BIRZEIT UNIVERSITY

Discrete Mathematic and Application Comp233

CHAPTER 8

Relations

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Relations on Sets

A more formal way to refer to the kind of relation defined on sets is to call it a **binary relation** because it is a subset of a **Cartesian product of two sets**.

At the end of this section we define an ***n-ary relation*** to be a subset of a Cartesian product of *n* sets, where *n* is any ***integer greater than or equal to two***.

Such a relation is the fundamental structure used in ***relational databases***. However, because we focus on binary relations in this text, when we use the term *relation* by itself, we will **mean binary relation**.

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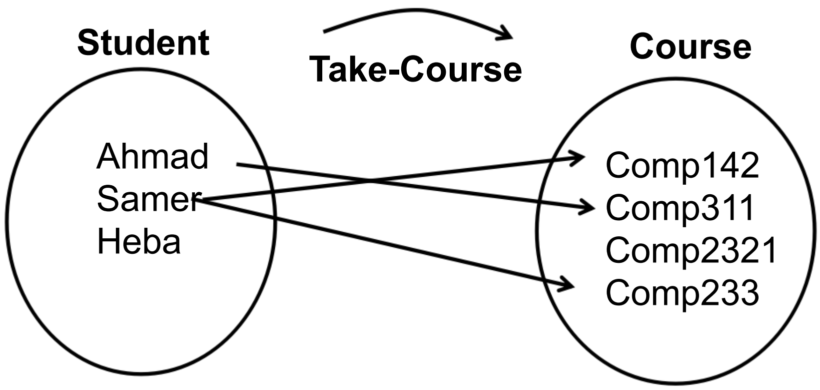
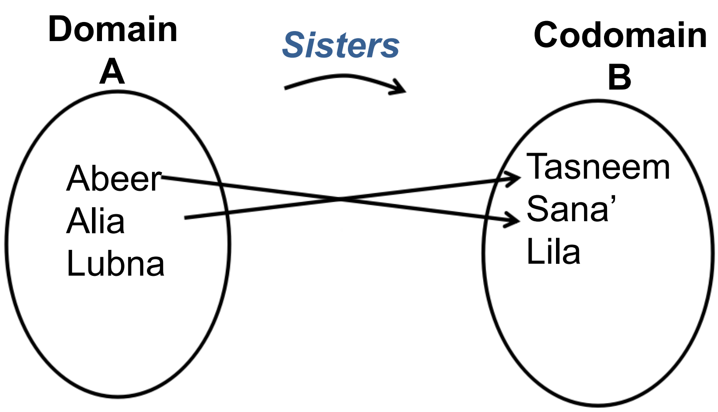
Example

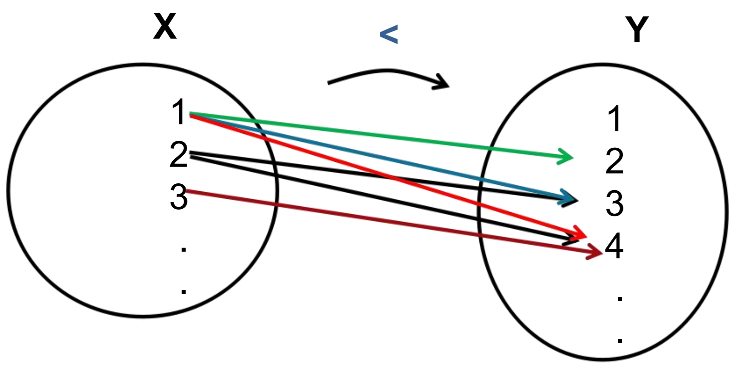


Example



$A \textit{ Sister } B = \{(\text{Alia}, \text{Tasneem}), (\text{Abeer}, \text{Sana})\}$

Example



Z+ → Z+

What is a Relation?

Definition

Let A and B be sets. A **(binary) relation R from A to B** is a subset of $A \times B$. Given an ordered pair (x,y) in $A \times B$, **<u>x is related to y</u>** by R, written $x\ R\ y$, if and only if, (x, y) is in R.

$$x\ R\ y \qquad (x,y) \in R$$

Example: a relation on a Power Set

Let $X = \{a, b, c\}$.
Then $P(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.
Define a relation **S** from $P(X)$ to **Z** as follows: For all sets A and B in $P(X)$ (for all subsets A and B of X),

A **S** $B \Leftrightarrow$ ***A* has at least as many elements as *B*.**

Is $\{a,b\}$ S $\{b,c\}$? Yes, both sets have two elements.

Is $\{a\}$ S $\emptyset$? Yes, $\{a\}$ has one element and $\emptyset$ has zero elements, and $1 \geq 0$.

Is $\{b,c\}$ S $\{a,b,c\}$? No, $\{b, c\}$ has two elements and $\{a, b, c\}$ has three elements and $2 < 3$

Is $\{c\}$ S $\{a\}$? Yes, both sets have one element

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Example – *The Congruence Modulo 2 Relation*

Define a relation E from **Z** to **Z** as follows: For all $(m, n) \in$ **Z** $\times$ **Z**, $m \; E \; n \Leftrightarrow m - n$ is even.

a. Is $4 \; E \; 0$? Is $2 \; E \; 6$? Is $3 \; E \; (-3)$? Is $5 \; E \; 2$?
b. List five integers that are related by E to 1.
c. Prove that if n is any odd integer, then $n \; E \; 1$.

Solution:
a. Yes, $4 \; E \; 0$ because $4 - 0 = 4$ and 4 is even.

Yes, $2 \; E \; 6$ because $2 - 6 = -4$ and -4 is even.

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Example 2 – *Solution* cont'd

Yes, 3 E (–3) because 3 – (–3) = 6 and 6 is even.

No, 5 $\not E$ 2 because 5 – 2 = 3 and 3 is not even.

b. There are many such lists. One is

1 because 1 – 1 = 0 is even,

3 because 3 – 1 = 2 is even,

5 because 5 – 1 = 4 is even,

–1 because –1 – 1 = –2 is even,

–3 because –3 – 1 = –4 is even.

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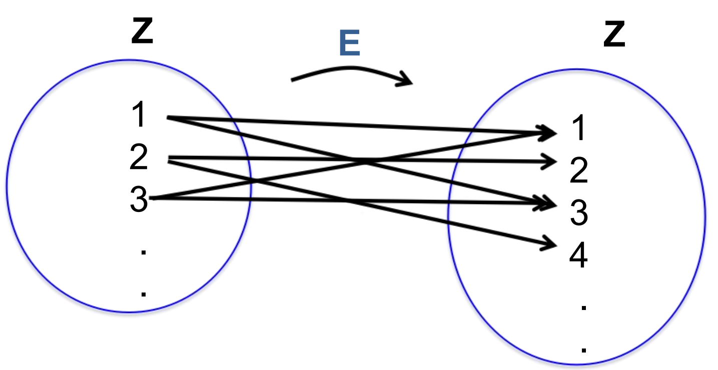
Example



Define a relation E from **Z** to **Z** as follows:

For all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$, $m\ E\ n \Leftrightarrow m - n$ is even.

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Example – *Solution*

cont'd

c. Proof:

Suppose n is any odd integer.

Then $n = 2k + 1$ for some integer k. Now by definition of E, $n\ E\ 1$ if, and only if, $n - 1$ is even.

But by substitution,

$$n - 1 = (2k + 1) - 1 = 2k,$$

and since k is an integer, **$2k$ is even.**

Hence $n\ E\ 1$ *[as was to be shown]*.

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Example – *Solution*

cont'd

It can be shown that integers m and n are related by E if, and only if, $m \bmod 2 = n \bmod 2$ (that is, both are even or both are odd).

When this occurs m and n are said to be **congruent تتطابق modulo 2**.

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The Inverse of a Relation

If R is a relation from A to B, then a relation R^{-1} from B to A can be defined by interchanging the elements of all the ordered pairs of R.

