subset of X and the set of inverse images in X of all the elements in a

subset of Y.

• Definition

If $f: X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

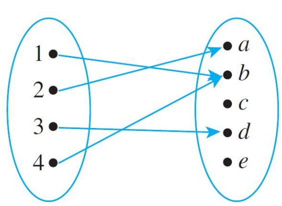
$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

and
$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of A**, and $f^{-1}(C)$ is called the **inverse image of C**.

Example 13 – *The Action of a Function on Subsets of a Set*

Let X = {1, 2, 3, 4} and Y = {a, b, c, d, e}, and define $F : X \to Y$ by the following arrow diagram:



$$F(A) = \{b\}$$

$$F(X) = \{a, b, d\}$$

$$F^{-1}(C) = \{1, 2, 4\}$$

$$F^{-1}(D) = \emptyset$$

Let A = {1, 4}, C = {a, b}, and D = {c, e}. Find $F(A)$, $F(X)$, $F^{-1}(C)$, and $F^{-1}(D)$.

One-to-One and Onto, Inverse Functions

In this section we discuss two important properties that functions may satisfy: the property of being *one-to-one* and the property of being *onto*.

Functions that satisfy both properties are called *one-to-one correspondences* or *one-to-one onto functions*.

When a function is a one-to-one correspondence, the elements of its domain and co-domain match up perfectly, and we can define an *inverse function* from the co-domain to the domain that "undoes" the action of the function.

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One-to-One Functions

We have noted earlier that a function may send several elements of its domain to the same element of its co-domain.

In terms of arrow diagrams, this means that two or more arrows that start in the domain can point to the same element in the co-domain.

On the other hand, if no two arrows that start in the domain point to the same element of the co-domain then the function is called *one-to-one* or *injective*.

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One-to-One Functions

For a one-to-one function, each element of the range is the image of at most one element of the domain.

• Definition

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently, $\quad \text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$

Symbolically,

$$F: X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

To obtain a precise statement of what it means for a function *not* to be one-to-one, take the negation of one of the equivalent versions of the definition above.

One-to-One Functions

Thus:

A function $F: X \to Y$ is *not* one-to-one $\Leftrightarrow$ $\exists$ elements x_1 and x_2 in X with $F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

That is, if elements x_1 and x_2 can be found that have the same function value but are not equal, then F is not one-to-one.

One-to-One Functions

In terms of arrow diagrams, a one-to-one function can be thought of as a function that separates points. That is, it takes distinct points of the domain to distinct points of the co-domain.

A function that is not one-to-one fails to separate points. That is, at least two points of the domain are taken to the same point of the co-domain.

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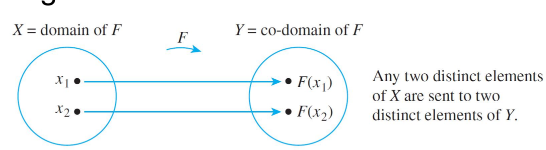
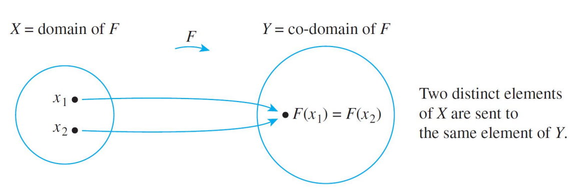
One-to-One Functions

This is illustrated in Figure 7.2.1



A One-to-One Function Separates Points

Figure 7.2.1 (a)



A Function That Is Not One-to-One Collapses Points Together

Figure 7.2.1 (b)

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Example 1 – *Identifying One-to-One Functions Defined on Finite Sets*

a. Do either of the arrow diagrams in Figure 7.2.2 define one-to-one functions?

