Chapter 4 review

1) Prove using the definition of even: For all integers n, if n is even then $(-1)^n = 1$.

Let $n \in \mathbb{Z}$ be even.

$\therefore \exists p \in \mathbb{Z}$ such that $n = 2p$ by the definition of even.

$\therefore (-1)^n = (-1)^{2p} = ((-1)^2)^p = 1^p = 1$

$\therefore (-1)^n = 1$

2) Prove using the definition of odd: The product of any two odd integers is odd.

Let $n, m \in \mathbb{Z}$ be odd

$\therefore \exists p, q \in \mathbb{Z}$ such that $n = 2p+1$ and $m = 2q+1$ by the definition of odd.

$$\begin{aligned} n \cdot m &= (2p+1)(2q+1) \\ &= 4pq + 2p + 2q + 1 \\ &= 2(2pq + p + q) + 1 \end{aligned}$$

$\therefore n \cdot m$ is odd.

3) Prove: For each integer n with $1 \leq n \leq 5$, $n^2 - n + 11$ is prime.

For $n=1$, $1^2 - 1 + 11 = 11$ is prime

For $n=2$, $2^2 - 2 + 11 = 13$ is prime

For $n=3$, $3^2 - 3 + 11 = 17$ is prime

For $n=4$, $4^2 - 4 + 11 = 23$ is prime

For $n=5$, $5^2 - 5 + 11 = 31$ is prime

$\therefore$ $\forall n \in \mathbb{Z}$ with $1 \leq n \leq 5$, $n^2 - n + 11$ is prime by exhaustion.

4) Show that: .123123123... is a rational number.

Let $x = .123123123\ldots$

$\therefore$ $1000x = 123.123123\ldots$

$\therefore$ $1000x - x = 123$

$\therefore$ $999x = 123$

$\therefore$ $x = \frac{123}{999}$

$\therefore$ $.123123\ldots = \frac{123}{999}$ is a rational number.

5) Prove using the definition of divides: For all integers a, b, and c, if a divides b and b divides c then a divides c.

Let $a, b, c \in \mathbb{Z}$. Assume $a|b$ and $b|c$

$\therefore$ $\exists\, p, q \in \mathbb{Z}$ such that $b = a \cdot p$ and $c = b \cdot q$ by the definition of divides

$\therefore$ $c = (a \cdot p) \cdot q$

$\therefore$ $c = a \cdot (p \cdot q)$

$\therefore$ $a|c$

6) Evaluate 50 **div** 4 and 50 **mod** 4.

$$50 = 12 \cdot 4 + 2$$

$\therefore$ 50 div 4 = 12 and 50 mod 4 = 2

7) Prove using the definition of **mod**: For every integer p, if p**mod**$5 = 2$ then $4p$**mod**$5 = 3$.

Let $p \in \mathbb{Z}$ such that $p \bmod 5 = 2$

$\therefore \exists n \in \mathbb{Z}$ such that $p = 5n + 2$ by the definition of mod

$\therefore 4p = 4 \cdot 5n + 2 \cdot 4$

$\therefore 4p = 4 \cdot 5n + 5 + 3$

$\therefore 4p = 5(4n+1) + 3$

$\therefore (4p) \bmod 5 = 3$

8 Prove: For all integers n, $n^2 - n$ is even. Let n be an integer

Case 1: Assume that n is even

$\therefore n^2$ is even because the product of two even integers is even.

$\therefore n^2 - n$ is even because the difference of two even integers is even.

Case 2: Assume that n is odd

$\therefore n^2$ is odd because the product of two odd integers is odd.

$\therefore n^2 - n$ is even because the difference of two odd integers is even.

$\therefore \forall$ integers n, $n^2 - n$ is even.

9 a) How do you prove a statement by contradiction?

Assume that the statement is false. Then use a logical argument to reach a contradiction. Conclude that the statement is true.

9 b) Prove: There is no greatest negative rational number.

Assume that there is a greatest negative rational number.

$\therefore \exists p \in \mathbb{Q}$ such that $p < 0$ and $\forall q \in \mathbb{Q}$ $q \leq p$.

$$p < 0 \Rightarrow p < p/2 < 0$$

$p/2 \in \mathbb{Q}$ because the quotient of two nonzero rational numbers is rational.

$\therefore$ p is not the greatest negative rational number. by proof by contradiction.

9 a) How do you prove "$\forall x \in D$, if $P(x)$ then $Q(x)$" by contraposition?

Let x be an arbitrary, but fixed element of D.

Assume that $Q(x)$ is false. Prove that $P(x)$ is False.

Conclude that $\forall x \in D$, if $P(x)$ then $Q(x)$.

9 b) Prove: For all integers a, b, and c, if $a \nmid bc$ then $a \nmid b$.

Let $a, b, c \in \mathbb{Z}$ be arbitrary, but fixed.

Assume that $a \mid b$.

$\therefore$ $\exists p \in \mathbb{Z}$ such that $b = a \cdot p$

$\therefore$ $bc = a \cdot (c \cdot p)$

$\therefore$ $a \mid bc$

$\therefore$ $\forall a, b, c \in \mathbb{Z}$ if $a \nmid bc$ then $a \nmid b$. By proof by contraposition