• Definition

Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

This definition can be written operationally as follows:

$$\boxed{\text{For all } x \in A \text{ and } y \in B, \quad (y, x) \in R^{-1} \quad \Leftrightarrow \quad (x, y) \in R.}$$

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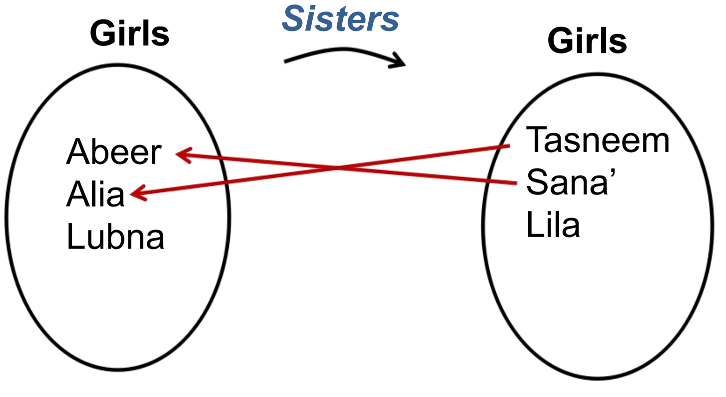
Example



$Sister^{-1}$ = {(Tasneem, Alia), (Sana, Abeer)}

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Example – *The Inverse of a Finite Relation*

Let A = {2, 3, 4} and B = {2, 6, 8} and let R be the "divides" relation from A to B :
For all $(x, y) \in A \times B$,

$$x \; R \; y \quad \Leftrightarrow \quad x \mid y \qquad\qquad x \text{ divides } y.$$

a. State explicitly which ordered pairs are in R and R^{-1}, and draw arrow diagrams for R and R^{-1}.

b. Describe R^{-1} in words.

Solution:

a. R = {(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)}

R^{-1} = {(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)}

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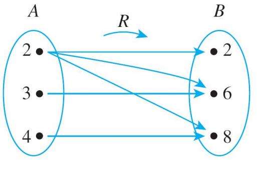
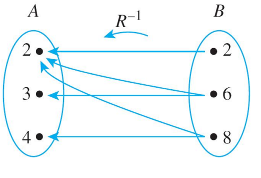
Example 4 – *Solution*

cont'd



To draw the arrow diagram for R^{-1}, you can copy the arrow diagram for R but reverse the directions of the arrows.



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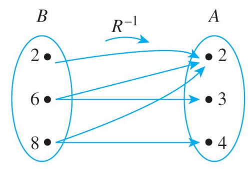
Example 4 – *Solution*

cont'd

Or you can redraw the diagram so that B is on the left.



b. R^{-1} can be described in words as follows:
For all $(y, x) \in B \times A$,

$y\ R^{-1}\ x$ y is a multiple of x.

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Directed Graph of a Relation

• Definition

A **relation on a set** A is a relation from A to A.

When a relation R is defined *on* a set A, the arrow diagram of the relation can be modified so that it becomes a **directed graph**.

Instead of representing A as two separate sets of points, represent A only once, and draw an arrow from each point of A to each related point.

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Directed Graph of a Relation

As with an ordinary arrow diagram,

For all points x and y in A,

there is an arrow from x to y $\Leftrightarrow$ $x\ R\ y$ $\Leftrightarrow$ $(x, y) \in R$.

If a point is related to itself, a loop is drawn that extends out from the point and goes back to it.

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Example 6 – *Directed Graph of a Relation*

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows: For all $x, y \in A$,

$$x\ R\ y \qquad 2 \mid (x - y).$$

Draw the directed graph of R.

Solution:

Note that $3\ R\ 3$ because $3 - 3 = 0$ and $2 \mid 0$ since $0 = 2 \cdot 0$. Thus there is a loop from 3 to itself.

Similarly, there is a loop from 4 to itself, from 5 to itself, and so forth, since the difference of each integer with itself is 0, and $2 \mid 0$.

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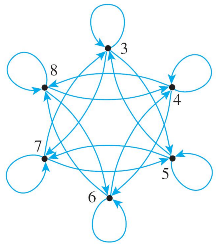
Example 6 – *Solution*

cont'd

Note also that 3 *R* 5 because 3 – 5 = –2 = 2 • (–1). And 5 *R* 3 because 5 – 3 = 2 = 2 • 1.

Hence there is an arrow from 3 to 5 and also an arrow from 5 to 3.

The other arrows in the directed graph, as shown below, are obtained by similar reasoning.



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N-ary Relations and Relational Databases

N-ary relations form the mathematical foundation for relational database theory.

A binary relation is a subset of a Cartesian product of two sets, similarly, an *n-ary* relation is a subset of a Cartesian product of *n* sets.

• **Definition**

Given sets $A_1, A_2, \ldots, A_n$, an ***n*-ary relation** R on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$. The special cases of 2-ary, 3-ary, and 4-ary relations are called **binary, ternary,** and **quaternary relations,** respectively.

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Example 7 – *A Simple Database*

The following is a radically simplified version of a database that might be used in a hospital.

Let A_1 be a set of positive integers, A_2 a set of alphabetic character strings, A_3 a set of numeric character strings, and A_4 a set of alphabetic character strings.

Define a quaternary relation R on $A_1 \times A_2 \times A_3 \times A_4$ as follows:

$(a_1, a_2, a_3, a_4) \in R$ a patient with patient ID number a_1, named a_2, was admitted on date a_3, with primary diagnosis a_4.

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Example 7 – *A Simple Database* cont'd

At a particular hospital, this relation might contain the following 4-tuples:

(011985, Jamal Ali, 020710, asthma)

(574329, Tala Ibraheem, 011410, pneumonia)

(466581, Mai Ahmad, 010310, appendicitis)

(008352, Samer Zaid, 112409, gastritis)

(011985, Jamal Ali, 021710, pneumonia)

(244388, Sarah Walid, 010310, broken leg)

(778400, Jamal Baker, 122709, appendicitis)

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Example 7 – *A Simple Database* cont'd

In discussions of relational databases, the tuples are normally thought of as being written in tables.

Each row of the table corresponds to one tuple, and the header for each column gives the descriptive attribute for the elements in the column.

Operations within a database allow the data to be manipulated in many different ways.

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Example 7 – *A Simple Database* cont'd

At a particular hospital, this relation might contain the following 4-tuples:

(011985, Jamal Ali, 020710, asthma)

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(244388, Sarah Walid, 010310, broken leg)

(778400, Jamal Baker, 122709, appendicitis)

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Example 7 – *A Simple Database* cont'd

For example, in the database language SQL, if the above database is denoted *S*, the result of the query

SELECT Patient_ID#, Name FROM *S* WHERE
Admission_Date = **010310**

would be a list of the ID numbers and names of all patients admitted on 01-03-10:

466581 Mai Ahmad,
244388 Sarah Walid.

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Example 7 – *A Simple Database* cont'd

This is obtained by taking the **intersection** of the set $A_1 \times A_2 \times \{010310\} \times A_4$ with the database and then **projecting** onto the first two coordinates.

Similarly, SELECT can be used to obtain a list of all admission dates of a given patient.