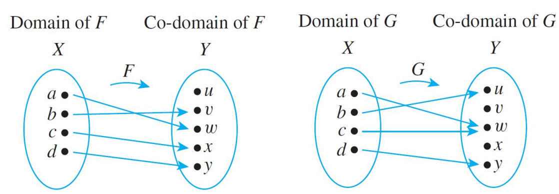

Figure 7.2.2

b. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, and $H(3) = d$.

Define $K: X \to Y$ as follows: $K(1) = d$, $K(2) = b$, and $K(3) = d$. Is either H or K one-to-one?

Example 1 – *Solution*

a. F is one-to-one but G is not.

F is one-to-one because no two different elements of X are sent by F to the same element of Y.

G is not one-to-one because the elements a and c are both sent by G to the same element of Y: $G(a) = G(c) = w$ but $a \neq c$.

Example 1 – *Solution*

cont'd

b. *H* is one-to-one but *K* is not.

H is one-to-one because each of the three elements of the domain of *H* is sent by *H* to a different element of the co-domain: $H(1) \neq H(2)$, $H(1) \neq H(3)$, and $H(2) \neq H(3)$. *K*, however, is not one-to-one because $K(1) = K(3) = d$ but $1 \neq 3$.

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One-to-One Functions on Infinite Sets

Now suppose *f* is a function defined on an infinite set *X*. By definition, *f* is one-to-one if, and only if, the following universal statement is true:

$$\forall x_1, x_2 \in X, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

Thus, to prove *f* is one-to-one, you will generally use the method of direct proof:

suppose x_1 and x_2 are elements of *X* such that $f(x_1) = f(x_2)$

and **show** that $x_1 = x_2$.

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Example 2 – *Proving or Disproving That Functions Are One-to-One*

Define *f*: **R** → **R** and *g*: **Z** → **Z** by the rules.

$$f(x) = 4x - 1 \quad \text{for all} \quad x \in \mathbf{R}$$

and $$g(n) = n^2 \quad \text{for all} \quad n \in \mathbf{Z}.$$

a. Is *f* one-to-one? Prove or give a counterexample.

b. Is *g* one-to-one? Prove or give a counterexample.

To show that *f* is *not* one-to-one, you will ordinarily

find elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

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Example 2 – *Solution*

It is usually best to start by taking a positive approach to answering questions like these. Try to prove the given functions are one-to-one and see whether you run into difficulty.

If you finish without running into any problems, then you have a proof. If you do encounter a problem, then analyzing the problem may lead you to discover a counterexample.

a. The function *f*: **R** → **R** is defined by the rule

$$f(x) = 4x - 1 \quad \text{for all real numbers } x.$$

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Example 2 – *Solution*

cont'd

To prove that *f* is one-to-one, you need to prove that

$$\forall \text{ real numbers } x_1 \text{ and } x_2, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

Substituting the definition of *f* into the outline of a direct proof, you

suppose x_1 and x_2 are any real numbers such that $4x_1 - 1 = 4x_2 - 1$,

and **show** that $x_1 = x_2$.

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Example 2 – *Solution*

cont'd

Can you reach what is to be shown from the supposition?

Of course. Just add 1 to both sides of the equation in the supposition and then divide both sides by 4.

This discussion is summarized in the following formal answer.

Answer to (a):

If the function $f: \mathbf{R} \to \mathbf{R}$ is defined by the rule $f(x) = 4x - 1$, for all real numbers x, then f is one-to-one.

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Example 2 – *Solution*

cont'd

Proof:
Suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$.
[We must show that $x_1 = x_2$.]

By definition of *f*,

$$4x_1 - 1 = 4x_2 - 1.$$

Adding 1 to both sides gives

$$4x_1 = 4x_2,$$

and dividing both sides by 4 gives

$$x_1 = x_2,$$

which is what was to be shown.

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Example 2 – *Solution*

cont'd

b. The function g: **Z** → **Z** is defined by the rule

$$g(n) = n^2 \quad \text{for all integers } n.$$

As above, you start as though you were going to prove that *g* is one-to-one.

