BIRZEIT UNIVERSITY

Discrete Mathematic and Application Comp233

CHAPTER 9

Counting AND Probability

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Introduction

Imagine and observing **whether 0, 1, or 2** heads are obtained. It would be natural to guess that each of these events occurs about one-third of the time, but in fact this is not the case.

Table 9.1.1 below shows actual data obtained from tossing two quarters 50 times.

Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	卌 卌 |	11	22%
1 head obtained	卌 卌 卌 卌 卌 ||	27	54%
0 heads obtained	卌 卌 ||	12	24%

Experimental Data Obtained from Tossing Two Quarters 50 Times

Table 9.1.1

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- 0, 1 or 2 heads?
- Does each of these events occur about 1/3 of the time?

1/4

No heads obtained

2/4

One head obtained

1/4

Two heads obtained

4

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Introduction

So if you repeatedly toss two balanced coins and record the number of heads, you should expect relative frequencies similar to those shown in Table 9.1.1.

Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	𝍸 𝍸 |	11	22%
1 head obtained	𝍸 𝍸 𝍸 𝍸 𝍸 ||	27	54%
0 heads obtained	𝍸 𝍸 ||	12	24%

Experimental Data Obtained from Tossing Two Quarters 50 Times
Table 9.1.1

To formalize this analysis and extend it to more complex situations, **we introduce the notions of random process, sample space, event and probability.**

Introduction

To say that a process is **random** means that when it takes place, **one outcome from some set of outcomes is sure** to occur, but it is **impossible to predict** with certainty which outcome that will be.

• Definition

A **sample space** is the set of all possible outcomes of a random process or experiment. An **event** is a subset of a sample space.

In case an experiment has finitely many outcomes and all outcomes are equally likely to occur, the ***probability*** of an event (set of outcomes) is just the ratio of the number of outcomes in the event to the total number of outcomes.

Introduction

Equally Likely Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability of** $\boldsymbol{E}$, denoted $\boldsymbol{P(E)}$, is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S}.$$

• Notation

For any finite set A, $N(A)$ denotes the number of elements in A.

With this notation, the equally likely probability formula becomes

$$P(E) = \frac{N(E)}{N(S)}.$$

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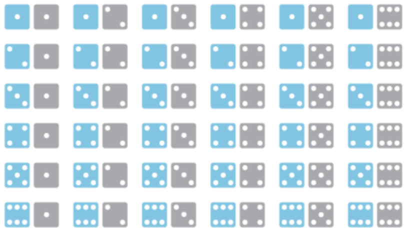
Example 2 – *Rolling a Pair of Dice*

A die is one of a pair of dice. It is a cube with six sides, each containing from one to six dots, called *pips*. Suppose a blue die and a gray die are rolled together, and the numbers of dots that occur face up on each are recorded.

The possible outcomes can be listed as follows, where in each case the die on the left is blue and the one on the right is gray.



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Example 2 – *Rolling a Pair of Dice* cont'd

A more compact notation identifies, say,  with the notation (2,4),  with (5,3), and so forth.

a. **Use the compact notation to write the sample space *S* of possible outcomes.**

b. **Use set notation to write the event *E* that the numbers showing face up have a sum of 6 and find the probability of this event.**

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Example 2 – *Solution*

a. *S* = {(1,1), (1,2), (1,3), (1,4), (1,5), (16),(21), (22), (23), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)}.

b. *E* = {(1,5), (2,4), (3,3), (4,2), (5,1)}.

Let *P*(*E*) be the probability of the event of the numbers whose sum is 6 $P(E) = \frac{N(E)}{N(S)} = \frac{5}{36}$.

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Counting the Elements of a List

More generally, if m and n are integers and $m \leq n$, how many integers are there from m through n? To answer this question, note that $n = m + (n - m)$, where $n - m \geq 0$ [since $n \geq m$].

Theorem 9.1.1 The Number of Elements in a List

If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

Example: how many integer numbers between 5 and 12

list:	5	6	7	8	9	10	11	12
	↕	↕	↕	↕	↕	↕	↕	↕
count:	1	2	3	4	5	6	7	8

So the answer is 8.

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Example 4 – *Counting the Elements of a Sublist*

Question . What is the probability that a randomly chosen three-digit integer between (100-999) that is divisible by 5?

100	101	102	103	104	105	106	107	108	109	110	...	994	995	996	997	998	999
↕					↕					↕			↕				
$5 \cdot 20$					$5 \cdot 21$					$5 \cdot 22$			$5 \cdot 199$				

By Theorem 9.1.1 the total number of integers from 100 through 999 is 999 – 100 + 1 = 900. By part (a), 180 of these are divisible by 5.

Hence the probability that a randomly chosen three-digit integer is divisible by 5 is 180/900 = 1/5.

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Example 1 – *Possibilities for Tournament Play*

Teams *A* and *B* are to play each other repeatedly until one wins two games in a row or a total of three games.

One way in which this tournament can be played is for *A* to win the first game, *B* to win the second, and *A* to win the third and fourth games. Denote this by writing *A*–*B*–*A*–*A*.

a. How many ways can the tournament be played?

b. Assuming that all the ways of playing the tournament are equally likely, what is the probability that five games are needed to determine the tournament winner?

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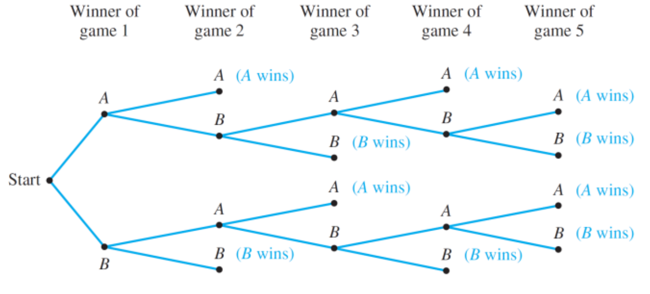
Example 1(a) – *Solution*

The possible ways for the tournament to be played are represented by the distinct paths from "root" (the start) to "leaf" (a terminal point) in the tree shown sideways in Figure 9.2.1.



The Outcomes of a Tournament

Figure 9.2.1

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Example 1(b) – *Solution*

cont'd

The fact that there are ten paths from the root of the tree to its leaves shows that there are ten possible ways for the tournament to be played.

1. A-A
2. A-B-A-A
3. A-B-A-B-A
4. A-B-A-B-B
5. A-B-B

6. B-A-A
7. B-A-B-A-A
8. B-A-B-A-B
9. B-A-B-B
10. B-B

the probability that five games are needed is

4/10 = 2/5 = 40%.

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The Multiplication Rule

Consider the following example. Suppose a computer installation has four input/output **units (*A*, *B*, *C*, and *D*)** and three **central processing units (*X*, *Y*, and *Z*).**

Any input/output unit can be paired with any central processing unit. How many ways are there to pair an input/output unit with a central processing unit?

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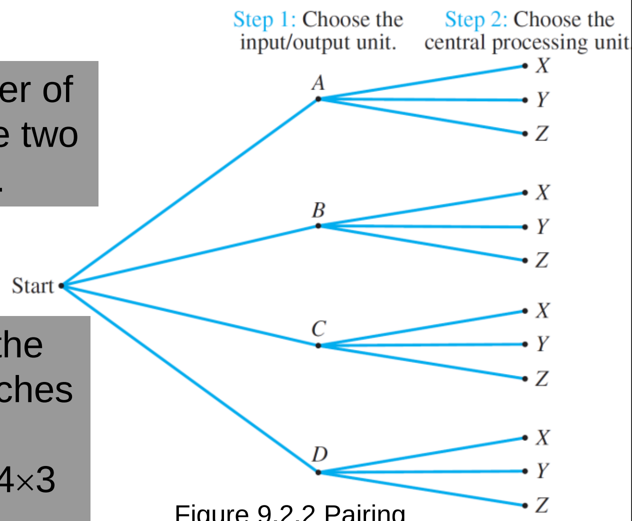
Possibility tree:

The total number of ways to pair the two types of units…



is the same as the number of branches of the tree:
$3 + 3 + 3 + 3 = 4 \times 3$
= **12**

Figure 9.2.2 Pairing Objects Using a Possibility Tree

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Theorem 9.2.1 The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,

the second step can be performed in n_2 ways (regardless of how the first step was performed),

:

the k^{th} step can be performed in n_k ways (regardless of how the preceding steps were performed),

Then the entire operation can be performed in

$n_1 \times n_2 \times n_3 \times \ldots \times n_k$ ways.

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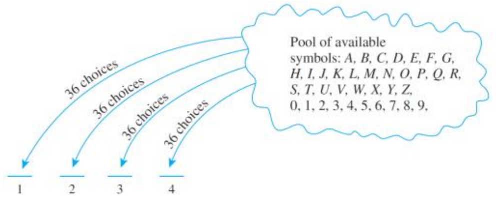
Example 2 – *Number of Personal Identification Numbers (PINs)*

A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits, with repetition allowed. How many different PINs are possible?

Solution:

Typical PINs are CARE, 3387, B32B, and so forth.

You can think of forming a PIN as a four-step operation to fill in each of the four symbols in sequence.



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Example 2 – *Solution*

cont'd

Step 1: Choose the first symbol.

Step 2: Choose the second symbol.

Step 3: Choose the third symbol.

Step 4: Choose the fourth symbol.

There is a fixed number of ways to perform each step, namely 36, regardless of how preceding steps were performed.

And so, by the multiplication rule, there are
$36 \cdot 36 \cdot 36 \cdot 36 = 36^4 = 1{,}679{,}616$ PINs in all.

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Example 4 – *Number of PINs without Repetition*

In Example 2 we formed PINs using four symbols, either letters of the alphabet or digits, and supposing that letters could be repeated. Now suppose that repetition is not allowed.

a. How many different PINs are there?

b. If all PINs are equally likely, what is the probability that a PIN chosen at random contains no repeated symbol?

Solution:

a. Again think of forming a PIN as a four-step operation: Choose the first symbol, then the second, then the third, and then the fourth.

Example 4 – *Solution*

cont'd

There are 36 ways to choose the first symbol, 35 ways to choose the second (since the first symbol cannot be used again), 34 ways to choose the third (since the first two symbols cannot be reused), and 33 ways to choose the fourth (since the first three symbols cannot be reused).