For Jamal Ali this list is

02-07-10 and

02-17-10

SELECT Admission_Date FROM *S*
WHERE Name = "Jamal Ali"

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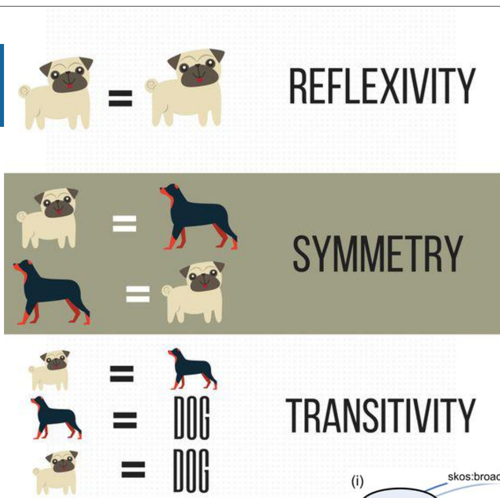
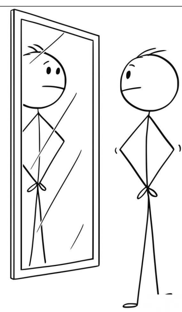
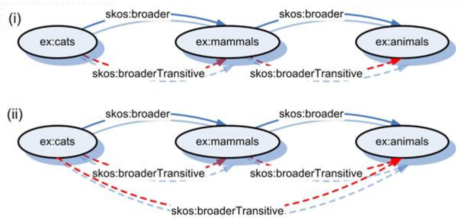
SECTION 8.2

Reflexivity, Symmetry, and Transitivity







Reflexivity, Symmetry, and Transitivity

Let $A = \{2, 3, 4, 6, 7, 9\}$ and define a relation R on A as follows: For all $x, y \in A$,

$$x\ R\ y \quad \Leftrightarrow \quad 3 \mid (x - y).$$

Then $2\ R\ 2$ because $2 – 2 = 0$, and $3 \mid 0$.

Similarly, $3\ R\ 3$, $4\ R\ 4$, $6\ R\ 6$, $7\ R\ 7$, and $9\ R\ 9$.

Also $6\ R\ 3$ because $6 – 3 = 3$, and $3 \mid 3$.

And $3\ R\ 6$ because $3 – 6 = –(6 – 3) = –3$, and $3 \mid (–3)$.

Similarly, $3\ R\ 9$, $9\ R\ 3$, $6\ R\ 9$, $9\ R\ 6$, $4\ R\ 7$, and $7\ R\ 4$.

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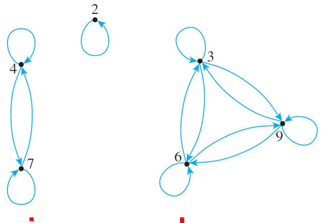
Reflexivity, Symmetry, and Transitivity

Thus the directed graph for R has the appearance shown at the right.



This graph has three important properties:

1. **Each point of the graph has an arrow looping around from it back to itself. (Reflexivity)**

2. In each case where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first. (**Symmetry**)

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Reflexivity, Symmetry, and Transitivity

3. In each case where there is an arrow going **from one point to a second and from the second point to a third, there is an arrow going from the first point to the third**. **That is**, there are no "**incomplete directed triangles**" in the graph. (**Transitivity)**

Properties (1), (2), and (3) correspond to properties of general relations called *reflexivity*, *symmetry*, and *transitivity*.

• Definition

Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for all $x \in A$, $x\ R\ x$.
2. R is **symmetric** if, and only if, for all $x, y \in A$, ***if*** $x\ R\ y$ then $y\ R\ x$.
3. R is **transitive** if, and only if, for all $x, y, z \in A$, ***if*** $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$.

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Reflexivity, Symmetry, and Transitivity

Because of the equivalence of the expressions $x\ R\ y$ and $(x, y) \in R$ for all x and y in A, the reflexive, symmetric, and transitive properties can also be written as follows:

1. R is reflexive	$\Leftrightarrow$	for all x in A, $(x, x) \in R$.
2. R is symmetric	$\Leftrightarrow$	for all x and y in A, ***if*** $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive	$\Leftrightarrow$	for all x, y and z in A, ***if*** $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

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Reflexivity, Symmetry, and Transitivity

In informal terms, properties (1)–(3) say the following:

1. **Reflexive:** Each element is related to itself.

2. **Symmetric:** If any one element is related to any other element, then the second element is related to the first.

3. **Transitive:** If any one element is related to a second and that second element is related to a third, then the first element is related to the third.

   Note that the definitions of reflexivity, symmetry, and transitivity are **universal statements**.

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Reflexivity, Symmetry, and Transitivity

This means that to prove a relation has one of the properties, you **use either the method of exhaustion or the method of generalizing from the generic particular**.

Now consider what it means for a relation *not* to have one of the properties defined previously. **We have known that the negation of a universal statement is existential.**

Hence if R is a relation on a set A, then

1. R is **not reflexive** there is an element x in A such that $x \not R x$ [*that is, such that* $(x, x) \notin R$].

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Reflexivity, Symmetry, and Transitivity

2. R is **not symmetric** ⇔ there are elements x and y in A such that $x \; R \; y$ but $y \not R \; x$ [that is, such that $(x, y) \in R$ but $(y, x) \notin R$].

3. R is **not transitive** ⇔ there are elements x, y and z in A such that $x \; R \; y$ and $y \; R \; z$ but $x \not R \; z$ [that is, such that $(x, y) \in R$ and $(y, z) \in R$ but $(x, z) \notin R$].

It follows that you can show that a relation does *not* have one of the **properties by finding a counterexample**.

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Example 1 – *Properties of Relations on Finite Sets*

Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$,
$S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$,
$T = \{(0, 1), (2, 3)\}$.

a. Is R reflexive? symmetric? transitive?

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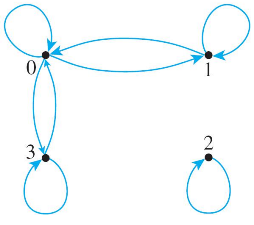
Example 1(a) – *Solution*

The directed graph of *R* has the appearance shown below.



R is reflexive: There is a loop at each point of the directed graph. This means that each element of *A* is related to itself, so *R* is reflexive.

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Example 1(a) – *Solution*

cont'd

R is symmetric: In each case where there is an arrow going **from one point** of the graph **to a second**, there is an arrow going **from the second point back to the first**.

This means that whenever one element of *A* is related by *R* to a second, then the second is related to the first. Hence *R* is symmetric.

R is not transitive: There is an arrow going from 1 to 0 and an arrow going from 0 to 3, but there is no arrow going from 1 to 3.

This means that there are elements of *A*—0, 1, and 3—such that $1\ R\ 0$ and $0\ R\ 3$ but $1 \not R\ 3$. Hence *R* is not transitive.

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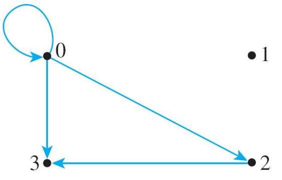
Example 1(b) – *Solution*

cont'd

b. Is *S* reflexive? symmetric? transitive?

The directed graph of *S* has the appearance shown below.



***S* is not reflexive**: There is no loop at 1, for example. Thus $(1, 1) \notin S$, and so *S* is not reflexive.

S is not symmetric: There is an arrow from 0 to 2 but not from 2 to 0. Hence $(0, 2) \in S$ but $(2, 0) \notin S$, and so *S* is not symmetric.

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Example 1(b) – *Solution*

cont'd

***S* is transitive**: There are three cases for which there is an arrow going from one point of the graph to a second and from the second point to a third.

Namely, there are arrows going from 0 to 2 and from 2 to 3; there are arrows going from 0 to 0 and from 0 to 2; and there are arrows going from 0 to 0 and from 0 to 3.

In each case there is an arrow going from the first point to the third. (Note again that the "first," "second," and "third" points need not be distinct.)

This means that whenever $(x, y) \in S$ and $(y, z) \in S$, then $(x, z) \in S$, for all $x, y, z \in \{0, 1, 2, 3\}$, and so *S* is transitive.

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Example 1(c) – *Solution*

cont'd

c. Is T reflexive? symmetric? transitive?

The directed graph of T has the appearance shown at right.

***T* is not reflexive:** There is no loop at 0, for example. Thus (0, 0) T, so $\notin$ is not reflexive.

***T* is not symmetric:** There is an arrow from 0 to 1 but not from 1 to 0. Thus $(0, 1) \in T$ but $(1, 0) \notin T$, and so T is not symmetric.

***T* is transitive:** The transitivity condition is vacuously true for T. To see this, observe that the transitivity condition says that

For all $x, y, z \in A$, if $(x, y) \in T$ and $(y, z) \in T$ then $(x, z) \in T$.