Substituting the definition of *g* into the outline of a direct proof, you

suppose n_1 and n_2 are integers such that $n_1^2 = n_2^2,$

and **try to show** that $n_1 = n_2$.

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Example 2 – *Solution* cont'd

Can you reach what is to be shown from the supposition? No! It is quite possible for two numbers to have the same squares and yet be different.

For example, $2^2 = (-2)^2$ but $2 \neq -2$.

Thus, in trying to prove that *g* is one-to-one, you run into difficulty.

But analyzing this difficulty leads to the discovery of a counterexample, which shows that *g* is not one-to-one.

Example 2 – *Solution* cont'd

This discussion is summarized as follows:

Answer to (b):

If the function $g: \mathbf{Z} \to \mathbf{Z}$ is defined by the rule $g(n) = n^2$, for all $n \in \mathbf{Z}$, then g is not one-to-one.

Counterexample:

Let $n_1 = 2$ and $n_2 = 2$. Then by definition of *g*,

$$g(n_1) = g(2) = 2^2 = 4 \quad \text{and also}$$

$$g(n_2) = g(-2) = (-2)^2 = 4.$$

Hence $$g(n_1) = g(n_2) \quad \text{but} \quad n_1 \neq n_2,$$

and so *g* is not one-to-one.

Application: Hash Functions

Imagine a set of student records, each of which includes the student's social security number, and suppose the records are to be stored in a table in which a record can be located if the social security number is known.

One way to do this would be to place the record with social security number *n* into position *n* of the table. However, since social security numbers have nine digits, this method would require a table with 999,999,999 positions.

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Application: Hash Functions

The problem is that creating such a table for a small set of records would be very wasteful of computer memory space.

Hash functions are functions defined from larger to smaller sets of integers, frequently using the *mod* function, which provide part of the solution to this problem.

We illustrate how to define and use a *hash* function with a very simple example.

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Example 3 – A Hash Function

Suppose there are no more than seven student records. Define a function *Hash* from the set of all social security numbers (ignoring hyphens) to the set {0, 1, 2, 3, 4, 5, 6} as follows:

$$Hash(n) = n \; mod \; 7 \quad \text{for all social security numbers } n.$$

To use your calculator to find *n mod* 7, use the formula
n mod 7 = *n* – 7 · (*n div* 7).

In other words, divide *n* by 7, multiply the integer part of the result by 7, and subtract that number from *n*. For instance, since 328343419/7 = 46906202.71 . . . ,

$$Hash(328\text{-}34\text{-}3419) = 328343419 - (7 \cdot 46906202) = 5.$$

Example 3 – A Hash Function cont'd

As a first approximation to solving the problem of storing the records, try to place the record with social security number *n* in position *Hash*(*n*).

For instance, if the social security numbers are 328-34-3419, 356-63-3102, 223-79-9061, and 513-40-8716, the positions of the records are as shown in Table 7.2.1.

0	356-63-3102
1	
2	513-40-8716
3	223-79-9061
4	
5	328-34-3419
6	

Table 7.2.1

Example 3 – A Hash Function cont'd

The problem with this approach is that *Hash* may not be one-to one; *Hash* might assign the same position in the table to records with different social security numbers. Such an assignment is called a **collision**.

When collisions occur, various **collision resolution methods** are used. One of the simplest is the following: If, when the record with social security number *n* is to be placed, position *Hash*(*n*) is already occupied, start from that position and search downward to place the record in the first empty position that occurs, going back up to the beginning of the table if necessary.

Example 3 – A Hash Function cont'd

To locate a record in the table from its social security number, *n*, you compute *Hash*(*n*) and search downward from that position to find the record with social security number *n*. If there are not too many collisions, this is a very efficient way to store and locate records.

Suppose the social security number for another record to be stored is 908-37-1011. Find the position in Table 7.2.1 into which this record would be placed.