Thus, the multiplication rule can be applied to conclude that there are $36 \cdot 35 \cdot 34 \cdot 33 = 1{,}413{,}720$ different PINs with no repeated symbol.

b. By part (a) there are 1,413,720 PINs with no repeated symbol, and by Example 2 there are 1,679,616 PINs in all.

Example 4 – *Solution*

cont'd

Thus the probability that a PIN chosen at random contains no repeated symbol is $\frac{1,413,720}{1,679,616} \cong .8417$. In other words, approximately 84% of PINs have no repeated symbol.

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When the Multiplication Rule Is Difficult or Impossible to Apply

Consider the following problem:

Three officers—a president, a treasurer, and a secretary—are to be chosen from among four people: Abeer, Basil, Carmen, and Dalia. Suppose that, for various reasons, Abeer **cannot be president** and **either Carmen or Dalia must be secretary**. How many ways can the officers be chosen?

It is natural to try to solve this problem using the multiplication rule. A person might answer as follows:

There are three choices for president (all except Abeer), three choices for treasurer (all except the one chosen as president), and two choices for secretary (Carmen or Dalia).

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When the Multiplication Rule Is Difficult or Impossible to Apply

Therefore, by the multiplication rule, there are 3 • 3 • 2 = 18 choices in all.

Unfortunately, this analysis is incorrect. The number of ways to choose the **secretary** varies **depending on who** is chosen for president and treasurer.

For instance, if Basil is chosen for president and Abeer for treasurer, then there **are two choices** for secretary: Carmen and Dalia.

But if Basil is chosen for president and Carmen for treasurer, then there is **just one** choice for secretary: Dalia.

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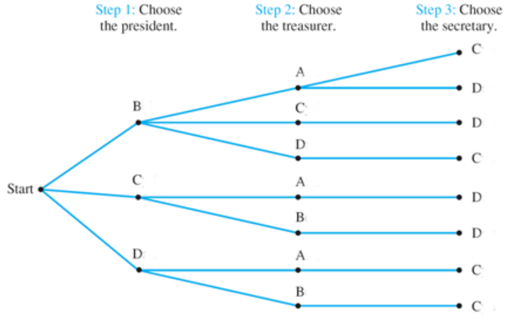
When the Multiplication Rule Is Difficult or Impossible to Apply

The clearest way to see all the possible choices is to construct the possibility tree, as is shown in Figure 9.2.3.



From the tree it is easy to see that there are only eight ways to choose a president, treasurer, and secretary so as to satisfy the given conditions.

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Example 7 – *A More Subtle Use of the Multiplication Rule*

Reorder the steps for choosing the officers in the previous example so that the total number of ways to choose officers can be **computed using the multiplication rule**.

Solution:

Step 1: Choose the secretary.

Step 2: Choose the president.

Step 3: Choose the treasurer.

Example 7 – *Solution*

cont'd

There are exactly two ways to perform step 1 (either Carmen or Dalia may be chosen), two ways to perform step 2 (neither Abeer nor the person chosen in step 1 may be chosen but either of the other two may), and two ways to perform step 3 (either of the two people not chosen as secretary or president may be chosen as treasurer).

Thus, by the multiplication rule, the total number of ways to choose officers is **$2 \cdot 2 \cdot 2 = 8$**.

Example 7 – *Solution*

cont'd

A possibility tree illustrating this sequence of choices is shown in Figure 9.2.4.

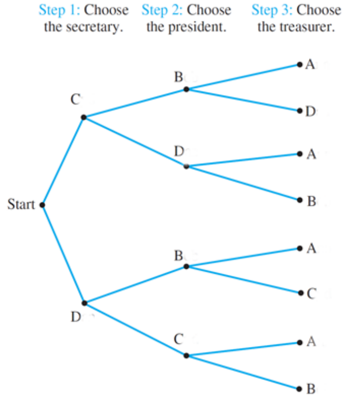

Figure 9.2.4

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Permutations

A **permutation** of a set of objects is an ordering of the objects in a row. For example, the set of elements *a*, *b*, and *c* has six permutations.

abc acb cba bac bca cab

In general, given a set of *n* objects, how many permutations does the set have? Imagine forming a permutation as an ***n*-step operation**:

Step 1: Choose an element to write first.

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Permutations

Step 2: Choose an element to write second.

⋮ ⋮

Step *n*: Choose an element to write *n*th.

Any element of the set can be chosen in step 1, so there are n ways to perform step 1.

Any element except that chosen in step 1 can be chosen in step 2, so there are $n - 1$ ways to perform step 2.

Permutations

In general, the number of ways to perform each successive step is one less than the number of ways to perform the preceding step.

At the point when the *n*th element is chosen, there is only one element left, so there is only one way to perform step n.

Hence, by the multiplication rule, there are

$$n(n-1)(n-2)\cdot\cdot\cdot 2\cdot 1 = n!$$

ways to perform the entire operation.

Permutations

In other words, there are *n*! permutations of a set of *n* elements. This reasoning is summarized in the following theorem.

Theorem 9.2.2

For any integer n with $n \geq 1$, the number of permutations of a set with n elements is $n!$.

Example 8 – *Permutations of the Letters in a Word*

a. How many ways can the letters in the word *COMPUTER* be arranged in a row?

8! = 40,320

b. How many ways can the letters in the word *COMPUTER* be arranged if the letters *CO* must remain next to each other (in order) as a unit?

CO	M	P	U	T	E	R

7! = 5,040.

c. If letters of the word *COMPUTER* are randomly arranged in a row, what is the **probability** that the letters *CO* remain next to each other (in order) as a unit?

$$P = \frac{7!}{8!} = \frac{1}{8}$$

Permutations of Selected Elements

Given the set {*a*, *b*, *c*}, there are six ways to select two letters from the set and write them in order.

ab ***ac*** ***ba*** ***bc*** ***ca*** ***cb***

Each such ordering of two elements of {*a*, *b*, *c*} is called a 2-*permutation* of {*a*, *b*, *c*}.

• Definition

An ***r*-permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $\boldsymbol{P(n, r)}$.

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Permutations of Selected Elements

Theorem 9.2.3

If n and r are integers and $1 \le r \le n$, then the number of r-permutations of a set of n elements is given by the formula

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) \qquad \text{first version}$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!} \qquad \text{second version.}$$

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Example 10 – *Evaluating r-Permutations*

a. Evaluate $P(5, 2)$.

b. How many 4-permutations are there of a set of seven objects?

c. How many 5-permutations are there of a set of five objects?

Solution:

a. $$P(5, 2) = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot \not{3} \cdot \not{2} \cdot \not{1}}{\not{3} \cdot \not{2} \cdot \not{1}}$$

$$= 20$$

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Example 10 – *Solution*

cont'd

b. The number of 4-permutations of a set of seven objects is

$$P(7, 4) = \frac{7!}{(7-4)!}$$

$$= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot \not{3} \cdot \not{2} \cdot \not{1}}{\not{3} \cdot \not{2} \cdot \not{1}}$$

$$= 7 \cdot 6 \cdot 5 \cdot 4$$

$$= 840.$$

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Example 10 – *Solution*

cont'd

c. The number of 5-permutations of a set of five objects is

$$P(5,5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5! = 120.$$

Note that the definition of 0! as 1 makes this calculation come out as it should, for the number of 5-permutations of a set of five objects is certainly equal to the number of permutations of the set.

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Example 12 – *Proving a Property of P (n, r)*

Prove that for all integers $n \geq 2$,

$$P(n,2) + P(n,1) = n^2.$$

Solution:

Suppose *n* is an integer that is greater than or equal to 2.

By Theorem 9.2.3,

$$P(n,2) = \frac{n!}{(n-2)!}$$

$$= \frac{n(n-1)(n-2)!}{(n-2)!}$$

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Example 12 – *Solution*

cont'd

$$= n(n-1)$$

and

$$P(n, 1) = \frac{n!}{(n-1)!}$$

$$= \frac{n \cdot (n-1)!}{(n-1)!} = n.$$

Hence

$$P(n, 2) + P(n, 1) = n \cdot (n-1) + n$$

$$= n^2 - n + n$$

$$= n^2,$$

which is what we needed to show.

Exercises

28. If the largest of 56 consecutive integers is 279, what is the smallest?

29. If the largest of 87 consecutive integers is 326, what is the smallest?

30. How many even integers are between 1 and 1,001?

31. How many integers that are multiples of 3 are between 1 and 1,001?

Exercises

14. Suppose that in a certain state, all automobile license plates have four uppercase letters followed by three digits.

a. How many different license plates are possible?

b. How many license plates could begin with *A* and end in 0?

c. How many license plates could begin with *TGIF*?

d. How many license plates are possible in which all the letters and digits are distinct?

e. How many license plates could begin with *AB* and have all letters and digits distinct?

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Exercises

33. Six people attend the theater together and sit in a row with exactly six seats.

a. How many ways can they be seated together in the row?

b. Suppose one of the six is a doctor who must sit on the aisle in case she is paged. How many ways can the people be seated together in the row with the doctor in an aisle seat?

c. Suppose the six people consist of three married couples and each couple wants to sit together with the older partner on the left. How many ways can the six be seated together in the row?

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Exercises

40. Prove that for every integer $n \geq 2$, $P(n+1, 3) = n^3 - n$.

41. Prove that for every integer $n \geq 2$,

$$P(n+1, 2) - P(n, 2) = 2P(n, 1).$$

42. Prove that for every integer $n \geq 3$,

$$P(n+1, 3) - P(n, 3) = 3P(n, 2).$$

43. Prove that for every integer $n \geq 2$, $P(n, n) = P(n, n-1)$.

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Counting Elements of Disjoint Sets: The Addition Rule

The basic rule underlying the calculation of the number of elements **in a union or difference or intersection** is the **addition rule.** ($A \cup B, A \cap B, A - B$)

This rule states that the number of elements in a union of **mutually disjoint finite** sets **equals the sum** of the number of elements in each of the component sets.