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Example 1(c) – *Solution*

cont'd

The only way for this **to be false** would be for there to exist elements of A that make the hypothesis true and the conclusion false.

That is, there would have to be elements x, y, and z in A such that $(x, y) \in T$ and $(y, z) \in T$ and $(x, z) \notin T$.

In other words, there would have to be two ordered pairs in T that have the potential to "link up" by having the *second* element of one pair be the *first* element of the other pair.

But the **only elements in *T* are (0, 1) and (2, 3), and these do not have the potential to link up**. **Hence the hypothesis is never true**. It follows that it is **impossible for T not to be transitive, and thus T is transitive.**

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Properties of Relations on Infinite Sets

Suppose a relation R is defined on an infinite set A. To **prove** the relation is reflexive, symmetric, or transitive, first write down what is to be proved. For **instance**, for **symmetry** you need to prove that

$$\forall\ x, y \in A, \text{ if } x\ R\ y \text{ then } y\ R\ x.$$

Then use the definitions of A and R to rewrite the statement for the particular case in question. For instance, for the **"equality"** relation on the set of real numbers, the rewritten statement is

$$\forall\ x, y \in R, \text{ if } x = y \text{ then } y = x.$$

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Properties of Relations on Infinite Sets

Sometimes the truth of the rewritten statement will be **immediately obvious** (as it is here).

At other times you will need to prove it using the method of **generalizing from the generic particular**.

We begin with the relation of equality, one of the simplest and yet most important relations.

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Example 2 – *Properties of Equality*

Define a relation *R* on **R** (the set of all real numbers) as follows: For all real numbers *x* and *y*.

$$\boxed{x\ R\ y \quad \Leftrightarrow \quad x = y.}$$

a. Is *R* reflexive?

b. Is *R* symmetric?

c. Is *R* transitive?

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Example 2(a) – *Solution*

***R* is reflexive:** *R* is reflexive if, and only if, the following statement is true:

For all $x \in$ **R**, $x\ R\ x$.

Since $x\ R\ x$ just means that $x = x$, this is the same as saying

For all $x \in$ **R**, $x = x$.

But this statement is **certainly true**; every real number is equal to itself.

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Example 2(b) – *Solution*

cont'd

***R* is symmetric:** *R* is symmetric if, and only if, the following statement is true:

For all $x, y \in$ **R**, if $x\ R\ y$ then $y\ R\ x$.

By definition of *R*, $x\ R\ y$ means that $x = y$ and $y\ R\ x$ means that $y = x$. Hence *R* is symmetric if, and only if,

For all $x, y \in$ **R**, if $x = y$ then $y = x$.

But this statement is certainly true; if one number is equal to a second, then the second is equal to the first.

Example 2(c) – *Solution*

cont'd

***R* is transitive:** *R* is transitive if, and only if, the following statement is true:

For all $x, y, z \in$ **R**, if $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$.

By definition of *R*, $x\ R\ y$ means that $x = y$, $y\ R\ z$ means that $y = z$, and $x\ R\ z$ means that $x = z$. Hence *R* is transitive if, and only if, the following statement is true:

For all $x, y, z \in$ **R**, if $x = y$ and $y = z$ then $x = z$.

But this statement is **certainly true**: If one real number equals a second and the second equals a third, then the first equals the third.

Example 4 – *Properties of Congruence Modulo 3*

Define a relation T on **Z** (the set of all integers) as follows: For all integers m and n,

$$m\ T\ n \quad \Leftrightarrow \quad 3 \mid (m - n).$$

This relation is called **congruence modulo 3**.

a. Is T reflexive?

b. Is T symmetric?

c. Is T transitive?

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Example 4(a) – *Solution*

***T* is reflexive:** To show that T is reflexive, it is necessary to show that

For all $m \in$ **Z**, $m\ T\ m$.

By definition of T, this means that

For all $m \in$ **Z**, $3 \mid (m - m)$.

Or, since $m - m = 0$, For all $m \in$ **Z**, $3 \mid 0$.

But this is true: $3 \mid 0$ since $0 = 3 \cdot 0$. Hence T is reflexive. This reasoning is formalized in the following proof.

Proof of Reflexivity: Suppose m is a particular but arbitrarily chosen integer. *[We must show that $m\ T\ m$.]* Now $m - m = 0$. But $3 \mid 0$ since $0 = 3 \cdot 0$. Hence $3 \mid (m - m)$. Thus, by definition of T, $m\ T\ m$ *[as was to be shown]*.

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Example 4(b) – *Solution*

cont'd

***T* is symmetric:** To show that T is symmetric, it is necessary to show that

For all $m, n \in$ **Z**, if $m\ T\ n$ then $n\ T\ m$.

By definition of T this means that

For all $m, n \in$ **Z**, if $3 \mid (m - n)$ then $3 \mid (n - m)$.

Is this true? Suppose m and n are particular but arbitrarily chosen integers such that $3 \mid (m - n)$.

Must it follow that $3 \mid (n - m)$? *[In other words, can we find an integer so that $n - m = 3 \cdot$ (that integer)?]*

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Example 4(b) – *Solution*

cont'd

By definition of "divides," since

$$3 \mid (m - n),$$

then $\quad m - n = 3k \quad$ for some integer k.

The crucial observation is that $n - m = -(m - n)$. Hence, you can multiply both sides of this equation by -1 to obtain

$$-(m - n) = -3k,$$

which is equivalent to $\quad n - m = 3(-k)$.

[Thus we have found an integer, namely $-k$, so that $n - m = 3 \cdot$ (that integer).]

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Example 4(b) – *Solution*

cont'd

Since $-k$ is an integer, this equation shows that

$$3 \mid (n - m).$$

It follows that T is symmetric.

The reasoning is formalized in the following proof.

Proof of Symmetry: Suppose m and n are particular but arbitrarily chosen integers that satisfy the condition $m\ T\ n$. *[We must show that $n\ T\ m$.]* By definition of T, since $m\ T\ n$ then $3 \mid (m - n)$. By definition of "divides," this means that $m - n = 3k$, for some integer k. Multiplying both sides by -1 gives $n - m = 3(-k)$. Since $-k$ is an integer, this equation shows that $3 \mid (n - m)$. Hence, by definition of T, $n\ T\ m$ *[as was to be shown]*.

Example 4(c) – *Solution*

cont'd

***T* is transitive:** To show that T is transitive, it is necessary to show that

For all $m, n, p \in$ **Z**, if $m\ T\ n$ and $n\ T\ p$ then $m\ T\ p$.

By definition of T this means that

For all $m, n, p \in$ **Z**,

if $3 \mid (m - n)$ and $3 \mid (n - p)$ then $3 \mid (m - p)$.

Is this true? Suppose m, n, and p are particular but arbitrarily chosen integers such that $3 \mid (m - n)$ and $3 \mid (n - p)$.

Must it follow that $3 \mid (m - p)$? *[In other words, can we find an integer so that $m - p = 3 \cdot$ (that integer)?]*

Example 4(c) – *Solution*

cont'd

By definition of "divides," since

$3 \mid (m - n)$ and $3 \mid (n - p)$,

then $m - n = 3r$ for some integer r,

and $n - p = 3s$ for some integer s.

The crucial observation is that $(m - n) + (n - p) = m - p$.

Add these two equations together to obtain

$$(m - n) + (n - p) = 3r + 3s,$$

which is equivalent to $m - p = 3(r + s)$.
[Thus we have found an integer so that $m - p = 3 \cdot$ *(that integer).]*

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Example 4(c) – *Solution*

cont'd

Since r and s are integers, $r + s$ is an integer. So this equation shows that

$$3 \mid (m - p).$$

It follows that T is transitive.

The reasoning is formalized in the following proof.

