Mustafa Jarrar: Lecture Notes in Discrete Mathematics.
Birzeit University, Palestine, 2015

Graphs and Trees

10.1 Introduction to Graphs

10.2 Introduction to Trees

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 10 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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Counting

10.1 Graphs

In this lecture:

- ❑ Part 1: **Concept and Terminology**
- ❑ Part 2: Directed Graphs
- ❑ Part 3: Examples of Graphs
- ❑ Part 4: Graphs Types (Simple, Complete, Bipartite, Sub graphs)
- ❑ Part 5: Node/Graph Degree
- ❑ Part 6: Handshake and other Graph Theorems

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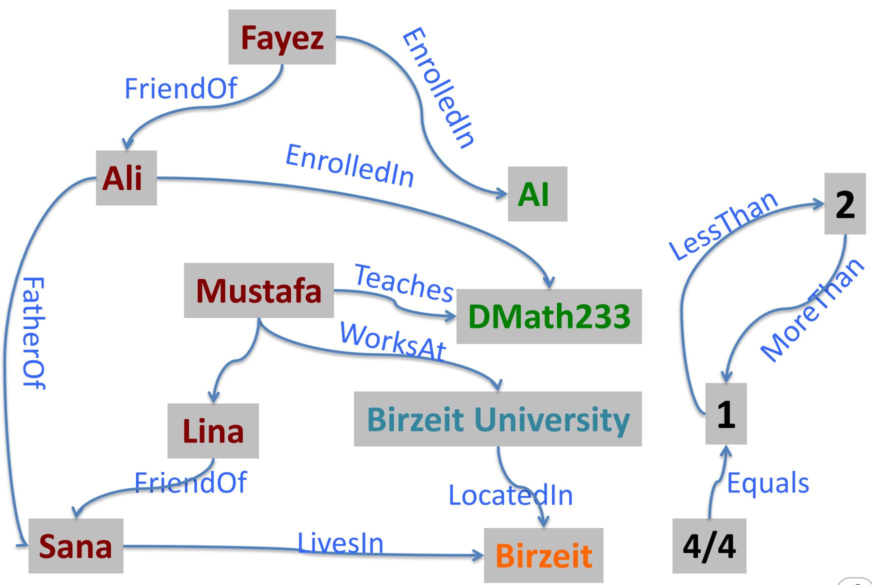
Example from Previous Chapter



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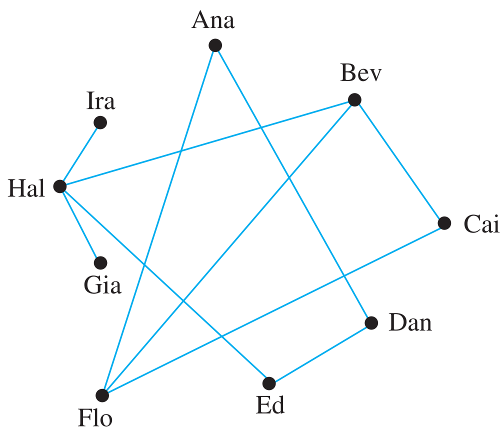
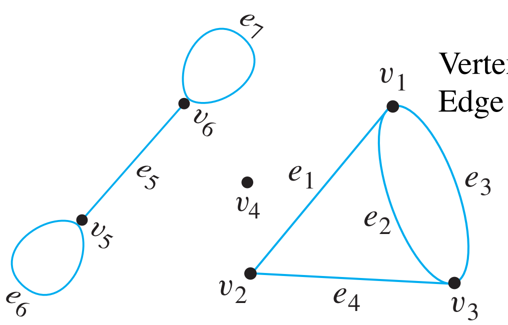
Example

Graph to show people who know each other



Basics

A graph G (شبكة) consists of two finite sets: a nonempty set V(G) of vertices (عقد) and a set E(G) of edges (علاقات), where each edge is associated with a set consisting of either one or two vertices called its endpoints.



