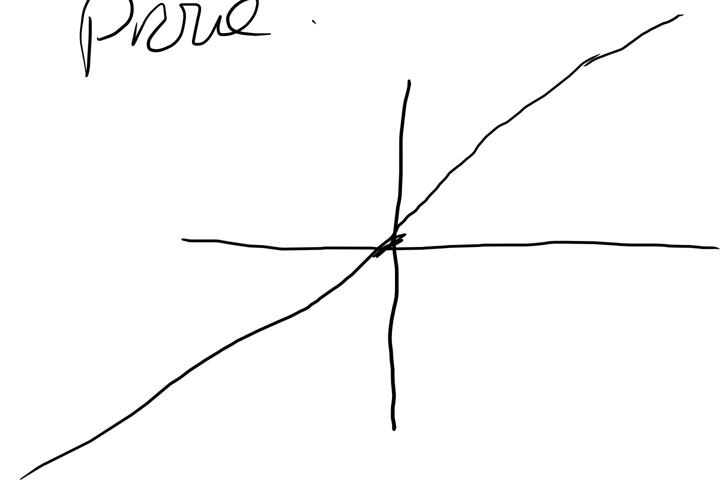
Properties of Relations on finite Sets

Suppose R is defined on finite set A. To prove
Relation is $\{$Reflexive, Symmetric, Transitive$\}$

$\Rightarrow$ First, write down what to prove.



Ex: $x = y$ set R

1) **Reflexive:** $\forall x \in A,\ xRx$.

$\forall x \in R,\ xRx$ is Reflexive

clear: $xRx \Rightarrow x = x$ True

Symmetric: $\forall x,y \in \mathbb{R}$ if xRy then yRx.

$xRy \iff x = y$, $y = x \Rightarrow yRx$

$\therefore$ Symmetric

(3) Transitive: $\forall x,y,z \in \mathbb{R}$ if xRy and yRz, then xRz

$\Rightarrow$ from Question $xRy \iff x = y$

$yRz \iff y = z$

$x = y$, $y = z$ $\therefore x = z$

xRz

Transitive

Ex. $mTn \iff 3 \mid (m-n)$ m, n integer $\in \mathbb{Z}$

(1) Reflexive: $\forall_n \in \mathbb{Z}$, nTn

$\iff$ $nTn \implies 3 \mid (n-n)$, $3 \mid 0$

$\therefore$ $3 \mid 3 \cdot 0$ $\therefore$ T is reflexive

(2) Symmetric: $\forall_{m,n} \in \mathbb{Z}$ if mTn then nTm

start $mTn \iff 3 \mid (m-n)$, we show

END $\to nT(m) \iff 3 \mid (n-m)$

$*$ — $m - n = 3k$ k is integer

$-(m-n) = -3k \implies n - m = \underbrace{3(-k)}_{\text{some integer } d}$

$n - m = 3d$

$\therefore$ nTm So it's Symmetric

Transitive: $\forall m,n,p \in \mathbb{Z}$ if mTn and nTp

then mTp.

$mTn \iff 3 \mid (m-n)$

$nTp \iff 3 \mid (n-p)$

$\boxed{\text{Show that } mTp \iff 3 \mid (m-p)}$

$m - \cancel{n} = 3k$ k is integer

$+ \quad \cancel{n} - p = 3d$ d is integer

$m - p = 3(k+d)$ Since integer pluse integer is integer

Suppose $t = k+d$ t is integer

$m - p = 3t$

$\frac{m-p}{3} = t \implies$ mRp $\#$ transitive

Ex: Assume that R, S are relations on a set A. Prove that if R and S is symmetric then $R \cap S$ is symmetric?

Symmetric: $\{ \forall x, y \in A$ if $(x,y) \in R \cap S$ then $(y,x) \in R \cap S \}$

from intersection $(x,y) \in R$ and $(x,y) \in S$

Case $(x,y) \in R \Rightarrow$ Since R is symmetric $(y,x) \in R$

Case $(x,y) \in S \Rightarrow$ Since S is symmetric so, $(y,x) \in S$

$\therefore (y,x) \in R \cap S$ from intersection

Transitive Closure of a Relation:

Ex: $(1,3), (3,4)$ in a relation R then pair $(1,4)$ must be in R, R is transitive

$\Rightarrow$ Roughly is transitive closure of R.