Proof of Transitivity: Suppose m, n, and p are particular but arbitrarily chosen integers that satisfy the condition $m \; T \; n$ and $n \; T \; p$. *[We must show that* $m \; T \; p$.*]* By definition of T, since $m \; T \; n$ and $n \; T \; p$, then $3 \mid (m - n)$ and $3 \mid (n - p)$. By definition of "divides," this means that $m - n = 3r$ and $n - p = 3s$, for some integers r and s. Adding the two equations gives $(m - n) + (n - p) = 3r + 3s$, and simplifying gives that $m - p = 3(r + s)$. Since $r + s$ is an integer, this equation shows that $3 \mid (m - p)$. Hence, by definition of T, $m \; T \; p$ *[as was to be shown]*.

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Exercise

- Assume R and S are relations on set A. Prove or disprove that:

if R and S are symmetric then $R \cap S$ is symmetric ?

Yes. To prove this we must show that for all x and y in A, if $(x, y) \in R \cap S$ then $(y, x) \in R \cap S$. So suppose (x, y) is a particular but arbitrarily chosen element in $R \cap S$. *[We must show that* $(y, x) \in R \cap S$.*]*

By definition of intersection, $(x, y) \in R$ and $(x, y) \in S$. In case $(x, y) \in R$, then $(y, x) \in R$ because R is symmetric, and hence $(y, x) \in R \cap S$

by definition of intersection. In case $(x, y) \in S$ then $(y, x) \in S$ because S is symmetric, and hence $(y, x) \in R \cap S$ by definition of union. Thus, in both cases, $(y, x) \in R \cap S$ *[as was to be shown]*.

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The Transitive Closure of a Relation

Generally speaking, a relation fails to be transitive because it fails to contain certain ordered pairs.

For example, if (1, 3) and (3, 4) are in a relation R, then the pair (1, 4) ***must* be in *R* if *R* is to be transitive**.

To obtain a transitive relation from one that is not transitive, it is necessary to add ordered pairs.

Roughly speaking, the relation obtained by adding the least number of ordered pairs to ensure transitivity is called the ***transitive closure* of the relation.**

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The Transitive Closure of a Relation

In a sense made precise by the formal definition, the transitive closure of a relation is the smallest transitive relation that contains the relation.

• Definition

Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the following three properties:

1. R^t is transitive.
2. $R \subseteq R^t$.
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$.

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Example 5 – *Transitive Closure of a Relation*

Let $A = \{0, 1, 2, 3\}$ and consider the relation R defined on A as follows:

$$R = \{(0, 1), (1, 2), (2, 3)\}.$$

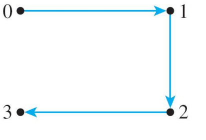
Find the transitive closure of R.

Solution:

Every ordered pair in R is in R^t, so

$$\{(0, 1), (1, 2), (2, 3)\} \subseteq R^t.$$

Thus the directed graph of R contains the arrows shown at the right.



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Example 5 – *Solution*

cont'd

Since there are arrows going from 0 to 1 and from 1 to 2, R^t must have an arrow going from 0 to 2.

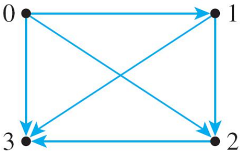
Hence $(0, 2) \in R^t$. Then $(0, 2) \in R^t$ and $(2, 3) \in R^t$, so since R^t is transitive, $(0, 3) \in R^t$.

Also, since $(1, 2) \in R^t$ and $(2, 3) \in R^t$, then $(1, 3) \in R^t$.

Thus R^t contains at least the following ordered pairs:

$$\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}.$$

But this relation is transitive; hence it equals R^t. Note that the directed graph of R^t is as shown at the right.



A={0,1,2,3}

Let $R = (0, 1), (0, 2), (1, 1), (1, 3), (2, 2), (3, 0)$.
Find R^t, the transitive closure of R.

$R^t = R \cup \{(0, 0), (0, 3), (1, 0), (3, 1), (3, 2), (3, 3), (1, 2)\}$

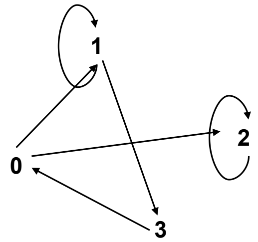
$= \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3), (2, 2), (3, 0), (3, 1), (3, 2), (3, 3)\}$

Exercise : Page 459/51

Let $A = \{0, 1, 2, 3\}$ and consider the relation R defined on A as follows:

$$R = \{(0, 1), (0, 2), (1, 1),(1,3),(2,2), (3,0)\}.$$

Find the transitive closure of *R*.



$R^t = \{ (1,0) , (0,3), (2,1),(2,3), (3,2),(3,1),(1,2), (0,0),(3,3)$
$(0, 1), (0, 2), (1, 1),(1,3),(2,2), (3,0)\}.$

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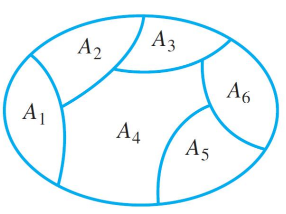
8.3The Relation Induced by a Partition

A **partition** of a set A is a finite or infinite collection of nonempty, **mutually disjoint subsets** whose union is A. The diagram of Figure 8.3.1 illustrates a partition of a set A by subsets $A_1, A_2, \ldots, A_6$.



$A_i \cap A_j = \emptyset$, whenever $i \neq j$
$A_i \cup A_2 \cup \cdots \cup A_6 = A$

A Partition of a Set

Figure 8.3.1

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The Relation Induced by a Partition

• Definition

Given a partition of a set A, the **relation induced by the partition,** R, is defined on A as follows: For all $x, y \in A$,

$$x \; R \; y \quad \Leftrightarrow \quad \text{there is a subset } A_i \text{ of the partition such that both } x \text{ and } y \text{ are in } A_i.$$

Example 1 – *Relation Induced by a Partition*

Let A = {0, 1, 2, 3, 4} and consider the **following partition of** A:

{0, 3, 4}, {1}, {2}.

Find the relation R induced by this partition.

Solution:

Since {0, 3, 4} is a subset of the partition,

0 R 3 because both 0 and 3 are in {0, 3, 4},

3 R 0 because both 3 and 0 are in {0, 3, 4},

Example 1 – *Solution* cont'd

0 R 4 because both 0 and 4 are in {0, 3, 4},

4 R 0 because both 4 and 0 are in {0, 3, 4},

3 R 4 because both 3 and 4 are in {0, 3, 4}, and

4 R 3 because both 4 and 3 are in {0, 3, 4}.

Also, *0 R 0* because both 0 and 0 are in {0, 3, 4}

3 R 3 because both 3 and 3 are in {0, 3, 4}, and

4 R 4 because both 4 and 4 are in {0, 3, 4}.

Example 1 – *Solution* cont'd

Since {1} is a subset of the partition,

1 R 1 because both 1 and 1 are in {1},

and since {2} is a subset of the partition,

2 R 2 because both 2 and 2 are in {2}.

Hence

R = {(0, 0), (0, 3), (0, 4), (1, 1), (2, 2), (3, 0), (3, 3), (3, 4), (4, 0), (4, 3), (4, 4)}.

The Relation Induced by a Partition

The fact is that ***a relation induced by a partition of a set satisfies all three properties***: reflexivity, symmetry, and transitivity.

Theorem 8.3.1

Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

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Definition of an Equivalence Relation

A relation on a set that satisfies the three properties of reflexivity, symmetry, and transitivity is called an ***equivalence relation***.

• Definition

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

Thus, according to Theorem 8.3.1, the relation induced by a partition is an **equivalence relation.**

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Example 2 – *An Equivalence Relation on a Set of Subsets*

Let X be the set of all **nonempty** subsets of {1, 2, 3}. Then

X = {{1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

Define a relation **R** on X as follows: For all A and B in X,

A **R** B the least element of A equals the least element of B.

Prove that **R** is an equivalence relation on X.

Example 2 – *Solution*

R is reflexive: Suppose A is a nonempty subset of {1, 2, 3}. *[We must show that A* **R** *A.]*

It is true to say that the least element of A equals the least element of A. Thus, by definition of R, A **R** A.