0	356-63-3102
1	
2	513-40-8716
3	223-79-9061
4	
5	328-34-3419
6	

Table 7.2.1

Example 3 – *Solution*

When you compute *Hash* you find that *Hash*(908-37-1011) = 2, which is already occupied by the record with social security number 513-40-8716.

Searching downward from position 2, you find that position 3 is also occupied but position 4 is free.

$$\begin{array}{ccccccc} 908\text{-}37\text{-}1011 & \xrightarrow{Hash} & 2 & \rightarrow & 3 & \rightarrow & 4 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{occupied} & & \text{occupied} & & \text{free} \end{array}$$

Therefore, you place the record with social security number *n* into position 4.

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Onto Functions

We have noted that there may be an element of the co-domain of a function that is not the image of any element in the domain.

On the other hand, *every* element of a function's co-domain may be the image of some element of its domain. Such a function is called *onto* or *surjective*. When a function is onto, its range is equal to its co-domain.

• Definition

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

$$F: X \rightarrow Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

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Onto Functions

To obtain a precise statement of what it means for a function *not* to be onto, take the negation of the definition of onto:

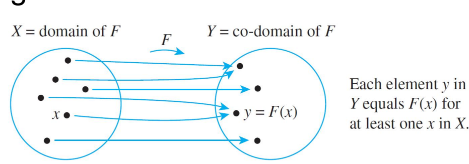
$$F: X \to Y \text{ is } \textit{not} \text{ onto} \quad \Leftrightarrow \quad \exists y \text{ in } Y \text{ such that } \forall x \in X, F(x) \neq y.$$

That is, there is some element in *Y* that is *not* the image of *any* element in *X*. In terms of arrow diagrams, a function is onto if each element of the co-domain has an arrow pointing to it from some element of the domain.

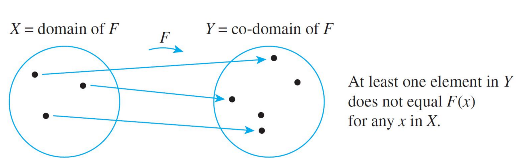
A function is not onto if at least one element in its co-domain does not have an arrow pointing to it.

Onto Functions

This is illustrated in Figure 7.2.3.



A Function That Is Onto
Figure 7.2.3 (a)



A Function That Is Not Onto
Figure 7.2.3 (b)

Example 4 – *Identifying Onto Functions Defined on Finite Sets*

a. Do either of the arrow diagrams in Figure 7.2.4 define onto functions?

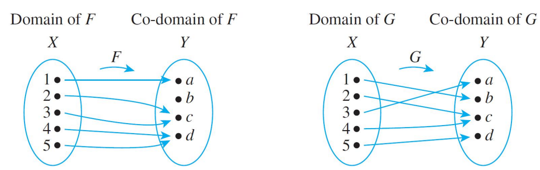


Figure 7.2.4

b. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$.
Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, $H(3) = c$, $H(4) = b$. Define K: X $\to Y$ as follows: $K(1) = c$, $K(2) = b$, $K(3) = b$, and $K(4) = c$. Is either H or K onto?

Example 4 – *Solution*

a. F is not onto because $b \neq F(x)$ for any x in X.

G is onto because each element of Y equals $G(x)$ for some x in X: $a = G(3)$, $b = G(1)$, $c = G(2) = G(4)$, and $d = G(5)$.

b. H is onto but K is not.

H is onto because each of the three elements of the co-domain of H is the image of some element of the domain of H: $a = H(2)$, $b = H(4)$, and $c = H(1) = H(3)$. K, however, is not onto because $a \neq K(x)$ for any x in $\{1, 2, 3, 4\}$.

Onto Functions on Infinite Sets

Now suppose *F* is a function from a set *X* to a set *Y*, and suppose *Y* is infinite. By definition, *F* is onto if, and only if, the following universal statement is true:

$$\forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Thus to prove *F* is onto, you will ordinarily use the method of generalizing from the generic particular:

suppose that *y* is any element of *Y*

and **show** that there is an element *X* of *X* with *F*(*x*) = *y*.