Def: Let A be a set and R is a relation on A. the transitive closure of R is the relation R^t on A satisfies the following

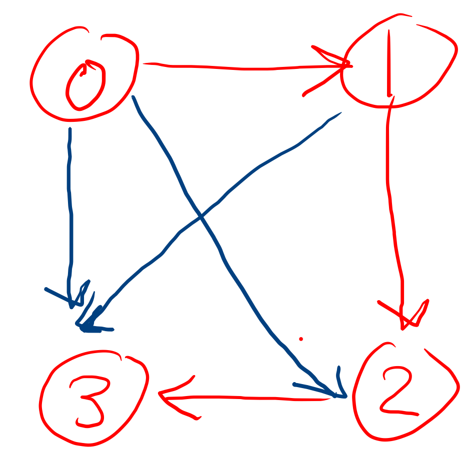
1. R^t is transitive
2. $R \subseteq R^t$
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$

Ex: $A = \{0, 1, 2, 3\}$, Consider R on A as follows $R = \{(0,1), (1,2), (2,3)\}$

Find the transitive Closure of R. Find R^t?

$R = \{(\overset{x}{0}, \overset{y}{1}), (\overset{y}{1}, \overset{z}{2}), (2,3)\}$.

$R^t = R \cup \{(0,2), (1,3), (0,3)\}$



Transitive ✓

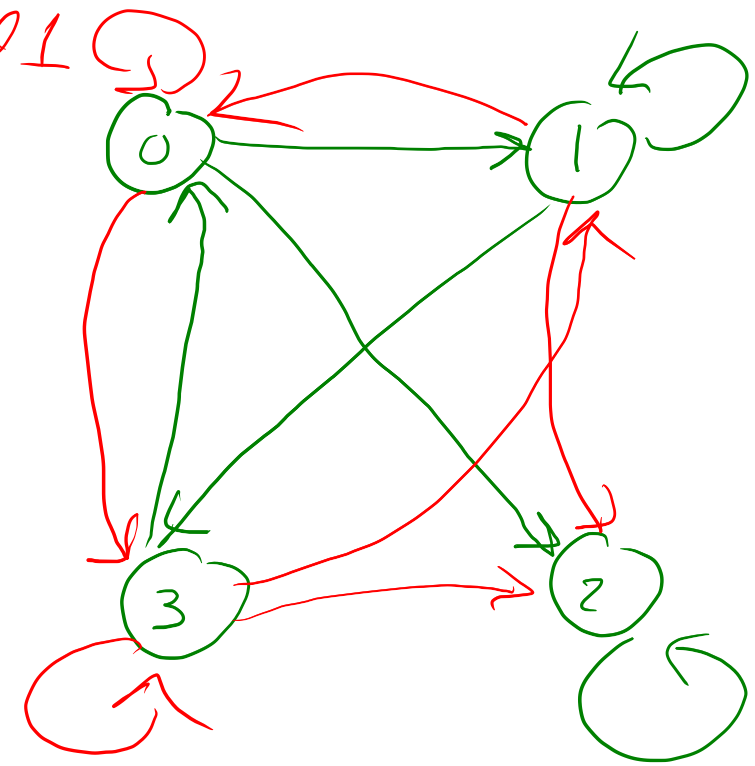
$A = \{0, 1, 2, 3\} \Rightarrow R = \{(0,1), (0,2), (1,1), (1,3),$
$(2,2), (3,0)\}$

Find R^t ?

Round 1

Round 2

$R \cup \{(0,3), (1,0), (3,1), (3,2),$
$(0,0), (1,2), (3,3),\}$



Transitive

$A = \{0, 1, 2, 3\}$

$R = \{(0,1), (0,2), (1,1), (1,3), (2,2), (3,0)\}$

Find R^t ?

Round 1

$R^t = R \cup \{(0,3), (1,0), (3,1), (3,2),$

Round 3 $(0,0), (1,2), (3,3)\}$