R is symmetric: Suppose A and B are nonempty subsets of {1, 2, 3} and A **R** B. *[We must show that B* **R** *A.]*

Since A **R** B, the least element of A equals the least element of B.

But this implies that the least element of B equals the least element of A, and so, by definition of **R**, B **R** A.

Example 2 – *Solution*

cont'd

R is transitive: Suppose A, B, and C are nonempty subsets of {1, 2, 3}, A **R** B, and B **R** C. *[We must show that A **R** C.]*

Since A **R** B, the least element of A equals the least element of B and since B **R** C, the least element of B equals the least element of C.

Thus the least element of A equals the least element of C, and so, by definition of **R**, A **R** C.

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Equivalence Classes of an Equivalence Relation

Suppose there is an equivalence relation on a certain set.
If a is any particular element of the set, then one can ask, "What is the subset of all elements that are related to a?" This subset is called the **equivalence class of a.**

• Definition

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of a**, denoted **[a]** and called the **class of a** for short, is the set of all elements x in A such that x is related to a by R.

In symbols:

$$[a] = \{x \in A \mid x\ R\ a\}$$

$$\text{for all } x \in A, \quad x \in [a] \quad \Leftrightarrow \quad x\ R\ a.$$

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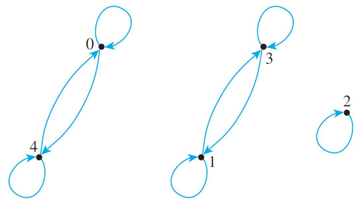
Example 5 – *Equivalence Classes of a Relation Given as a set of Ordered Pairs*

Let A = {0, 1, 2, 3, 4} and define a relation R on A as follows:

R = {(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)}.

The directed graph for R is as shown below. As can be seen by inspection, R is an equivalence relation on A. **Find the distinct equivalence classes of *R*.**



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Example 5 – *Solution*

First find the equivalence class of every element of A.

$$[0] = \{x \in A \mid x\ R\ 0\} = \{0, 4\}$$

$$[1] = \{x \in A \mid x\ R\ 1\} = \{1, 3\}$$

$$[2] = \{x \in A \mid x\ R\ 2\} = \{2\}$$

$$[3] = \{x \in A \mid x\ R\ 3\} = \{1, 3\}$$

$$[4] = \{x \in A \mid x\ R\ 4\} = \{0, 4\}$$

Note that [0] = [4] and [1] = [3]. Thus the *distinct* equivalence classes of the relation are

{0, 4}, {1, 3}, and {2}.

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Equivalence Classes of an Equivalence Relation

The first lemma says that if two elements of A are related by an equivalence relation R, then their equivalence classes are the same.

> **Lemma 8.3.2**
>
> Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If $a\ R\ b$, then $[a] = [b]$.

This lemma says that if a certain condition is satisfied, then $[a] = [b]$. Now $[a]$ and $[b]$ are *sets*, and two sets are equal if, and only if, each is a subset of the other.

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Equivalence Classes of an Equivalence Relation

Hence the proof of the lemma consists of two parts: first, a proof that $[a] \subseteq [b]$ and second, a proof that $[b] \subseteq [a]$.

To show each subset relation, it is necessary to show that every element in the left-hand set is an element of the
right-hand set.

The second lemma says that any **two equivalence classes** of an equivalence relation are either **mutually disjoint or identical**.

> **Lemma 8.3.3**
>
> If A is a set, R is an equivalence relation on A, and a and b are elements of A, then
>
> $$\text{either} \quad [a] \cap [b] = \emptyset \quad \text{or} \quad [a] = [b].$$

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Equivalence Classes of an Equivalence Relation

The statement of Lemma 8.3.3 has the form

if p then (q or r),

where p is the statement "A is a set, R is an equivalence relation on A, and a and b are elements of A," q is the statement "$[a] \cap [b] = \emptyset$," and r is the statement "$[a] = [b]$."

Theorem 8.3.4 The Partition Induced by an Equivalence Relation

If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A, and the intersection of any two distinct classes is empty.

Congruence Modulo *n*

Example 10 – *Equivalence Classes of Congruence Modulo3*

Let *R* be the relation of congruence modulo 3 on the set**Z** of all integers. That is, for all integers *m* and *n*,

$$\boxed{m\ R\ n \quad \Leftrightarrow \quad 3 \mid (m-n) \quad \Leftrightarrow \quad m \equiv n \pmod 3).}$$

Describe the distinct equivalence classes of*R*.

Solution:
For each integer *a*,

$$[a] = \{x \in \mathbf{Z} \mid x\ R\ a\}$$
$$= \{x \in \mathbf{Z} \mid 3 \mid (x-a)\}$$

Example 10 – *Solution* cont'd

$$= \{x \in \mathbf{Z} \mid x - a = 3k, \text{ for some integer } k\}.$$

Therefore,

$$[a] = \{x \in \mathbf{Z} \mid x = 3k + a, \text{ for some integer } k\}.$$

In particular,

$$[0] = \{x \in \mathbf{Z} \mid x = 3k + 0, \text{ for some integer } k\}$$
$$= \{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\}$$
$$= \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\},$$
$$[1] = \{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\}$$
$$= \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\},$$

Example 10 – *Solution*

cont'd

$$[2] = \{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}$$
$$= \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}.$$

Now since 3 *R* 0, then by Lemma 8.3.2,

$$[3] = [0].$$

More generally, by the same reasoning,

$$\boxed{[0] = [3] = [-3] = [6] = [-6] = \ldots, \text{ and so on.}}$$

Similarly,

$$\boxed{[1] = [4] = [-2] = [7] = [-5] = \ldots, \text{ and so on.}}$$

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Example 10 – *Solution*

cont'd

And

$$\boxed{[2] = [5] = [-1] = [8] = [-4] = \ldots, \text{ and so on.}}$$

Notice that every integer is in class [0], [1], or [2]. Hence the **distinct** equivalence classes are

$$\boxed{\{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\},}$$

$$\boxed{\{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\}, \quad \text{and}}$$

$$\boxed{\{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}.}$$

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Example 10 – *Solution*

cont'd

In words, **the three classes of congruence modulo 3 are**

(1) the set of all integers that are divisible by 3,

(2) the set of all integers **that leave a remainder of 1** when divided by 3, and

(3) the set of all integers that **leave a remainder of 2** when divided by 3.

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Congruence Modulo *n*

• **Definition**

Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is any element a such that $[a] = S$.

• **Definition**

Let m and n be integers and let d be a positive integer. We say that ***m* is congruent to *n* modulo *d*** and write

$$\boldsymbol{m \equiv n \ (\text{mod } d)}$$

if, and only if, $d \mid (m - n)$.

Symbolically: $m \equiv n \ (\text{mod } d) \quad \Leftrightarrow \quad d \mid (m - n)$

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Example 11 – *Evaluating Congruences*

Determine which of the following congruences are true and which are false.

a. $12 \equiv 7 \pmod 5$ **b.** $6 \equiv -8 \pmod 4$ **c.** $3 \equiv 3 \pmod 7$

Solution:

a. True. 12 – 7 = 5 = 5 · 1. Hence 5 | (12 – 7), and so 12 ≡ 7 (mod 5).

b. False. 6 – (–8) = 14, and $4 \nmid 14$ because 14 ≠ 4 · *k* for any integer *k*. Consequently, $6 \not\equiv -8 \pmod 4$.

c. True. 3 – 3 = 0 = 7 · 0. Hence 7 | (3 – 3), and so 3 ≡ 3 (mod 7).

A Definition for Rational Numbers

Example 12 – *Rational Numbers Are Really Equivalence Classes*

Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero. Symbolically,

$$A = \mathbf{Z} \times (\mathbf{Z} - \{0\}).$$

Define a relation R on A as follows: For all $(a, b), (c, d) \in A$,

$$\boxed{(a, b)\ R\ (c, d) \quad \Leftrightarrow \quad ad = bc.}$$

The fact is that R is an equivalence relation.

a. Prove that R is equivalance .

b. Describe the distinct equivalence classes of *R*.