To prove *F* is *not* onto, you will usually

find an element *y* of *Y* such that *y* ≠ *F*(*x*) for *any x* in *X*.

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Example 5 – *Proving or Disproving That Functions Are Onto*

Define *f*: **R** → **R** and *h*: **Z** → **Z** by the rules

$$f(x) = 4x - 1 \quad \text{for all } x \in \mathbf{R}$$

And

$$h(n) = 4n - 1 \quad \text{for all } n \in \mathbf{Z}.$$

a. Is *f* onto? Prove or give a counterexample.

b. Is *h* onto? Prove or give a counterexample.

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Example 5 – *Solution*

a. The best approach is to start trying to prove that *f* is onto and be alert for difficulties that might indicate that it is not. Now *f*: **R** → **R** is the function defined by the rule

$$f(x) = 4x - 1 \quad \text{for all real numbers } x.$$

To prove that *f* is onto, you must prove

$$\forall y \in Y, \exists x \in X \text{ such that } f(x) = y.$$

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Example 5 – *Solution*

cont'd

Substituting the definition of *f* into the outline of a proof by the method of generalizing from the generic particular, you

suppose *y* is a real number

and **show** that there exists a real number *x* such that $y = 4x - 1$.

Scratch Work: If such a real number *x* exists, then

$$4x - 1 = y$$

$$4x = y + 1 \quad \text{by adding 1 to both sides}$$

$$x = \frac{y+1}{4} \quad \text{by dividing both sides by 4.}$$

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Example 5 – *Solution*

cont'd

Thus *if* such a number *x* exists, it must equal (y + 1)/4. Does such a number exist? Yes.

To show this, let $x = (y + 1)/4$, and then made sure that

(1) *x* is a real number and that

(2) *f* really does send *x* to *y*.

The following formal answer summarizes this process.

Answer to (a):

If $f: \mathbf{R} \to \mathbf{R}$ is the function defined by the rule $f(x) = 4x - 1$ for all real numbers x, then f is onto.

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Example 5 – *Solution*

cont'd

Proof:

Let $y \in \mathbf{R}$. *[We must show that* $\exists x$ *in* **R** *such that* $f(x) = y$*.]* Let $x = (y + 1)/4$.

Then *x* is a real number since sums and quotients (other than by 0) of real numbers are real numbers. It follows that

$$f(x) = f\left(\frac{y+1}{4}\right) \quad \text{by substitution}$$

$$= 4\cdot\left(\frac{y+1}{4}\right) - 1 \quad \text{by definition of } f$$

$$= (y+1) - 1 = y \quad \text{by basic algebra.}$$

[This is what was to be shown.]

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Example 5 – *Solution*

cont'd

b. The function h: **Z** → **Z** is defined by the rule

$$h(n) = 4n - 1 \quad \text{for all integers } n.$$

To prove that h is onto, it would be necessary to prove that

$$\forall \text{ integers } m, \exists \text{ an integer } n \text{ such that } h(n) = m.$$

Substituting the definition of h into the outline of a proof by the method of generalizing from the generic particular, you

suppose m is any integer

and **try to show** that there is an integer n with $4n - 1 = m$.

Example 5 – *Solution*

cont'd

Can you reach what is to be shown from the supposition? No! If $4n - 1 = m$, then

$$n = \frac{m+1}{4} \quad \text{by adding 1 and dividing by 4.}$$

But n must be an integer. And when, for example, $m = 0$, then

$$n = \frac{0+1}{4} = \frac{1}{4},$$

which is *not* an integer

Thus, in trying to prove that h is onto, you run into difficulty, and this difficulty reveals a counterexample that shows h is not onto.

Example 5 – *Solution*

cont'd

This discussion is summarized in the following formal answer.