Theorem 9.3.1 The Addition Rule

Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$. Then

$$N(A) = N(A_1) + N(A_2) + \cdots + N(A_k).$$

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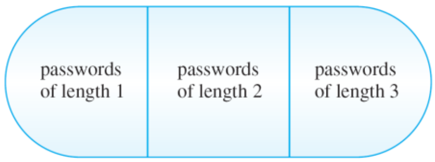
Example 1 – *Counting Passwords with Three or Fewer Letters*

A computer access password consists of from one to three letters chosen from the 26 in the alphabet with repetitions allowed. **How many different passwords are possible?**

Solution:

The set of all passwords can be partitioned into subsets consisting of those of **length 1, those of length 2, and those of length 3** as shown in Figure 9.3.1.



Set of All Passwords of Length ≤ 3

Figure 9.3.1

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Example 1 – *Solution*

cont'd

By the addition rule, the total number of passwords equals the number of passwords of length 1, plus the number of passwords of length 2, plus the number of passwords of length 3.

number of passwords of length 1 $= 26$ because there are 26 letters in the alphabet

Now the

number of passwords of length 2 $= 26^2$ because forming such a word can be thought of as a two-step process in which there are 26 ways to perform each step

number of passwords of length 3 $= 26^3$ because forming such a word can be thought of as a three-step process in which there are 26 ways to perform each step.

Hence the total number of passwords $= 26 + 26^2 + 26^3$

$= 18{,}278.$

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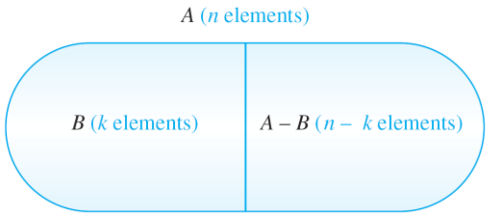
The Difference Rule

An important consequence of the addition rule is the fact that if the number of elements in **a set *A* and the number in a subset *B* of *A* are both known, then the number of elements that are in *A* and not in *B* can be computed.**



Theorem 9.3.2 The Difference Rule

If A is a finite set and B is a subset of A, then

$$N(A - B) = N(A) - N(B).$$

The Difference Rule

The difference rule holds for the following reason: If B is a subset of A, then the two sets B and $A - B$ have no elements in common and $B \cup (A - B) = A$. Hence, by the addition rule,

$$N(B) + N(A - B) = N(A).$$

Subtracting $N(B)$ from both sides gives the equation

$$N(A - B) = N(A) - N(B).$$

Example 3 – *Counting PINs with Repeated Symbols*

A typical PIN (personal identification number) is a sequence of any **four symbols chosen from the 26** letters in the alphabet and **the ten digits**, **with repetition allowed**.

a. How many PINs contain repeated symbols?

b. If all PINs are equally likely, what is the probability that a randomly chosen PIN contains a repeated symbol?

Example 3(a) – *Solution*

There are **36^4 = 1,679,616** PINs when repetition is allowed, and there are **36 • 35 • 34 • 33** = **1,413,720** PINs when repetition is not allowed.

Thus, by the difference rule, there are

1,679,616 – 1,413,720 = 265,896

PINs that contain **at least one repeated symbol**.

Thus, by the equally likely probability formula, the probability that a randomly chosen PIN contains a repeated symbol is $\frac{265{,}896}{1{,}679{,}616} \cong 0.158 = 15.8\%.$

The Difference Rule

$$Prove\ that\ P(S - A) = 1 - P(A)$$

$$\begin{aligned} P(S-A) &= \frac{N(S-A)}{N(S)} && \text{by definition of probability in the equally likely case} \\ &= \frac{N(S)-N(A)}{N(S)} && \text{by the difference rule} \\ &= \frac{N(S)}{N(S)} - \frac{N(A)}{N(S)} && \text{by the laws of fractions} \\ &= 1 - P(A) && \text{by definition of probability in the equally likely case} \end{aligned}$$

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The Difference Rule

We know that the probability that a PIN chosen at random contains no repeated symbol is $P(A) = \frac{1{,}413{,}720}{1{,}679{,}616} \cong .8417$.

And hence

$$\begin{aligned} P(S-A) &\cong 1 - 0.842 \\ &\cong 0.158 \\ &= 15.8\% \end{aligned}$$

Formula for the Probability of the Complement of an Event

If S is a finite sample space and A is an event in S, then

$$P(A^c) = 1 - P(A).$$

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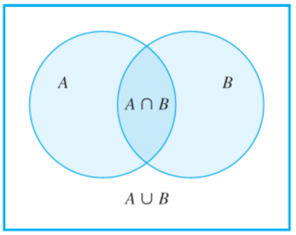
The Inclusion/Exclusion Rule

The addition rule says how many elements are in a union of sets if the sets are mutually disjoint. Now consider the question of how to determine the number of elements in a **union of sets when some of the sets OVERLAPED**.

What is Number of Elements in $A \cup B$??



$$N(A \cup B) = N(A) + N(B) - N(A \cap B).$$

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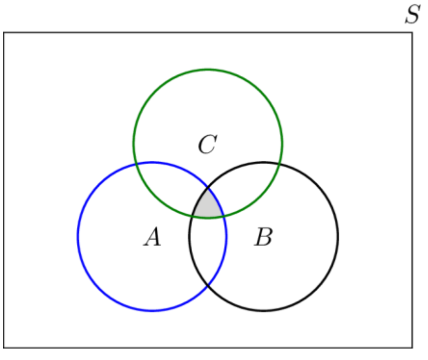
What is Number of Elements in $A \cup B \cup C$?



Theorem 9.3.3 The Inclusion/Exclusion Rule for Two or Three Sets

If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

and

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$

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Example

Given 50 students:

30 took pre-calculus;
18 took calculus;
26 took Java;
9 took pre-calculus & calculus;
16 took pre-calculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

How many students did not take any of the three courses?

50−47 = 3.

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57

Given 50 students:

30 took pre-calculus;
18 took calculus;
26 took Java;
9 took pre-calculus & calculus;
16 took pre-calculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

➢ **How many students took all three courses?**

P = the set of students who took precalculus
C = the set of students who took calculus
J = the set of students who took Java.

$$N(P \cup C \cup J) =$$

$$N(P) + N(C) + N(J) - N(P \cap C) - N(P \cap J) - N(C \cap J) + N(P \cap C \cap J)$$

$$47 = 30 + 26 + 18 - 9 - 16 - 8 + N(P \cap C \cap J).$$

$$N(P \cap C \cap J) = 6.$$

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Cont..

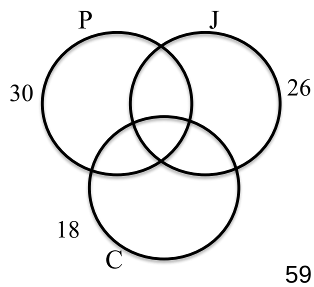
Given 50 students:

30 took pre-calculus;
18 took calculus;
26 took Java;
9 took pre-calculus & calculus;
16 took pre-calculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

How many students took pre-calculus and calculus but not Java?

= (N(P∩C)) – (N(P∩C∩J) =?

9 – 6 = 3



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Cont..

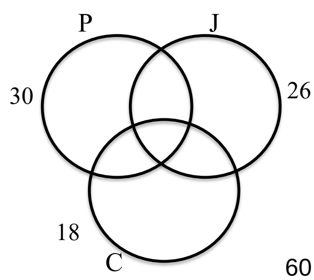
Given 50 students:

30 took precalculus;
18 took calculus;
26 took Java;
9 took precalculus & calculus;
16 took precalculus & Java;
8 took calculus & Java;
47 took at least 1of the 3 courses.

How many students took precalculus but neither calculus nor Java?

N(P) – (N(P∩C)) – N(P∩J)) + N(P∩C∩J) =?

30 – 9 – 16 + 6
= 11



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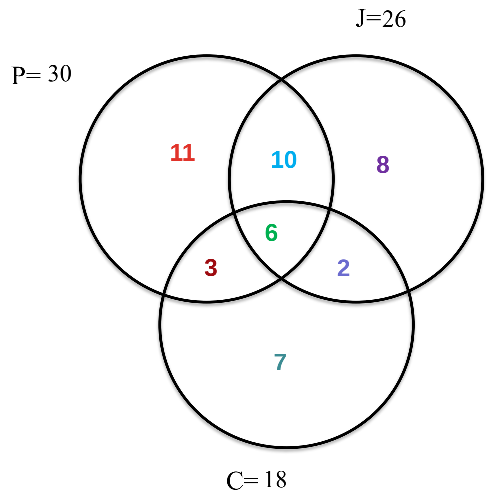

Example 6 – *Counting Elements of a General Union*

a. How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?

b. How many integers from 1 through 1,000 are neither multiples of 3 nor multiples of 5?

Solution:

a. Let A = the set of all integers from 1 through 1,000 that are multiples of 3.

Let B = the set of all integers from 1 through 1,000 that are multiples of 5.

Example 6 – *Solution*

cont'd

Then

$A \cup B$ = the set of all integers from 1 through 1,000 that are multiples of 3 or multiples of 5

and

$A \cap B$ = the set of all integers from 1 through 1,000 that are multiples of both 3 and 5

= the set of all integers from 1 through 1,000 that are multiples of 15.

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Example 6 – *Solution*

cont'd

Because every third integer from 3 through 999 is a multiple of 3, each can be represented in the form 3*k*, for some integer *k* from 1 through 333.

Hence there are 333 multiples of 3 from 1 through 1,000, and so ***N*(*A*) = 333.**

1	2	3	4	5	6	...	996	997	998	999
		↕			↕		↕			↕
		$3 \cdot 1$			$3 \cdot 2$		$3 \cdot 332$			$3 \cdot 333$

64

64

Example 6 – *Solution*

cont'd

Similarly, each multiple of 5 from 1 through 1,000 has the form 5*k*, for some integer *k* from 1 through 200.

1	2	3	4	5	6	7	8	9	10	. . .	995	996	997	998	999	1,000
				↕					↕		↕					↕
				$5 \cdot 1$					$5 \cdot 2$		$5 \cdot 199$					$5 \cdot 200$

Thus there are 200 multiples of 5 from 1 through 1,000 and ***N(B) = 200.***

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Example 6 – *Solution*

cont'd

Finally, each multiple of 15 from 1 through 1,000 has the form 15*k*, for some integer *k* from 1 through 66 (since 990 = 66 • 15).