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Example 12(a) – *Solution*

Suppose (a, b), (c, d), and (e, f) are particular but arbitrarily chosen elements of A such that $(a, b)\ R\ (c, d)$ and
$(c, d)\ R\ (e, f)$.

[We must show that for all $(a, b), (c, d), (e, f) \in A$,
if $(a, b)\ R\ (c, d)$ *and* $(c, d)\ R\ (e, f)$, *then* $(a, b)\ R\ (e, f)$.*]*

[We must show that $(a, b)\ R\ (e, f)$.*]* By definition of R,

$$(1)\ \ ad = bc \quad \text{and} \quad (2)\ \ cf = de.$$

Since the second elements of all ordered pairs in A are nonzero, $b \neq 0$, $d \neq 0$, and $f \neq 0$.

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Example 12(a) – *Solution*

cont'd

Multiply both sides of equation (1) by f and both sides of equation (2) by b to obtain

$$(1')\ \ adf = bcf \quad \text{and} \quad (2')\ \ bcf = bde.$$

Thus

$$adf = bde$$

and, since $d \neq 0$, it follows from the cancellation law for multiplication that

$$af = be.$$

It follows, by definition of *R*, that (*a*, *b*) *R* (*e*, *f*) *[as was to be shown].*

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Example 12(b) – *Solution*

cont'd

There is one equivalence class for each distinct rational number.

Each equivalence class consists of all ordered pairs (*a*, *b*) that, if written as fractions *a*/*b*, would equal each other.

The reason for this is that the condition for two rational numbers to be equal is the same as the condition for two ordered pairs to be related.

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Example 12(b) – *Solution*

cont'd

For instance, the class of (1, 2) is

$$[(1, 2)] = \{(1, 2), (-1, -2), (2, 4), (-2, -4), (3, 6), (-3, -6), \ldots\}$$

since $\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6}$ and so forth.

Exercises:

44. Let $A = \mathbf{Z}^+ \times \mathbf{Z}^+$. Define a relation R on A as follows: For every (a, b) and (c, d) in A,

$$(a, b)\ R\ (c, d) \quad \Leftrightarrow \quad a + d = c + b.$$

a. Prove that R is reflexive.
b. Prove that R is symmetric.
H **c.** Prove that R is transitive.
d. List five elements in $[(1, 1)]$.
e. List five elements in $[(3, 1)]$.
f. List five elements in $[(1, 2)]$.
g. Describe the distinct equivalence classes of R.

Solution

DEFINITIONS

A relation R on a set A is **reflexive** if $(a, a) \in R$ for every element $a \in A$.
A relation R on a set A is **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$
A relation R on a set A is **transitive** if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$

A relation R is an **equivalence relation** if the relation R is transitive, symmetric and reflexive.

The **equivalence class** of a is the set of all elements that are relation to a.
Notation: $[a]_R$

$$[a] = \{x \in A | x\ R\ a\}$$

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SOLUTION

$$A = \mathbf{Z} \times \mathbf{Z}$$
$$R = \{((a, b), (c, d)) \in A \times A | a + d = b + c\}$$

(a) To proof: R is reflexive

PROOF

Let $(a, b) \in A = \mathbf{Z} \times \mathbf{Z}$.

By the commutative property of addition:

$$a + b = b + a$$

However, this then implies that $(a, b)\ R\ (a, b)$ by definition of R.

Since $(a, b)\ R\ (a, b)$ for all $(a, b) \in A$, R is reflexive.
□

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(b) To proof: R is symmetric

PROOF

Let $((a, b), (c, d)) \in R$.

By the definition of R:

$$a + d = b + c$$

By the commutative property of addition:

$$d + a = c + b$$

Interchanging the left and right side of the equality does not change the equality:

$$c + b = d + a$$

By definition of R:

$$((c, d), (a, b)) \in R$$

Since $((a, b), (c, d)) \in R$ implies $((c, d), (a, b)) \in R$, R is symmetric.
□

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(c) To proof: R is transitive

PROOF

Let $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$.

By the definition of R:

$$a + d = b + c$$
$$c + f = d + e$$

Let us solve the first equation to a and the second equation to f:

$$a = b + c - d$$
$$f = d + e - c$$

Let us add the previous two equations:

$$a + f = (b + c - d) + (d + e - c) = b + e$$

By definition of R:

$$((a, b), (e, f)) \in R$$

Since $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$ implies $((a, b), (e, f)) \in R$, R is transitive.
□

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(d) Elements in $[(1,1)]$ are elements $(x,y) \in A$ that are related to $(1,1)$.

By the definition of R:

$$(a,b) \in [(1,1)] \Leftrightarrow a+1 = 1+b \Leftrightarrow a = b$$

Thus all elements of the form (a,a) with $a \in \mathbf{Z}^+$ are elements in $[(1,1)]$.

For example, five possible elements of $[(1,1)]$ are then $(1,1),(2,2),(3,3),(4,4),(5,5),\ldots$

(e) Elements in $[(3,1)]$ are elements $(x,y) \in A$ that are related to $(3,1)$.

By the definition of R:

$$(a,b) \in [(3,1)] \Leftrightarrow a+1 = 3+b \Leftrightarrow a = b+2$$

Thus all elements of the form $(b+2,b)$ with $b \in \mathbf{Z}^+$ are elements in $[(3,1)]$.

For example, five possible elements of $[(3,1)]$ are then $(3,1),(4,2),(5,3),(6,4),(7,5),\ldots$

(f) Elements in $[(1,2)]$ are elements $(x,y) \in A$ that are related to $(1,2)$.

By the definition of R:

$$(a,b) \in [(1,2)] \Leftrightarrow a+2 = 1+b \Leftrightarrow a+1 = b$$

Thus all elements of the form $(a,a+1)$ with $a \in \mathbf{Z}^+$ are elements in $[(1,2)]$.

For example, five possible elements of $[(1,2)]$ are then $(1,2),(2,3),(3,4),(4,5),(5,6),\ldots$

(g) Let $(a,b) \in A$ and let $n = a-b$

If $n = 0$, then the equivalence class is $[(1,1)] = \{(x,x) \in A \times A\}$ (see part (d)).

If $n < 0$, then the equivalence class is $[(n+1,1)] = \{(x+n,x) \in A \times A\}$ (see part (e)).

If $n > 0$, then the equivalence class is $[(1,1+n)] = \{(x,x+n) \in A \times A\}$ (see part (f)).

(a) Proof that R is reflexive by using the commutative property of addition and the definition of R
(b) Proof that R is symmetric by using the definition of R and the commutative property of addition
(c) Proof that R is transitive by using the definition of R
(d) Answers could vary. For example: $(1,1),(2,2),(3,3),(4,4),(5,5)$
(e) Answers could vary. For example: $(3,1),(4,2),(5,3),(6,4),(7,5)$
(f) Answers could vary. For example: $(1,2),(2,3),(3,4),(4,5),(5,6)$
(g) The different equivalence classes are $[(1,1)] = \{(x,x) \in A \times A\}$, $[(1+n,1)] = \{(x+n,x) \in A \times A\}$ with $n \in \mathbf{Z}^+$ and $[(1,1+n)] = \{(x,x+n) \in A \times A\}$ with $n \in \mathbf{Z}^+$

Exercises:

In 19–31, (1) prove that the relation is an equivalence relation, and (2) describe the distinct equivalence classes of each relation.