Answer to (b):

If the function $h: \mathbf{Z} \to \mathbf{Z}$ is defined by the rule $h(n) = 4n - 1$ for all integers n, then h is not onto.

Counterexample:

The co-domain of *h* is **Z** and $0 \in \mathbf{Z}$. But $h(n) \neq 0$ for any integer *n*.

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Example 5 – *Solution*

cont'd

For if $h(n) = 0$, then

$$4n - 1 = 0 \quad \text{by definition of } h$$

which implies that

$$4n = 1 \quad \text{by adding 1 to both sides}$$

and so

$$n = \frac{1}{4} \quad \text{by dividing both sides by 4.}$$

But 1/4 is not an integer. Hence there is no integer *n* for which $f(n) = 0$, and thus *f* is not onto.

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Relations between Exponential and Logarithmic Functions

For positive numbers $b \neq 1$, the **exponential function with base *b***, denoted $\exp_b$, is the function from **R** to $\mathbf{R}^+$ defined as follows:

For all real numbers *x*,

$$\exp_b(x) = b^x$$

where $b^0 = 1$ and $b^{-x} = 1/b^x$.

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Relations between Exponential and Logarithmic Functions

When working with the exponential function, it is useful to recall the laws of exponents from elementary algebra.

Laws of Exponents

If b and c are any positive real numbers and u and v are any real numbers, the following laws of exponents hold true:

$$b^u b^v = b^{u+v} \quad 7.2.1$$

$$(b^u)^v = b^{uv} \quad 7.2.2$$

$$\frac{b^u}{b^v} = b^{u-v} \quad 7.2.3$$

$$(bc)^u = b^u c^u \quad 7.2.4$$

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Relations between Exponential and Logarithmic Functions

Equivalently, for each positive real number *x* and real number *y*,

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

It can be shown using calculus that both the exponential and logarithmic functions are one-to-one and onto.

Therefore, by definition of one-to-one, the following properties hold true:

For any positive real number b with $b \neq 1$,

$$\text{if } b^u = b^v \text{ then } u = v \quad \text{for all real numbers } u \text{ and } v, \tag{7.2.5}$$

and

$$\text{if } \log_b u = \log_b v \text{ then } u = v \quad \text{for all positive real numbers } u \text{ and } v. \tag{7.2.6}$$

Relations between Exponential and Logarithmic Functions

These properties are used to derive many additional facts about exponents and logarithms. In particular we have the following properties of logarithms.

Theorem 7.2.1 Properties of Logarithms

For any positive real numbers b, c and x with $b \neq 1$ and $c \neq 1$:

a. $\log_b(xy) = \log_b x + \log_b y$

b. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

c. $\log_b(x^a) = a \log_b x$

d. $\log_c x = \dfrac{\log_b x}{\log_b c}$

One-to-One Correspondences

Consider a function $F: X \to Y$ that is both one-to-one and onto. Given any element x in X, there is a unique corresponding element $y = F(x)$ in Y (since F is a function).

Also given any element y in Y, there is an element x in X such that $F(x) = y$ (since F is onto) and there is only one such x (since F is one-to-one).

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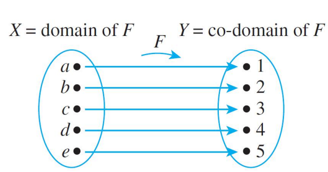
One-to-One Correspondences

Thus, a function that is one-to-one and onto sets up a pairing between the elements of X and the elements of Y that matches each element of X with exactly one element of Y and each element of Y with exactly one element of X.

Such a pairing is called a *one-to-one correspondence* or *bijection* and is illustrated by the arrow diagram in Figure 7.2.5.



An Arrow Diagram for a One-to-One Correspondence

Figure 7.2.5

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One-to-One Correspondences

One-to-one correspondences are often used as aids to counting. The pairing of Figure 7.2.5, for example, shows that there are five elements in the set *X*.

• Definition

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.