1	2	. . .	15	. . .	30	. . .	975	. . .	990	. . .	999	1,000
			↕		↕		↕		↕			
			$15 \cdot 1$		$15 \cdot 2$		$15 \cdot 65$		$15 \cdot 66$			

Hence there are 66 multiples of 15 from 1 through 1,000, and ***N(A ∩ B) = 66.***

66

Example 6 – *Solution*

cont'd

It follows by the inclusion/exclusion rule that

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$
$$= 333 + 200 - 66$$
$$= 467.$$

Thus, 467 integers from 1 through 1,000 are multiples of 3 or multiples of 5.

b. There are 1,000 integers from 1 through 1,000, and by part (**a**), 467 of these are multiples of 3 or multiples of 5.

Thus, by the set difference rule, there are **1,000 – 467 = 533** that are neither multiples of 3 nor multiples of 5.

The Inclusion/Exclusion Rule

Note that the solution to part (**b**) of Example 6 hid a use of De Morgan's law.

The number of elements that are neither in *A* nor in *B* is **$N(A^c \cap B^c)$, and by De Morgan's law, $A^c \cap B^c = (A \cup B)^c$.**

So **$N((A \cup B)^c)$** was then calculated using the set difference rule: **$N((A \cup B)^c) = N(U) - N(A \cup B)$,** where the universe *U* was the set of all integers from 1 through 1,000.

11. a. How many ways can the letters of the word *QUICK* be arranged in a row?
    b. How many ways can the letters of the word *QUICK* be arranged in a row if the Q and the U must remain next to each other in the order QU?
    c. How many ways can the letters of the word *QUICK* be arranged in a row if the letters QU must remain together but may be in either the order QU or the order UQ?

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H 15. Identifiers in a certain database language must begin with a letter, and then the letter may be followed by other characters, which can be letters, digits, or underscores (_). However, 82 keywords (all consisting of 15 or fewer characters) are reserved and cannot be used as identifiers. How many identifiers with 30 or fewer characters are possible? (Write the answer using summation notation and evaluate it using a formula from Section 5.2.)

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H **★21.** Six new employees, two of whom are married to each other, are to be assigned six desks that are lined up in a row. If the assignment of employees to desks is made randomly, what is the probability that the married couple will have nonadjacent desks? (*Hint:* First find the probability that the couple will have adjacent desks, and then subtract this number from 1.)

★22. Consider strings of length n over the set $\{a, b, c, d\}$.

a. How many such strings contain at least one pair of adjacent characters that are the same?

b. If a string of length ten over $\{a, b, c, d\}$ is chosen at random, what is the probability that it contains at least one pair of adjacent characters that are the same?

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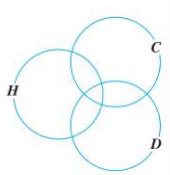
33. A college conducted a survey to explore the academic interests and achievements of its students. It asked students to place checks beside the numbers of all the statements that were true of them. Statement #1 was "I was on the honor roll last term," statement #2 was "I belong to an academic club, such as the math club or the Spanish club," and statement #3 was "I am majoring in at least two subjects." Out of a sample of 100 students, 28 checked #1, 26 checked #2, and 14 checked #3, 8 checked both #1 and #2, 4 checked both #1 and #3, 3 checked both #2 and #3, and 2 checked all three statements.

a. How many students checked at least one of the statements?

b. How many students checked none of the statements?

c. Let H be the set of students who checked #1, C the set of students who checked #2, and D the set of students who checked #3. Fill in the numbers for all eight regions of the diagram below.



Sample of Students

d. How many students checked #1 and #2 but not #3?

e. How many students checked #2 and #3 but not #1?

f. How many students checked #2 but neither of the other two?

34. A study was done to determine the efficacy of three different drugs—A, B, and C—in relieving headache pain. Over the period covered by the study, 50 subjects were given the chance to use all three drugs. The following results were obtained:

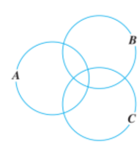
21 reported relief from drug A.
21 reported relief from drug B.
31 reported relief from drug C.
9 reported relief from both drugs A and B.
14 reported relief from both drugs A and C.
15 reported relief from both drugs B and C.
41 reported relief from at least one of the drugs.

Note that some of the 21 subjects who reported relief from drug A may also have reported relief from drugs B or C. A similar occurrence may be true for the other data.

a. How many people got relief from none of the drugs?

b. How many people got relief from all three drugs?

c. Let A be the set of all subjects who got relief from drug A, B the set of all subjects who got relief from drug B, and C the set of all subjects who got relief from drug C. Fill in the numbers for all eight regions of the diagram below.



Sample of Subjects

d. How many subjects got relief from A only?

72

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35. An interesting use of the inclusion/exclusion rule is to check survey numbers for consistency. For example, suppose a public opinion polltaker reports that out of a national sample of 1,200 adults, 675 are married, 682 are from 20 to 30 years old, 684 are female, 195 are married and are from 20 to 30 years old, 467 are married females, 318 are females from 20 to 30 years old, and 165 are married females from 20 to 30 years old. Are the polltaker's figures consistent? Could they have occurred as a result of an actual sample survey?

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9.5 Counting Subsets of a Set: Combinations

Given a set *S* with *n* elements, how many subsets of size *r* can be chosen from *S*?

The number of subsets of size *r* that can be chosen from *S* equals the number of subsets of size *r* that *S* has.

Each individual subset of size *r* is called an ***r-combination* of the set.**

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Example 1 – 3-*Combinations*

Let *S* = {Abeer, Basil, Carmen, Dalia}. Each committee consisting of three of the four people in *S* is a 3-combination of *S*.

a. List all such 3-combinations of *S*.

b. What is $\binom{4}{3}$

Solution:

a. Each 3-combination of *S* is a subset of *S* of size 3. But each subset of size 3 can be obtained by leaving out one of the elements of *S*.

The 3-combinations are

{Basil, Carmen, Dalia} Leave out Abeer

Example 1 – *Solution* cont'd

{Abeer, Carmen, Dalia} Leave out Basil

{Abeer, Basil, Dalia} Leave out Carmen

{Abeer, Basil, Carmen} Leave out Dalia

b. Because $\binom{4}{3}$ the number of 3-combinations of a set with four elements, by part (**a**), $\binom{4}{3}$ = 4.

Counting Subsets of a Set: Combinations

• Definition

Let n and r be nonnegative integers with $r \le n$. An ***r*-combination** of a set of n elements is a subset of r of the n elements. As indicated in Section 5.1, the symbol

$$\binom{n}{r},$$

which is read "n choose r," denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

We have known that calculators generally use symbols like $C(n, r)$, ${}_nC_r$, $C_{n,r}$, or ${}^nC_r$ instead of $\binom{n}{r}$.

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Counting Subsets of a Set: Combinations

There are **two distinct methods** that can be used to select ***r*** objects from a set of ***n*** elements.

In an **ordered selection (*r*-permutation)**, it is not only what elements are chosen but also the order in **which they are chosen that matters.**

Use a Permutation. Order is important!

To find the number of Permutations of n items chosen r at a time, you can use the formula

$${}_nP_r = \frac{n!}{(n-r)!} \quad \text{where } 0 \le r \le n.$$

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Combinations

In an **unordered selection *r*-combination**, on the other hand, it is only the identity of the chosen elements that matters. Two unordered selections are said to be the same if they consist of the same elements, **regardless of the order** in which the elements are chosen.

Use a Combination because order does not matter!

To find the number of Combinations of n items chosen r at a time, you can use the formula

$$ {}_nC_r = \frac{n!}{r!(n-r)!} \quad \text{where } 0 \le r \le n. $$

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Example 2 – *Unordered Selections*

How **many unordered selections of two** elements can be made from the set {0, 1, 2, 3}?

Solution:

An **unordered selection of two elements** from {0, 1, 2, 3} is the same as a 2-combination, or subset of size 2, taken from the set.

These can be listed systematically:

{0, 1}, {0, 2}, {0, 3} subsets containing 0

{1, 2}, {1, 3} subsets containing 1 but not already listed

{2,3} subsets containing 2 but not already listed.

Since this listing exhausts all possibilities, there are six subsets in all.

Thus $\binom{4}{2}$ = 6, which is the number of unordered selections of two elements from a set of four.

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Counting Subsets of a Set: Combinations

The reasoning used in Example 3 applies in the general case as well.

Theorem 9.5.1

The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n, r)}{r!} \quad \text{first version}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{second version}$$

where n and r are nonnegative integers with $r \leq n$.

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Example 3 – *Relation between Permutations and Combinations*

Write all 2-permutations of the set {0, 1, 2, 3}. Find an equation relating the number of **2-permutations**, $P(4, 2)$, and the number of 2-combinations, $\binom{4}{2}$, and solve this equation for $\binom{4}{2}$.

Solution:

According to Theorem 9.2.3, the number of 2-permutations of the set {0, 1, 2, 3} is $P(4, 2)$, which equals

$$\frac{4!}{(4-2)!} = \frac{4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 12.$$

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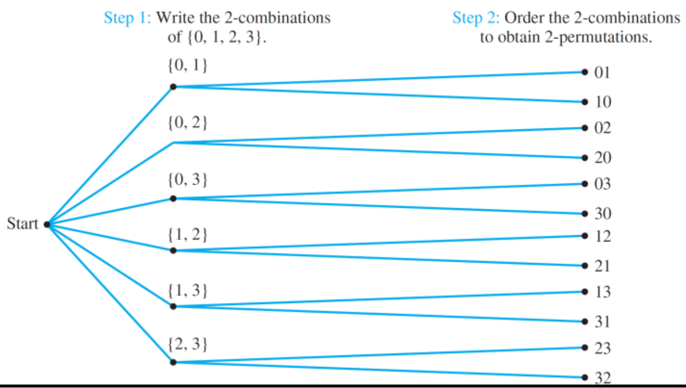
Example 3 – *Solution*

cont'd

Now the act of constructing a 2-permutation of {0, 1, 2, 3} can be thought of as a two-step process:

Step 1: Choose a **subset of two elements** from {0, 1, 2, 3}.