19. A is the set of all students at your college.

a. R is the relation defined on A as follows: For every x and y in A,

$$x\,R\,y \quad \Leftrightarrow \quad x \text{ has the same major (or double major) as } y.$$

(Assume "undeclared" is a major.)

b. S is the relation defined on A as follows: For every $x, y \in A$,

$$x\,S\,y \quad \Leftrightarrow \quad x \text{ is the same age as } y.$$

20. E is the relation defined on $\mathbf{Z}$ as follows:

$$\text{For every } m, n \in \mathbf{Z}, \quad m\,E\,n \quad \Leftrightarrow \quad 4 \mid (m - n).$$

H **21.** R is the relation defined on $\mathbf{Z}$ as follows:

$$\text{For every } m, n \in \mathbf{Z}, \quad m\,R\,n \quad \Leftrightarrow \quad 7m - 5n \text{ is even.}$$

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SOLUTION

(a)

A = set of all students at your college

$R = \{(x, y) \in A \times A | x \text{ has the same major as } y\}$

(1) To proof: R is an equivalence relation

PROOF

Reflexive

Since an individual always has the same major as itself, $(x, x) \in R$ for all $x \in A$ and thus R is indeed reflexive.

Symmetric

Let us assume that $(x, y) \in R$.

By the definition of R: x and y have the same major.

But then y and x also have the same major:

$$(y, x) \in R$$

Since $(x, y) \in R$ implies $(y, x) \in R$, R is symmetric.

Transitive

Let us assume that $(x, y) \in R$ and $(y, z) \in R$.

By the definition of R:

x and y have the same major

y and z have the same major

But then x and z also have the same major:

$$(x, z) \in R$$

Since $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$, R is transitive.

Equivalence relation

Since R is reflexive, symmetric and transitive, R is also an equivalence relation.

□

(2) Each equivalence class will contain students who have the same major or students who have the same double major.

Thus there is an equivalence class per major and per double major.

(b)

A = set of all students at your college
$R = \{(x, y) \in A \times A | x \text{ has the same age as } y\}$

(1) To proof: R is an equivalence relation

PROOF

Reflexive

Since an individual always has the same age as itself, $(x, x) \in R$ for all $x \in A$ and thus R is indeed reflexive.

Symmetric

Let us assume that $(x, y) \in R$.

By the definition of R: x and y have the same age.

But then y and x also have the same age:

$$(y, x) \in R$$

Since $(x, y) \in R$ implies $(y, x) \in R$, R is symmetric.

Transitive

Let us assume that $(x, y) \in R$ and $(y, z) \in R$.

By the definition of R:

$$x \text{ and } y \text{ have the same age}$$
$$y \text{ and } z \text{ have the same age}$$

But then x and z also have the same age:

$$(x, z) \in R$$

Since $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$, R is transitive.

Equivalence relation

Since R is reflexive, symmetric and transitive, R is also an equivalence relation.
□

(2) Each equivalence class will contain students who have the same age.

Thus there is an equivalence class per age.

23. Let P be a set of parts shipped to a company from various suppliers. S is the relation defined on P as follows: For every $x, y \in P$,

$x \, S \, y \iff x$ has the same part number and is shipped from the same supplier as y.

24. Let A be the set of identifiers in a computer program. It is common for identifiers to be used for only a short part of the execution time of a program and not to be used again to execute other parts of the program. In such cases, arranging for identifiers to share memory locations makes efficient use of a computer's memory capacity. Define a relation R on A as follows: For all identifiers x and y,

$x \, R \, y \iff$ the values of x and y are stored in the same memory location during execution of the program.

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SOLUTION

A = Set of identifiers in a computer program
$R = \{(x, y) \in A \times A |\ x$ and y are stored in the same memory location$\}$

(1) To proof: R is an equivalence relation

PROOF

Reflexive

Let $x \in A$

Since identifier x is always stored in the same memory location as itself, $(x, x) \in R$.

Since $(x, x) \in R$ for all $x \in A$, R is indeed reflexive.

Symmetric

Let us assume that $(x, y) \in R$. By the definition of R:

x and y are stored in the same memory location

However, if x and y are stored in the same memory location, then y and x are also stored in the same memory location.

$$(y, x) \in R$$

Since $(x, y) \in R$ implies $(y, x) \in R$, R is symmetric.

Transitive

Let us assume that $(x, y) \in R$ and $(y, z) \in R$.

By the definition of R:

x and y are stored in the same memory location
y and z are stored in the same memory location

However, this then implies that x, y and z are all stored in the same memory location, thus x and z are stored in the same memory location.

$$(x, z) \in R$$

Since $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$, R is transitive.

Equivalence relation

Since R is reflexive, symmetric and transitive, R is also an equivalence relation.
□

(2) Two identifiers are related if they are stored in the same memory location during the execution of the program.

Each equivalence class then contains all identifiers that are stored in the same memory location during the execution of the program.

23. Let P be a set of parts shipped to a company from various suppliers. S is the relation defined on P as follows: For every $x, y \in P$,

$x\ S\ y \iff x$ has the same part number and is shipped from the same supplier as y.

24. Let A be the set of identifiers in a computer program. It is common for identifiers to be used for only a short part of the execution time of a program and not to be used again to execute other parts of the program. In such cases, arranging for identifiers to share memory locations makes efficient use of a computer's memory capacity. Define a relation R on A as follows: For all identifiers x and y,

$x\ R\ y \iff$ the values of x and y are stored in the same memory location during execution of the program.

H **26.** D is the relation defined on $\mathbf{Z}$ as follows: For every $m, n \in \mathbf{Z}$,

$$m\ D\ n \iff 3 \mid (m^2 - n^2).$$

1) For any $m \in \mathbb{Z}$,

$$m^2 - m^2 = 0 \Rightarrow 3/m^2 - m^2 \Rightarrow mDm$$

$$\Rightarrow D \text{ is Reflexive}$$

For $m, n \in \mathbb{Z}$ such that mDn.

$$\begin{aligned} mDn \Rightarrow 3/m^2 - n^2 \Rightarrow m^2 - n^2 = 3k \quad \text{For some } k \in \mathbb{Z} \\ \Rightarrow n^2 - m^2 = 3(-k) \quad -k \in \mathbb{Z} \\ \Rightarrow 3/n^2 - m^2 \\ \Rightarrow nDm \end{aligned}$$

For $m, n, p \in \mathbb{Z}$ such that mDn , nDp. Therefore:

$$\begin{aligned} 3/m^2 - n^2 \quad \& \quad 3/n^2 - p^2 \\ \Rightarrow m^2 - n^2 = 3k \quad \& \quad n^2 - p^2 = 3l \quad \text{for some } k, l \in \mathbb{Z} \\ \Rightarrow m^2 - n^2 + n^2 - p^2 = 3k + 3l \\ \Rightarrow m^2 - p^2 = 3(k + l) \quad \& \quad k + l \in \mathbb{Z} \\ \Rightarrow mDp \\ \Rightarrow D \quad \text{is transitive} \end{aligned}$$

So, D is reflexive , symmetric and transitive.

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2) To find equivalence classes.
By division algorithm, for any $m \in \mathbb{Z}, \exists$ unique qr such that,

$$m = 3q + r \quad 0 \le r < 3$$

$$\begin{aligned} \Rightarrow m = 3q \quad \text{or} \quad m = 3q + 1 \quad \text{or} \quad m = 3q + 2 \\ \Rightarrow m^2 = (3q)^2 \quad \text{or} \quad m^2 = (3q+1)^2 \quad \text{or} \quad m^2 = (3q+2)^2 \\ \Rightarrow m^2 = 3(3q^2) \quad \text{or} \quad m^2 = (3q)^2 + 2(3q) + 1^2 \quad \text{or} \quad m^2 = 3(3q^2) + 3(4q) + 3 + 1^2 \\ \Rightarrow m^2 - 0^2 = 3(3q^2) \quad \text{or} \quad m^2 - 1^2 = 3(3q^2 + 2q) \quad \text{or} \quad m^2 - 1^2 = 3(3q^2 + 4q) \end{aligned}$$

$$\Rightarrow 3/m^2 - 0^2 \quad \text{or} \quad 3/m^2 - 1^2 \Rightarrow \boxed{\text{Equiv. Classes are } [0],[1]}$$

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