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Example 10 – *A Function of Two Variables*

Define a function

F: **R** × **R** → **R** × **R** as follows: For all $(x, y) \in$ **R** × **R**,

$$F(x, y) = (x + y, x - y).$$

Is *F* a one-to-one correspondence from **R** × **R** to itself?

Solution:

The answer is yes. To show that *F* is a one-to-one correspondence, you need to show both that *F* is one-to-one and that *F* is onto.

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Example 10 – *Solution*

cont'd

Proof that *F* is one-to-one:
Suppose that (x_1, y_1) and (x_2, y_2) are any ordered pairs in **R** × **R** such that

$$F(x_1, y_1) = F(x_2, y_2).$$

[We must show that $(x_1, y_1) = (x_2, y_2)$.*]* By definition of *F*,

$$(x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2).$$

For two ordered pairs to be equal, both the first and second components must be equal. Thus x_1, y_1, x_2, and y_2 satisfy the following system of equations:

$$x_1 + y_1 = x_2 + y_2 \qquad (1)$$

$$x_1 - y_1 = x_2 - y_2 \qquad (2)$$

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Example 10 – *Solution*

cont'd

Adding equations (1) and (2) gives that

$$2x_1 = 2x_2, \quad \text{and so} \quad x_1 = x_2.$$

Substituting $x_1 = x_2$ into equation (1) yields

$$x_1 + y_1 = x_1 + y_2, \quad \text{and so} \quad y_1 = y_2.$$

Thus, by definition of equality of ordered pairs, $(x_1, y_1) = (x_2, y_2)$. *[as was to be shown].*

Scratch Work for the Proof that *F* is onto: To prove that *F* is onto, you suppose you have any ordered pair in the co-domain **R** × **R**, say (u, v), and then you show that there is an ordered pair in the domain that is sent to (u, v) by *F*.

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Example 10 – *Solution*

cont'd

To do this, you suppose temporarily that you have found such an ordered pair, say (*r*, *s*). Then

$$F(r, s) = (u, v)$$ because you are supposing that F sends (r, s) to (u, v),

and

$$F(r, s) = (r + s, r - s)$$ by definition of F.

Equating the right-hand sides gives

$$(r + s, r - s) = (u, v).$$

By definition of equality of ordered pairs this means that

$$r + s = u \quad (1)$$

$$r - s = v \quad (2)$$

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Example 10 – *Solution*

cont'd

Adding equations (1) and (2) gives

$$2r = u + v, \quad \text{and so} \quad r = \frac{u+v}{2}.$$

Subtracting equation (2) from equation (1) yields

$$2s = u - v, \quad \text{and so} \quad s = \frac{u-v}{2}.$$

Thus, if *F* sends (*r*, *s*) to (*u*, *v*), then $r = (u + v)/2$ and $s = (u - v)/2$.

To turn this scratch work into a proof, you need to make sure that

(1) $\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$ is in the domain of *F*, and

(2) that *F* really does send $\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$ to (*u*, *v*).

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Example 10 – *Solution*

cont'd

Proof that *F* is onto:
Suppose (u, v) is any ordered pair in the co-domain of F.
[We will show that there is an ordered pair in the domain of F that is sent to (u, v) by F.]

Let $r = \frac{u+v}{2}$ and $s = \frac{u-v}{2}$.

Then (r, s) is an ordered pair of real numbers and so is in the domain of F. In addition:

$$F(r,s) = F\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \quad \text{by definition of } F$$

$$= \left(\frac{u+v}{2} + \frac{u-v}{2}, \frac{u+v}{2} - \frac{u-v}{2}\right) \quad \text{by substitution}$$

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Example 10 – *Solution*

cont'd

$$= \left(\frac{u+v+u-v}{2}, \frac{u+v-u+v}{2}\right)$$

$$= \left(\frac{2u}{2}, \frac{2v}{2}\right)$$

$$= (u, v) \quad \text{by algebra.}$$

[This is what was to be shown.]