Vertex Set=$\{v_1, v_2, v_3, v_4, v_5, v_6\}$
Edge Set = $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

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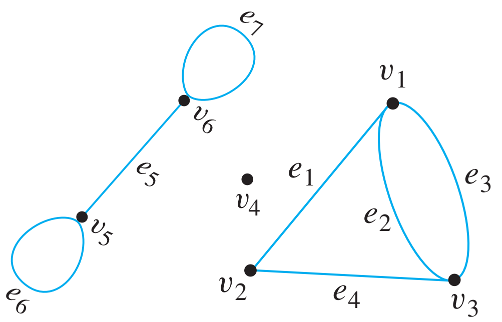

edge-endpoint function:

Edge	Endpoints
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_1, v_3\}$
e_4	$\{v_2, v_3\}$
e_5	$\{v_5, v_6\}$
e_6	$\{v_5\}$
e_7	$\{v_6\}$

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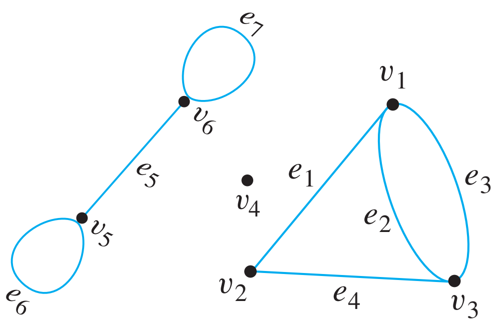

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e_6	$\{v_5\}$
e_7	$\{v_6\}$

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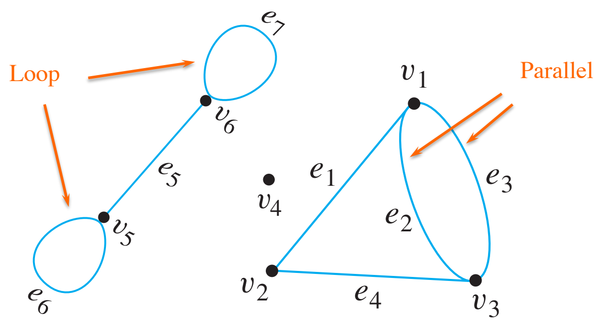
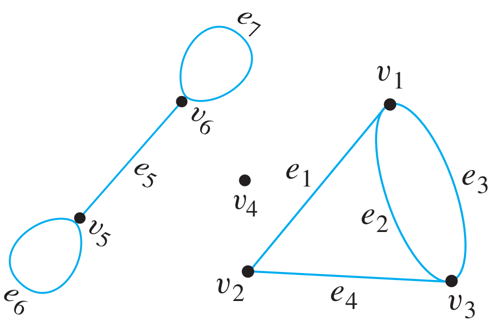
Basics

An edge with just one endpoint is called a **loop**, and two or more distinct edges with the same set of endpoints are said to be **parallel**.



Basics

Two vertices that are connected by an edge are called adjacent (متجاورة); and a vertex that is an endpoint of a loop is said to be adjacent to itself. An edge is said to be incident (ساقطة) on each of its endpoints, and two edges incident on the same endpoint are called adjacent. A vertex on which no edges are incident is called isolated.



e_1, e_2, and e_3 are incident on v_1.

v_2 and v_3 are adjacent to v_1.

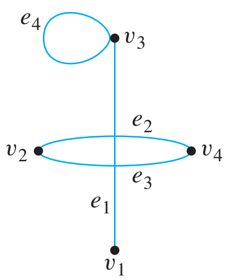
e_2, e_3, and e_4 are adjacent to e_1

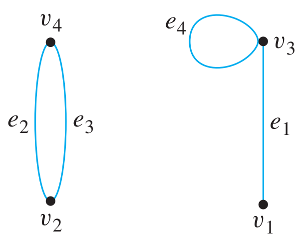
v_5 and v_6 are adjacent to themselves.

v_4 is an isolated vertex.

Same Graphs

Two Drawings represent the same graph if the have same edge-endpoint function





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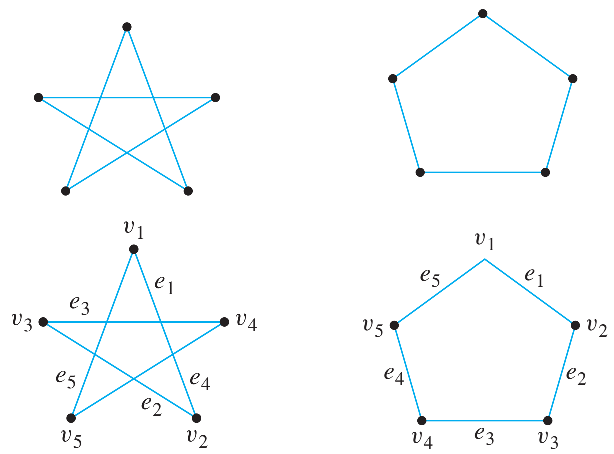
Same Graphs

