Equivlance Classes:

Def: Suppose that A is a set on Relation R is an equivlance relation on A. For each element a in A. the equivlance class of a, denoted by [a] and called the class of a.

$$[a] = \{ x \in A \mid xRa \}$$

$\forall x,\ x \in [a] \iff xRa$

$$\left\{ \begin{array}{l} \text{Reflexive} \\ \text{Symmetric} \\ \text{Transitive} \end{array} \right\} \Rightarrow \text{Relation is Equivlance}$$

$A = \{0, 1, 2, 3, 4\}$ and define a relation R on set A as follows:

$$R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}$$

Find distinct equivalance class of R ?

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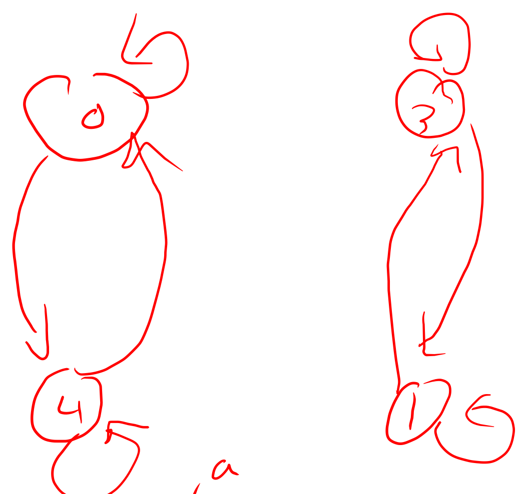


$[2] = \{x \in A \mid x R 2\}$
$= \{2\}$

$[3] = \{x \in A \mid x R 3\}$
$= \{1, 3\}$

$[4] = \{x \in A \mid x R 4\}$
$= \{0, 4\}$

a a

$[0] = \{x \in A \mid x R 0\} = \{0, 4\}$

$[1] = \{x \in A \mid x R 1\} = \{1, 3\}$

distinct classes

✓ $[0] = [4]$
✓ $[1] = [3]$
✓ $[2]$

Lemma: (1) equivalance (two elements in A) same class

(2) $[a] \cap [b] = \phi$

Congruence Modulo n

$\in \mathbb{Z}$ $\left\{\begin{matrix} R \\ T \\ S \end{matrix}\right\} \Rightarrow$ equivalence relation

$$\boxed{mRn \iff 3 \mid (m-n) \iff m \equiv n \pmod 3}$$

$$mRn \iff 3 \mid (m-n) \Rightarrow \text{eq. Relation}$$

Find distinct classes?

$m-n=3k$

$\boxed{m=3k+n} \Rightarrow \boxed{x=3k+a}$ ch. 4

$\Leftrightarrow$

$$[a] = \{x \in \mathbb{Z} \mid xRa\}$$ definition

$$[0] = \{x \in \mathbb{Z} \mid xR0\}$$

$$[a] = \{x \in \mathbb{Z} \mid x = 3k + a\}$$

integer

$$[0] = \{x \in \mathbb{Z} \mid x = 3k + 0\} = \{x \in \mathbb{Z} \mid x = 3k\}$$

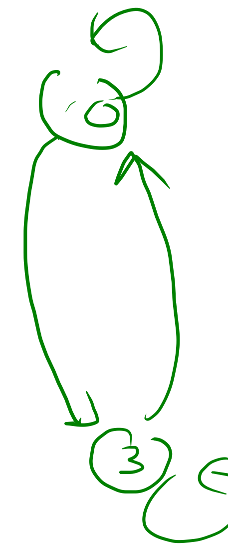
$$= \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$$

$$[1] = \{x \in \mathbb{Z} \mid x R 1\}$$
$$= \{x \in \mathbb{Z} \mid x = 3k+1\}$$
$$= \{\ldots, -5, -2, 1, 4, 7, \ldots\}$$

$0 R 3, \; 3 R 0$

$\downarrow$

$3 \mid (m-n)$



$$[2] = \{x \in \mathbb{Z} \mid x R 2\}$$
$$= \{x \in \mathbb{Z} \mid x = 3k+2\}$$
$$= \{\ldots, -7, -4, -1, 2, 5, 8, \ldots\}$$