Step 2: Choose an **ordering** for the two-element subset.



Example 3 – *Solution*

cont'd

Solving the equation for $\binom{4}{2}$ gives

$$\binom{4}{2} = \frac{P(4, 2)}{2!}$$

We know that $P(4, 2) = \frac{4!}{(4-2)!}$.

Hence, substituting yields

$$\binom{4}{2} = \frac{\frac{4!}{(4-2)!}}{2!}$$

$$= \frac{4!}{2!(4-2)!}$$

$$= 6.$$

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Practice: To play a particular card game, each player is dealt five cards from a standard deck of 52 cards. How many different hands are possible?

$$_{52}C_5 = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} =$$

$$\frac{52*51*50*49*48}{5*4*3*2*1} = 2{,}598{,}960$$

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Practice:

A student must answer 3 out of 5 essay questions on a test. In how many different ways can the student select the questions?

Practice: A student must answer 3 out of 5 essay questions on a test. In how many different ways can the student select the questions?

$$ {}_5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5*4}{2*1} = 10 $$

Example 4 – *Calculating the Number of Teams*

Consider again the problem of choosing five members from a group of twelve to work as a team on a special project. How many distinct five-person teams can be chosen?

Solution:

The number of distinct five-person teams is the same as the number of subsets of size 5 (or 5-combinations) that can be chosen from the set of twelve. This number is $\binom{12}{5}$.

By Theorem 9.5.1,

$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot 7!} = 11 \cdot 9 \cdot 8 = 792.$$

Thus there are 792 distinct five-person teams.

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Counting Subsets of a Set: Combinations

The formula for the number of *r*-combinations of a set can be applied in a wide variety of situations. Let us illustrate this in the next example.

Before we begin the next example, a remark on the phrases *at least* and *at most* is in order:

The phrase **at least *n*** means "*n* or more."
The phrase **at most *n*** means "*n* or fewer."

For instance, if a set consists of three elements and you are to choose at least two, you will choose two or three; if you are to choose at most two, you will choose none, or one, or two.

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Example 7 – *Teams with Members of Two Types*

Suppose the group of twelve consists of five men and seven women.

a. How many five-person teams can be chosen that consist of three men and two women?

b. How many five-person teams contain at least one man?

c. How many five-person teams contain at most one man?

Solution:

a. To answer this question, think of forming a team as a two-step process:

Step 1: Choose the men.

Example 7 – *Solution*

cont'd

Step 2: Choose the women.

There are $\binom{5}{3}$ ways to choose the three men out of the five and $\binom{7}{2}$ ways to choose the two women out of the seven.

Hence, by the product rule,

$$\begin{bmatrix}\text{number of teams of five that} \\ \text{contain three men and two women}\end{bmatrix} = \binom{5}{3}\binom{7}{2} = \frac{5!}{3!2!} \cdot \frac{7!}{2!5!}$$

$$= \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{2} \cdot 1 \cdot \cancel{2} \cdot 1}$$

$$= 210.$$

Example 7 – *Solution*

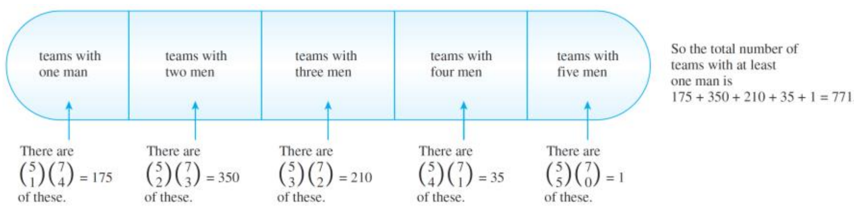
cont'd

b. How many five-person teams contain at least one man?

b. This question can also be answered either by the addition rule or by the difference rule. The solution by the difference rule is shorter and is shown first.

Total= $\binom{5}{1}\binom{7}{4}+\binom{5}{2}\binom{7}{3}+\binom{5}{3}\binom{7}{2}+\binom{5}{4}\binom{7}{1}+\binom{5}{5}\binom{7}{0}$

=175+350+210+35+1=771



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93

Example 7 – *Solution*

cont'd

The number of teams in each subset of the partition is calculated using the method illustrated in part (**a**). There are

$\binom{5}{1}\binom{7}{4}$ teams with one man and four women

$\binom{5}{2}\binom{7}{3}$ teams with two men and three women

$\binom{5}{3}\binom{7}{2}$ teams with three men and two women

$\binom{5}{4}\binom{7}{1}$ teams with four men and one woman

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Example 7 – *Solution*

cont'd

$$\binom{5}{5}\binom{7}{0} \text{ teams with five men and no women.}$$

Hence, by the addition rule,

$$\begin{bmatrix}\text{number of teams with} \\ \text{at least one man}\end{bmatrix}$$

$$= \binom{5}{1}\binom{7}{4} + \binom{5}{2}\binom{7}{3} + \binom{5}{3}\binom{7}{2} + \binom{5}{4}\binom{7}{1} + \binom{5}{5}\binom{7}{0}$$

$$= \frac{5!}{1!4!}\cdot\frac{7!}{4!3!} + \frac{5!}{2!3!}\cdot\frac{7!}{3!4!} + \frac{5!}{3!2!}\cdot\frac{7!}{2!5!} + \frac{5!}{4!1!}\cdot\frac{7!}{1!6!} + \frac{5!}{5!0!}\cdot\frac{7!}{0!7!}$$

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Example 7 – *Solution*

cont'd

$$= \frac{5\cdot\cancel{4!}\cdot7\cdot\cancel{6}\cdot5\cdot\cancel{4!}}{\cancel{4!}\cdot\cancel{3\cdot2}\cdot\cancel{4!}} + \frac{5\cdot\overset{2}{\cancel{4}}\cdot\cancel{3!}\cdot7\cdot\cancel{6}\cdot5\cdot\cancel{4!}}{\cancel{3!}\cdot\cancel{2}\cdot\cancel{4!}\cdot\cancel{3\cdot2}} + \frac{5\cdot\overset{2}{\cancel{4}}\cdot\cancel{3!}\cdot7\cdot\overset{3}{\cancel{6}}\cdot\cancel{5!}}{\cancel{2}\cdot\cancel{3!}\cdot\cancel{5!}\cdot\cancel{2}}$$

$$+ \frac{5\cdot\cancel{4!}\cdot7\cdot\cancel{6!}}{\cancel{4!}\cdot\cancel{6!}} + \frac{\cancel{5!}\cdot\cancel{7!}}{\cancel{5!}\cdot\cancel{7!}}$$

$$= 175 + 350 + 210 + 35 + 1$$

$$= 771.$$

96

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Example 7 – *Solution*

cont'd

Now a team with no men consists entirely of five women chosen from the seven women in the group, so there are $\binom{7}{5}$ such teams. Also, by Example 4, the total number of five-person teams is $\binom{12}{5} = 792$.

Hence, by the difference rule,

$$\begin{bmatrix}\text{number of teams}\\ \text{with at least}\\ \text{one man}\end{bmatrix} = \begin{bmatrix}\text{total number}\\ \text{of teams}\\ \text{of five}\end{bmatrix} - \begin{bmatrix}\text{number of teams}\\ \text{of five that do not}\\ \text{contain any men}\end{bmatrix}$$

$$= \binom{12}{5} - \binom{7}{5} = 792 - \frac{7!}{5!\cdot 2!}$$

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Example 7 – *Solution*

cont'd

$$= 792 - \frac{7\cdot \overset{3}{\cancel{6}}\cdot \cancel{5!}}{\cancel{5!}\cdot \cancel{2}\cdot 1} = 792 - 21 = 771.$$

This reasoning is summarized in Figure 9.5.5.

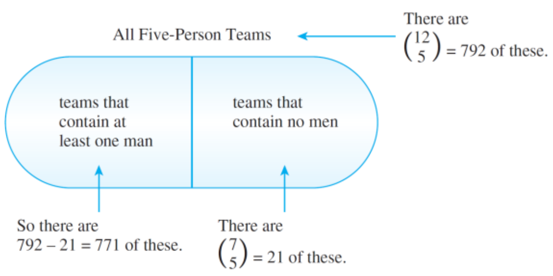

Figure 9.5.5

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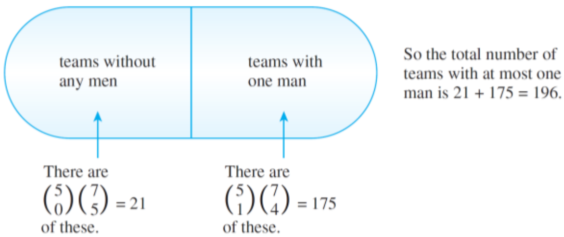
Example 7 – *Solution*

cont'd

c. How many five-person teams contain at most one man?

c. As shown in Figure 9.5.7, the set of teams containing at most one man can be partitioned into the set that does not contain any men and the set that contains exactly one man.

Hence, by the addition rule,



Teams with At Most One Man

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Example 7 – *Solution*

cont'd

$$\begin{bmatrix}\text{number of teams}\\ \text{with at}\\ \text{most one man}\end{bmatrix} = \begin{bmatrix}\text{number of}\\ \text{teams without}\\ \text{any men}\end{bmatrix} + \begin{bmatrix}\text{number of}\\ \text{teams with}\\ \text{one man}\end{bmatrix}$$

$$= \binom{5}{0}\binom{7}{5} + \binom{5}{1}\binom{7}{4}$$

$$= 21 + 175$$

$$= 196.$$

This reasoning is summarized in Figure 9.5.7.

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Example 10 – *Permutations of a Set with Repeated Elements*

Consider various ways of ordering the letters in the word *MISSISSIPPI*:

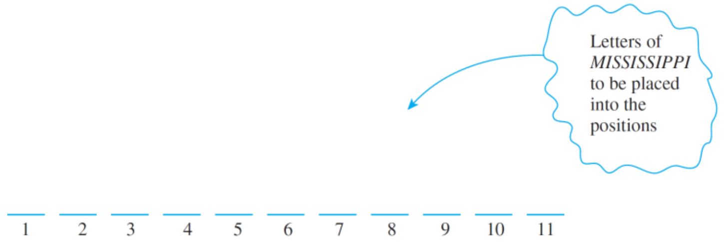
IIMSSPISSIP, *ISSSPMIIPIS*, *PIMISSSSIIP*, and so on.