Label these graphs to make them the same graph



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Counting

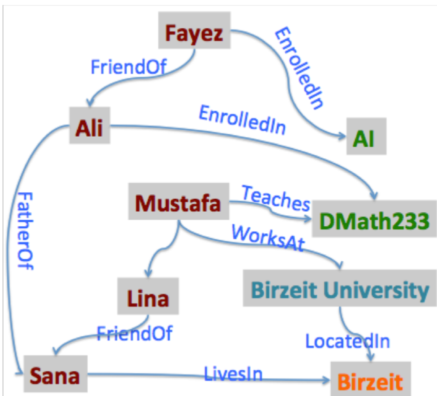
10.1 Graphs

In this lecture:

- ❑ Part 1: Concept and Terminology
- ❑ Part 2: **Directed Graphs**
- ❑ Part 3: Examples of Graphs
- ❑ Part 4: Graphs Types (Simple, Complete, Bipartite, Sub graphs)
- ❑ Part 5: Node/Graph Degree
- ❑ Part 6: Handshake and other Graph Theorems

Directed Graph

A **directed graph, or digraph,** consists of two finite sets: a nonempty set *V* (*G*) of vertices and a set *D*(*G*) of directed edges, where each is associated with an ordered pair of vertices called its **endpoints.** If edge *e* is associated with the pair (v, w) of vertices, then *e* is said to be the (**directed**) **edge** from v to w.



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Road Graphs

Nodes are road conjunctions, edges are roads

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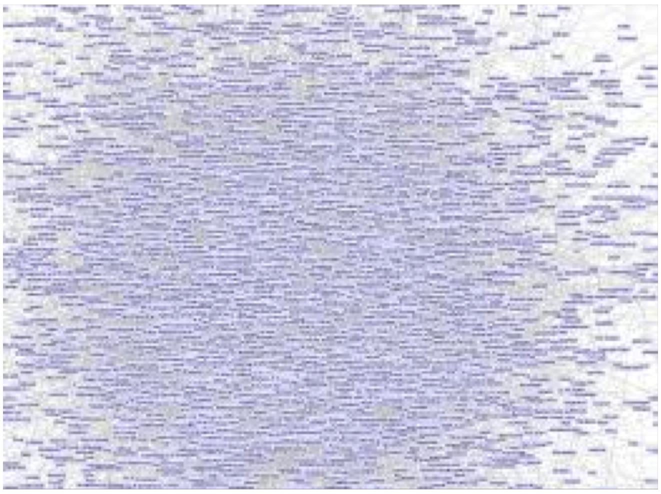
Internet Graph



Nodes pages/bookmarks, edges are URLs

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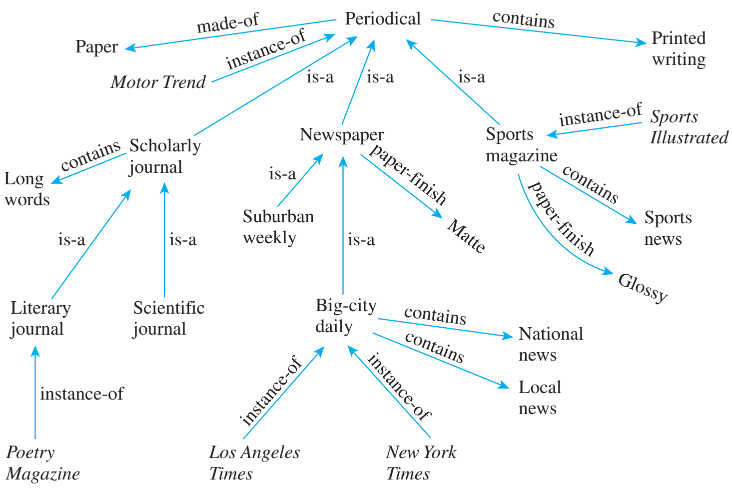
Knowledge Representation Graphs



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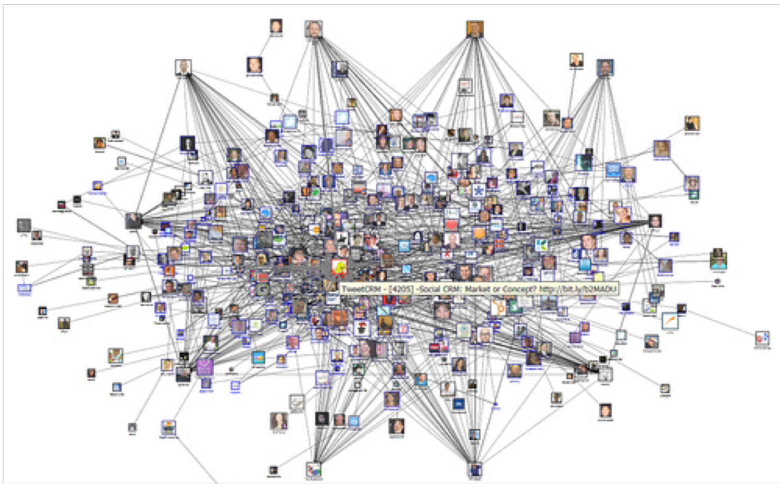
Social Graphs