Using Lemma 1:

$$\begin{cases} [0] = [3] = [6] = [-6] = [-3] \\ [1] = [4] = [-2] = [7] = [-5] \\ [2] = [5] = [8] = [-1], \ldots \end{cases}$$

$$\boxed{n = 3q + r \quad 0 \le r < d}$$ ← ch.4

$0 \le r < 3$

$[0], [1], [2]$

$$\boxed{m R n \iff 5 \mid m - n}$$ → ch. 4

$\Leftrightarrow \quad n = 5q + r \quad 0 \le r < 5$

$m - n = 5k$

$$\boxed{m = 5k + n}$$

$\swarrow \quad [0] \; [1], [2] \; [3] \; [4]$

Def: $m \equiv n \pmod{d} \iff d \mid (m-n)$

Ex: $12 \equiv 7 \bmod 5$

$5 \mid 12-7 \implies 5 \mid 5$ ✓

Ex: $6 \equiv -8 \bmod 4$ (x)

$4 \mid 6 - -8 \implies 4 \nmid 14$

Rational Numbers:

Let $A = \mathbb{Z} \times (\mathbb{Z} - \{0\})$ Define a Relation

R on set A as follows: $\forall\ (a,b), (c,d) \in A$

$$(a,b)\, R\, (c,d) \iff ad = bc \qquad b \neq 0,\ d \neq 0$$

Prove that R is equivalance Relation.

Reflexive: ✓ (Homework)

Symmetric: ✓ ≤

Transitive

Transitive: R is transitive $\forall (a,b),(c,d),(e,f) \in A$

if $(a,b) R (c,d)$ and $(c,d) R (e,f)$

then $(a,b) R (e,f)$?

$\Leftrightarrow$ $(a,b) R (c,d) \Longleftrightarrow [ad = bc] \times f$ — 1

$(c,d) R (e,f) \Longleftrightarrow [cf = de] \times b$ — 2

Show $[(a,b) R (e,f)]$

$adf = bcf$

$cfb = deb$

$d \neq 0$

$= \quad a\cancel{d}f = \cancel{d}eb$

$af = eb \implies (a,b) R (e,f)$ #

Classes

definition

$ad = bc$

$$[a] = \{ x \in A \mid x R a \}$$

$$[(a,b)] = \{ (x,y) \in A \mid (x,y) R (a,b) \}$$

$$[(1,2)] = \{ (x,y) \in A \mid (x,y) R (1,2) \}$$

Find elements in class

$$\Rightarrow \quad 2x = y \quad , \quad \boxed{y = 2x}$$

$$= \{ (1,2), (2,4), (3,6), (4,8) \ldots \ldots$$
$$(-1,-2), (-2,-4), \ldots \}$$

$(a,b)\, R\, (c,d) \iff a+d=c+b$

1) Prove that R is equivalence? $\{R, T, S\}$

2) Find distinct classes.

Homework } Next Exam

Ex: $m D n \iff 3 \mid m^2 - n^2$

1) Prove that D relation is equivalance

2) Find equivalance classes.

Reflexive: (1) D is reflexive. $m D m$

$\Leftrightarrow \quad 3 \mid m^2 - m^2 \Rightarrow 3 \mid 0$ ✓

$\therefore$ reflexive

(2) Symmetric: D is symmetric. if $m D n$ then $n D m$.

$\Leftrightarrow \quad m D n \iff 3 \mid m^2 - n^2$

$-1(m^2 - n^2 = 3k) \Rightarrow n^2 - m^2 = -3k$

$= 3(-k)$

$n^2 - m^2 = 3\alpha$

$\therefore 3 \mid n^2 - m^2$ ✓ Symmetric

Transitive: D is transitive mDn and nDp then mDp

$\hookrightarrow$ $mDn \iff 3 \mid m^2 - n^2$.

$nDp \iff 3 \mid n^2 - p^2$

① $m^2 - \cancel{n^2} = 3d$

② + $\cancel{n^2} - p^2 = 3k$ ✓ t

$m^2 - p^2 = 3(d+k)$

$m^2 - p^2 = 3t \implies 3 \mid m^2 - p^2$

$\therefore$ D is Transitive.