How many distinguishable orderings are there?

Note : {4 S, 1 M, 4 I, 2 P}

Example 10 – *Solution*

Imagine placing the 11 letters of *MISSISSIPPI* one after another into 11 positions.



Because copies of the same letter cannot be distinguished from one another, once the positions for a certain letter are known, then all copies of the letter can go into the positions in any order.

Example 10 – *Solution*

cont'd

It follows that constructing an ordering for the letters can be thought of as a four-step process:

Step 1: Choose a subset of four positions for the *S*'s.

Step 2: Choose a subset of four positions for the *I*'s.

Step 3: Choose a subset of two positions for the *P*'s.

Step 4: Choose a subset of one position for the *M*.

Since there are 11 positions in all, there are $\binom{11}{4}$ subsets of four positions for the *S*'s.

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Example 10 – *Solution*

cont'd

Once the four *S*'s are in place, there are seven positions that remain empty, so there are $\binom{7}{4}$ subsets of four positions for the *I*'s. After the *I*'s are in place, there are three positions left empty, so there are $\binom{3}{2}$ subsets of two positions for the *P*'s.

That leaves just one position for the *M*. But 1 = $\binom{1}{1}$. Hence by the multiplication rule,

$$\begin{bmatrix}\text{number of ways to} \\ \text{position all the letters}\end{bmatrix} = \binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1}$$

$$= \frac{11!}{4!7!}\cdot\frac{7!}{4!3!}\cdot\frac{3!}{2!1!}\cdot\frac{1!}{1!0!} = \frac{11!}{4!\cdot 4!\cdot 2!\cdot 1!} = 34{,}650.$$

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9.6 Counting Subsets of a Set: Combinations

The reasoning used in this example can be used to derive the following general theorem.

Theorem 9.5.2 Permutations with sets of Indistinguishable Objects

Suppose a collection consists of n objects of which

n_1 are of type 1 and are indistinguishable from each other
n_2 are of type 2 and are indistinguishable from each other
⋮
n_k are of type k and are indistinguishable from each other,

and suppose that $n_1 + n_2 + \cdots + n_k = n$. Then the number of distinguishable permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!n_3!\cdots n_k!}.$$

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Example: Which formula to use???

How many different strings can be made by reordering the letters of the string SUCCESS?

Solution:

- 3 S's, 2 C's, 1 E and 1 U
- C(7,3) to place to S's
- C(4,2) to place the C's
- C(2,1) to place the E
- C(1,1) to place the U
- C(7,3)C(4,2)C(2,1)C(1,1) $= \frac{7!}{3!4!} * \frac{4!}{2!2!} * \frac{2!}{1!1!} * \frac{1!}{0!0!}$

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Practice:

A basketball team consists of two centers, five forwards, and four guards. In how many ways can the coach select a starting line up of one center, two forwards, and two guards?

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Combinations

Practice: A basketball team consists of two centers, five forwards, and four guards. In how many ways can the coach select a starting line up of one center, two forwards, and two guards?

Center: Forwards: Guards:

$${}_2C_1 = \frac{2!}{1!1!} = 2 \quad {}_5C_2 = \frac{5!}{2!3!} = \frac{5*4}{2*1} = 10 \quad {}_4C_2 = \frac{4!}{2!2!} = \frac{4*3}{2*1} = 6$$

$${}_2C_1 * {}_5C_2 * {}_4C_2$$

Thus, the number of ways to select the starting line up is 2*10*6 = 120.

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6. A student council consists of 15 students.

a. In how many ways can a committee of six be selected from the membership of the council?

b. Two council members have the same major and are not permitted to serve together on a committee. How many ways can a committee of six be selected from the membership of the council?

c. Two council members always insist on serving on committees together. If they can't serve together, they won't serve at all. How many ways can a committee of six be selected from the council membership?

d. Suppose the council contains eight men and seven women.

(i) How many committees of six contain three men and three women?

(ii) How many committees of six contain at least one woman?

e. Suppose the council consists of three freshmen, four sophomores, three juniors, and five seniors. How many committees of eight contain two representatives from each class?

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(a)

(15 choose 6)

(b)

Assume that those two people are called x and y.

(13 choose 5) + (13 choose 5) + (13 choose 5)
Choose x, but not y + choose y but not x + choose none

OR

(2 choose 1) x (13 choose 5) + (13 choose 6)
Choose one of x, y but not the other one + choose none

(c)

Assume that those two people are called x and y.

(13 choose 4) + (13 choose 6)
Choose x and y + choose none

(d)(i)

(8 choose 3) x (7 choose 3)

(d)(ii)

[(7 choose 1) x (8 choose 5)] + [(7 choose 2) x (8 choose 4)] + [(7 choose 3) x (8 choose 3)] +[(7 choose 4) x (8 choose 2)] + [(7 choose 5) x (8 choose 1)] + [(7 choose 6)(8 choose 0)] = (15 choose 6) - [(7 choose 0) x (8 choose 6)]

At least one woman means one woman or two women or three women or … 6 women. This is the same as all possible ways minus the case of choosing no women at all

(e)

(3 choose 2) x (4 choose 2) x (3 choose 2) x (5 choose 2)

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7. A computer programming team has 13 members.

a. How many ways can a group of seven be chosen to work on a project?

b. Suppose seven team members are women and six are men.

(i) How many groups of seven can be chosen that contain four women and three men?

(ii) How many groups of seven can be chosen that contain at least one man?

(iii) How many groups of seven can be chosen that contain at most three women?

c. Suppose two team members refuse to work together on projects. How many groups of seven can be chosen to work on a project?

d. Suppose two team members insist on either working together or not at all on projects. How many groups of seven can be chosen to work on a project?

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20. **a.** How many distinguishable ways can the letters of the word *MILLIMICRON* be arranged in order?

b. How many distinguishable orderings of the letters of *MILLIMICRON* begin with *M* and end with *N*?

c. How many distinguishable orderings of the letters of *MILLIMICRON* contain the letters *CR* next to each other in order and also the letters *ON* next to each other in order?

17. Ten points labeled *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H*, *I*, *J* are arranged in a plane in such a way that no three lie on the same straight line.

a. How many straight lines are determined by the ten points?

b. How many of these straight lines do not pass through point *A*?

c. How many triangles have three of the ten points as vertices?

d. How many of these triangles do not have point *A* as a vertex?

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Laith's Hypothesis and approved by Rawan

*H**28.* A student council consists of three freshmen, four sophomores, four juniors, and five seniors. How many committees of eight members of the council contain at least one member from each class?

Total= $\binom{3}{1}\binom{4}{1}\binom{4}{1}\binom{5}{1}\binom{12}{4}$

$=5X4X4X3X\frac{12!}{4!8!}$=5X4X4X3X495

=**118,800**

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*H**28.* A student council consists of three freshmen, four sophomores, four juniors, and five seniors. How many committees of eight members of the council contain at least one member from each class?

Total of students are 16 students, we want to choose 8 from 16 if there is no restrictions

3: Freshmen
4:Sphomores
4:Juniors
5:Senior
Total :16
Selection Council : 8

Total= $\binom{16}{8}=\frac{16!}{8!8!}=12,870$

N(Without F) = $\binom{13}{8}=\frac{13!}{5!8!}=1287$ (16-3)

N(Without Sp) = $\binom{12}{8}=\frac{12!}{4!8!}=495$

N(Without J) = $\binom{12}{8}=\frac{12!}{4!8!}=495$

N(Without S) = $\binom{11}{8}=\frac{11!}{3!8!}=165$

N(Without F & Sp) = $\binom{9}{8}=\frac{9!}{1!8!}=9$ (16-3-4)

N(Without F & J) = $\binom{7}{8}=0$ (16-5-4)

N(Without F & S) = $\binom{8}{8}=\frac{8!}{0!8!}=1$ (16-3-5)

N(Without Sp & J) = $\binom{8}{8}=\frac{8!}{0!8!}=1$ (16-3-5)

N(Without Sp & S) = $\binom{7}{8}=0$ (16-4-5)

N(Without J & S) = $\binom{7}{8}=0$ (16-4-5)

Total=12870-1287-495-495-165+9+9+1=10447

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Some Advice about Counting

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Some Advice about Counting

Students learning counting techniques often ask, "**How do I know what to multiply and what to add**? **When do I use the multiplication rule and when do I use the addition rule?"**

Unfortunately, these questions have no easy answers.

You should construct a model that would allow you to continue counting the objects one by one if you had **enough time.**

If you can imagine the elements to be counted as being obtained through a **multistep process** then you can use the **multiplication rule**.

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Some Advice about Counting

The total number of **elements will be the product** of the number of ways to perform each step. If, however, you can imagine the set of elements to be counted as being broken up **into disjoint subsets**, then you can use the **addition rule.**

The total number of elements in the set will be the sum of the number of elements in each subset.

One of **the most common mistakes** students make is to **count certain possibilities more than once.**

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Example 11 – *Double Counting*

Consider again the problem of Example 7(**b**). A group consists of **five men** and **seven women**. How many teams of five contain at **least one man**?

Incorrect Solution
Imagine constructing the team as a two-step process:

Step 1: Choose a subset of one man from the five men.

Step 2: Choose a **subset of four others from** the remaining eleven people.

Hence, by the multiplication rule, there are $\binom{5}{1} \cdot \binom{11}{4}$ = 1,650 **five-person teams that contain at least one man**.

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Example 11 – *Double Counting* cont'd

Analysis of the Incorrect Solution
The problem with the solution is that some teams are **counted more than once**. Suppose the men are Anwr, Belall, Kareem, Dia', and Emad and the women are Fatima, Ghada, Huda, Inas, Julia, Kmila, and Lura.

According to the method described previously, one possible outcome of the two-step process is as follows:

Outcome of step 1: Anwr

Outcome of step 2: Belall, Ghada, Inas, and Julia.