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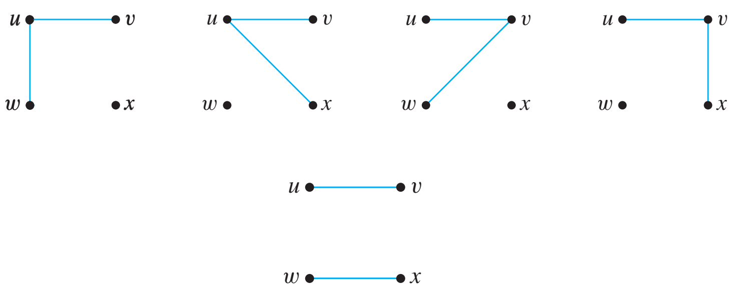
Simple Graphs

• Definition and Notation

A **simple graph** is a graph that does not have any loops or parallel edges. In a simple graph, an edge with endpoints v and w is denoted $\{v, w\}$.

Example:
Draw all simple graphs with the four vertices $\{u, v, w, x\}$ and two edges, one of which is $\{u, v\}$.

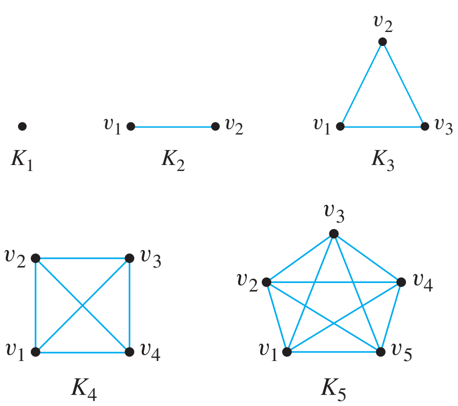


Complete Graphs

• Definition

Let n be a positive integer. A **complete graph on n vertices,** denoted K_n, is a simple graph with n vertices and exactly one edge connecting each pair of distinct vertices.

All pairs of vertices are connected by edges.



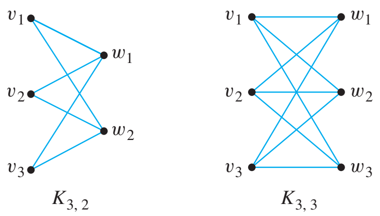
Bipartite Graphs (Bigraphs)

Vertex set can be separated into two subsets: Each vertex in one of the subsets is connected by exactly one edge to each vertex in the other subset, but not to vertices in its own subset.

• Definition

Let m and n be positive integers. A **complete bipartite graph on** (m, n) **vertices,** denoted $K_{m,n}$, is a simple graph with distinct vertices $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_n$ that satisfies the following properties: For all $i, k = 1, 2, \ldots, m$ and for all $j, l = 1, 2, \ldots, n$,

1. There is an edge from each vertex v_i to each vertex w_j.
2. There is no edge from any vertex v_i to any other vertex v_k.
3. There is no edge from any vertex w_j to any other vertex w_l.



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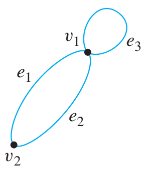
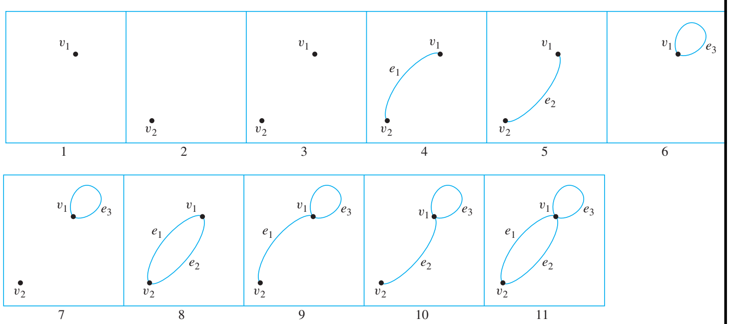
Subgraphs

• Definition

A graph H is said to be a **subgraph** of a graph G if, and only if, every vertex in H is also a vertex in G, every edge in H is also an edge in G, and every edge in H has the same endpoints as it has in G.