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Inverse Functions

If F is a one-to-one correspondence from a set X to a set Y, then there is a function from Y to X that "undoes" the action of F; that is, it sends each element of Y back to the element of X that it came from. This function is called the *inverse function* for F.

Theorem 7.2.2

Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $\boldsymbol{F^{-1}: Y \to X}$ that is defined as follows:

Given any element y in Y,

$$\boldsymbol{F^{-1}(y)} = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

In other words,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

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Inverse Functions

The proof of Theorem 7.2.2 follows immediately from the definition of one-to-one and onto.

Given an element y in Y, there is an element x in X with $F(x) = y$ because F is onto; x is unique because F is one-to-one.

• Definition

The function F^{-1} of Theorem 7.2.2 is called the **inverse function** for F.

Note that according to this definition, the logarithmic function with base $b > 0$ is the inverse of the exponential function with base b.

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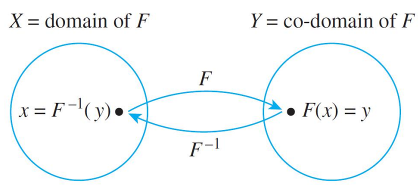
Inverse Functions

The diagram that follows illustrates the fact that an inverse function sends each element back to where it came from.



Example 13 – *Finding an Inverse Function for a Function Given by a Formula*

The function f: **R** → **R** defined by the formula

$$f(x) = 4x - 1 \quad \text{for all real numbers } x$$

was shown to be one-to-one in Example 2 and onto in Example 5. Find its inverse function.

Solution:

For any *[particular but arbitrarily chosen]* y in **R**, by definition of f^{-1},

$f^{-1}(y)$ = that unique real number x such that $f(x) = y$.

Example 13 – *Solution*

cont'd

But

$$
\begin{aligned}
& f(x) = y \\
\Leftrightarrow \quad & 4x - 1 = y && \text{by definition of } f \\
\Leftrightarrow \quad & x = \frac{y+1}{4} && \text{by algebra.}
\end{aligned}
$$

Hence

$$f^{-1}(y) = \frac{y+1}{4}.$$

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Inverse Functions

The following theorem follows easily from the definitions.

Theorem 7.2.3

If X and Y are sets and $F: X \to Y$ is one-to-one and onto, then $F^{-1}: Y \to X$ is also one-to-one and onto.

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Example 14 – *Finding an Inverse Function for a Function of Two Variables*

Define the inverse function F^{-1} : **R** × **R** → **R** × **R** for the one-to-one correspondence given in Example 10.

Solution:

The solution to Example 10 shows that

$$F\left(\frac{u+v}{2}, \frac{u-v}{2}\right) = (u, v).$$

Because F is one-to-one, this means that $\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$ is the unique ordered pair in the domain of F that is sent to (u, v) by F.

Thus, F^{-1} is defined as follows: For all $(u, v) \in$ **R** × **R**,

$$F^{-1}(u, v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right).$$

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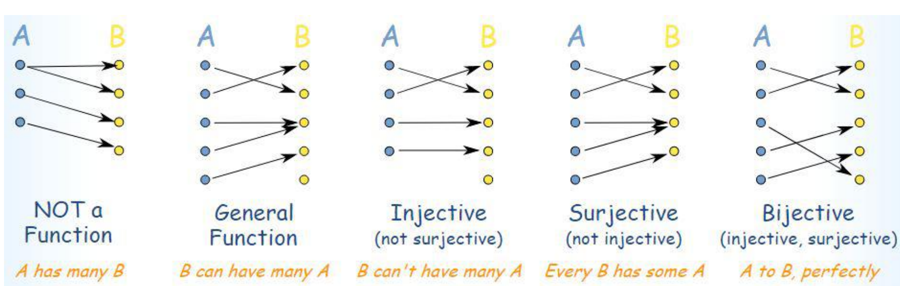

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