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Example 11 – *Solution* cont'd

In this case the team would be {Anwr, Belall Ghada, Inas, and Julia}. But another possible outcome is

Outcome of step 1: Belall

Outcome of step 2: Anwr, Ghada, Inas, and Julia.

which also gives the team {Belall ,Anwr, Ghada, Inas, and Julia}.

Thus this one team is given by two different branches of the possibility tree, and so it is **counted twice**.

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SECTION 9.6

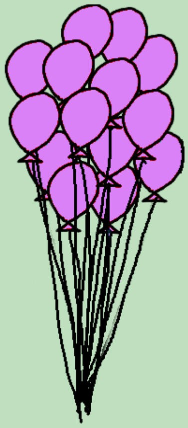
r-Combinations with Repetition Allowed

r-Combinations with Repetition Allowed

In this section we ask: How many ways are there to choose *r* elements without regard to order from a set of *n* elements if *repetition is allowed*? A good way to imagine this is to visualize the *n* elements as categories of objects from which multiple selections may be made.

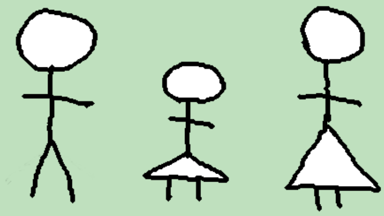
• Definition

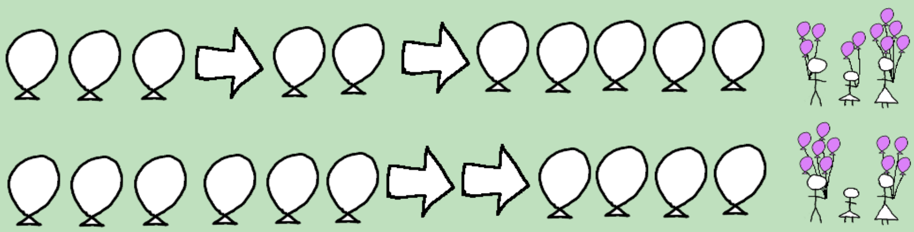
An ***r*-combination with repetition allowed**, or **multiset of size *r***, chosen from a set X of n elements is an unordered selection of elements taken from X with repetition allowed. If $X = \{x_1, x_2, \ldots, x_n\}$, we write an r-combination with repetition allowed, or multiset of size r, as $[x_{i_1}, x_{i_2}, \ldots, x_{i_r}]$ where each x_{i_j} is in X and some of the x_{i_j} may equal each other.



This is again a stars-and-bars problem.

(Imagine we want to buy 10 balloons of three children)





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Example 1 – *r-Combinations with Repetition Allowed*

Write a complete list to find the **number of 3-combinations** with **repetition allowed**, or **multisets of size 3**, that can be **selected from {1, 2, 3, 4}.** Observe that because the order in which the elements are **chosen does not matter**, the elements of each selection may be written in increasing order, and writing the elements in increasing order will ensure that no combinations are overlooked.

Solution:

[1, 1, 1] ; [1, 1, 2]; [1, 1, 3]; [1, 1, 4]   all combinations with 1, 1

[1, 2, 2] ; [1, 2, 3]; [1, 2, 4];   all additional combinations with 1, 2

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Example 1 – *Solution*

cont'd

[1, 3, 3] ; [1, 3, 4]; [1, 4, 4]; — all additional combinations with 1, 3 or 1, 4

[2, 2, 2] ; [2, 2, 3]; [2, 2, 4]; — all additional combinations with 2, 2

[2, 3, 3] ; [2, 3, 4]; [2, 4, 4]; — all additional combinations with 2, 3 or 2, 4

[3, 3, 3] ; [3, 3, 4]; [3, 4, 4]; — all additional combinations with 3, 3 or 3, 4

[4, 4, 4] — the only additional combination with 4, 4

20 way

Thus there are <u>twenty</u> 3-combinations with repetition allowed. <u>It's like string problem ?</u>

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Given ***n objects****, we want to* ***select r objects*** *with replacement?* $\binom{n+r-1}{r}$

Example:
We want to select 3 objects from 4 objects (4 choose 3)

1 2 3 4, 2 , 4

2 4 4

$=\binom{5}{3}$

1 2 3 4, 1 , 3

1 3 4

$=\binom{5}{3}$

We have 20 types of dunats, we want to take at home 12 pice of different types. how many different type can be there

1 2 3 4, ……. 19 ,20 , 2

2

$$=\binom{20+12-1}{12}=\binom{31}{12}$$

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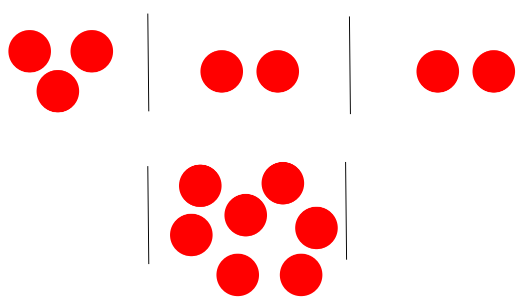
Given the following equation is
$x1 + x2 + x3 = 7$, *how many posible solution are there?*



How many divisors we need? (2 *divisors*) , *note it's* " + ")

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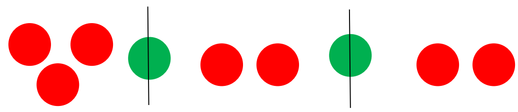
$How\ many\ divisors\ we\ need?\ (\ 2\ divisors\ ,note\ it's\ "+")$

$Given\ the\ following\ equation\ is$
$x1 + x2 + x3 = 7, how\ many\ posible\ solution\ are\ there?$



$Number\ of\ divisors = (n-1) = 3 - 1 = \ 2\ divisors$

r=7, n=3

$\binom{n+r-1}{r}$ $=\binom{7+3-1}{7}$=$\binom{9}{7}$

Or equivalent $\binom{n+r-1}{n-1} = \binom{9}{2}$

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We have 3 shelves and we have to distribute 9 books over these shelves, shelf number must have more than 1 book how many possible ways can we distritute these books?

$x1 + x2 + x3 = 9,$
$x1 \geq 0, x2 \geq 0\ , x3 > 1$

$x1 + x2 + x3 = 9,$
$x1 \geq 0, x2 \geq 0\ , x3 \geq 2$

r=7, n=3

$x1 + x2 + x3 = 9$
$\acute{x}3 = x3 - 2$
$x1 \geq 0, x2 \geq 0\ , x3 \geq 0$

$\binom{n+r-1}{r}$=$\binom{7+3-1}{7}$=$\binom{9}{7}$

Or equivalent $\binom{n+r-1}{n-1} = \binom{9}{2}$

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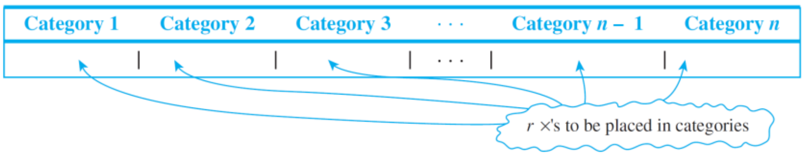
r-Combinations with Repetition Allowed

The number of strings of $n - 1$ vertical bars and r crosses is the number of ways to choose r positions, into which to place the r crosses, out of a total of $r + (n - 1)$ positions, leaving the remaining positions for the vertical bars.



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r-Combinations with Repetition Allowed

But by Theorem 9.5.1, this number is $\binom{r+n-1}{r}$.

Theorem 9.5.1

The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n, r)}{r!} \quad \text{first version}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{second version}$$

where n and r are nonnegative integers with $r \le n$.

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r-Combinations with Repetition Allowed

This discussion proves the following theorem.

Theorem 9.6.1

The number of r-combinations with repetition allowed (multisets of size r) that can be selected from a set of n elements is

$$\binom{r+n-1}{r}.$$

This equals the number of ways r objects can be selected from n categories of objects with repetition allowed.

Previous example : choose 3 from {1,2,3,4} with reptation

Total= $\binom{n+r-1}{r} = \binom{4+3-1}{3} = \binom{6}{3} = \frac{6!}{3!3!} = \frac{6X5X4X3!}{3!3!} = 20$

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Example

A) In a party we wants to buy 15 cans of drinks. He shops at a store that sells 5 different types of soft drinks.

How many different selections of cans of 15 soft drinks can he make?

Can be represented by a string of $5 - 1 = 4$ vertical bars (to separate the categories of soft drinks) and 15 crosses (to represent the cans selected). For instance,

$$\times \times \times \mid \times \times \times \times \times \times \times \mid \quad \mid \times \times \times \mid \times \times$$

$$\binom{15+5-1}{15} = \binom{19}{15} = \frac{19 \cdot \overset{6}{\cancel{18}} \cdot 17 \cdot \overset{2}{\cancel{16}} \cdot \cancel{15!}}{\cancel{15!} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 3{,}876.$$

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Example

B) A person giving a party wants to buy15 cans of drinks. He shops at a store that sells 5 different types of soft drinks.

*If Strawberry is one of the types of soft drink, how many different selections include at **least 6 cans of Strawberry**?*

Thus we need to select **9 cans** from the **5 types**.
The nine additional cans can be represented as 9 ×'s and 4 |'s.

$$\binom{9+4}{9} = \binom{13}{9} = \frac{13 \cdot \overset{5}{12} \cdot 11 \cdot 10 \cdot 9!}{9! \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 715.$$

C. If the store has only five cans of *Strawberry*, *, how many different selections ?*

Total = 3876-715=3161

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Example 3 – *Solution*

cont'd

The table below illustrates this for $n = 5$.

Category 1		2		3		4		5	Result of the Selection
	|		|	× ×	|		|	×	(3, 3, 5)
×	|	×	|		|	×	|		(1, 2, 4)

Thus the number of such triples is the same as the number of strings of $(n – 1)$ /'s and 3 ×'s, which is

$$\binom{3+(n-1)}{3} = \binom{n+2}{3} = \frac{(n+2)!}{3!(n+2-3)!}$$

$$= \frac{(n+2)(n+1)n(n-1)!}{3!(n-1)!} = \frac{n(n+1)(n+2)}{6}.$$

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The Number of Integral Solutions of an Equation

How many solutions **are there to the equation** $x1 + x2 + x3 + x4 = 10$ if $x1$, $x2$, $x3$, and $x4$ are **nonnegative integers**?