Subgraphs



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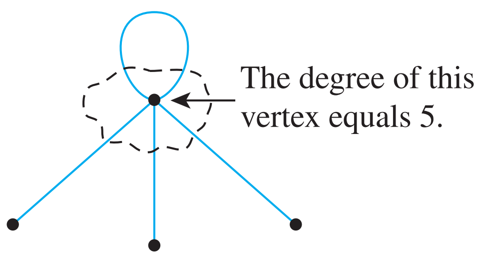
The concept of Degree

• Definition

Let G be a graph and v a vertex of G. The **degree of** v, denoted **deg(**v**)**, equals the number of edges that are incident on v, with an edge that is a loop counted twice. The **total degree of** G is the sum of the degrees of all the vertices of G.



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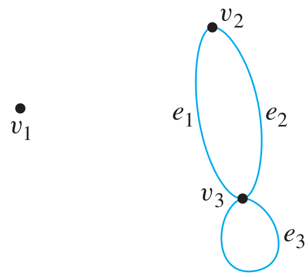
The concept of Degree



$\deg(v_1) = 0$ since no edge is incident on v_1 (v_1 is isolated).

$\deg(v_2) = 2$ since both e_1 and e_2 are incident on v_2.

$\deg(v_3) = 4$ since $e1$ and $e2$ are incident on v_3 and the loop e_3 is also incident on v_3 (and contributes 2 to the degree of v_3).

Total degree of $G = \deg(v_1) + \deg(v_2) + \deg(v_3) = 0 + 2 + 4 = 6$

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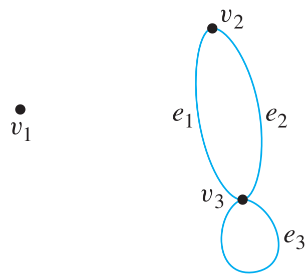
The concept of Degree



Can we calculate the graph degree directly?!

Yes, it is (2 . number of edges)

Total degree of $G = \deg(v_1) + \deg(v_2) + \deg(v_3) = 0 + 2 + 4 = 6$

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The Handshake Theorem

Theorem 10.1.1 The Handshake Theorem

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G. Specifically, if the vertices of G are $v_1, v_2, \ldots, v_n$, where n is a nonnegative integer, then

$$\begin{aligned}\text{the total degree of } G &= \deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) \\ &= 2 \cdot (\text{the number of edges of } G).\end{aligned}$$

Handshaking at a party: if the numbers experienced by each person are added together, the sum will equal twice the total number of handshakes.

Corollary 10.1.2

The total degree of a graph is even.

Proof:

By Theorem 10.1.1 the total degree of G equals 2 times the number of edges, which is an integer, and so the total degree of G is even.

Exercise

Draw a graph with the specified properties or show that no such graph exists.

➔ A graph with four vertices of degrees 1, 1, 2, and 3

No such graph is possible.
By Corollary 10.1.2, the total degree of a graph is even.
But a graph with four vertices of degrees 1, 1, 2, and 3 would have a total degree of 1 + 1 + 2 + 3 = 7, which is odd.

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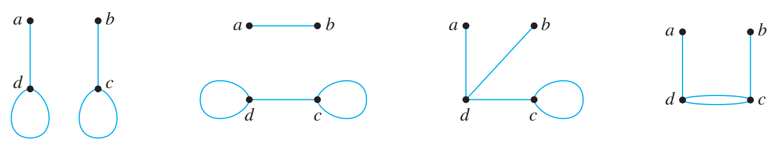
Exercise

Draw a graph with the specified properties or show that no such graph exists.

➔ A graph with four vertices of degrees 1, 1, 3, and 3

Any of these:



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Exercise

Draw a graph with the specified properties or show that no such graph exists.

➔ A simple graph with four vertices of degrees 1, 1, 3, and 3

There is no simple graph with four vertices of degrees 1, 1, 3, and 3.

Exercise (Acquaintance graphs)

Is it possible in a group of nine people for each to be friends with exactly five others?

No

Otherwise,
the degree of the graph would 5 for each node, thus = 5 . 9 = 45
And 45 is odd, so it is not possible.

Number of Vertices with Odd Degree

Proposition 10.1.3

In any graph there are an even number of vertices of odd degree.

Exercise:
Is there a graph with ten vertices of degrees 1, 1, 2, 2, 2, 3, 4, 4, 4, and 6?

➔**No**

(1+1+2+2+2+3+4+4+4+6 = 29) which is odd