Categories x_1	x_2	x_3	x_4	Solution to the equation $x_1 + x_2 + x_3 + x_4 = 10$
× × |	× × × × × |	|	× × ×	$x_1 = 2,\quad x_2 = 5,\quad x_3 = 0,\quad \text{and} \quad x_4 = 3$
× × × × |	× × × × × × |	|		$x_1 = 4,\quad x_2 = 6,\quad x_3 = 0,\quad \text{and} \quad x_4 = 0$

$$\text{Total}=\binom{n+r-1}{r}=\binom{10+4-1}{10}=\binom{13}{10}=\frac{13!}{3!10!}=286$$

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Additional Constraints on the Number of Solutions

How many integer solutions are there to the equation $x1 + x2 + x3 + x4 =$ **10 if each $x_i \geq 1$?**

one is distributed for four categories, then distribute the remaining six cross others categories

$$\text{Total}=\binom{n+r-1}{r}=\binom{6+4-1}{6}=\binom{9}{6}=\frac{9!}{6!9!}=84$$

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Example 3 – *Counting Triples (i, j, k) with $1 \le i \le j \le k \le n$*

If n is a positive integer, how many triples of integers from 1 through n can be formed in which the elements of the triple are written in increasing order but are not necessarily distinct? In other words, how many triples of integers (i, j, k) are there with $1 \le i \le j \le k \le n$?

Solution:

Any triple of integers (i, j, k) with $1 \le i \le j \le k \le n$ can be represented as a **string of $n - 1$ vertical bars** and three crosses (**"r" in equation**), with the positions of the crosses indicating which three integers from 1 to n are included in the triple.

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Example 4 – *Counting Iterations of a Loop*

How many times will the innermost loop be iterated when the algorithm segment below is implemented and run? (Assume n is a positive integer.)

```
for k := 1 to n
    for j := 1 to k
        for i := 1 to j
            [Statements in the body of the inner loop,
            none containing branching statements that lead
            outside the loop]
        next i
    next j
next k
```

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Example 4 – *Solution*

Construct a trace table for the values of *k*, *j*, and *i* for which the statements in the body of the innermost loop are executed. (See the table that follows.) Because ***i*** goes from ***1 to j***, it is always the case that ***i* ≤ *j*.**

k	1	2 →	→	→	3 →	→	→	→	→	→	⋯	*n* →	→	→	→	→	→	→
j	1	1	2 →	→	1	2 →	→	3 →	→	→	⋯	1	2 →	→	⋯	*n* →	→	→
i	1	1	1	2	1	1	2	1	2	3	⋯	1	1	2	⋯	1	⋯	*n*

Similarly, because ***j*** goes from ***1 to k***, it is always the case that ***j* ≤ *k*.**

To focus on the details of the table construction, consider what happens when ***k* = *3*.** In this case, ***j*** takes each value 1, 2, and 3.

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Example 4 – *Solution*

cont'd

When *j* = 1, *i* can only take the value 1 (because *i* ≤ *j*). When *j* = 2, *i* takes each value 1 and 2 (again because *i* ≤ *j*). When *j* = 3, *i* takes each value 1, 2, and 3 (yet again because *i* ≤ *j*).

Observe that there is one iteration of the innermost loop for each column of this table, and there is one column of the table for each triple of integers **(*i*, *j*, *k*) with 1 ≤ *i* ≤ *j* ≤ *k* ≤ *n*.**

<u>But Example 3</u> showed that the number of such triples is **[*n*(*n* + 1)(*n* + 2)]/6**. Thus there are **[*n*(*n* + 1)(*n* + 2)]/6** iterations of the innermost loop.

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Which Formula to Use?

Earlier we have discussed four different ways of choosing *k* elements from *n*. The order in which the choices are made may or may not matter, and repetition may or may not be allowed. The following table summarizes which formula to use in which situation.

	Order Matters	**Order Does Not Matter**
Repetition Is Allowed	n^k	$\binom{k+n-1}{k}$
Repetition Is Not Allowed	$P(n,k)$	$\binom{n}{k}$

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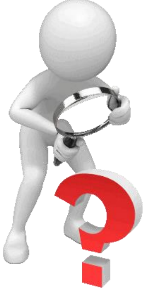
Exercises

In 8 and 9, how many times will the innermost loop be iterated when the algorithm segment is implemented and run? Assume *n*, *m*, *k*, and *j* are positive integers.

8. **for** $m := 1$ **to** n
  for $k := 1$ **to** m
    for $j := 1$ **to** k
      for $i := 1$ **to** j
        [Statements in the body of the inner loop, none containing branching statements that lead outside the loop]
      next i
    next j
  next k
next m



Total= $\binom{n+4-1}{4} = \binom{n+3}{4} = \frac{(n+3)!}{4!(n-1)!}$

$= \frac{n(n+3)(n+2)(n+1)(n-1)!}{4!(n-1)!} = \frac{n(n+3)(n+2)(n+1)}{24}$

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9. **for** $k := 1$ **to** n
  for $j := k$ **to** n
    for $i := j$ **to** n
      [Statements in the body of the inner loop, none containing branching statements that lead outside the loop]
    next i
  next j
next k

how many inner statements executed in the kth loop؟

we might observe that there are exactly as many triples of integers **(i, j, k)** with $1 \le k \le j \le i \le n$, with **repetition allowed**.

$$\text{Total}= \binom{n+3-1}{3} = \binom{n+2}{3} = \frac{(n+2)!}{3!(n+1)!} = \frac{n(n+2)(n+1)(n-1)!}{3!(n-1)!} = \frac{n(n+2)(n+1)}{6}$$

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```
#include <stdio.h>
int main()
{ int a, b, d,i, j, k, l, m,n; long int c;

n = 5, a = 0, b=0, c = 1, d = 0;
  for  (i= 1;i<=n; i++)
    for (j= 1;j<=i; j++)
    {
          { for (k= 1;k<=j; k++)
              for (l= 1;l<=k; l++)
                for (m= 1;m<=l; m++)
                  {
                   a++;
                   b+=a;
                  }
            }

           d++;
          c*=(10*d);
    }
 printf("a=%d,b=%d,c=%d, d=%d\n",a,b,c,d);
      return 0;
}
```

a=126,b=8001,c=1013,076,743,680,000, d=15

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how many inner statements executed in the kth loop, we might observe that there are exactly as many triples of integers (i, j, k, l, m) with $1 \le i \le j \le k \le l \le m \le 5$ as there are 5 − combinations of integers from 1 through 5 with ***repetition allowed***.

Hence, a= $\binom{5+5-1}{5}\binom{9}{5}$ = *126*

b=$\sum_{i=1}^{126} i = \frac{i(i+1)}{2} = \frac{126(127)}{2}$=8001

Output of d,

there are exactly as many double's of integers (i, j) with 1 ≤ i ≤ j ≤ 5 as there are 2-combinations of integers from 1 through 5 with **repetition allowed.**

Hence, d= $\binom{5+2-1}{2}\binom{6}{2} = \frac{6!}{2!4!} = \frac{6*5*4!}{2*4!}$=15.

Output of C: 10× 15! = $10\prod_{1}^{15} d$

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What is the output? (5 Marks Bonus)

```
#include <stdio.h>
int func(int n);
int main()
{ int a, i, n;

     n = 99999, a = 0;
  for  (i= 1;i<=n; i++)
          if( func(i))
             a++;

 printf("a=%d\n",a);
     return 0;
}


}
```

```
int func(int x){
 int i=0,sum=0;
 while(x!=0)
     {
      sum+=x%10;
       x/=10;
      }
 if(sum ==10)
    return 1;

return 0;}
```

Output : 996

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Exercises

*15. For how many integers from 1 through 99,999 is the sum of their digits equal to 10?

We need to put 10 objects into 5 groups (digits) but there are 5 ways to put All 10 objects into the same position (0+0+0+0+10) not allowed, so subtract these 5 undesirable outcomes

Total= $\binom{5+10-1}{10}$-5= $\binom{14}{10} - 5 = \frac{(14)!}{4!(10)!} - 5 = 1001 - 5 = 996$

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H 17. a. A store sells 8 colors of balloons with at least 30 of each color. How many different combinations of 30 balloons can be chosen?

b. If the store has only 12 red balloons but at least 30 of each other color of balloon, how many combinations of balloons can be chosen?

c. If the store has only 8 blue balloons but at least 30 of each other color of balloon, how many combinations of balloons can be chosen?

d. If the store has only 12 red balloons and only 8 blue balloons but at least 30 of each other color of balloon, how many combinations of balloons can be chosen?

Total= $\binom{r+n-1}{r}$= $\binom{37}{30} = \frac{37!}{7!30!} = 10{,}295{,}472$

Let us first select 13 red balloons. We need to select the remaining r=30-13=17 balloons from n=8 kinds of balloons

Total (**with picking at least 13 red**) = $\binom{17+8-1}{17}$= $\binom{24}{17} = \frac{24!}{7!17!} = 346{,}104$

Total (**at most 12 red balloon**)=10,295,472$-346{,}104 = 9{,}949{,}368$

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Exercises (c)

Let select 9 blue balloons. We need to select the remaining r=30-9=21 balloons from n=8 kinds of balloons

Total (with at least 9 blue) = $\binom{21+8-1}{21} = \binom{28}{21} = 1{,}184{,}040$

Total (with at most 8 blue) = $10{,}295{,}472 - 1{,}184{,}040 = 9{,}111{,}432$

D) Let select 9 blue balloons. And 13 red We need to select the remaining r=30-22=8 balloons from n=8 kinds of balloons

Total (with at least 9B and 13R) = $\binom{8+8-1}{8} = \binom{15}{8} = 6435$

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$N(A1 \cup A2) = N(A1) + N(A2) - N(A1 \cap A2)$

Total (with at most 8B and 12R) =346,104 + 1,184,040 -6435 =1,523,709

d) Answer : = **$10,295,472$ -1,523,709 =<u>8,771,763</u>**

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With my best wishes for brilliant success

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